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A CHARACTERIZATION OF PERFECT RINGS

VLASTIMIL B. DLAB

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J. P. Jans has shown that if a ring R is right perfect, then a certain torsion in the category $\text{Mod } R$ of left R -modules is closed under taking direct products. Extending his method, J. S. Alin and E. P. Armendariz showed later that this is true for every (hereditary) torsion in $\text{Mod } R$. Here, we offer a very simple proof of this result. However, the main purpose of this paper is to present a characterization of perfect rings along these lines: A ring R is right perfect if and only if every (hereditary) torsion in $\text{Mod } R$ is fundamental (i.e., derived from "prime" torsions) and closed under taking direct products; in fact, then there is a finite number of torsions, namely 2^n for a natural number n . Finally, examples of rings illustrating that the above characterization cannot be strengthened are provided. Thus, an example of a ring R_1 is given which is not perfect, although there are only fundamental torsions in $\text{Mod } R_1$, and only $4 = 2^2$ of these. Furthermore, an example of a ring R_{2*} is given which is not perfect and which, at the same time, has the property that there is only a finite number (namely, 3) of (hereditary) torsions in $\text{Mod } R_{2*}$ all of which are closed under taking direct products. Moreover, the ideals of R_{2*} form a chain (under inclusion) and $\text{Rad } R_{2*}$ is a nil idempotent ideal; all the other proper ideals are nilpotent and R_{2*} can be chosen to have a (unique) minimal ideal.

In what follows, R stands always for a ring with unity, \mathcal{L} for the set of all left ideals of R and $\text{Mod } R$ for the category of all (unital) left R -modules and R -homomorphisms. Given $L \in \mathcal{L}$ and $\rho \in R$, $L:\rho$ denotes the (right) ideal-quotient of L by ρ , i.e., the left ideal of all $\chi \in R$ such that $\chi\rho \in L$. We shall call a subset \mathcal{H} of \mathcal{L} a Q -set if it is closed with respect to this operation, i.e., if $K \in \mathcal{H}$ and $\rho \in R$ implies $K:\rho \in \mathcal{H}$; obviously, \mathcal{L} and $\{R\}$ are the greatest and the least Q -sets, respectively. Thus, a topologizing idempotent filter (briefly, a T -set) of left ideals of P. Gabriel [4] is a Q -set \mathcal{H} satisfying, in addition to the filter properties, also the following "radical" condition: If L is a left ideal of R such that $L:\kappa \in \mathcal{H}$ for every element κ of $K \in \mathcal{H}$, then $L \in \mathcal{H}$, as well.

By a *torsion* T in $\text{Mod } R$ we shall always understand a hereditary torsion; thus, a torsion T in $\text{Mod } R$ is a full subcategory of $\text{Mod } R$ such that

- (a) T is closed under taking submodules,
- (b) for every $M \in \text{Mod } R$, there is the greatest submodule (the T -torsion part) $T(M)$ of M belonging to T and

(c) $T(M/T(M)) = 0$ for every $M \in \text{Mod } R$.

As a consequence, every torsion in $\text{Mod } R$ is closed under taking quotients, direct sums and inductive limits. There is a one-to-one correspondence between the torsions in $\text{Mod } R$ and the T -sets of left ideals of R :

If \mathcal{K} is a T -set, then the class $T(\mathcal{K})$ of all R -modules whose elements have orders from \mathcal{K} is a torsion in $\text{Mod } R$; on the other hand, if T is a torsion in $\text{Mod } R$, then the T -set $\mathcal{K}(T) = \{L \mid L \in \mathcal{L} \text{ and } R \text{ mod } L \in T\}$ satisfies $T = T[\mathcal{K}(T)]$. Given an R -module M , let us always denote the T -torsion part of it by $T(M)$.

Thus, given a torsion T , we can define the following two-sided ideals I_T and $J_T \cong I_T$ of R :

$$I_T = \bigcap_{L \in \mathcal{K}(T)} L$$

and

$$J_T/I_T = T(R/I_T).$$

Using this notation, we can prove easily

PROPOSITION 1. *The following four statements are equivalent:*

- (i) *A torsion T in $\text{Mod } R$ is closed under taking direct products.*
- (ii) *For every subset \mathcal{S} of $\mathcal{K}(T)$,*

$$\bigcap_{L \in \mathcal{S}} L \in \mathcal{K}(T).$$

(iii) $I_T \in \mathcal{K}(T)$

(iv) $J_R = R$.

Proof. The equivalence of (ii), (iii), and (iv) is trivial. Also the implication (ii) \rightarrow (i) follows easily; for, the order of an element of a direct product is evidently the intersection of the orders of its components. Finally, in order to show that (i) \rightarrow (iv), we consider the monogenic submodule of the direct product

$$\prod_{L \in \mathcal{K}(T)} R \text{ mod } L$$

generated by the element whose components are generators of $R \text{ mod } L$; it is obviously R -isomorphic to R/I_T .

PROPOSITION 2. *Let every proper (i.e., $\neq R$) two-sided ideal J of R satisfy the following condition: There is $\kappa \notin J$ such that, for every $\rho \in R$ with $\rho\kappa \notin J$, there exists $\sigma \in R$ with $\sigma\rho\kappa = \kappa$. Then every torsion in $\text{Mod } R$ is closed under taking direct products.*

Proof. Let T be a torsion and J_T the two-sided ideal defined above. Assume that $J_T \neq R$. Thus, there exists $\kappa \in J_T$ with the properties stated in our assumption. Since

$$\bigcap_{L \in \mathcal{K}(T)} L = I_T \subseteq J_T ,$$

there is $L_0 \in \mathcal{K}(T)$ such that $\kappa \in L_0$. Hence

$$L_0 : \kappa = (R\kappa \cap L_0) : \kappa \subseteq J_T : \kappa ,$$

and therefore $J_T : \kappa \in \mathcal{K}(T)$, in contradiction to the fact that R/J_T has no nonzero element of order belonging to $\mathcal{K}(T)$. Consequently, $J_T = R$ and Proposition 2 follows in view of Proposition 1.

THEOREM A. *If a ring R satisfies the minimum condition on principal left ideals, i.e., if R is right perfect (cf. H. Bass [2]), then every torsion in $\text{Mod } R$ is closed undertaking direct products.*

Proof. Given an ideal $J \neq R$, consider the (nonempty) set of all principal left ideals which are not contained in J ; take a minimal element K of this set, $\kappa \in K \setminus J$ and apply Proposition 2.

REMARK 1. We can see easily that if R satisfies the minimum condition on principal left ideals, then every R -module M has a nonzero socle; the latter property is, in turn, obviously equivalent to either of the following two statements:

- (i) Every monogenic R -module has a nonzero socle.
- (ii) For every proper left ideal L of R , there is $\rho \in R \setminus L$ such that $L : \rho \neq R$ is maximal in R .

Before we proceed to establish the characterization of perfect rings, let us introduce the following convenient notation and terminology. Denote by $\mathcal{W} \subseteq \mathcal{L}$ the Q -set of all maximal left ideals of R (R itself included). Obviously, for every $W \in \mathcal{W}$, $W \neq R$, the subset

$$\{W : \rho \mid \rho \in R\}$$

is a minimal Q -set contained in \mathcal{W} . Denoting by \mathcal{W}_ω , $\omega \in \Omega$, all such (distinct) minimal Q -sets, it is easy to see that $\{\mathcal{W}_\omega \mid \omega \in \Omega\}$ is a covering of \mathcal{W} , i.e.,

$$\mathcal{W} = \bigcup_{\omega \in \Omega} \mathcal{W}_\omega \quad \text{and} \quad \mathcal{W}_{\omega_1} \cap \mathcal{W}_{\omega_2} = \{R\} \quad \text{for} \quad \omega_1 \neq \omega_2 .$$

Furthermore, for every $\Omega_1 \subseteq \Omega$, put

$$\mathcal{W}_{\Omega_1} = \bigcap_{\omega \in \Omega_1} \mathcal{W}_\omega ;$$

of course, $\mathcal{W} = \mathcal{W}_\Omega$ and $\mathcal{W}_\omega = \mathcal{W}_{(\omega)}$ for each $\omega \in \Omega$. Now, for every $\Omega_1 \subseteq \Omega$, denote the smallest T -set containing \mathcal{W}_{Ω_1} by $\mathcal{W}_{\Omega_1}^*$. It can be easily shown (cf. [3]) that $\mathcal{W}_{\Omega_1}^*$ is the unique T -set \sim -equivalent to \mathcal{W}_{Ω_1} in the sense that, for every proper left ideal $L \in \mathcal{W}_{\Omega_1}^*$,

$$\{L: \rho \mid \rho \in R\} \cap \mathcal{W}_{\Omega_1}^* \neq \{R\}.$$

As a consequence,

$$\mathcal{W}_{\Omega_1}^* \cap \mathcal{W} = \mathcal{W}_{\Omega_1}.$$

Let us call the torsions $T(\mathcal{W}_\omega^*)$, $\omega \in \Omega$, the *prime torsions* in $\text{Mod } R$ and, more generally, torsions $T(\mathcal{W}_{\Omega_1}^*)$ corresponding to the subsets Ω_1 of Ω , the *fundamental torsions* (i.e., derived from prime ones) in $\text{Mod } R$.

On the basis of the above characterization of the T -sets $\mathcal{W}_{\Omega_1}^*$, one can derive very easily the following well-known

PROPOSITION 3. *For any ring R , all the fundamental torsions $T(\mathcal{W}_{\Omega_1}^*)$ in $\text{Mod } R$ are distinct and form a lattice ideal of the complete lattice of all torsions in $\text{Mod } R$, which is isomorphic to the lattice 2^Ω of all subsets of Ω .*

Proof. In order to complete the proof we need only to show that every torsion T in $\text{Mod } R$ contained in $T(\mathcal{W}^*)$ is fundamental. But this follows from the fact that the T -set $\mathcal{K}(T) \subseteq \mathcal{W}^*$ is evidently \sim -equivalent to $\mathcal{K}(T) \cap \mathcal{W}$ and since $\mathcal{K}(T) \cap \mathcal{W} = \mathcal{W}_{\Omega_0}$ for a suitable $\Omega_0 \subseteq \Omega$, we have, in view of the fact that there is unique T -set \sim -equivalent to \mathcal{W}_{Ω_0} ,

$$\mathcal{K}(T) = \mathcal{W}_{\Omega_0}^*,$$

as required.

REMARK 2. We can see easily that the assertion that every torsion in $\text{Mod } R$ is fundamental is equivalent to the assertion that $\mathcal{W}^* = \mathcal{L}$, which in turn is equivalent to any of the statements of the previous Remark 1 (for, $\mathcal{W}^* \sim \mathcal{W}$).

Now, let us formulate the following

THEOREM B. *Let R be a ring such that every fundamental torsion in $\text{Mod } R$ is closed under taking direct products. Then $R/\text{Rad } R$ is semisimple (i.e., artinian); in particular, Ω is finite.*

Proof. For each $\omega \in \Omega$, put

$$W_\omega^0 = \bigcap_{W \in \mathcal{W}} W$$

and notice that the intersection

$$\text{Rad } R = \bigcap_{\omega \in \Omega} W_{\omega}^0$$

is, according to Proposition 3, irredundant. For, \mathscr{W}_{ω}^* (for each $\omega \in \Omega$) and \mathscr{W}^* are the smallest T -sets containing W_{ω}^0 and $\text{Rad } R$, respectively.

In order to prove our theorem, it is sufficient to show that the socle of $R/\text{Rad } R$ is the whole quotient ring $R/\text{Rad } R$; for, $R/\text{Rad } R$ is a ring with unity. First, observe that, in view of the fact that $\text{Rad } R \in \mathscr{W}^*$, the socle of $R/\text{Rad } R$ is essential in $R/\text{Rad } R$ in the sense that it intersect every nonzero left ideal of $R/\text{Rad } R$ nontrivially. Write

$$S/\text{Rad } R = \text{Socle } (R/\text{Rad } R)$$

with the two-sided ideal $S \supseteq \text{Rad } R$ of R and assume

$$S \neq R .$$

Then, there is a (proper) maximal left ideal W of R such that

$$S \subseteq W \subset R ;$$

and, $W \in \mathscr{W}_{\omega_1}$ for a suitable $\omega_1 \in \Omega$. Moreover, clearly

$$S \subseteq W_{\omega_1}^0 .$$

Hence, since $\bigcap_{\omega \in \Omega} W_{\omega}^0$ is irredundant,

$$\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_{\omega}^0 \neq \left(\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_{\omega}^0 \right) \cap W_{\omega_1}^0 = \text{Rad } R ;$$

on the other hand, since $\text{Rad } R \subseteq S \subseteq W_{\omega_1}^0$,

$$\left(\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_{\omega}^0 \right) \cap S = \text{Rad } R ,$$

and thus

$$\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_{\omega}^0 = \text{Rad } R ,$$

a contradiction.

The proof of the theorem is completed.

Now, the main result of the present paper, namely the characterization of perfect rings, follows straight forward from Theorem A, Remarks 1 and 2, Theorem B and the fact that a (right) perfect rings can be characterized as a ring R with unity such that every (left) R -module has a nonzero socle and that $R/\text{Rad } R$ is artinian (H. Bass [2]):

COROLLARY. *A ring R is right perfect if and only if all torsions in $\text{Mod } R$ are fundamental and are closed under taking direct products.*

In conclusion, let us remark that the above characterization cannot be strengthened, even if we take into account the additional condition that there is a finite number of fundamental torsions in $\text{Mod } R$ (the fact which is a consequence of our characterization). To show this, we present the following two examples of rings (which can easily be generalized):

EXAMPLE 1. Let N be the set of all natural numbers, F a field. Denote by $R_1 = R_1(\aleph_0, F)$ the ring of all countable “bounded” matrices over F , i.e., the ring of all functions $f: N \times N \rightarrow F$ satisfying the condition that there is a natural number n_f such that

$$f(i, j) = 0 \quad \text{for } i \neq j, i > n_f \text{ or } j > n_f$$

and

$$f(i, i) = f(n_f + 1, n_f + 1) \quad \text{for all } i > n_f,$$

with matrix addition and multiplication. It is easy to verify that, for every $n \in N$,

$$C_n = \{f \mid f \in R_1 \text{ and } f(i, j) = 0 \text{ for } j \neq n\}$$

are minimal left ideals in R_1 and that the socle

$$S = \bigoplus_{n \in N} C_n$$

of R_1 is a (two-sided) maximal ideal in R_1 ; obviously, $R_1/S \cong F$. Furthermore, $\mathscr{W}'_1 = \{S, R_1\}$ is a minimal Q -set of left ideals of R_1 . Also, for every $n \in N$, the left ideals

$$W_n = \{f \mid f \in R_1 \text{ and } f(i, n) = 0\}$$

are maximal in R_1 and belong to the same minimal Q -set \mathscr{W}'_2 . It is easy to see that the set of all maximal left ideals of R_1

$$\mathscr{W} = \mathscr{W}'_1 \cup \mathscr{W}'_2$$

and that there are 4 torsions in $\text{Mod } R$, all of them fundamental, namely

$$0 = T(\{R\}), T(\mathscr{W}'_1^*), T(\mathscr{W}'_2^*) \text{ and } \text{Mod } R = T(\mathscr{W}^*).$$

Only $T(\mathscr{W}'_2^*)$ is not closed under taking direct products. Of course, R_1 is not perfect.

EXAMPLE 2. Denote by Q^+ the set of all nonnegative rational

numbers endowed with the usual order \leq . Let F be a field. Denote by $R_2 = R(Q^+, F)$ the ring of all functions $f: Q^+ \rightarrow F$ such that the support

$$\text{Sup } f = \{r \mid r \in Q^+ \text{ and } f(r) \neq 0\}$$

is contained in a well-ordered (with respect to \leq) subset of Q^+ which has no finite limit point, with the addition and multiplication defined by

$$(f_1 + f_2)(r) = f_1(r) + f_2(r)$$

and

$$(f_1 * f_2)(r) = \sum_{\substack{t \in Q^+ \\ t \leq r}} f_1(t) \cdot f_2(r - t),$$

respectively.

It is a matter of routine to verify that R_2 is a (commutative) ring. Now, for every $f \in R_2$, denote by r_f the least nonzero rational number such that $f(r_f) \neq 0$. Moreover, for every $t \in Q^+$, denote by $f^{(t)}$ the function of R_2 defined by

$$f^{(t)}(r) = \begin{cases} 1 & \text{for } r = t, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can see easily that, for every $f \in R_2$,

$$f = f^{(r_f)} * \bar{f},$$

where $\bar{f}(r) = f(r + r_f)$ for $r \in Q^+$ (and thus, $r_{\bar{f}} = 0$). First, we are going to prove the following

LEMMA. *If $\bar{f} \in R_2$ such that $r_{\bar{f}} = 0$, then there is $\bar{g} \in R_2$ satisfying*

$$\bar{f} * \bar{g} = f^{(0)} \text{ (= unity of } R_2 \text{)}.$$

Proof. In order to ease the technical difficulties of the proof, observe first that having a well-ordered subset S of Q^+ with no finite limit point, we can consider the subsemigroup \bar{S} of Q^+ generated by S : \bar{S} is again well-ordered and has no finite limit point. Hence, we may consider, for a moment, that our function \bar{f} is defined on a well-ordered subsemigroup \bar{S} of Q^+ with no limit point and try to find \bar{g} defined on the same set \bar{S} , i.e., with $\text{Sup } \bar{g} \subseteq \bar{S}$. Write

$$S = \{r_i\}_{i=0}^\infty \text{ with } 0 = r_0 < r_1 < r_2 < \dots < r_n < \dots.$$

Let us proceed by induction: Denoting by \bar{g}_1 the function defined by

$$\bar{g}_1(0) = [\bar{f}(0)]^{-1}, \bar{g}_1(r_1) = -[\bar{f}(0)]^{-2} \cdot f(r_1) \text{ and } \bar{g}_1(r) = 0 \text{ otherwise,}$$

we can see easily that

$$\bar{f} * \bar{g}_1 = f^{(0)} + h_1,$$

where

$$\text{Sup } \bar{g}_1 \subseteq \{r_i\}_{i=0}^1 \text{ and } \text{Sup } h_1 \subseteq \{r_i\}_{i=2}^\infty.$$

Assuming that, for a natural $n \geq 1$, we have $\bar{g}_n \in R_2$ and $h_n \in R_2$ with

$$\text{Sup } \bar{g}_n \subseteq \{r_i\}_{i=0}^n \text{ and } \text{Sup } h_n \subseteq \{r_i\}_{i=n+1}^\infty$$

such that

$$\bar{f} * \bar{g}_n = f^{(0)} + h_n,$$

let us define

$$\bar{g}_{n+1} = \bar{g}_n + g_{n+1},$$

where

$$g_{n+1}(r_{n+1}) = -[\bar{f}(0)]^{-1} h_n(r_{n+1}) \text{ and } g_{n+1}(r) = 0 \text{ otherwise.}$$

Then,

$$\bar{f} * \bar{g}_{n+1} = \bar{f} * \bar{g}_n + \bar{f} * g_{n+1} = f^{(0)} + h_n + \bar{f} * g_{n+1}$$

and, writing

$$h_{n+1} = h_n + \bar{f} * g_{n+1},$$

we can easily check that

$$\text{Sup } h_{n+1} \subseteq \{r_i\}_{i=n+2}^\infty.$$

For,

$$h_{n+1}(r) = (\bar{f} * g_{n+1})(r) = \sum_{0 \leq t \leq r} \bar{f}(t) g_{n+1}(r-t) = 0 \text{ for } r < r_{n+1}$$

and

$$\begin{aligned} h_{n+1}(r_{n+1}) &= h_n(r_{n+1}) + \sum_{0 \leq t \leq r_{n+1}} \bar{f}(t) g_{n+1}(r_{n+1} - t) \\ &= h_n(r_{n+1}) + \bar{f}(0) g_{n+1}(r_{n+1}) = h_n(r_{n+1}) - h_n(r_{n+1}) = 0, \end{aligned}$$

as required.

Finally, to complete the proof of our lemma, denote by \bar{g} the function defined by

$$\bar{g}(r) = \begin{cases} g_i(r_i) & \text{for } r = r_i, i = 0, 1, \dots \\ 0 & \text{elsewhere.} \end{cases}$$

Then,

$$\bar{f} * \bar{g} = f^{(0)} ;$$

for, if $i = 1, 2, \dots$

$$\begin{aligned} (\bar{f} * \bar{g})(r_i) &= (\bar{f} * [\bar{g}_i + (\bar{g} - \bar{g}_i)])(r_i) \\ &= (\bar{f} * \bar{g}_i)(r_i) + [\bar{f} * (\bar{g} - \bar{g}_i)](r_i) \\ &= (f^{(0)} + h_i)(r_i) + [\bar{f} * (\bar{g} - \bar{g}_i)](r_i) \\ &= 0 + \sum_{0 \leq t \leq r_i} \bar{f}(t)(\bar{g} - \bar{g}_i)(r_i - t) \\ &= 0 . \end{aligned}$$

As a consequence, $f \in R_2$ is a unit in R_2 if and only if $r_f = 0$. Moreover, for every $r \in Q^+$, there exist two ideals

$$\bar{I}_r = \{f \mid f \in R_2 \text{ and } r_f \geq r\}$$

and

$$I_r = \{f \mid f \in R_2 \text{ and } r_f > r\} ;$$

these are all ideals of R_2 . Notice that,

$$I_r \subset \bar{I}_r$$

and that

$$r_1 < r_2 \text{ implies } I_{r_1} \supset \bar{I}_{r_2} ;$$

in particular,

$$\bar{I}_0 = R_2 \text{ and } I_0 = \text{Rad } R_2 .$$

It is also easy to see that there are no divisors of zero in $[R_2]$ and that

$$(\text{Rad } R_2)^2 = \text{Rad } R_2 .$$

For, if $f \in \text{Rad } R_2$, then $r_f > 0$ and obviously,

$$f = f^{((1/2)r_f)} * g ,$$

where

$$g(r) = f\left(r + \frac{1}{2}r_f\right) \text{ for } r \in Q^+ ;$$

here, both $f^{((1/2)r_f)}$ and g evidently belong to $\text{Rad } R_2$.

Finally, given a positive rational number q , define

$$R_{2q} = R_2 / I_q$$

(similarly, we can consider $\bar{R}_{2q} = R_2 / \bar{I}_q$). It is easy to see that

$$\text{Rad } R_{2q} \cong I_0/I_q$$

satisfies again

$$(\text{Rad } R_{2q})^2 = \text{Rad } R_{2q} ,$$

but that every other proper ideal (which is isomorphic to either I_r/I_q or I_r/I_q for $r \geq q$) is nilpotent; besides,

$$\text{Socle } (R_{2q}) \cong \bar{I}_q/I_q .$$

Thus, there are only three torsions in $\text{Mod } R_{2q}$, namely

$$0 = T(\{R\}), T(\{R_{2q}, \text{Rad } R_{2q}\}) \quad \text{and} \quad \text{Mod } R_{2q} = T(\mathcal{L}^{R_{2q}}) .$$

All of them are evidently closed under taking direct products; but, only the first two are fundamental. And, R_{2q} is not perfect.

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