ON SOME EXTREME SIMPLEXES

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Let \( A \) be a fixed point in \( n \)-dimensional Euclidean space. Let \( B_1, B_2, \ldots, B_{n+1} \) be the vertices of a simplex \( S_n \) of \( n \) dimensions, that is, the \( n + 1 \) vertices do not lie on a \( (n - 1) \)-dimensional subspace. Let \( d_i \), assumed to be positive, be the distance of \( B_i \) from \( A \), and let \( \ell_{ij} \) be the cosine of the angle between the straight lines \( AB_i \) and \( AB_j \) for \( i, j = 1, 2, \ldots, n + 1 \). Let \( \pi_i \) denote the \( (n - 1) \)-dimensional hyperplane passing through all the vertices of \( S_n \) except \( B_i \), let \( p_i \), assumed positive, be the perpendicular distance of \( \pi_i \) from \( A \), and let \( m_{ij} \) denote the cosine of the angle between the normals from \( A \) to \( \pi_i \) and \( \pi_j \) for \( i, j = 1, 2, \ldots, n + 1 \). The present paper deals with the following problems.

(a) An expression for the content of \( S_n \), \( C(S_n) \) say, in terms of \( d_i \) and \( \ell_{ij} \) for \( i, j = 1, 2, \ldots, n + 1 \) is first obtained. Then leaving \( d_1, d_2, \ldots, d_{n+1} \) fixed, values of \( \ell_{ij} \), say \( \ell_{ij}^* \), are determined in such a manner that \( C(S_n) \) is a maximum, and the maximum value of \( C(S_n) \) is obtained for the two cases that arise: (i) when \( A \) is inside \( S_n \), (ii) when \( A \) is outside \( S_n \). The latter case does not arise when \( d_1 = d_2 = \cdots = d_{n+1} \).

(b) An expression for \( C(S_n) \) is obtained in terms of \( p_i \) and \( m_{ij} \), \( i, j = 1, 2, \ldots, n + 1 \). Then leaving \( p_1, p_2, \ldots, p_{n+1} \) fixed, values for \( m_{ij} \), say \( m_{ij}^* \), are determined in such a manner that \( C(S_n) \) is a minimum, and such \( C(S_n) \) is computed for the two cases that arise depending on (i) whether \( A \) is inside \( S_n \) or (ii) \( A \) is outside \( S_n \). The latter case does not arise when

\[
p_1 = p_2 = \cdots = p_{n+1}.
\]

The results are stated below.

(a) The content of \( S_n \), \( \max C(S_n) \) and \( \ell_{ij}^* \) are given by

\[
(1.1) \quad n!C(S_n) = |(\ell_{ij}d_id_j + 1)|^{1/2}
\]

\[
(1.2) \quad \max (n!C(S_n))^2 = -u^{-1} \prod_{i=1}^{n+1} (d_i^2 - u)
\]

\[
(1.3) \quad \ell_{ij}^* = u/(d_id_j) \quad \text{for} \quad i, j = 1, 2, \ldots, n + 1; \quad i \neq j,
\]

where \( u \) satisfies the equation

\[
(1.4) \quad 1 + u \sum_{i=1}^{n+1} (d_i^2 - u)^{-1} = 0.
\]

The unique negative root for \( u \) in (1.4) corresponds to the case when \( A \) is inside \( S_n \). When the relation

\[
d_1 = d_2 = \cdots = d_{n+1}
\]

is not satisfied, the smallest positive root for \( u \) in (1.4) corresponds to the case when \( A \) is outside \( S_n \). Other roots for \( u \) in (1.4), if any, are inadmissible.
(b) The content $C(S_n)$, $\min (C(S_n))$ and $m_{ij}^*$ are given by

\begin{equation}
(n!C(S_n))^2 = |(p_ip_j + m_{ij})| \prod_{i=1}^{n+1} |M_{ii}|
\end{equation}

where $|M_{ii}|$ is the cofactor of $m_{ii}$ in $|(m_{ij})|$ and

\begin{equation}
\min (n!C(S_n))^2 = -v^{-1}n^{2n} \prod_{i=1}^{n+1} (p_i^* - v)
\end{equation}

and

\begin{equation}
m_{ij}^* = v(p_ip_j) \text{ for } i \neq j; i, j = 1, 2, \ldots, n + 1;
\end{equation}

where $v$ satisfies the equation

\begin{equation}
1 + v \sum_{i=1}^{n+1} (p_i^* - v)^{-1} = 0.
\end{equation}

The unique negative root for $v$ in (1.8) corresponds to the case when $A$ is inside $S_n$. When the relation

\[ p_1 = p_2 = \cdots = p_{n+1} \]

is not satisfied, the smallest positive root for $v$ in (1.8) corresponds to the case when $A$ is outside $S_n$. All other roots, if any, are inadmissible.

When $d_1 = d_2 = \cdots = d_{n+1}$, we obtain the special result that the largest simplex inscribed in a sphere of $n$-dimensions is a regular one, while when $p_1 = p_2 = \cdots = p_{n+1}$ the smallest simplex circumscribing a sphere is a regular one.

The coordinates of $B_i$ referred to a $n$-dimensional Cartesian coordinate system with origin at $A$ will be denoted by $(x_{i,1}, x_{i,2}, \ldots, x_{i,n})$. $(x_1, x_2, \ldots, x_n)$ will denote a general point in the $n$-space.

2. Extremal simplex determined by the distance of vertices. The content of $S_n$ is given by (Sommerville, p. 124) $n!C(S_n) = |V|$ where

\begin{equation}
V = \begin{bmatrix}
x_{1,1} & \cdots & x_{1,n} & 1 \\
x_{2,1} & \cdots & x_{2,n} & 1 \\
\vdots & & \vdots & \vdots \\
x_{n+1,1} & \cdots & x_{n+1,n} & 1
\end{bmatrix}
\end{equation}

so that $(n!C(S_n))^2 = |VV'| = |(w_{ij})|$ say, where

\begin{equation}
w_{ij} = 1 + s_{ij} \text{ for } i, j = 1, 2, \ldots, n + 1; \text{ and}
\end{equation}

\begin{equation}
(s_{ij}) = \begin{bmatrix}
x_{1,1} & \cdots & x_{1,n} & x_{1,1} & \cdots & x_{1,n} \\
x_{2,1} & \cdots & x_{2,n} & x_{2,1} & \cdots & x_{2,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
x_{n+1,1} & \cdots & x_{n+1,n} & x_{n+1,1} & \cdots & v_{n+1,n}
\end{bmatrix},
\end{equation}

\begin{equation}
= (l_{ij}d_i d_j).
\end{equation}
Hence we have proved (1.1).

We note that \( s_{ii} = d_i^2 \) for \( i = 1, 2, \ldots, n + 1 \). From (2.3) we also note that the rank of \((s_{ij})\) is less than \( n + 1 \) so that \( |(s_{ij})| = 0 \) and \((s_{ij})\) is semi-positive definite. Further we note that both \((s_{ij})\) and \((w_{ij})\) are symmetric matrices and since \( B_i, \ldots, B_{n+1} \) do not lie on a \((n - 1)\)-dimensional subspace, we must have \(|(w_{ij})| \neq 0\). In fact, \(|(w_{ij})| > 0\) since \((w_{ij})\) is positive definite. Our problem of maximizing \( C(S_n) \) with respect to the \( l_{ij} \), \( i \neq j \), for given values of \( d_i, d_j > 0 \), may be re-stated as follows.

We must maximize \(|(w_{ij})|\) over the class of symmetric matrices \((s_{ij})\) or \((w_{ij})\) with respect to \( s_{ij}, i, j = 1, \ldots, n + 1 \), subject to the conditions: \(|(s_{ij})| = 0 \) and \( s_{ii} = d_i^2 \) for \( i = 1, \ldots, n + 1 \). Further \((s_{ij})\) should be semipositive definite and \(|w_{ij}| \neq 0\).

Let \( \theta \) and \( \mu_1, \ldots, \mu_{n+1} \) be Lagrange multipliers. We seek the extreme values of the function \( L \) with respect to \( s_{ij}, i, j = 1, \ldots, n + 1 \), where

\[
L = |w_{ij}| - \theta |s_{ij}| + \sum_{i=1}^{n+1} \mu_i (s_{ii} - d_i^2).
\]

Hence \( s_{ij} \) must satisfy

\[
\frac{1}{2} \frac{\partial L}{\partial s_{ij}} = |W_{ij}| - \theta |S_{ij}| = 0 \quad \text{for} \quad i \neq j, i, j, = 1, \ldots, n + 1
\]

and

\[
\frac{\partial L}{\partial s_{ii}} = |W_{ii}| - \theta |S_{ii}| + \mu_i = 0 \quad \text{for} \quad i = 1, \ldots, n + 1;
\]

where \(|W_{kl}|\) and \(|S_{kl}|\) denote co-factors of \( w_{kl} \) and \( s_{kl} \) in \(|(w_{ij})|\) and \(|(s_{ij})|\) respectively.

This implies that

\[
\sum_{j=1}^{n+1} w_{kj} \frac{1}{2} \frac{\partial L}{\partial s_{ij}} + w_{ki} \frac{\partial L}{\partial s_{ii}} = 0
\]

so that

\[
\sum_{j=1}^{n+1} w_{kj} |W_{ij}| - \theta \sum_{j=1}^{n+1} w_{kj} |S_{ij}| + \mu_i w_{ki} = 0.
\]

Let \( k \neq i \); then using (2.2), \( w_{kj} = 1 + s_{kj} \) and by the well-known property that expansions in terms of alien co-factors vanish identically (Aitken, p. 51) we finally obtain

\[
-\theta \sum_{j=1}^{n+1} |S_{ij}| + \mu_i w_{ki} = 0
\]

so that \( s_{ki} = w_{ki} - 1 = \theta/\mu_i \sum_{j=1}^{n+1} |S_{ij}| - 1 \), for all \( k \neq i \). Since the above expression for \( s_{ki} \) is constant for values of \( k = 1, \ldots, n + 1 \), \( k \neq i \), we conclude that the elements of the \( i \)th column of \((s_{ij})\), except
\(s_{ij} = d_i^2\), must be equal. Since \(s_{ij}\) is a symmetric matrix, the above property extends to the rows of \((s_{ij})\) and it is easily seen that the extreme values of \(L\) correspond to values \(s_{ij}^*\) of \(s_{ij}\) where

\[
(2.5) \quad s_{ij}^* = u \text{ for } i \neq j, \ i, j = 1, \ldots, n + 1
\]

while

\[
(2.6) \quad s_{ii}^* = d_i^2, \ i = 1, \ldots, n + 1.
\]

Now \(u\) can be determined from the relation \(|s_{ij}| = 0\) so that we must have

\[
(2.6) \quad \begin{vmatrix} d_1^2 & u & \cdots & u \\ u & d_2^2 & \cdots & u \\ \vdots & \vdots & \ddots & \vdots \\ u & u & \cdots & d_{n+1}^2 \end{vmatrix} = 0.
\]

Let us define the determinant

\[
(2.7) \quad D_k(x; a_1, \ldots, a_k) = \begin{vmatrix} a_1 & x & \cdots & x \\ x & a_2 & \cdots & x \\ \cdots & \cdots & \cdots & \cdots \\ x & x & \cdots & a_k \end{vmatrix}.
\]

From the relation due to Grabeiri (1874) (see Muir, vol. 3, 4, p. 110), or by subtracting the first row of the above determinant from the remaining rows and by the use of Cauchy expansion in terms of the first row and first column, we have

\[
(2.8) \quad D_k(x; a_1, \ldots, a_k) = \left(1 + x \sum_{i=1}^{k} (a_i - x)^{-1}\right) \prod_{i=1}^{k} (a_i - x).
\]

Hence from (2.6) \(u\) must satisfy the equation

\[
(2.9) \quad \left(1 + u \sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right) \prod_{i=1}^{n+1} (d_i^2 - u) = 0.
\]

From (2.2) and (2.5) the extreme value of \((n!C(S_n))^2\) for any \(u\) satisfying (2.9) is equal to

\[
(2.10) \quad D_{n+1}(1+u; 1 + d_1^2, \ldots, 1 + d_{n+1}^2) = \left(1 + (1 + u) \sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right) \prod_{i=1}^{n+1} (d_i^2 - u)
\]

by the use of (2.9).

Since \(u = 0\) does not satisfy (2.6), we immediately obtain from
(2.9) that the expression (2.10) is equal to

$$-u^{n+1} \prod_{i=1}^{n+1} (d_i^2 - u)$$

which is the extreme value of \((n!C(s_n))^2\) in terms of \(u\). In order that the content is nonzero we must have \(u \neq d_i^2\) for \(i = 1, \ldots, n + 1\). This statement along with (2.9) implies that \(u\) must satisfy the equation

$$1 + u \sum_{i=1}^{n+1} (d_i^2 - u)^{-1} = 0.$$  

The roots for \(u\), temporarily assuming that \(d_1, \ldots, d_{n+1}\) are distinct, can be located by Descartes rule of signs by checking the signs of the left-hand-side of (2.12) for values of \(u\), equal to \(-\infty, 0, +\infty\) and in the neighborhood of \(d_i^2, i = 1, \ldots, n + 1\). Relabelling \(d_i\) such that \(d_1 < d_2 < \cdots < d_{n+1}\), it is easily verified that all the roots for \(u\) are real, say \(u_1, \ldots, u_{n+1}\) and may be labelled in such a manner that

$$u_i < 0 < d_1^2 < u_2 < d_2^2 < \cdots < u_{n+1} < d_{n+1}^2.$$  

Consider the characteristic roots of \((s_{ij}^2)\) given by \(|s_{ij}^2 - \lambda I| = 0\). By (2.5) and (2.7) \(\lambda\) must satisfy

$$D_{n+1}(u; d_i^2 - \lambda, \cdots, d_{n+1}^2 - \lambda) = 0.$$  

Hence from (2.9)

$$\left(1 + u \sum_{i=1}^{n+1} (d_i^2 - \lambda - u)^{-1}\right) \prod_{i=1}^{n+1} (d_i^2 - \lambda - u) = 0.$$  

By similar method as used to obtain (2.13) we find that the roots for \(\lambda\) may be so labelled that \(\lambda_1 = 0\) and

$$d_i^2 < \lambda_{i+1} + u < d_{i+1}^2 \quad i = 1, \ldots, n.$$  

In order that all the roots for \(\lambda\) are nonnegative it is easily seen that the relation

$$d_2^2 - u > \lambda_2 \geq 0$$

must be satisfied so that we must have \(u < d_2^2\). From (2.13) we find that the only admissible roots for \(u\) are \(u_1\) and \(u_2\).

To establish (1.4) it only remains to show that \(u_1\) corresponds to the case when \(A\) is inside the extremal simplex whereas \(u_2\) corresponds to the case when \(A\) is outside the extremal simplex.

Consider the equation of \(\pi_i\), passing through all the vertices of \(S_n\) except \(B_i\) having the coordinates \((x_i, \cdots, x_{i,n})\), given by

$$L_i(x_1, \cdots, x_n) = 0,$$

where
Now $A$ and $B_i$ lie on the same side of $\pi_i$ if and only if $L_i(x_{i,1}, \ldots, x_{i,n})$. $L_i(0, \ldots, 0) > 0$ while $A$ and $B_i$ lie on opposite sides of $\pi_i$ if and only if $L_i(x_{i,1}, \ldots, x_{i,n})$. $L_i(0, \ldots, 0) < 0$.

Now by direct multiplication of the determinant $L_i(x_{i,1}, \ldots, x_{i,n})$ with the transpose of the determinant $L_i(0, 0, \ldots, 0)$ we obtain

$$L_i(0, 0, \ldots, 0) = \begin{vmatrix} x_{i,1} & \cdots & x_{i,n} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{i-1,1} & \cdots & x_{i-1,n} & 1 \\ x_1 & \cdots & x_n & 1 \\ x_{i+1,1} & \cdots & x_{i+1,n} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} & 1 \end{vmatrix}.$$

We now assume that $S_n$ is an extremal simplex so that from (2.5) $s_{\nu\nu} = d_{i}^2$, $\nu = 1, \ldots, n+1$ and $s_{\nu k} = u$, $\nu \neq k$, $\nu, k = 1, \ldots, n+1$. Then in the last determinant each entry in the $i$-th column is 1, the $j$-th diagonal entry is $d_{i}^2 + 1$ for $j \neq i, j = 1, \ldots, n+1$ while the remaining entries are $1 + u$. Subtracting $(1 + u)$ times the $i$-th column from the remaining columns we immediately obtain

$$L_i(x_{i,1}, \ldots, x_{i,n}) \cdot L_i(0, 0, \ldots, 0) = (d_{i}^2 - u)^{-1} \prod_{j=1}^{n+1} (d_{j}^2 - u)$$

Since from (2.11) the numerator of the last expression is positive, we find that $A$ and $B_i$ lie on the same side of $\pi_i$ if and only if

$$-u^{-1}(d_{i}^2 - u) > 0,$$

while they lie on opposite sides of $\pi_i$ if and only if $-u^{-1}(d_{i}^2 - u) < 0$.

Since $-u^{-1}(d_{i}^2 - u) < 0$ and $-u^{-1}(d_{1}^2 - u_i) > 0$, it is readily checked that we have proved (1.2), (1.3) and (1.4) in the case when $d_i, \ldots, d_{n+1}$ are distinct.

Necessary modifications are easily made when some or all of the $d_i$ are not distinct.

Finally we remark that the simplex corresponding to $u_i$ has larger
content than that for $u_2$. This is because
\[ d_i - u_i > d_2 - u_2 > 0 \quad \text{for} \quad i = 2, \ldots, n - 1 \]
and
\[ -u_i^{-1}(d_i^2 - u_i) = 1 - \frac{d_i^2}{u_i} > 1 - d_i/u_2 = -u_2^{-1}(d_2^2 - u_2), \]
so that
\[ (2.15) \quad -u_i^{-1} \prod_{i=1}^{n+1} (d_i^2 - u_i) > -u_2^{-1} \prod_{i=1}^{n+1} (d_i^2 - u_2). \]
We also note that when $d_1 = d_2 = \cdots = d_{n+1}$ (1.4) has a unique negative root for $u$ and the point $A$ corresponding to this value of $u$ must lie inside the extremal simplex.

3. Simplex determined by distances of faces. We recall that the $(n-1)$-dimensional hyperplane $\pi_i$ passes through all the vertices of $S_n$ except $B_i$. The distance of $\pi_i$ from $A$ is $p_i$. The point $B_i$ does not lie on $\pi_i$ but does lie on all the remaining $n$ hyperplanes
\[ \pi_j, j \neq i, j = 1, \ldots, n + 1. \]

Let $\pi_i$ be given by (in normal form)
\[ (3.1) \quad \pi_i: e_{i,1}x_1 + e_{i,2}x_2 + \cdots + e_{i,n}x_n = e_{i,n+1} \]
where for notational convenience we have written
\[ (3.2) \quad p_i = e_{i,n+1}, \]
and $e_{i,1}, \ldots, e_{i,n}$ are the direction cosines of the normal to $\pi_i$, so that we have
\[ (3.3) \quad \sum_{j=1}^{k} e_{i,j}e_{k,j} = m_{ik}; \quad i, k = 1, 2, \ldots, n + 1; \quad m_{ii} = 1. \]
The notations used in this section will be listed first and some relations needed later will be established in order to avoid future digression.

We define the $(n+1) \times (n+1)$ matrix $E$ in double suffix notation as
\[ (3.4) \quad E = (e_{i,j}) \]
and $E_{i,j}$ will denote the co-factor of $e_{i,j}$ in $E$. We also define the $(n+1) \times (n+1)$ matrix $M$ as
\[ (3.5) \quad M = (m_{ij}) \]
and $M_{ii}$ as co-factor of $m_{ij}$ in $M$.

Let $\sigma_i$ denote the signature of $|E_{i,n+1}|$ so that

$$
(3.6) \quad \sigma_i = \begin{cases} 
1 & \text{if } |E_{i,n+1}| > 0 \\
-1 & \text{if } |E_{i,n+1}| < 0 
\end{cases} \quad \text{for } i = 1, \ldots, n + 1.
$$

We remark here that $E_{i,n+1}$ is nonsingular. This is because

$$
\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_{n+1}
$$

have one and only one point in common, namely $(x_{i,1}, \ldots, x_{i,n})$. Since $\pi_i$ does not pass through the above common point, it is easily seen that the matrix $E$ is also nonsingular, so that

$$
(3.7) \quad |E| \neq 0 \text{ and } |E_{i,n+1}| \neq 0, i = 1, \ldots, n + 1.
$$

Furthermore it is easily seen that

$$
(3.8) \quad |E_{i,n+1}| = \sigma_i |E_{i,n+1}E_i^{t,n+1}|^{1/2} = \sigma_i |M_{ii}|^{1/2} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{for } i = 1, \ldots, n + 1
$$

where the radical above as well as all radicals appearing in this paper will be always taken as positive. Hence from (3.2) and (3.4) we have

$$
(3.9) \quad |E| = \sum_{i=1}^{n+1} p_i |E_{i,n+1}| = \sum_{i=1}^{n+1} \sigma_i p_i |M_{ii}|^{1/2} = \rho \text{ (say)}.
$$

$D$ will denote the diagonal matrix

$$
(3.10) \quad D = \text{Diag}. (p_1, \ldots, p_{n+1})
$$

and let

$$
(3.11) \quad R = (r_{ij}) = D^{-1}MD^{-1}
$$

so that $r_{ii} = p_i^{-2}$ for $i = 1, \ldots, n + 1$. Since

$$
M = \begin{bmatrix}
e_{1,1} & \cdots & e_{1,n} \\
\vdots & \ddots & \vdots \\
e_{n+1,1} & \cdots & e_{n+1,n}
\end{bmatrix} \begin{bmatrix}
e_{1,1} & \cdots & e_{1,n} \\
\vdots & \ddots & \vdots \\
e_{n+1,1} & \cdots & e_{n+1,n}\end{bmatrix}'
$$

we also remark that $M$ and consequently $R$ are symmetric positive semi-definite matrices, so that $|M| = 0$ and $|R| = 0$.

Finally, it follows that

$$
(3.12) \quad |M_{ii}| = |R_{ii}| \left(\prod_{j=1}^{n+1} p_j^2\right)/p_i.
$$

To obtain the content $C(S_n)$, we will use the formula (2.1). Since $(x_{i,1}, \ldots, x_{i,n})$ lies on $\pi_j; j \neq i, j = 1, \ldots, n + 1$, we may directly solve for $x_{i,j}$ from the following $n$ linear equations:
A simple calculation shows that (see (3.4))

\[ x_{i,j} = (-1)^{n-j}(-1)^{i+j} |E_{i,j}|/((-1)^{n+1+i} |E_{i,n+1}|) \, . \]

Hence we obtain

\[ x_{i,j} = -|E_{i,j}|/|E_{i,n+1}| \, ; \, i, j = 1, \cdots, n + 1 \, . \]

Substituting these values in \(|V|\) of (2.1) and factoring out \(-1\) from each of the first \(n\) columns of \(V\) and also factoring out \(|E_{i,n+1}|^{-1}\) from the \(i\)th row of \(V\) for \(i = 1, \cdots, n + 1\), we readily obtain

\[
n!C(S_n) = (-1)^n \left| \text{Adj } E \right| \prod_{i=1}^{n+1} |E_{i,n+1}| \]

(3.13)

\[
= (-1)^n |E|^n \prod_{i=1}^{n+1} |E_{i,n+1}| \]

where \(|\text{Adj } E|\) is the adjoint determinant of \(|E|\). In order to avoid the ambiguity of sign in \(C(S_n)\) we consider \((n!C(S_n))^2\) instead and from (3.9) and (3.12) we obtain

\[
(n!C(S_n))^2 = |E|^{2n} \prod_{i=1}^{n+1} |E_{i,n+1}|^2
\]

\[
= \left( \sum_{i=1}^{n+1} \sigma_i p_i |M_{ii}|^{1/2} \right)^{2n} \prod_{i=1}^{n+1} |M_{ii}|^{1/2}
\]

\[= \left( \sum_{i=1}^{n+1} \sigma_i |R_{ii}|^{1/2} \right)^{2n} \prod_{i=1}^{n+1} |R_{ii}|^{1/2} \, . \]

Our problem of minimization is equivalent to minimizing

\[
ln \left[ \left( \sum_{i=1}^{n+1} \sigma_i |R_{ii}|^{1/2} \right)^{2n} \prod_{i=1}^{n+1} |R_{ii}|^{1/2} \right] \]

with respect to \(r_{i,j} \, ; \, i, j = 1, \cdots, n + 1\), subject to the restriction that \(r_{i,i} = p_i^{-2} \, ; \, i = 1, \cdots, n + 1\) and \(|R| = 0\) over the class of symmetric matrices \(R\).

Let \(\lambda, \mu_1, \cdots, \mu_{n+1}\) be Lagrange multipliers and we seek the extreme value of

\[
L = ln \left( \sum_{i=1}^{n+1} \sigma_i |R_{ii}|^{1/2} \right)^2 - \frac{1}{n} \sum_{i=1}^{n+1} ln |R_{ii}| - \lambda |R| + \sum_{i=1}^{n+1} \mu_i (r_{i,i} - p_i^{-2}) \, . \]
\[ r_{ij} \] must satisfy:

\[
\frac{\partial L}{\partial r_{ij}} = -\frac{n+1}{\sum_{\nu=1}^{n+1} |R_{\nu\nu}|^{1/2}} \sigma_{\nu} \frac{\partial |R_{\nu\nu}|^{1/2}}{\partial r_{ij}} - \frac{1}{n} \sum_{\nu=1}^{n+1} \frac{1}{R_{\nu\nu}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ij}} - \lambda \frac{\partial |R_{\nu\nu}|}{\partial r_{ij}} = 0 ,
\]

\( i \neq j, i, j = 1, \ldots, n + 1 \)

and

\[
\frac{\partial L}{\partial r_{ii}} = \frac{n+1}{\sum_{\nu=1}^{n+1} |R_{\nu\nu}|^{1/2}} \sigma_{\nu} \frac{\partial |R_{\nu\nu}|^{1/2}}{\partial r_{ii}} - \frac{1}{n} \sum_{\nu=1}^{n+1} \frac{1}{R_{\nu\nu}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ii}} - \lambda \frac{\partial |R_{\nu\nu}|}{\partial r_{ii}} + \mu = 0
\]

where \( \rho \) is as defined in (3.9).

These equations reduce to

\[
\frac{1}{2} \frac{\partial L}{\partial r_{ij}} = \sum_{\nu=1}^{n+1} (\rho^{-1} \sigma_{\nu} |R_{\nu\nu}|^{-1/2} - n^{-1} |R_{\nu\nu}|^{-1}) |R_{\nu\nu_{ij}}| - \lambda |R_{ij}| = 0
\]

for \( i \neq j; i, j = 1, \ldots, n + 1 \)

and

\[
\frac{\partial L}{\partial r_{ii}} = \sum_{\nu=1}^{n+1} (\rho^{-1} \sigma_{\nu} |R_{\nu\nu}|^{-1/2} - n^{-1} |R_{\nu\nu}|^{-1}) |R_{\nu\nu_{ii}}| - \lambda |R_{ii}| + \mu = 0
\]

where \( |R_{\nu\nu_{ij}}| \) is the co-factor of \( r_{ij} \) in \( |R_{\nu\nu}| \).

Hence the minimizing values of \( r_{ij}, r_{ij}^{\star} \), say, must satisfy the equations in \( r_{ij} \):

\[ r_{ii} = p_i^2 \]

and

\[
(3.14) \quad \sum_{j=1}^{n+1} r_{ij} \frac{\partial L}{\partial r_{ij}} + r_{ii} \frac{\partial L}{\partial r_{ii}} = 0
\]

and

\[
(3.15) \quad \sum_{j=1}^{n+1} r_{ij} \frac{\partial L}{\partial r_{ij}} + r_{ij} \frac{\partial L}{\partial r_{ii}} = 0 .
\]

After obvious simplification (3.14) yields

\[
\sum_{\nu} (\rho^{-1} \sigma_{\nu} |R_{\nu\nu}|^{-1/2} - n^{-1} |R_{\nu\nu}|^{-1}) |R_{\nu\nu}| + \mu_i p_i^{-2} = 0 ,
\]

or

\[
(3.16) \quad \mu_i = p_i^2 \rho^{-1} \sigma_{\nu} R_{ii} .
\]

From (3.15) we obtain for \( k \neq i \),
After some calculations we obtain

\[(3.18) \quad r_{ki} = \mu_i^{-1}(\sigma_k | R_{kk} |^{-1/2} - n^{-1} | R_{kk} |^{-1}) R_{ik} \quad .\]

It is easily seen from (3.11) that \(| R_{ik} | = \sigma_i \sigma_k | M_{ik} | \) and

\[M_{ik} = | E_{i,n+1} | | E_{k,n+1} | \]

and hence from (3.8),

\[| R_{ik} | = \sigma_{ik} | R_{ii} |^{1/2} | R_{kk} |^{1/2} \]

so that substituting for \( \mu_i \) from (3.16) in (3.18) we obtain

\[(3.19) \quad \rho_i r_{ki} = 1 - n^{-1} \rho \sigma_k | R_{kk} |^{-1/2} .\]

In obtaining (3.18) from (3.17), we illustrate the case for \( i = 1, \ n + 1 = 4 \) and \( k = 2 \), for the expression, for example:

\[
\sum_{j=1}^{n+1} \sum_{\nu \neq i,j} \sigma_{\nu} | R_{\nu \nu} |^{-1/2} r_{kj} | R_{\nu \nu \iota j} |
\]

\[
= r_{21} (\sigma_2 | R_{22} |_{11} | R_{22} |^{-1/2} + \sigma_3 | R_{33} |_{11} | R_{33} |^{-1/2} + \sigma_4 | R_{44} |_{11} | R_{44} |^{-1/2})
\]

\[
+ r_{23} (\sigma_3 | R_{23} |_{12} | R_{23} |^{-1/2} + \sigma_4 | R_{44} |_{12} | R_{44} |^{-1/2})
\]

\[
+ r_{24} (\sigma_4 | R_{24} |_{13} | R_{24} |^{-1/2} + \sigma_3 | R_{33} |_{13} | R_{33} |^{-1/2})
\]

\[
= \sigma_3 | R_{21} | | R_{22} |^{-1/2} .
\]

The last expression is obtained from the coefficients of \(| R_{aa} |^{-1/2} \); the coefficients of \(| R_{aa} |^{-1/2} \) or \(| R_{11} |^{-1/2} \) are easily seen to vanish identically, since they represent expansion by alien co-factors.

In the summation appearing in (3.17) only the term with \( \nu = k \) survives;

\[
\sum_{j=1}^{n+1} r_{kj} | R_{kk \iota j} |
\]

is the expansion of the determinant obtained by replacing the elements of the \( i \)-th row of \(| R | \) by those of the \( k \)-th row of \(| R | \) with the \( k \)-th row and \( k \)-th column deleted. Transferring the elements \( r_{ki} \) appearing in the \( i \)-th row to the \( k \)-th row, there results the minor of \( r_{ki} \) in \(| R | \). Hence multiplying by \((-1)^{i-k}\) and \((-1)^{i+k}\) we obtain \(| R_{ki} | \). It is thus seen that

\[
\sum_{j=1}^{n+1} r_{kj} | R_{kk \iota j} | = | R_{ki} | = | R_{ik} | .
\]
From (3.19) it is easily checked that we have

\[(3.20)\]

\[p_i^* p_j^* r_{ik} = p_i^* p_j^* r_{jk},\]

for all \(i, j = 1, \ldots, n + 1, \) with \(i \neq k, j \neq k.\)

Since the matrix

\[(p_i^* r_{ij} p_j^*) = D^2 R D^2 = D^2 D^{-1} M D^{-1} D^2 = D M D = (p_i m_{ij} p_j)\]

is symmetric, and (3.20) implies that nondiagonal elements of each row or column of this matrix are equal we conclude, (in a manner analogous to (2.5)) that \(r_{ii}^* = p_i^{-2}, i = 1, \ldots, n + 1\) and

\[p_i^* r_{ij}^* p_j^* = p_i p_j m_{ij}^* = v,\]

say, for \(i \neq j; i, j = 1, \ldots, n + 1\) so that

\[(3.21)\]

\[\begin{cases} 
  m_{ii}^* = 1 & \text{for } i = 1, \ldots, n + 1 \\
  m_{ij}^* = \frac{v}{p_i p_j} & \text{for } i \neq j; i, j = 1, \ldots, n + 1. 
\end{cases}\]

We obtain values of \(v\) by equating \(|r_{ii}^*| = 0\) or equivalently by setting \(|D M D| = |(p_i p_j m_{ij}^*)| = 0\), where \(p_i p_j m_{ij}^* = v, i \neq j\) and \(p_i^* m_{ii}^* = p_i^*\), and it is seen from (2.7) that \(v\) must satisfy

\[D_{n+1}(v; p_i^*, \ldots, p_{n+1}^*) = 0.\]

and hence

\[(3.22)\]

\[\left(1 + v \sum_{i=1}^{n+1} (p_i^2 - v)^{-1}\right) \prod_{i=1}^{n+1} (p_i^2 - v) = 0.\]

We also note from (3.13), (3.8), (3.9) and (3.12) that

\[(n! C(S_n))^2 = \rho^{2n} \prod_{i=1}^{n+1} (|R_{ii}|^{-1} r_{ii}^*)\]

\[(3.23)\]

\[= p_i^* |R_{ii}|^{-1} \prod_{i=2}^{n+1} (\rho |R_{ii}|^{-1})^2.\]

But from (3.19) we have

\[\rho \sigma_k |R_{kk}|^{-1/2} = n(1 - p_i^* r_{kk}^*)\]

so that \(\rho \sigma_k |R_{ii}|^{-1/2} = n(p_i^2 - v)/p_i^2\), from (3.21). Also from (3.21), since \(r_{ij}^* = v/(p_i^* p_j^*)\) and \(r_{ii}^* = p_i^{-2}\) it is easily seen that

\[|R_{ii}| \prod_{i=2}^{n+1} p_i^2 = D_n(v; p_i^2, \ldots, p_{n+1}^2)\]

\[= \left(1 + v \sum_{i=2}^{n+1} (p_i^2 - v)^{-1}\right) \prod_{i=2}^{n+1} (p_i^2 - v)\]

\[= (p_i^2 - v)^{-1} (-v(p_i^2 - v)^{-1} + \ldots).\]
\[ + 1 + v \sum_{i=1}^{n+1} (p_i^2 - v)^{-1} \prod_{i=1}^{n+1} (p_i^2 - v) \]

\[ = -v(p_i^2 - v)^{-1} \prod_{i=1}^{n+1} (p_i^2 - v) \text{ from (3.22).} \]

Substituting in (3.23) we readily find that

\[ (n!C(S_n))^2 = v^{-1}n^{2n} \prod_{i=1}^{n+1} (p_i^2 - v) . \]

Thus (1.6) is proved.

In order that \( S_n \) is nondegenerate \( v \neq p_i^2, i = 1, \cdots, n + 1 \). Hence from (3.22) \( v \) must satisfy

\[ 1 + v \sum_{i=1}^{n+1} (p_i^2 - v)^{-1} = 0 . \]

Thus we have exactly the same equation as (2.12) with \( d_i \) replaced by \( p_i \) and \( u \) replaced by \( v \). By exactly the same argument that follows (2.12) we conclude that, when \( p_i, \cdots, p_{n+1} \) are distinct, if the roots of (3.25) are so labelled that the unique negative root of (3.25) is \( v_1 \) and the smallest positive root for \( v \) is \( v_2 \) and if the \( p_i \) are labelled so that \( p_1 \) is the smallest and \( p_2 \) the second smallest \( p_i, i = 1, \cdots, n + 1 \), we have the two eligible roots of (3.26) as \( v_1 \) and \( v_2 \) satisfying

\[ v_1 < 0 < p_1^2 < v_2 < p_2^2 . \]

It remains to prove that \( v_1 \) corresponds to the case when \( A \) is inside \( S_n \) while \( v_2 \) corresponds to the case when \( A \) is outside \( S_n \).

We will prove that, for the extremal simplexes obtained above, the vertex \( B_i \) and the fixed point \( A \) lie on the same side of \( \pi_i \) if

\[ p_i^2 - v > 0 \]

while \( A \) and \( B_i \) lie on opposite sides if \( p_i^2 - v < 0 \).

Let

\[ L_i(x_1, \cdots, x_n) = e_i, x_1 + \cdots + e_{i,n} x_n - e_{i,n+1} . \]

Then \( L_i(0, \cdots, 0) = -e_{i,n+1} = -p_i \), and

\[ L_i(x_1, \cdots, x_n) \]

\[ = -\sum_{j=1}^{n+1} e_{i,j} |E_{i,j}| / |E_{i,n+1}| \text{ (by virtue of (3.5))} \]

\[ = -|E| / |E_{i,n+1}| \]

\[ = -p_i \rho/\sigma_i |R_{i} i|^{i/2} \text{ (from (3.8) and (3.12))} \]

\[ = -np_i (1 - p_i^2 r_{ki}^x) \text{ (from (3.19))} \]

\[ = -np_i (1 - v/p_i^2) \text{ (from (3.21))} \]
Hence $L_i(0, \ldots, 0) \cdot L_i(x_{i,i}, \ldots, x_{i,n}) = n(p_i^2 - v)$. Now the equation of $\pi_i$ is $L_i(x_i, \ldots, x_n) = 0$. Hence $p_i^2 - v > 0$ implies that $A$ and $B_i$ lie on the same side of $\pi_i$ while $p_i^2 - v < 0$ implies that $A$ and $B_i$ lie on opposite sides of $\pi_i$. Since $p_i^2 - v_i$ is positive for $i = 1, \ldots, n + 1$ we conclude from (3.26) that corresponding to $v_i$, $A$ is inside $S_n$. Also from (3.26) we find $p_i^2 - v_i$ is negative so that corresponding to $v_i$ the point $A$ lies outside $S_n$. Hence it is readily checked that we have proved (1.5), (1.6), (1.7) and (1.8).

Finally, using an argument analogous to that used to obtain (2.15) we find that

$$-v_1^{-1} \prod_{i=1}^{n+1} (p_i - v_i) > -v_2^{-1} \prod_{i=1}^{n+1} (p_i^2 - v_2)$$

so that from (3.24) we conclude that the content of $S_n$ corresponding to $v_1$ is greater than the content of $S_n$ corresponding to $v_2$.

Obvious modifications in the foregoing proofs are easily made when some or all the $p_1, \ldots, p_{n+1}$ are equal.

When $p_1 = p_2 = \ldots = p_{n+1}$, (3.25) has a unique negative solution for $v$ and in this case $A$ must lie inside the extremal simplex.

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