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**TORSION CLASSES AND PURE SUBGROUPS**

BARRY J. GARDNER

## TORSION CLASSES AND PURE SUBGROUPS

B. J. GARDNER

In this note we obtain a classification of the classes  $\mathcal{F}$  of abelian groups satisfying the following closure conditions:

(i) If  $\{A_\mu \mid \mu \in M\} \subseteq \mathcal{F}$ , then  $\mathcal{F}$  contains the direct sum  $\sum A_\mu$ .

For a short exact sequence

$$(*) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

(ii)  $C \in \mathcal{F}$  if  $B \in \mathcal{F}$

(iii)  $B \in \mathcal{F}$  if  $A, C \in \mathcal{F}$

(iv)  $A \in \mathcal{F}$  if  $B \in \mathcal{F}$  and  $(*)$  is pure.

Classes satisfying (i), (ii) and (iii) are called *torsion classes* (of abelian groups) and were first studied by Dickson [2], who classified those which contain only torsion groups and showed that the general classification problem reduces, essentially, to that for torsion classes determined (in the sense of §2 below) by torsion-free groups. The torsion classes which are closed under taking subgroups (called *strongly-complete Serre classes*) can be described quite simply ([1], [2], [10]). A possible approach to the general problem is to investigate torsion classes closed under taking the subgroups corresponding to *proper classes* of monomorphisms as used in relative homological algebra (see for example [8], pp. 367 *et seqq.*), and herein lies the motivation for the present paper.

1. **Notation.** "Group" means "abelian group" throughout.  $h(x)$  denotes the height of an element of a torsion-free group  $\tau(x)$  its type and  $\tau(X)$  the type of a rational group  $X$ . An  $S$ -group, where  $S$  is a set of primes, is a group whose elements have orders belonging to the multiplicative semigroup  $S^*$  generated by  $S$ . A group  $A$  is  $p$ -divisible for a prime  $p$  if  $pA = A$  and  $S$ -divisible if  $p$ -divisible for each  $p \in S$ .  $\mathcal{F}_0, \mathcal{F}_0$  are the classes of all torsion and torsion-free groups respectively. For a group  $A, A_t$  is the torsion subgroup,  $A_p$  its  $p$ -primary component. The direct sum (or discrete direct sum) of a set of groups  $\{A_\mu \mid \mu \in M\}$  is denoted by  $\sum A_\mu$ , the direct product (or complete direct sum) by  $\sum^* A_\mu$  and an element of either by  $(a_\mu)$ .  $[A, B]$  is the group of homomorphisms from a group  $A$  to a group  $B$ . If  $a$  is an element of a torsion-free group  $A$ ,  $[a]$  denotes the cyclic subgroup it generates,  $[a]_*$  the smallest pure subgroup containing it.  $Z$  is the group of integers,  $Q$  the (additive) group of rational numbers,  $Z(p)$  the cyclic group of order  $p$ ,  $Z(p^\infty)$  the quasicyclic  $p$ -group. For

a set  $S$  of primes,  $Q(S)$  is the subgroup  $\{m/n \mid m \in \mathbf{Z}, n \in S^*\}$  of  $\mathbf{Q}$  and for a prime  $p$ ,  $Q(p) = \{m/p^n \mid m, n \in \mathbf{Z}, n \geq 0\}$ .  $I_p$  is the group or ring of  $p$ -adic integers.

For unexplained terms see [4].

**2. Torsion classes.** We begin by listing some properties of torsion classes for later use.

For a class  $\mathcal{C}$  of groups we write  $T(\mathcal{C})$  for the torsion class determined by  $\mathcal{C}$ , i.e. the smallest torsion class  $\mathcal{T}$  with  $\mathcal{C} \subseteq \mathcal{T}$  but if  $\mathcal{C}$  has a single member  $C$ ,  $T(C)$  rather than  $T(\{C\})$  will be used.

**T1.**  $A \in T(\mathcal{C})$  if and only if  $[A, B] = 0$  whenever  $[C, B] = 0$  for all  $C \in \mathcal{C}$ . [3].

$T(\mathcal{C})$  is also the lower radical class determined by  $\mathcal{C}$ , in the sense of Kurosh [7]-Shul'geifer [9], so by the simplified version of the Kurosh construction which applies in an abelian category, we obtain

**T2.**  $A \in T(\mathcal{C})$  if and only if every nonzero homomorphic image  $B$  of  $A$  has a nonzero subgroup which is a homomorphic image of some  $C \in \mathcal{C}$ , i.e.,  $[C, B] \neq 0$ .

A torsion class  $\mathcal{T}$  will be called a  $t$ -torsion class if it contains only torsion groups.

**T3.** Let  $S_1, S_2$  be disjoint sets of primes and let  $\mathcal{T}$  be the class of all groups of the form  $A_1 \oplus A_2$ , where  $A_1$  is an  $S_1$ -group and  $A_2$  a divisible  $S_2$ -group. Then  $\mathcal{T}$  is the  $t$ -torsion class

$$T(\{Z(p) \mid p \in S_1\} \cup \{Z(p^\infty) \mid p \in S_2\}).$$

Any  $t$ -torsion class is uniquely represented in this way. [2].

**T4.** Let  $\mathcal{T}$  be a torsion class and  $p$  a prime. Then either  $Z(p) \in \mathcal{T}$  or every group in  $\mathcal{T}$  is  $p$ -divisible [2].

**PROPOSITION 2.1.** If  $\mathcal{T}$  is a torsion class containing a torsion-free group  $A$ , then  $Z(p^\infty) \in \mathcal{T}$  for every prime  $p$ .

*Proof.* If  $Z(p) \in \mathcal{T}$ , then  $\mathcal{T}$  contains all  $p$ -groups (T3); if not, then  $A$  is  $p$ -divisible, so  $\tau([a]_*) \geq \tau(Q(p))$  for any nonzero  $a \in A$ . Thus  $A/[a]$  has a subgroup and therefore a direct summand isomorphic to  $Z(p^\infty)$ , i.e.  $Z(p^\infty)$  is a homomorphic image of  $A$ .

**T5.** A torsion class  $\mathcal{T}$  contains a group  $A$  if and only if  $A_i$

and  $A/A_t \in \mathcal{T}$  [2].

T6. Any torsion class  $\mathcal{T}$  satisfies the equality

$$\mathcal{T} = T([\mathcal{T} \cap \mathcal{T}_0] \cup [\mathcal{T} \cap \mathcal{T}_0]) .$$

[2].

T7.  $T(Q(S))$  is the class of  $S$ -divisible groups, for any set  $S$  of primes. (Cf. [2], Proposition 4.1.)

3. A simplification of the problem. As a first step, we show that every torsion class closed under taking pure subgroups is either a  $t$ -torsion class or is determined by rational and torsion groups. A class of the latter kind will be called an *r.t.-torsion class*.

PROPOSITION 3.1. All  $t$ -torsion classes are closed under taking pure subgroups.

*Proof.* Let  $S_1, S_2$  be disjoint sets of primes. If  $A_1$  is an  $S_1$ -group and  $A_2$  a divisible  $S_2$ -group, then clearly any pure subgroup of  $A_1 \oplus A_2$  is the direct sum of an  $S_1$ -group and a divisible  $S_2$ -group.

THEOREM 3.2. A torsion class  $\mathcal{T}$  is closed under taking pure subgroups if and only if  $\mathcal{T} \cap \mathcal{T}_0$  is.

*Proof.* Let  $A'$  be a pure subgroup of  $A \in \mathcal{T}$ , and consider the induced diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_t & \longrightarrow & A' & \longrightarrow & A'/A'_t \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow g \\ 0 & \longrightarrow & A_t & \longrightarrow & A & \longrightarrow & A/A_t \longrightarrow 0 \end{array}$$

with exact rows and columns, where  $g$  is defined by  $g(a' + A'_t) = a' + A_t$ .  $A'_t$  is pure in  $A'$  and hence in  $A$ . Therefore  $A'_t$  is pure in  $A_t$  so by Proposition 3.1,  $A'_t \in \mathcal{T} \cap \mathcal{T}_0$ . The kernel of  $g$  is  $A' \cap A_t/A'_t = 0$ . If, for some nonzero  $n \in \mathbb{Z}$ ,  $a' \in A'$  and  $a \in A$  we have  $g(a' + A'_t) = n(a + A_t)$ , then  $m(a' - na) = 0$  for some nonzero  $m \in \mathbb{Z}$ , i.e.  $ma' = mna$ . Since  $A'$  is pure in  $A$ , there exists  $a'' \in A'$  with  $ma' = mna''$ . But then  $g(a' + A'_t) = ng(a'' + A'_t)$ , so that  $g$  is a pure monomorphism. Thus if  $\mathcal{T} \cap \mathcal{T}_0$  is closed under taking pure subgroups,  $A'/A'_t \in \mathcal{T} \cap \mathcal{T}_0$

so  $A' \in \mathcal{T}$  and  $\mathcal{T}$  is therefore closed under taking pure subgroups. The converse is obvious.

**THEOREM 3.3.** *If a torsion class  $\mathcal{T}$  is closed under taking pure subgroups, then*

$$\mathcal{T} = T([\mathcal{T} \cap \mathcal{T}_0] \cup \overline{\mathcal{T}})$$

where  $\overline{\mathcal{T}}$  is the class of rational groups in  $\mathcal{T}$ .

The proof uses the following lemmas:

**LEMMA 3.4.** *For  $\mathcal{T}$  and  $\overline{\mathcal{T}}$  as in Theorem 3.3,  $\mathcal{T} \cap \mathcal{T}_0 = T(\overline{\mathcal{T}}) \cap \mathcal{T}_0$ .*

*Proof.* Clearly  $\mathcal{T} \cap \mathcal{T}_0 \supseteq T(\overline{\mathcal{T}}) \cap \mathcal{T}_0$ . Let  $A$  be any group in  $\mathcal{T} \cap \mathcal{T}_0$ . Then  $A$  is a homomorphic image of  $\sum [a]_*$  where the sum extends over all  $a \in A$  and each  $[a]_* \in \mathcal{T}$ , so  $A \in T(\overline{\mathcal{T}})$ .

**LEMMA 3.5.** *For any two classes  $\mathcal{C}_1, \mathcal{C}_2$  of groups,  $T(\mathcal{C}_1 \cup \mathcal{C}_2) = T(T[\mathcal{C}_1] \cup T[\mathcal{C}_2])$ .*

To complete the proof of Theorem 3.3, we observe that

$$\begin{aligned} \mathcal{T} &= T([\mathcal{T} \cap \mathcal{T}_0] \cup [\mathcal{T} \cap \mathcal{T}_0]) = T([\mathcal{T} \cap \mathcal{T}_0] \cup [T(\overline{\mathcal{T}}) \cap \mathcal{T}_0]) \\ &\subseteq T([\mathcal{T} \cap \mathcal{T}_0] \cup T(\overline{\mathcal{T}})) = T([\mathcal{T} \cap \mathcal{T}_0] \cup \overline{\mathcal{T}}) \subseteq \mathcal{T}. \end{aligned}$$

We conclude this section by showing that not every r.t. torsion class is closed under taking pure subgroups.

**PROPOSITION 3.6.** *Let  $\mathcal{T}$  be a torsion class closed under taking pure subgroups and  $\Gamma$  the set of types of rational groups in  $\mathcal{T}$ . If  $\gamma, \delta \in \Gamma$ , then  $\gamma \cap \delta \in \Gamma$ .*

*Proof.* Let  $X$  and  $Y$  be rational groups with  $\tau(X) = \gamma$  and  $\tau(Y) = \delta$ . Then  $X \oplus Y$  has elements and therefore pure rational subgroups of type  $\gamma \cap \delta$ .

Thus for example if  $p$  and  $q$  are distinct primes,  $T(\{Q(p), Q(q)\})$  is not closed under taking pure subgroups since  $\tau(Q(p)) \cap \tau(Q(q)) = \tau(Z)$  and  $[Q(p), Z] = 0 = [Q(q), Z]$ .

**4. The main results.** In this section we obtain an explicit characterization of the torsion classes closed under taking pure subgroups.

**LEMMA 4.1.** *Let  $X$  be a rational group and  $S = \{p \text{ prime} \mid X \text{ is}$*

$p$ -divisible}. Then  $I_p \in T(X)$  whenever  $p \notin S$ .

*Proof.* Let  $P$  be the set of primes distinct from  $p$ . Then  $I_p \in T(Q(P))(T7)$ . Also, there is a short exact sequence

$$0 \longrightarrow X \longrightarrow Q(P) \longrightarrow \sum Z(q^\infty) \longrightarrow 0$$

where  $q$  ranges over  $P - S$ . Since  $\sum Z(q^\infty) \in T(X)$ (Proposition 2.1), it follows that  $T(X)$  contains  $Q(P)$  and hence  $I_p$ .

The main result can now be stated.

**THEOREM 4.2.** *A torsion class  $\mathcal{T}$  is closed under taking pure subgroups if and only if either*

(i)  *$\mathcal{T}$  is a  $t$ -torsion class*

or (ii)  *$\mathcal{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ , where  $P$  and  $S$  are sets of primes with  $P \subseteq S$ .*

The proof of Theorem 4.2 will be accomplished in several stages. We first prove

**LEMMA 4.3.** *Let  $\{X_\mu \mid \mu \in M\}$  be a set of rational groups. Let  $A = \sum X_\mu$  and  $S = \{p \text{ prime} \mid A \text{ is } p\text{-divisible}\}$ . Then  $T(\{X_\mu \mid \mu \in M\})$  contains  $\sum^* A_i, i = 1, 2, 3, \dots$ , where each  $A_i = A$ .*

*Proof.* Let  $f: \sum^* A_i \rightarrow Y$  be a nonzero epimorphism. We show that  $[X_\mu, Y] \neq 0$  for at least one value of  $\mu$ .

If  $Y_p \neq 0$  for some  $p$ , then since  $Y$  is  $S$ -divisible, so is  $Y_p$ . If  $p \in S$ ,  $Y_p$  is therefore a direct sum of copies of  $Z(p^\infty)$  so by Proposition 2.1,  $Y_p \in T(X_\mu)$  for each  $\mu$  and *a fortiori*  $[X_\mu, Y] \neq 0$  for all  $\mu$ . If  $p \notin S$ , then at least one  $X_\mu$  is  $p$ -reduced, whence  $[X_\mu, Y_p] \neq 0$ .

If  $Y$  is torsion-free, there are two possibilities. If  $f((a_i)) \neq 0$  for some  $(a_i)$  with almost all  $a_i = 0$ , then  $f$  induces a nonzero map from some  $A_i$ , and hence from some  $X_\mu$ , into  $Y$ , while if  $f((a_i)) = 0$  whenever  $a_i = 0$  for almost all values of  $i$ , then  $f$  factorizes as

$$\begin{array}{ccc} \sum^* A_i & \xrightarrow{f} & Y \\ \downarrow & \nearrow & \\ \sum^* A_i / \sum A_i & & \end{array}$$

where the other maps are epimorphisms.  $\sum^* A_i / \sum A_i$  is algebraically compact (see [6]), and also torsion-free, since  $\sum A_i$  is a pure subgroup of  $\sum^* A_i$ . Thus  $\sum^* A_i / \sum A_i$  is the direct sum of a divisible group and a (reduced) cotorsion group [5]; so, therefore, is  $Y$ , which being torsion-

free is algebraically compact [5]. Since  $Y$  is  $S$ -divisible, it has the form  $D \oplus \sum^* R_p$ ,  $p \in S$  where each  $R_p$  is *inter alia* a reduced  $I_p$ -module and  $D$  is divisible. If  $D \neq 0$  then for each  $\mu \in M$  there are monomorphisms  $X_\mu \rightarrow Q \rightarrow D$ . If  $D = 0$ , let  $R_p \neq 0$ . Then at least one  $X_\mu$  is  $p$ -reduced, so by Lemma 4.1,  $I_p \in T(X_\mu)$ . Since there is an epimorphism (an  $I_p$ -epimorphism) from a direct sum of copies of  $I_p$  to  $R_p$ , we have  $R_p \in T(I_p) \subseteq T(X_\mu)$ , so  $[X_\mu, R_p] \neq 0$  and the proof is complete.

The next step is to show when  $T(\{X_\mu \mid \mu \in M\})$  is closed under taking pure subgroups.

LEMMA 4.4. *With the notation of Lemma 4.3, if  $T(\{X_\mu \mid \mu \in M\})$  is closed under taking pure subgroups, it contains  $Q(S)$ .*

*Proof.* Let  $p_1, p_2, p_3, \dots$  be the natural enumeration of the primes, and let  $J = \{i \mid p_i \in S\}$ . For each  $j \in J$ , choose  $a_j \in A$  with  $h_j(a_j) = 0$ , where  $h_j$  denotes height at  $p_j$ . For example, let  $a_j = (x_{j\mu})$  with  $x_{j\mu} \in X_\mu$  satisfying the following conditions: (i)  $x_{j\lambda} \neq 0$  for some  $\lambda \in M$  for which  $X_\lambda$  is  $p_j$ -reduced; (ii)  $h_j(x_{j\lambda}) = 0$ ; (iii)  $x_{j\mu} = 0$  for  $\mu \neq \lambda$ . For a natural number  $i \notin J$ , let  $a_i$  be an arbitrary element of  $A$ , and regard the resulting  $(a_i)$  as an element of a group  $\sum^* A_i$ ,  $i = 1, 2, 3, \dots$ . Then  $h((a_i)) = \prod_{i=1}^\infty h(a_i)$ . In particular,  $h_j((a_i)) = 0$ . Therefore, since  $\sum^* A_i$  is  $S$ -divisible, the height of  $(a_i)$  at a prime  $p$  is infinite if  $p \in S$  and zero otherwise, i.e.,  $\tau((a_i)) = \tau(Q(S))$  and  $\sum^* A_i$  has a pure subgroup isomorphic to  $Q(S)$ . By Lemma 4.3 and assumption, therefore,  $Q(S) \in T(\{X_\mu \mid \mu \in M\})$ .

Since each  $X_\mu$  is  $S$ -divisible and  $T(Q(S))$  is the class of all  $S$ -divisible groups (T7) we have

COROLLARY 4.5. *With the notation of Lemma 4.3, if  $T(\{X_\mu \mid \mu \in M\})$  is closed under taking pure subgroups, it is the class of all  $S$ -divisible groups.*

*Proof of Theorem 4.2.* Let  $\mathcal{S}$  be a torsion class closed under taking pure subgroups. If  $\mathcal{S}$  is not a  $t$ -torsion class, let  $\Gamma$  be the set of types of rational groups in  $\mathcal{S}$  and for each  $\gamma \in \Gamma$  let  $X_\gamma$  be a rational group of type  $\gamma$ . Then

$$\begin{aligned} \mathcal{S} &= T([\mathcal{S} \cap \mathcal{S}_0] \cup \{X_\gamma \mid \gamma \in \Gamma\}) && \text{(Theorem 3.3)} \\ \text{and } \mathcal{S} \cap \mathcal{S}_0 &= T(\{X_\gamma \mid \gamma \in \Gamma\}) \cap \mathcal{S}_0 && \text{(Lemma 3.4).} \end{aligned}$$

By Theorem 3.2,  $T(\{X_\gamma \mid \gamma \in \Gamma\})$  is closed under taking pure subgroups and therefore, by Corollary 4.5, is the class of all  $S$ -divisible groups, where  $S$  is the set of all primes dividing  $\sum X_\gamma$ . Thus  $\mathcal{S} =$

$T([\mathcal{T} \cap \mathcal{S}] \cup \{Q(S)\})$ . Let  $P = \{p \in S \mid Z(p) \in \mathcal{T}\}$ . Since  $T(Q(S)) \subseteq \mathcal{T}$ ,  $\mathcal{T}$  contains the groups  $Z(p^\infty)$  for all primes  $p$  as well as  $Z(p)$  for primes  $p \notin S$ . Thus by T3 and Lemma 3.5

$$\begin{aligned} \mathcal{T} &= T(\{Z(p) \mid p \notin S\} \cup \{Z(p) \mid p \in P\} \cup \{Z(p^\infty) \mid \text{all } p\} \cup \{Q(S)\}) \\ &= T(\{Z(p) \mid p \in P\} \cup \{Q(S)\}) . \end{aligned}$$

Conversely, that any class  $\mathcal{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$  with  $P \subseteq S$  is closed under taking pure subgroups follows from Theorem 3.2, Lemma 3.4 and the observation that  $T(Q(S))$  is closed under taking pure subgroups. By Proposition 3.1, the proof is now complete.

Note that by T1, for a torsion class  $\mathcal{T}$  which is not a  $t$ -torsion class, the representation  $\mathcal{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$  is unique. We conclude with a characterization of the groups in such a class:

PROPOSITION 4.6. *A group  $A$  belongs to  $\mathcal{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$  where  $P$  and  $S$  are sets of primes with  $P \subseteq S$ , if and only if there is a short exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

where  $A'$  is a  $P$ -group and  $A''$  is  $S$ -divisible.

*Proof.* Let  $A \in \mathcal{T}$  and  $A' = \sum A_p$  where the sum extends over all  $p \in P$ ,  $A'' = A/A'$ . Then  $A''$  has no  $P$ -component and belongs to  $\mathcal{T}$  (T5) so therefore has divisible  $S$ -component. Thus  $A''$  is  $S$ -divisible.  $A''/A''_i$  is torsion-free and belongs to  $\mathcal{T}$ . If not  $S$ -divisible, it has a non-zero  $S$ -reduced torsion free homomorphic image  $B$ . But then  $B \in \mathcal{T}$  and  $[Q(S), B] = 0 = [Z(p), B]$  for each  $p \in P$  and this contradicts T1, so  $A''/A''_i$  is  $S$ -divisible, whence  $A''$  is also. The converse is obvious.

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