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HOMOTOPY GROUPS OF PL-EMBEDDING SPACES

LAWRENCE STANISLAUS HUSCH, JR.

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L. S. HUSCH

Let N be a compact PL - n -manifold, and let M be a PL - m -manifold without boundary. Two of the major problems in PL -topology are to determine conditions such that (1) any continuous map of N into M can be homotoped to a PL -embedding, and (2) two homotopic PL -embeddings are PL -isotopic.

If $C(N, M)$ is the space of continuous maps of N into M with the compact open topology, and if $PL(N, M)$ is the subspace of PL -embeddings, one can consider the map $i_{\#}: \Pi_0(PL(N, M)) \rightarrow \Pi_0(C(N, M))$ induced by inclusion. If (1) is true, then $i_{\#}$ is onto; if (2) is true, then $i_{\#}$ is one-to-one. In this paper, we investigate the higher homotopy groups of $PL(N, M)$ and $C(N, M)$.

Irwin has shown that if N is a closed manifold, $m \geq n + 3$, then sufficient conditions for (1) are that N is $(2n - m)$ -connected and M is $(2n - m + 1)$ -connected. By raising the connectivities of N and M by one, Zeeman [7] proved (2).

By using Proposition 1 of Morlet [4] and Irwin [3], one can easily show the following theorem by using techniques similar to the proof of Theorem 2 below.

THEOREM 1. *Let N be a closed $(2n + s + 1 - m)$ -connected PL - n -manifold and let M be a $(2n + s + 2 - m)$ -connected PL - m -manifold without boundary, $m \geq n + 3$. The homomorphism $i_{\#}: \Pi_s(PL(N, M)) \rightarrow \Pi_s(C(N, M))$ induced by inclusion is an isomorphism; if the connectivities of N and M are lowered by one, then $i_{\#}$ is onto.*

An analogous theorem in the differential case has been proved by J. P. Dax [1], [2].

If N has a nonempty boundary, then Dancis, Hudson and Tindell (independently and unpublished) have shown that if N has a k -dimensional spine with $m \geq \{n + 3, n + k\}$, this is a sufficient condition for (1). If $m \geq \{n + 3, n + k + 1\}$, they obtain (2). We generalize.

THEOREM 2. *Let N be a compact PL - n -manifold with k -spine K , $k < n$, and let M be a PL - m -manifold without boundary. If $m \geq n + k + s + 1$, the homomorphism $i_{\#}: \Pi_s(PL(N, M)) \rightarrow \Pi_s(C(N, M))$ induced by inclusion is an isomorphism; if $m \geq n + k + s$, $i_{\#}$ is onto.*

Note that the codimension 3 restriction is eliminated. In § 3,

we obtain some consequences of this theorem and its proof.

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In this paper, we shall consider $PL(N, M)$ and $C(C, M)$ as Δ -sets (-i.e., as semisimplicial complexes in which the degeneracy maps are ignored). In § 1, we list the basic definitions and results on Δ -sets which we shall use. One may use either Rourke and Sanderson [6] or Morlet [5]. [Morlet uses the terminology "quasisimplicial" set.]

We shall assume familiarity with either [1] or [7] and shall use terminology therein with one exception. When referring to piecewise linear maps or manifolds, we shall always use the prefix " PL ".

Let X and Y be polyhedra. In this paper p_1 and p_2 will always denote projections of $X \times Y$ onto the first and second factors respectively. An isotopy between X and Y will be represented as a family of embeddings $f_t: X \rightarrow Y$, $t \in I = [0, 1]$.

1. Δ -sets. Let Δ^n denote the standard n -simplex with ordered vertices v_0, v_1, \dots, v_n . The i -th face map $\partial_i: \Delta^{n-1} \rightarrow \Delta^n$ is the order preserving simplicial embedding which omits v_i . Δ is the category whose objects are Δ^n , $n = 0, 1, \dots$ and whose morphisms are generated by the face maps. A Δ -set (Δ -group) is a contravariant functor from Δ to the category of sets (groups). A Δ -map between Δ -sets (Δ -groups) is a natural transformation between the functors.

If X is a Δ -set, $X^k = X(\Delta^k)$ is the set of k -simplexes and the maps $\partial_i = X(\partial_i)$ are called face maps. We shall be interested in pointed Δ -sets in which we distinguish a simplex $*^k \in X^k$ for each k and designate $* \subset X$ as the sub- Δ -set of X consisting of these simplexes and maps ∂_i defined by $\partial_i *^k = *^{k-1}$.

With each ordered simplicial complex K , we associate a Δ -set, also designated by K , whose k -simplexes are order-preserving simplicial embeddings of Δ^k into K .

Let $A_{n,i} = \text{Cl}(\text{bdry } \Delta^n - \partial_i \Delta^{n-1})$. A Δ -set X is called a Kan Δ -set if every Δ -map $f: A_{n,i} \rightarrow X$ can be extended to a Δ -map $f_i: \Delta^n \rightarrow X$.

If X is a Kan Δ -set and P is a polyhedron, a map $f: P \rightarrow X$ is a Δ -map $f: K \rightarrow X$ where K is an ordered triangulation of P . $f_0, f_1: P \rightarrow X$ are homotopic if there is a map $F: P \times I \rightarrow X$ such that $F|P \times \{i\} = f_i$, $i = 0, 1$. $[P; X]$ denotes the set of homotopy classes. We shall need the following two propositions which are proved by Rourke and Sanderson.

PROPOSITION 1. Any homotopy class in $[P; X]$ is represented by a Δ -map $f: K \rightarrow X$ where K is any ordered triangulation of P .

PROPOSITION 2. Let Q be a subpolyhedron of P and let

$h: Q \times I \cup P \times \{0\} \rightarrow X$ be a Δ -map to a Kan Δ -set X ; then h extends to a Δ -map $h': P \times I \rightarrow X$.

If X is a pointed Kan Δ -set, then the n -th homotopy group of X , $\Pi_n X = [I^n, \text{bdry } I^n; X, *]$, the homotopy classes of Δ -maps of pairs, where I^n is the PL - n -cell.

$C(N, M)(PL(N, M))$ is made into a Δ -set by defining the k -simplexes to be maps (PL -embeddings) $f: N \times \Delta^k \rightarrow M \times \Delta^k$ such that $p_2 f = p_2$ and defining $\partial_i f = f|N \times \partial_i \Delta^k$.

PROPOSITION 3. $C(N, M)$ and $PL(N, M)$ are Kan Δ -sets.

Proof. Let $f: A_{n,i} \rightarrow PL(N, M)$ be a Δ -map. f can then be considered as a PL -embedding

$$f: N \times A_{n,i} \longrightarrow M \times A_{n,i}$$

such that $p_2 f = p_2$. Using the fact that the pair $(A_{n,i} \times I, A_{n,i} \times \{0\})$ is PL -homeomorphic to $(\Delta^n, A_{n,i})$, one can easily construct the desired extension.

2. Proof of Theorem 1. The following two propositions are generalizations to product spaces of the simplicial approximation and general position theorems. They can be proved similarly.

PROPOSITION 4. Let M and Y be PL -manifolds and let $P \subseteq Q$ be compact polyhedra. Suppose $f: Q \rightarrow M \times Y$ is a continuous map such that $f|P$ is PL . There exists a homotopy $h_t: M \times Y \rightarrow M \times Y$, $t \in I$, such that

- (i) $p_2 h_t = p_2$ for $t \in I$;
- (ii) $h_t f|P = f$ for $t \in I$;
- (iii) $h_1 f: Q \rightarrow M \times Y$ is PL .

PROPOSITION 5. Let M and Y be PL -manifolds and let $P \subseteq Q$ be compact polyhedra. Suppose $f: Q \rightarrow M \times Y$ is a PL -map such that $f|P$ is a PL -embedding. There exists a PL -homotopy $h_t: M \times Y \rightarrow M \times Y$, $t \in I$, such that

- (i) $p_2 h_t = p_2$ for $t \in I$;
- (ii) $h_t f|P = f$ for $t \in I$;
- (iii) the singular set of $h_1 f$ has dimension $\leq 2 \dim Q - \dim(M \times Y)$;
- (iv) the branch set of $h_1 f$ has dimension $< 2 \dim Q - \dim(M \times Y)$.

The following two constructions are needed frequently in the following propositions.

PROPOSITION 6. Let N be a PL - n -manifold with k -spine K . Let

P be a polyhedron in N such that $\dim P + \dim K + 1 \leq \dim N$. There exists a PL -isotopy H_t of N , $t \in I$, such that $H_0 = \text{identity}$ and $H_1(N) \cap P = \emptyset$.

Proof. By general position, we can find a PL -ambient isotopy L_t of N so that $L_1K \cap P = \emptyset$. Let N' be a regular neighborhood of L_1K in N such that $N' \cap P = \emptyset$. Note that L_1K is also a spine of N . Hence, by the uniqueness theorem of regular neighborhoods, there is a PL -isotopy H_t of N , $t \in I$, such that $H_0 = \text{identity}$ and $H_1(N) = N'$.

CONSTRUCTION α . Let I_+^s be a PL -cell in the interior of I^s and let U be a neighborhood of $\text{Cl}(I^s - I_+^s)$ in I^s . Let U_0, U_1 be regular neighborhoods of $\text{Cl}(I^s - I_+^s)$ in I^s such that $U_0 \subseteq \text{int } U_1$ and $U_1 \subseteq U$. Let $\varphi: S^{s-1} \times I \rightarrow \text{Cl}(U_1 - U_0)$ be a PL -homeomorphism such that $\varphi(S^{s-1} \times \{i\}) = \text{bdry } U_i \cap \text{int } I^s$, $i = 0, 1$.

PROPOSITION 7. Let N, K, M be as in Theorem 2 with $m \geq n + k + s$. Let $f: N \times I^s \rightarrow M \times I^s$ be a PL -map such that $p_2f = p_2$ and such that there exists a neighborhood U of $\text{Cl}(I^s - I_+^s)$ such that $f|N \times U$ is a PL -embedding, then there exists a PL -homotopy $f_t: N \times I^s \rightarrow M \times I^s$ and a neighborhood V of $\text{Cl}(I^s - I_+^s)$ in I^s such that

- (i) $f_0 = f$, $p_2f_t = p_2$, $t \in I$;
- (ii) $f_t|V = f$, $t \in I$;
- (iii) $f_1: N \times I^s \rightarrow M \times I^s$ is a PL -embedding.

Proof. By Proposition 5, we can assume that the singular set T of f has dimension $\leq 2(n + s) - (m + s)$, the branch set $B \subset T$ of f has dimension $< 2(n + s) - (m + s)$, and that $f|K \times I^s$ is a PL -embedding. By Proposition 6, there is a PL -isotopy H_t of N such that $H_0 = \text{identity}$ and $H_1(N) \cap p_1B = \emptyset$. Hence there is no loss of generality in assuming that $f|p_1^{-1}(H_1(N)) \times I^s$ is a PL -embedding.

Let U_0, U_1 and φ be as in construction α . Define $F_t: N \times I^s \rightarrow N \times I^s$, $t \in I$, by

$$F_t(x, y) = \begin{cases} (H_t(x), y) & y \in \text{Cl}(I^s - U_1) \\ (x, y) & y \in U_0 \\ (H_{tt_0}(x), y) & y \in \text{Cl}(U_1 - U_0), y = \varphi(y_0, t_0). \end{cases}$$

Let $f_t = fF_t$ and $V = U_0$.

The following is the theorem of Dancis, Hudson and Tindell mentioned in the introduction. We include the proof for completeness.

PROPOSITION 8. Let N, K, M be as in Theorem 2 with $m \geq n + k$. There exists a PL -embedding $f: N \rightarrow M$.

Proof. Let $f': N \rightarrow M$ be a continuous map and approximate f' by a PL-map f'' such that f''/K is a PL-embedding and f'' is in general position. Let $B \subset S$ be the branch and singular set of f'' respectively. By Proposition 6, there is a PL-isotopy $H_t, t \in I$, of N such that $H_1(N) \cap S = S \cap K$. Let $f = f''H_1$.

REMARK. We shall make $PL(N, M)$ and $C(N, M)$ into pointed Δ -sets by defining the basepoint complex $*$ as follows. Let $*^s(x, y) = (f(x), y), x \in N, y \in \Delta^s$ where f is defined in Proposition 8. The face operators are defined naturally.

The proof of the following proposition is well known.

PROPOSITION 9. Let N, M, K be as in Theorem 2 with $m \geq n + k$. Let $g: N \times I^s \rightarrow M \times I^s$ represent an s -simplex in $PL(N, M)(C(N, M))$ such that

$$g|N \times \text{bdry } I^s = *^s|N \times \text{bdry } I^s,$$

g is homotopic rel $\text{bdry } I^s$ in $PL(N, M)(C(N, M))$ to $g': N \times I^s \rightarrow M \times I^s$ such that for some neighborhood U of $\text{Cl}(I^s - I^s_+)$ in $I^s, g'|N \times U = *^s|N \times U$.

PROPOSITION 10. Let N, M, K be as in Theorem 2 with $m \geq n + k + s + 1$ and let $F_i: N \times I^s \rightarrow M \times I^s$ be a PL-homotopy such that

- (i) F_i are PL-embeddings, $i = 0, 1$;
- (ii) $p_2 F_t = p_2, t \in I$;
- (iii) there exists a neighborhood U of $\text{Cl}(I^s - I^s_+)$ in I^s such that $F_t|N \times U = *^s$.

Then there exists a PL-isotopy $G_i: N \times I_s \rightarrow M \times I^s$ such that

- (i) $G_i = F_i$ for $i = 0, 1$;
- (ii) $p_2 G_t = p_2, t \in I$;
- (iii) there exists a neighborhood V of $\text{Cl}(I^s - I^s_+)$ in I^s such that $G_t|N \times V = *^s$.

Proof. Note that there is no loss of generality in assuming that there is an $\varepsilon > 0$ so that F_t are PL-embeddings, $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$. However, now this is a restatement of Proposition 7.

The proof of Theorem 2 now follows easily from the above propositions.

3. Applications. One of the immediate consequences of Theorem 2 is a partial generalization of Hudson's "concordance implies isotopy"

theorem [2]. (See also Proposition 1 of [4].)

COROLLARY 1. *Let N be a compact PL - n -manifold with k -spine K , $k < n$, and let M be a PL - m -manifold without boundary. Let $f: N \times I^s \rightarrow M \times I^s$ be a PL -embedding such that $p_2 f|N \times \text{bdry } I^s = p_2$. Then if $m \geq n + k + s$, there exists a PL -embedding $F: N \times I^s \rightarrow M \times I^s$ such that $F|N \times \text{bdry } I^s = f$ and $p_2 F = p_2$. If $m \geq n + k + s + 1$, f and F can be chosen to be isotopic rel $N \times \text{bdry } I^s$.*

Let X be an s -dimensional polyhedron and let $p: E \rightarrow X$ and $q: F \rightarrow X$ be PL -fiber bundles with fibers N and M respectively with structure groups $\text{Aut}(N)$ and $\text{Aut}(M)$ where

- (i) N is a PL - n -manifold with k -spine, $k < n$;
- (ii) M is a PL - m -manifold without boundary;
- (iii) $\text{Aut}(N)$ and $\text{Aut}(M)$ are the groups of PL -automorphisms of N and M , respectively.

By triangulating X and by using the propositions above together with induction on the dimension of the simplexes of X , one can easily prove the following.

COROLLARY 2. *If $f: E \rightarrow F$ is a continuous bundle map (-i.e., $qf = p$) and $m \geq n + k + s$, then f is homotopic through bundle maps to a PL -bundle map which is an embedding of E into F . If $m \geq n + k + s + 1$; any two PL -bundle embeddings of E into F are isotopic through bundle maps.*

A PL_m -bundle is a PL -bundle $q: F \rightarrow X$ whose fiber is Euclidean m -space R^m and whose structural group is the PL -automorphisms of R^m mod the origin.

COROLLARY 3. *Let N be a PL - n -manifold with k -spine, $k < n$; let $p: E \rightarrow X^s$ be a PL -fiber bundle with N as fiber and $\text{Aut}(N)$ as structural group. If $m \geq n + k + s$, then for any PL_m -bundle $q: F \rightarrow X$, there exists a PL -bundle map $f: E \rightarrow F$ which is an embedding. If $m \geq n + k + s + 1$, then any such two PL -bundle embeddings are isotopic through bundle maps.*

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Vol. 33, No. 1

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Mir Maswood Ali, <i>On some extremal simplexes</i>	1
Silvio Aurora, <i>On normed rings with monotone multiplication</i>	15
Silvio Aurora, <i>Normed fields which extend normed rings of integers</i>	21
John Kelly Beem, <i>Indefinite Minkowski spaces</i>	29
T. F. Bridgland, <i>Trajectory integrals of set valued functions</i>	43
Robert Jay Buck, <i>A generalized Hausdorff dimension for functions and sets</i>	69
Vlastimil B. Dlab, <i>A characterization of perfect rings</i>	79
Edward Richard Fadell, <i>Some examples in fixed point theory</i>	89
Michael Benton Freeman, <i>Tangential Cauchy-Riemann equations and uniform approximation</i>	101
Barry J. Gardner, <i>Torsion classes and pure subgroups</i>	109
Vinod B. Goyal, <i>Bounds for the solution of a certain class of nonlinear partial differential equations</i>	117
Fu Cheng Hsiang, <i>On C_1 summability factors of Fourier series at a given point</i>	139
Lawrence Stanislaus Husch, Jr., <i>Homotopy groups of PL-embedding spaces</i>	149
Daniel Ralph Lewis, <i>Integration with respect to vector measures</i>	157
Marion-Josephine Lim, <i>$\mathcal{L} - 2$ subspaces of Grassmann product spaces</i>	167
Stephen J. Pierce, <i>Orthogonal groups of positive definite multilinear functionals</i>	183
W. J. Pugh and S. M. Shah, <i>On the growth of entire functions of bounded index</i>	191
Siddani Bhaskara Rao and Ayyagari Ramachandra Rao, <i>Existence of triconnected graphs with prescribed degrees</i>	203
Ralph Tyrrell Rockafellar, <i>On the maximal monotonicity of subdifferential mappings</i>	209
R. Shantaram, <i>Convergence of a sequence of transformations of distribution functions. II</i>	217
Julianne Souček, <i>Rings of analytic functions</i>	233
Ted Joe Suffridge, <i>The principle of subordination applied to functions of several variables</i>	241
Wei-lung Ting, <i>On secondary characteristic classes in cobordism theory</i>	249
Pak-Ken Wong, <i>Continuous complementors on B^*-algebras</i>	255
Miyuki Yamada, <i>On a regular semigroup in which the idempotents form a band</i>	261