TANGENTIAL CAUCHY-RIEMANN EQUATIONS AND UNIFORM APPROXIMATION

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A smooth (C^∞) function on a smooth real submanifold M of complex Euclidean space C^n is a CR function if it satisfies the Cauchy-Riemann equations tangential to M. It is shown that each CR function admits an extension to an open neighborhood of M in C^n whose \bar{z}-derivatives all vanish on M to a prescribed high order, provided that the system of tangential Cauchy-Riemann equations has minimal rank throughout M. This result is applied to show that on a holomorphically convex compact set in M each CR function can be uniformly approximated by holomorphic functions.

1. Extension and approximation of CR functions. Each point p of a smooth real submanifold M of C^n has a complex tangent space H_pM. It is the largest complex-linear subspace of the ordinary real tangent space T_pM; evidently H_pM = T_pM \cap iT_pM. Its complex dimension is the complex rank of M at p. The theorem of linear algebra relating the real dimensions of T_pM, iT_pM and their sum and intersection shows that if M has real codimension k its complex rank is not less than n − k.

DEFINITION 1.1. M is a CR manifold if its complex rank is constant. It is generic if in addition this rank is minimal; that is, equal to the larger of 0 and n − k. A smooth function f on M is a CR function if ker \bar{\partial}_pf \supset H_pM for each p in M.

Here f is assumed to be extended in a smooth manner to an open neighborhood of M and \bar{\partial}_pf is regarded as the conjugate complex-linear part of the ordinary Fréchet differential d_pf. Since the condition on \bar{\partial}_pf is independent of the extension chosen, the definition makes sense. Computational equivalents to it and some elaboration are given in §2. A more comprehensive treatment of these ideas is found in the paper by S. Greenfield [1]. It should be mentioned that his definition [1, Definition II. A.1] of CR manifolds also requires that the distribution p \mapsto H_pM be involutive. That assumption is not needed here.

If M is a complex submanifold of C^n, then it is CR with complex rank equal to its complex dimension. It is not generic if it has positive codimension. Of course the CR functions on M are just its holomorphic functions.

At the other extreme, every real hypersurface is a generic CR
manifold of complex rank $n - 1$. These frequently have no nontrivial complex submanifolds, which is true for example of the usual $2n - 1$ sphere in $\mathbb{C}^n$.

$M$ is a generic $CR$ manifold if its complex rank is everywhere zero, which is the totally real [5] case.

An example of a proper generic $CR$ submanifold which is neither totally real nor a hypersurface can of course only be found if $n \geq 3$. There is one in $\mathbb{C}^3$, a 4-sphere $S^4$ given as the intersection of the usual 5-sphere and a real hyperplane transverse to it. Let

$$
\rho_1 = |z_1|^2 + |z_2|^2 + |z_3|^2 - 1
$$

and $\rho_2 = z_1 + \overline{z}_2$, where $z_1, z_2, z_3$ are the usual coordinates for $\mathbb{C}^3$, and let $S^4 = \{\rho_1 = \rho_2 = 0\}$. It follows from (2.2) below that $S^4$ has the requisite properties. Furthermore, $S^4$ has no nontrivial complex submanifolds (since the 5-sphere does not).

**Theorem 1.2.** If $f$ is a $CR$ function on a generic $CR$ manifold $M$ in $\mathbb{C}^n$ and $m$ is a nonnegative integer, then there is an extension of $f$ to a smooth function $f_m$ on an open set $U \ni M$ such that $\bar{\partial}f_m$ vanishes on $M$ to order $m$ in all directions.

This result is known [3, Lemma 4.3] and [5, Lemma 3.1] when $M$ is totally real. It is also proved in [2, Th. 2.3.2'] when $M$ is a real hypersurface. A local version which does not require that $M$ be generic is proved in [5, Lemma 3.3].

Theorem 1.2 plays a key role in a program outlined by L. Hörmander for showing that $CR$ functions can be uniformly approximated by holomorphic functions. The basic idea is to take a compact set $K$ in $M$ and a given $CR$ function $f$ on $M$ and find a solution $g$ of $\bar{\partial}g = \bar{\partial}f$ with $\sup_K |g|$ small. Then $u = f - g$ is holomorphic and approximates $f$ uniformly on $K$ with error no larger than $\sup_K |g|$.

In Hörmander’s implementation of this idea, Theorem 1.2 implies that a certain bound on an $L^2$ norm of the Sobolev type is imposed on $\bar{\partial}g$. The existence of solutions to $\bar{\partial}g = \bar{\partial}f$ subject to the same a priori bound [2] and a Sobolev inequality are used to estimate $\sup_K |g|$. This proof appears in [3] and [5] for the cases cited above. Since the only step of it which depends on the complex rank of $M$ is the conclusion of Theorem 1.2, this proof will, without further modification, yield a result on uniform approximation.

**Theorem 1.3.** If $M$ is a closed generic $CR$ submanifold of a domain of holomorphy $U$ in $\mathbb{C}^n$ and $K$ is a compact subset of $M$ holomorphically convex with respect to $U$, then each smooth $CR$ func-
tion on \( M \) is a uniform limit on \( K \) of functions holomorphic on \( U \).

In fact, the same method in conjunction with Theorem 1.2 will prove the stronger statement that approximation holds in the \( C^\infty \) topology; c.f. [5, Th. 6.1]. One merely replaces \( \sup_K |g| \) by a \( C^k \) norm of \( g \) on \( K \).

In the totally real case, it is known that the holomorphic convexity of any given compact subset \( K \) with respect to some domain of holomorphy is a consequence of the absence of complex tangent vectors. This follows from the fact [3, Th. 3.1] and [5, Corollary 4.2] that each \( K \) has arbitrarily small tubular neighborhoods which are domains of holomorphy. However, the case of the \( 2n - 1 \) sphere in \( \mathbb{C}^n \) shows that in the presence of complex tangent vectors holomorphic convexity must be assumed. When there is complex tangency, the problem of determining holomorphic convexity of a given compact subset of \( M \) is very difficult, even for the examples mentioned above.

It should be remarked that in Definition 1.1 and Theorem 1.2 \( \mathbb{C}^n \) may be replaced by any complex manifold, and if this manifold is Stein [2], it may replace \( U \) in Theorem 1.3. No significant modification of the exposition is required.

2. **CR manifolds and functions.** Each real-linear map \( L: \mathbb{C}^n \to \mathbb{C}^k \) is uniquely expressible as a sum \( L = S + T \) where \( S, T: \mathbb{C}^n \to \mathbb{C}^k \), \( S \) is complex linear, and \( T \) is conjugate complex linear. If \( J: v \to iv \), a direct computation shows that \( S = \frac{1}{2}(L - JLJ) \) and \( T = \frac{1}{2}(L + JLJ) \). Applying this result to the Fréchet differential \( d_p\rho \) of a smooth map \( \rho: \mathbb{C}^n \to \mathbb{C}^k \) at \( p \) there results

\[
d_p\rho = \partial_p\rho + \bar{\partial}_p\rho
\]

in which \( \partial_p\rho \) is the complex linear part of \( d_p\rho \) and \( \bar{\partial}_p\rho \) the conjugate complex linear part.

Each point of \( M \) has an open neighborhood \( U \) in \( \mathbb{C}^n \) on which there exists a smooth map \( \rho = (\rho_1, \cdots, \rho_k): U \to \mathbb{R}^k \) with maximal rank \( k \) on \( U \) and satisfying

\[
M \cap U = \{ z \in U : \rho(z) = 0 \}.
\]

Regarding \( \mathbb{R}^k \) as contained in \( \mathbb{C}^k \) in the usual way, and applying the remarks above to Definition 1.1, it follows that \( M \) is CR if and only if \( \bar{\partial}\rho \) has constant complex rank on \( M \cap U \), and is generic exactly when this rank is maximal. When \( k \geq n \) this means that \( H_pM = 0 \), which is the totally real case. The case of interest here is \( k \leq n \), when \( M \) is generic if and only if \( \bar{\partial}\rho \) has complex rank \( k \) on \( M \cap U \). Henceforth, it is assumed that \( k \leq n \). Since it is clear that \( \bar{\partial}\rho = (\bar{\partial}\rho_1, \cdots, \bar{\partial}\rho_k) \) it
follows that the condition

\[(2.2) \quad \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \quad \text{has no zeros on} \quad M \cap U \]

is necessary and sufficient that \( M \) be a generic \( CR \) manifold.

From Definition 1.1 and (2.2) it follows that a smooth function \( f \) on \( M \) is \( CR \) if and only if

\[(2.3) \quad \bar{\partial}f \wedge \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k = 0 \quad \text{on} \quad M. \]

Equivalently, since \{\( \bar{\partial}\rho_i, \cdots, \bar{\partial}\rho_k \)\} is, at points of \( M \), by virtue of (2.2) part of a basis for the space of conjugate-linear functionals on \( \mathbb{C}_n \), there exist smooth functions \( h_i, \cdots, h_k \) on \( U \) such that

\[(2.4) \quad \bar{\partial}f = \sum_{j=1}^{k} h_j \bar{\partial}\rho_j + O(\rho) . \]

Here \( O(\rho) \) denotes a form which vanishes on \( M \cap U \). It is a standard result [4, Lemma 2.1] that if \( g \) is a smooth \( O(\rho) \)-form there exist smooth forms \( g_i, \cdots, g_k \) such that

\[(2.5) \quad g = \sum_{j=1}^{k} \rho_j g_j . \]

More generally, \( O(\rho^m) \) will denote a smooth form on \( U \) which vanishes on \( M \cap U \) to order \( m \). Induction on \( m \) using (2.5) shows that if \( g \) is such a form there are smooth forms \( g_\alpha \) on \( U \) satisfying

\[(2.6) \quad g = \sum_{|\alpha| = m} \rho^\alpha g_\alpha , \]

in which the standard multi-index notation has been used. Thus \( \alpha = (\alpha_1, \cdots, \alpha_k) \) is a \( k \)-tuple of nonnegative integers, \( |\alpha| = \alpha_1 + \cdots + \alpha_k \), and \( \rho^\alpha = \rho_1^{\alpha_1} \cdots \rho_k^{\alpha_k} \). The coefficients \( g_\alpha \) are not unique on \( U \), but the fact that they are determined on \( M \cap U \) will be essential.

**Lemma 2.1.** If smooth forms \( g, g_\alpha \) are related on \( U \) by

\[ g = \sum_{|\alpha| = m} \rho^\alpha g_\alpha + O(\rho^{m+1}) \]

then for each \( \alpha, D^{\alpha}g | M \cap U = \alpha! g_\alpha | M \cap U \). In particular, if \( g = 0 \) on \( U \) then each \( g_\alpha | M \cap U = 0 \).

Here \( D^{\alpha} = D_1^{\alpha_1} \cdots D_k^{\alpha_k}, \) where \( D_j \) denotes differentiation with respect to \( \rho_j \) and \( \alpha! = \alpha_1! \cdots \alpha_k! \).

**Proof.** The statement is local and since \( \rho \) has rank \( k \), the proof can be reduced to the case where each \( \rho_j = x_j \), the \( j \)th ordinary Euclidean coordinate function. Then the lemma follows from the gen-
eral Leibniz formula
\[ D^\alpha(fg) = \sum_{\gamma \subseteq \alpha} \binom{\alpha}{\gamma} D^\gamma f \cdot D^{\alpha - \gamma} g \]
with \( f = x^\alpha \), noting that \( D^\gamma x^\alpha = 0 \) on \( M \cap U \) if \( \gamma < \alpha \) and \( D^\alpha x^\alpha = \alpha! \). Here \( \binom{\alpha}{\gamma} = \alpha! / \gamma! (\alpha - \gamma)! \) and \( \gamma < \alpha \) means that \( \gamma_j < \alpha_j \) for some \( j \).

3. Proof of Theorem 1.2. The proof is an induction on \( m \) in which \( f_{m+1} \) is obtained by subtraction of an \( O(\rho^{m+1}) \) function from \( f_m \). Similar procedures have been used in [2, Th. 2.3.2'], [3, Lemma 4.3], and [5, Lemmas 3.1 and 3.3]. The one used here borrows ideas from all of these. Since the totally real generic cases where \( k \leq n \) are treated in [3] and [5], it will be assumed that \( k \geq n \). However, the proof below can be read with \( k \geq n \), with some slight modifications.

In the presence of complex tangent vectors, the only known result is local in nature [5, Lemma 3.3]. Its proof refers to a particular local coordinate system for \( C^n \) and uses an initial extension \( f_0 \) which is independent of the coordinates normal to \( M \). This feature is clearly not preserved by the patching construction intended here, so an arbitrary extension of \( f \) must be admitted at each step. This introduces remainder terms of the form \( O(\rho^m) \), and it is necessary to keep an accurate account of their effects.

To begin the induction, extend a given CR function \( f \) from \( M \) to a smooth function \( f_0 \) on an open set \( U \supset M \).

First assume that the representation (2.1) holds on \( U \). Then \( \tilde{\partial} f_0 \) is of the form (2.4) and if \( u = \sum_{j=1}^k \rho_j h_j \) it is clear that \( \tilde{\partial} (f_0 - u) = O(\rho) \).

In general \( U \) has a locally finite cover by open sets \( U_j \) on each of which there exists a defining function \( \rho_j \) presenting \( M \cap U_j \) as in (2.1) and an \( O(\rho_j) \) function \( u_j \) satisfying \( \tilde{\partial} (f_0 - u_j) = O(\rho_j) \) on \( U_j \). If \( \{ \varphi_j \} \) is a partition of unity subordinate to \( \{ U_j \} \) and
\[(3.1) \quad u = \sum \varphi_j u_j \]
then
\[(3.2) \quad \tilde{\partial} (f_0 - u) = \sum \varphi_j \tilde{\partial} (f_0 - u_j) - \sum \varphi_j \partial u_j \cdot \partial \varphi_j . \]

By construction each term of either sum in (3.2) vanishes on \( M \). Therefore so does \( \tilde{\partial} f_1 \) if \( f_1 = f_0 - u \).

For the inductive step assume that \( m > 0 \) and \( f \) has an extension \( f_m \) to \( U \) such that \( \tilde{\partial} f_m \) vanishes on \( M \) to order \( m \). A global modification of \( f_m \) will again be obtained by patching local ones, so the construction is again begun by assuming that \( M \) is globally presented by (2.1).
Then by (2.6) there are smooth $(0, 1)$ forms $g_a$ such that

$$
(3.3) \quad \overline{\partial}f_m = \sum_{|\alpha| = m} \rho^a g_a .
$$

Hence

$$
(3.4) \quad 0 = \overline{\partial}^2 f_m = \sum_{|\alpha| = m} \sum_{j=1}^k \alpha_j \rho^{a_j-1} \overline{\partial}\rho_j \wedge g_a + O(\rho^m) ,
$$

in which $\alpha - j$ denotes $(\alpha_i, \ldots, \alpha_j - 1, \ldots, \alpha_k)$ if $\alpha_j > 0$. Wedge this equation with $\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k$ (missing) to show that for each $j$

$$
(3.5) \quad 0 = \sum_{|\alpha| = m} \alpha_j \rho^{a_j-1} \overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k \wedge g_a + O(\rho^m) .
$$

Now for fixed $j$, the map $\alpha \rightarrow \alpha - j$ is a one-to-one correspondence of $\{\alpha: |\alpha| = m \text{ and } \alpha_j > 0\}$ with $\{\beta: |\beta| = m - 1\}$. Therefore (3.5) may be rewritten as

$$
0 = \sum_{|\beta| = m-1} (\beta_j + 1) \rho^{a_j} \overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k \wedge g_{\beta + j} + O(\rho^m) .
$$

and Lemma 2.1 applied to deduce that $g_{\beta + j} \wedge \overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k = 0$ on $M$. Since this holds for every $j$ and $\beta$, it follows from the linear independence of $\overline{\partial}\rho_1, \ldots, \overline{\partial}\rho_k$ on $M$ that for each $\alpha$, $|\alpha| = m$, and each $j, 1 \leq j \leq k$, there is a function $h_{\alpha j}$ such that

$$
(3.6) \quad g_{\alpha} = \sum_{j=1}^k h_{\alpha j} \overline{\partial}\rho_j + O(\rho) .
$$

When substituted for $g_{\alpha}$ in (3.3) and (3.4) this relation yields

$$
(3.7) \quad \overline{\partial}f_m = \sum_{|\alpha| = m} \sum_{j=1}^k \rho^a h_{\alpha j} \overline{\partial}\rho_j + O(\rho^{m+1})
$$

and

$$
(3.8) \quad 0 = \sum_{|\alpha| = m} \sum_{i,j=1}^k \alpha_j \rho^{a_j} h_{\alpha i} \overline{\partial}\rho_j \wedge \overline{\partial}\rho_i + O(\rho^m) .
$$

The expression (3.7) suggests modifying $f_m$ by

$$
(3.9) \quad (n + 1) \overline{\partial}u = \sum_{\alpha, j} \rho^a h_{\alpha j} \overline{\partial}\rho_j + \sum_{\alpha, j} \sum_{i=1}^k \rho_j \alpha_i \rho^{a_i-1} h_{\alpha j} \overline{\partial}\rho_i + \sum_{\alpha, j} \rho^a \rho_j \overline{\partial}h_{\alpha j} .
$$
The first term of this is $\bar{\partial} f_m$. The second is

$$\sum_{i,j=1}^{k} \rho_j \left( \sum_{| \alpha | = m} \alpha_i \rho^{a-i} h_{a \alpha} \right) \bar{\partial} \rho_i,$$

which will be shown to equal $n \bar{\partial} f_m + O(\rho^{m+1})$.

To that end, for each $i < j$, wedging (3.8) with

$$\bar{\partial} \rho_i \wedge \cdots \wedge \hat{\bar{\partial} \rho_i} \wedge \cdots \wedge \hat{\bar{\partial} \rho_j} \wedge \cdots \wedge \bar{\partial} \rho_k$$

($\bar{\partial} \rho_i$ and $\bar{\partial} \rho_j$ are missing) gives the symmetry relation

$$0 = \sum_{| \alpha | = m} (\alpha_i \rho^{a-i} h_{a \alpha} - \alpha_i \rho^{a-i} h_{a \alpha}) + O(\rho^m).$$

Using this in (3.10) it becomes

$$\sum_{i,j=1}^{k} \rho_j \left( \sum_{| \alpha | = m} \alpha_i \rho^{a-i} h_{a \alpha} \right) \bar{\partial} \rho_i + O(\rho^{m+1})$$

which when the summation over $j$ is performed first is

$$\sum_{| \alpha | = m} \sum_{i=1}^{k} \alpha_i \rho^{a-i} h_{a \alpha} \bar{\partial} \rho_i + O(\rho^{m+1}).$$

Noting that $\sum_{j=1}^{k} \alpha_j = n$ completes the argument that the second term of (3.9) is $n \bar{\partial} f_m + O(\rho^{m+1})$. Therefore $\bar{\partial} u = \bar{\partial} f_m + O(\rho^{m+1})$.

Thus on each $U_i$ there is a function $u_i = O(\rho^{m+1})$ such that $\bar{\partial}(f_m - u_i) \mid U_i = O(\rho^{m+1})$. With $u$ defined again by (3.1) and $f_{m+1} = f_m - u$ it follows as before from (3.2) that $\bar{\partial} f_{m+1}$ vanishes on $M$ to order $m + 1$. This completes the proof.

4. Remarks. We know of no nongeneric examples where Theorem 1.2 fails. However, when $M$ is not generic, the above proof breaks down at the inductive step from $m = 1$ to $m = 2$: Since $\bar{\partial} \rho$ does not have maximal rank it may be assumed that there is an integer $l < k$ such that $\bar{\partial} \rho_i \wedge \cdots \wedge \hat{\bar{\partial} \rho_i} \wedge \cdots \wedge \bar{\partial} \rho_j$ has no zeros on $M$ but $\bar{\partial} \rho_i \wedge \cdots \wedge \bar{\partial} \rho_j = 0$ on $M$ if $j > l$. Thus there are more unknowns $g_a$ than equations available from (3.4). There are very simple cases where this occurs:

**Example 4.1.** If the usual coordinates of $C^2$ are denoted $z_1, z_2$ and $M = \{z: z_2 = 0\}$ then the function $f = z_2 \bar{z}_1$ is $CR$, for $\bar{\partial} f = z_2 d\bar{z}_1$. The most general function $u$ vanishing to second order on $M$ is by (the complex analogue of (2.5)) of the form

$$u = z_2^l g_1 + z_2 z_1 d\bar{z}_1 + z_2 \bar{z}_2 \bar{\partial} g_2 + 2 z_2 \bar{z}_2 d\bar{z}_2 + \bar{z}_2 \bar{\partial} g_3.$$

for suitable smooth functions $g_1$, $g_2$, and $g_3$. Therefore

$$\bar{\partial} u = z_2^l \bar{\partial} g_1 + z_2 g_1 d\bar{z}_2 + z_2 z_1 \bar{\partial} g_2 + 2 z_2 \bar{z}_2 d\bar{z}_2 + \bar{z}_2 \bar{\partial} g_3.$$

Each of these terms either vanishes to second order on $M$ or is linearly independent of $\bar{\partial}f$. Therefore no such $u$ will satisfy $\bar{\partial}(f - u) = O(\rho^2)$.

However since $f$ is zero on $M$, it obviously satisfies the conclusion of Theorem 1.2. In fact, if $M$ is a complex manifold, each CR function $f$ is holomorphic, so if $U$ is a domain of holomorphy Theorem 1.2 for $U$ and $M \cap U$ follows from Cartan's Theorem $B$ [2], which implies that $f$ has a holomorphic extension to $U$. Moreover, standard results in several complex variables show that Theorem 1.3 is true for any complex manifold $M$. Thus Theorem 1.2 and a consequent Theorem 1.3 may still hold in the nongeneric case, but some new ideas for proof are necessary.

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