

# Pacific Journal of Mathematics

**ON  $C, 1$  SUMMABILITY FACTORS OF FOURIER SERIES AT A  
GIVEN POINT**

FU CHENG HSIANG

# ON $|C, 1|$ SUMMABILITY FACTORS OF FOURIER SERIES AT A GIVEN POINT

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Let  $f(x)$  be a function integrable in the sense of Lebesgue over the interval  $(-\pi, \pi)$  and periodic with period  $2\pi$ . Let its Fourier series be

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &\equiv \sum_{n=0}^{\infty} A_n(x). \end{aligned}$$

Whittaker proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha \quad (\alpha > 0)$$

is summable  $|A|$  almost everywhere. Prasad improved this result by showing that the series

$$\sum_{n=n_0}^{\infty} A_n(x) \left/ \left( \prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon} \right. \quad (\log^k n_0 > 0)$$

is summable  $|A|$  almost everywhere.

In this note, the author is interested particularly in the  $|C, 1|$  summability factors of the Fourier series at a given point  $x_0$ .

Write

$$\begin{aligned} \varphi(t) &= f(x_0 + t) + f(x_0 - t) - 2f(x_0), \\ \Phi(t) &= \int_0^t |\varphi(u)| du. \end{aligned}$$

The author establishes the following theorems.

**THEOREM 1.** If

$$\Phi(t) = O(t) \quad (t \rightarrow +0),$$

then the series

$$\sum_{n=1}^{\infty} A_n(x_0)/n^\alpha$$

is summable  $|C, 1|$  for every  $\alpha > 0$ .

**THEOREM 2.** If

$$\Phi(t) = O \left\{ \frac{t}{\prod_{\mu=1}^k \log^\mu \frac{1}{t}} \right\}$$

as  $t \rightarrow +0$ , then the series

$$\sum_{n=n_0}^{\infty} \frac{A_n(x_0)}{\left( \prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon}}$$

is summable  $|C, 1|$  for every  $\varepsilon > 0$ .

A series  $\sum a_n$  is said to be absolutely summable (A) or summable  $|A|$ , if the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is of bounded variation in the interval  $\langle 0, 1 \rangle$ . Let  $\sigma_n^\alpha$  denote the  $n$ th Cesàro mean of order  $\alpha$  of the series  $\sum a_n$ , i.e.,

$$\sigma_n^\alpha = \frac{1}{(\alpha)_n} \sum_{k=0}^n (\alpha)_k a_{n-k}, \quad (\alpha)_k = \Gamma(k + \alpha + 1) / \Gamma(k + 1) \Gamma(\alpha + 1).$$

If the series

$$\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha|$$

converges, then we say that the series  $\sum a_n$  is absolutely summable  $(C, \alpha)$  or summable  $|C, \alpha|$ . It is known that [2] *if a series is summable  $|C|$ , it is also summable  $|A|$ , but not conversely.*

2. Suppose that  $f(x)$  is a function integrable in the sense of Lebesgue and periodic with period  $2\pi$ . Let its Fourier series be

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &\equiv \sum A_n(x). \end{aligned}$$

Whittaker [4] proved that the series

$$\sum_{n=1}^{\infty} A_n(x) / n^\alpha \tag{α > 0}$$

is summable  $|A|$  almost everywhere. Prasad [4] improved this result by showing that the series

$$\sum_{n=n_0}^{\infty} A_n(x) / \left( \prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\epsilon} (\log^k n_0 > 0),$$

where  $\log^k n = \log(\log^{k-1} n)$ ,  $\log^2 = \log(\log n)$ , is summable  $|A|$  almost everywhere.

Let  $(\lambda_n)$  be a convex and bounded sequence, Chow [1] demonstrated that the series

$$\sum A_n(x) \lambda_n$$

is summable  $|C, 1|$  almost everywhere, if the series  $\sum n^{-1} \lambda_n$  converges.

In this note, we are interested particularly in the  $|C, 1|$  summability factors of the Fourier series at a given point. For a fixed point  $x_0$ , we write

$$\varphi(t) = \varphi_{x_0}(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0),$$

and

$$\Phi(t) = \int_0^t |\varphi(u)| du .$$

We are going to establish the following

**THEOREM 1.** *If*

( i )  $\qquad\qquad\qquad \Phi(t) = O(t)$

as  $t \rightarrow +0$ , then the series

$$\sum_{n=1}^{\infty} \frac{A_n(x_0)}{n^\alpha}$$

is summable  $|C, 1|$  for every  $\alpha > 0$ .

3. The following lemmas are required.

**LEMMA 1 [3].** *Let  $\alpha > -1$  and let  $\tau_n^\alpha$  be the  $n$ th Cesàro mean of order  $\alpha$  of the sequence  $\{na_n\}$ , then*

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) .$$

**LEMMA 2.** *Write*

$$S_n(t) = \sum_{k=0}^n (n + 2 - k) \cos (n + 2 - k)t ,$$

then

$$S_n(t) = O \begin{cases} nt^{-1} & (nt \geq 1) , \\ n^2 & (\text{for all } t) . \end{cases}$$

In fact, we have

$$\begin{aligned} S_n(t) &= I \left\{ \frac{d}{dt} e^{i(n+2)t} \sum_{k=0}^n e^{-ikt} \right\} \\ &= I \left\{ \frac{d}{dt} \left( \frac{e^{i(n+2)t}}{1 - e^{-it}} - \frac{e^{it}}{1 - e^{-it}} \right) \right\} \\ &= I \left\{ (n + 2) \frac{ie^{i(n+2)t}}{1 - e^{-it}} - \frac{ie^{i(n+2)t}}{(1 - e^{-it})^2} \right. \\ &\quad \left. - \frac{ie^{it}}{1 - e^{-it}} + \frac{i}{(1 - e^{-it})^2} \right\} \\ &= O(nt^{-1}) + O(t^{-2}) \\ &= O(nt^{-1}) , \end{aligned}$$

if  $nt \leq 1$ . This proves the lemma. From this lemma, we can easily derive the following

LEMMA 3.

$$\left| \frac{1}{n+1} \left\{ \sum_{\nu=1}^n S_{\nu}(t) \Delta \frac{1}{(\nu+2)^{\alpha}} \right\} \right| \leq \begin{cases} \frac{A}{th^{\alpha}} + \frac{A}{nt^{2-\alpha}} & (t \geq 1), \\ An^{1-\alpha} & (\text{for all } t). \end{cases}$$

By Lemma 2, for  $nt \geq 1$ , we write

$$\begin{aligned} \frac{1}{n+1} \left\{ \sum_{\nu=1}^n S_{\nu}(t) \Delta \frac{1}{(\nu+2)^{\alpha}} \right\} &= \frac{1}{n+1} \left\{ \sum_{\nu=1}^{[t^{-1}]^{-1}} + \sum_{\nu=[t^{-1}]^{-1}+1}^n \right\} + O\left(\frac{1}{nt^{2-\alpha}}\right) \\ &= \frac{1}{n} O\left(\sum_{\nu=1}^{[t^{-1}]^{-1}} \nu^{1-\alpha}\right) + \frac{1}{nt} O\left(\sum_{\nu=1}^n \frac{1}{\nu^{\alpha}}\right) \\ &\quad + O\left(\frac{1}{nt^{2-\alpha}}\right) \\ &= O\left(\frac{1}{nt^{2-\alpha}}\right) + O\left(\frac{1}{tn^{\alpha}}\right), \end{aligned}$$

and for all  $t$ ,

$$\begin{aligned} \frac{1}{n+1} \left\{ \sum_{\nu=1}^n S_{\nu}(t) \Delta \frac{1}{(\nu+2)^{\alpha}} \right\} &= \frac{1}{n+1} O\left\{ \sum_{\nu=1}^n \nu^2 \frac{1}{\nu^{1+\alpha}} \right\} \\ &= \frac{1}{n+1} O\left\{ \sum_{\nu=1}^n \nu^{1-\alpha} \right\} \\ &= O(n^{1-\alpha}). \end{aligned}$$

This proves the lemma.

4. We have

$$A_n(x_0) = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \cos ntdt.$$

Let  $\tau_n(x_0)$  be the  $n$ th Cesàro mean of first order of the sequence  $\{nA_n(x_0)/n^{\alpha}\}$ , then

$$\frac{\pi}{2} \tau_n(x_0) = \int_0^{\pi} \varphi(t) \frac{1}{n+1} \sum_{\nu=0}^n \frac{(\nu+2) \cos(\nu+2)t}{(\nu+2)^{\alpha}} dt.$$

Abel's transformation gives

$$\begin{aligned} \frac{\pi}{2} \tau_n(x_0) &= \int_0^{\pi} \varphi(t) \frac{1}{n+1} \left\{ \sum_{\nu=0}^n S_{\nu}(t) \Delta \frac{1}{(\nu+2)^{\alpha}} \right\} dt \\ &\quad + \int_0^{\pi} \varphi(t) \frac{1}{n+1} \cdot \frac{S_n(t)}{(n+3)^{\alpha}} dt \\ &= I_{1n} + I_{2n}, \end{aligned}$$

say. Thus, on writing

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{3n} + I_{4n} ,$$

say, we see that

$$I_{3n} = O\left(n^{1-\alpha} \int_0^{1/n} |\varphi| dt\right) = O(n^{-\alpha}) ,$$

by condition (i) of the theorem.

$$I_{4n} = O\left\{\frac{1}{n^\alpha} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\} + O\left\{\frac{1}{n} \int_{1/n}^{\pi} \frac{|\varphi|}{t^{2-\alpha}} dt\right\} .$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt = O(1) + O\left\{\int_{1/n}^{\pi} \frac{dt}{t}\right\} = O(\log n) ,$$

and

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t^{2-\alpha}} dt \leq n^{1-\alpha} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = O(n^{1-\alpha} \log n) .$$

It follows that

$$I_{4n} = O\{\log n/n^\alpha\} .$$

As before, we write

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n} ,$$

say. Then,

$$I_{5n} = O\left(n^{1-\alpha} \int_0^{1/n} |\varphi| dt\right) = O(n^{-\alpha}) .$$

And

$$I_{6n} = O\left\{n^{-\alpha} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\} = O\{\log n/n^\alpha\} ,$$

by the similar arguments as in the estimation of the integral  $I_{4n}$ . By Lemma 1, we have to establish the convergence of  $\sum |\tau_n(x_0)|/n$ . And from the above analysis, it concludes that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\tau_n(x_0)|}{n} &\leq \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \{|I_{3n}| + |I_{4n}| + |I_{5n}| + |I_{6n}|\} \\ &= O\left\{\sum_{n=1}^{\infty} \frac{\log n}{n^{1+\alpha}}\right\} = O(1) . \end{aligned}$$

This proves Theorem 1.

5. Let  $\tau_n(x_0)$  be the  $n$ th Cesàro mean of first order of the sequence

$$\left\{ nA_n(x_0) / \left( \prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon} \right\} \quad (\varepsilon > 0),$$

where  $k$  is a positive integer. Abel's transformation gives

$$\begin{aligned} \frac{\pi}{2} \tau_n(x_0) &= \int_0^\pi \varphi(t) \frac{1}{n+1} \left\{ \sum_{\nu=0}^n S_\nu(t) \Delta \frac{1}{\left\{ \prod_{\mu=1}^{k-1} \log^\mu (\nu+2) \right\} \{ \log^k (\nu+2) \}^{1+\varepsilon}} \right. \\ &\quad \left. + \int_0^\pi \varphi(t) \frac{1}{n+1} \cdot \frac{S_n(t)}{\left\{ \prod_{\mu=1}^{k-1} \log^\mu (n+3) \right\} \{ \log^k (n+3) \}^{1+\varepsilon}} \right. \\ &= I_{1n} + I_{2n}, \end{aligned}$$

say. As before, we write

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^\pi = I_{3n} + I_{4n},$$

say, and

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^\pi = I_{5n} + I_{6n},$$

say. Since, for  $\nu \geq n_0$ ,

$$\left| \Delta \frac{1}{\left( \prod_{\mu=1}^{k-1} \log^\mu \nu \right) (\log^k \nu)^{1+\varepsilon}} \right| \leq \frac{A}{\nu \left( \prod_{\mu=1}^{k-1} \log^\mu \nu \right) (\log^k \nu)^{1+\varepsilon}},$$

we obtain

$$\begin{aligned} &\left| \frac{1}{n+1} \left\{ \sum_{\nu=0}^n S_\nu(t) \Delta \frac{1}{\left( \prod_{\mu=1}^{k-1} \log^\mu (\nu+2) \right) (\log^k (\nu+2))^{1+\varepsilon}} \right\} \right| \\ &\leq \begin{cases} \frac{A}{t \left( \prod_{\mu=0}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon}} + \frac{A}{t^2 \left( \prod_{\mu=1}^{k-1} \log^\mu \frac{1}{t} \right) (\log^k \frac{1}{t})^{1+\varepsilon}} & (nt \geq 1), \\ \frac{An}{\left( \prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon}} & (\text{for all } t). \end{cases} \end{aligned}$$

Now, if

$$\phi(t) = O \left\{ \frac{t}{\left( \prod_{\mu=1}^k \log^\mu \frac{1}{t} \right)} \right\}$$

as  $t \rightarrow +0$ , then

$$\begin{aligned} I_{3n} &= O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} \int_0^{1/n} |\varphi| dt\right\} \\ &= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\}. \end{aligned}$$

$$\begin{aligned} I_{4n} &= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\} \\ &\quad + O\left\{\frac{1}{n} \int_{1/n}^{\pi} \frac{|\varphi|}{t^2 \left(\prod_{\mu=1}^{k-1} \frac{1}{t}\right) \left(\log^k \frac{1}{t}\right)^{1+\varepsilon}} dt\right\}. \end{aligned}$$

But since

$$\begin{aligned} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt &= \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt \\ &= O(1) + O\left\{\int_{1/n}^{\pi} \frac{dt}{t \left(\prod_{\mu=1}^k \log^{\mu} \frac{1}{t}\right)}\right\} \\ &= O(1) + O\{\log^{k+1} n\}, \end{aligned}$$

and

$$\begin{aligned} \int_{1/n}^{\pi} \frac{|\varphi|}{t^2 \left(\prod_{\mu=1}^{k-1} \log^{\mu} \frac{1}{t}\right) \left(\log^k \frac{1}{t}\right)^{1+\varepsilon}} dt &= O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\} \\ &= O\left\{\frac{n \log^{k+1} n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\}, \end{aligned}$$

we obtain

$$I_{4n} = O\left\{\frac{\log^{k+1} n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\}.$$

Finally,

$$\begin{aligned} I_{5n} &= O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} \int_0^{1/n} |\varphi| dt\right\} \\ &= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}}\right\}, \end{aligned}$$



$$\begin{aligned}
 I_{6n} &= O\left\{ \frac{1}{\left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}} \int_{1/n}^\pi \frac{|\varphi|}{t} dt \right\} \\
 &= O\left\{ \frac{\log^{k+1} n}{\left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}} \right\}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \sum_{n=1}^\infty \frac{|\tau_n(x_0)|}{n} &= O\left\{ \sum_{n=n_0}^\infty \frac{\log^{k+1} n}{n \left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}} \right\} + O(1) \\
 &= O(1).
 \end{aligned}$$

Hence, we establish

**THEOREM 2.** *If*

$$\text{(ii)} \quad \Phi(t) = O\left\{ \frac{t}{\prod_{\mu=1}^k \log^\mu \frac{1}{t}} \right\}$$

as  $t \rightarrow +0$ , then the series

$$\sum_{n=n_0}^\infty \frac{A_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}} \quad (\log^k n_0 > 0)$$

is summable  $|C, 1|$  for every  $\varepsilon > 0$ .

6. For the conjugate series

$$\sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x),$$

we can derive two analogous theorems. Write, for a fixed  $x = x_0$ ,

$$\Psi(t) = \int_0^t |\psi(u)| du \equiv \int_0^t |f(x_0 + u) - f(x_0 - u)| du.$$

We have the following

**THEOREM 3.** *If*

$$\text{(iii)} \quad \Psi(t) = O(t)$$

as  $t \rightarrow +0$ , then the series

$$\sum_{n=1}^\infty \frac{B_n(x_0)}{n^\alpha}$$

is summable  $|C, 1|$  for every  $\alpha > 0$ .

THEOREM 4. If

$$(iv) \quad \Psi(t) = O\left\{\frac{t}{\prod_{\mu=1}^k \log^{\mu} \frac{1}{t}}\right\}$$

as  $t \rightarrow +0$ , then the series

$$\sum_{n=n_0}^{\infty} \frac{B_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\varepsilon}} \quad (\log^k n_0 > 0)$$

is summable  $|C, 1|$  for every  $\varepsilon > 0$ .

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Received February 28, 1969. This paper was written and completed during the author's stay at Berkeley in 1968.

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