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ORTHOGONAL GROUPS OF POSITIVE DEFINITE MULTILINEAR FUNCTIONALS

STEPHEN J. PIERCE

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Let V be a finite dimensional vector space over the real numbers R and let $T: V \rightarrow V$ be a linear transformation. If $\varphi: \times_1^m V \rightarrow R$ is a real multilinear functional and

$$\varphi(Tx_1, \dots, Tx_m) = \varphi(x_1, \dots, x_m),$$

$x_1, \dots, x_m \in V$, T is called an isometry with respect to φ . We say φ is positive definite if $\varphi(x, \dots, x) > 0$ for all nonzero $x \in V$. In this paper we prove that if φ is positive definite and T is an isometry with respect to φ , then all eigenvalues of T have modulus one and all elementary divisors of T over the complex numbers are linear.

Let V be an n -dimensional vector space over the real numbers R . Let $T: V \rightarrow V$ be a linear transformation of V . The following theorem [1, Th. 3] is easy to prove:

THEOREM 1. *There exists a positive definite symmetric quadratic form $\varphi: V \times V \rightarrow R$ such that*

$$(1) \quad \varphi(Tx, Ty) = \varphi(x, y), \quad x, y \in V$$

if and only if

1. *all eigenvalues of T have modulus 1;*
- (2) 2. *all elementary divisors of T over the complex numbers C are linear.*

Moreover, if T satisfies (2), then there is a positive definite symmetric φ such that (1) holds.

Theorem 1 can also be expressed in matrix theoretic terms. If A is a real $n \times n$ positive definite symmetric matrix and X is any automorph of A ;

$$(3) \quad X^T A X = A,$$

then X satisfies (2); moreover, if an $n \times n$ matrix X satisfies (2), then there is a positive definite symmetric A such that (3) holds.

Let $\varphi: \times_1^m V \rightarrow R$ be a real multilinear functional. Let H be a subgroup of the symmetric group S_m . If

$$(4) \quad \varphi(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = \varphi(x_1, \dots, x_m)$$

for all $\sigma \in H$ and all $x_i \in V, i = 1, \dots, m$, then φ is said to be symmetric with respect to H . If

$$(5) \quad \varphi(Tx_1, \dots, Tx_m) = \varphi(x_1, \dots, x_m)$$

for all $x_1, \dots, x_m \in V$, T is called an isometry of V with respect to φ . (Note that if $m > 2$, (5) has no matrix analogue). Let $\Omega_m(H, T)$ be the set of all φ satisfying (4) and (5). Clearly $\Omega_m(H, T)$ is a subspace of the vector space of all multilinear functionals symmetric with respect to H . We say φ is positive definite if

$$(6) \quad \varphi(x, \dots, x) > 0$$

for all nonzero x in V . The set of all positive definite φ in $\Omega_m(H, T)$ is denoted by $P_m(H, T)$. It is clear that $P_m(H, T)$ is a (possibly empty) convex cone in $\Omega_m(H, T)$.

The following result [1] was proved as a partial generalization of Theorem 1.

THEOREM 2. *Let $T: V \rightarrow V$ be linear. If $P_m(H, T)$ is nonempty, then*

(a) *m is even*

(b) *every eigenvalue γ of T has modulus 1*

(c) *elementary divisors of T corresponding to $\gamma = \pm 1$ are linear.*

Conversely, if m is even, all eigenvalues of T are ± 1 , and all elementary divisors of T are linear, then $P_m(H, T)$ is nonempty.

We conjectured that if $P_m(H, T)$ is nonempty, then (c) can be replaced by (c') "all elementary divisors of T over the complex field are linear." This would provide a complete generalization of Theorem 2, and thus justify (6) as a definition of a positive definite multilinear functional. The purpose of this paper is to prove this conjecture.

THEOREM 3. *If $P_m(H, T)$ is nonempty, then*

(a) *m is even*

(b) *all eigenvalues of T have modulus 1*

(c') *all elementary divisors of T over C are linear.*

Conversely, if (a), (b), and (c') hold, then $P_m(H, T)$ is nonempty.

2. Proof of Theorem 3. Assume that $P_m(H, T)$ is nonempty. Parts (a) and (b) follow from Theorem 2. We now prove two lemmas.

LEMMA 1. *If γ is an eigenvalue of T and $(x - \gamma)^k, k > 1$, is a nonlinear elementary divisor of T corresponding to γ , then $\gamma^m \neq 1$ for any integer m .*

Proof. Since T is a real transformation, it has a real elementary divisor

$$(7) \quad [(x - \gamma)(x - \bar{\gamma})]^k .$$

(By Theorem 2, γ cannot be real in this case.) Let W be the invariant subspace of T determined by (7), and let S be the restriction of T to W . Then S is an isometry of W with respect to φ , and hence S^r is also an isometry for any integer r . Now if $\gamma^r = 1$, then all eigenvalues of S^r are 1, and hence Theorem 2 implies that all elementary divisors of S^r are linear. Therefore, S^r is the identity on W , and thus, the elementary divisors of S are linear, a contradiction.

LEMMA 2. *If Theorem 3 is true for the case $H = S_m$, then it is true for any subgroup H of S_m .*

Proof. Let H be a subgroup of S_m and let $\varphi \in P_m(H, T)$. For each $\sigma \in S_m$, define

$$(8) \quad \varphi_\sigma(x_1, \dots, x_m) = \varphi(x_{\sigma(1)}, \dots, x_{\sigma(m)}) ,$$

$x_1, \dots, x_m \in V$. In general, φ_σ is not symmetric with respect to H , but φ_σ is positive definite and T is an isometry with respect to φ_σ . Set

$$(9) \quad \psi = \sum_{\sigma \in S_m} \varphi_\sigma .$$

Clearly ψ is positive definite, and T is an isometry with respect to ψ . Moreover, for any $\tau \in S_m$, and $x_1, \dots, x_m \in V$,

$$\begin{aligned} \psi(x_{\tau(1)}, \dots, x_{\tau(m)}) &= \sum_{\sigma \in S_m} \varphi_\sigma(x_{\tau(1)}, \dots, x_{\tau(m)}) \\ &= \sum_{\sigma \in S_m} \varphi(x_{\tau\sigma(1)}, \dots, x_{\tau\sigma(m)}) \\ &= \sum_{\mu \in S_m} \varphi(x_{\mu(1)}, \dots, x_{\mu(m)}) \\ &= \sum_{\mu \in S_m} \varphi_\mu(x_1, \dots, x_m) \\ &= \psi(x_1, \dots, x_m) . \end{aligned}$$

Thus $\psi \in P_m(S_m, T)$, and hence the elementary divisors of T are linear. This proves Lemma 2.

We may assume henceforth that $H = S_m$ and abbreviate $P_m(S_m, T)$ to P_m . If P_m is nonempty, and T has a nonlinear elementary divisor over C corresponding to the eigenvalue $\gamma = a + ib$ ($b \neq 0$), then there exist four linearly independent vectors v_1, \dots, v_4 in V such that

$$\begin{aligned}
 (10) \quad T v_1 &= a v_1 - b v_2 \\
 T v_2 &= b v_1 + a v_2 \\
 T x_3 &= v_2 + a v_3 - b v_4 \\
 T v_4 &= b v_3 + a v_4 .
 \end{aligned}$$

Let \bar{V} be the extension of V to an n -dimensional space over C , i.e., \bar{V} consists of all vectors of the form $x + iy$, $x, y \in V$. By linear extension, we regard T as a linear transformation of \bar{V} , and by multilinear extension, φ becomes a complex valued multilinear functional on $\times_1^m \bar{V}$. Equation (5) still holds in \bar{V} , but φ is no longer positive definite. Set

$$\begin{aligned}
 (11) \quad e_1 &= v_1 + i v_2, e_2 = v_1 - i v_2 \\
 e_3 &= v_3 + i v_4, e_4 = v_3 - i v_4 .
 \end{aligned}$$

From (10) and (11),

$$\begin{aligned}
 (12) \quad T e_1 &= \gamma e_1, \quad T e_2 = \bar{\gamma} e_2 \\
 T e_3 &= \gamma e_3 + v_2, \quad T e_4 = \bar{\gamma} e_4 + v_2 .
 \end{aligned}$$

By Lemma 1, γ is not a root of unity; thus,

$$\begin{aligned}
 (13) \quad \varphi(e_1, \dots, e_1, e_2, \dots, e_2) &= \varphi(T e_1, \dots, T e_1, T e_2, \dots, T e_2) \\
 &= \gamma^k \bar{\gamma}^{m-k} \varphi(e_1, \dots, e_1, e_2, \dots, e_2) \\
 &= 0,
 \end{aligned}$$

unless $k = m - k$, where k is the number of times e_1 occurs in (13). With $r = m/2$, we set

$$\varphi(e_1, \dots, e_1, e_2, \dots, e_2) = \nu .$$

Now $\nu \neq 0$; otherwise

$$\begin{aligned}
 (14) \quad \varphi(v_1, \dots, v_1) &= 2^{-m} \varphi(e_1 + e_2, \dots, e_1 + e_2) \\
 &= 0,
 \end{aligned}$$

contradicting (6). (Note that we are using the assumption that φ is symmetric with respect to S_m ; this gives us a convenient way of sorting expressions such as those on the right side of (14).)

Let $\mu = \varphi(v_1, \dots, v_1, e_3)$. Using (13) and (14), we compute,

$$\begin{aligned}
 \mu &= 2^{-m+1} \varphi(e_1 + e_2, \dots, e_1 + e_2, e_3) \\
 &= 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \dots, \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + v_2) \\
 &= 2^{-m+1} \varphi\left(\gamma e_1 + \bar{\gamma} e_2, \dots, \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + \frac{e_1 - e_2}{2i}\right) \\
 &= -2^{-m} i \binom{m-1}{r} (\bar{\gamma} - \gamma) \nu + \gamma 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \dots, \gamma e_1 + \bar{\gamma} e_2, e_3)
 \end{aligned}$$

$$\begin{aligned}
 &= -2^{-m}i\binom{m-1}{r}(\bar{\gamma} - \gamma)\nu + \gamma 2^{-m+1} \\
 &\quad \varphi\left(\gamma^2 e_1 + \bar{\gamma}^2 e_2, \dots, \gamma^2 e_1 + \bar{\gamma}^2 e_2, \gamma e_3 + \frac{e_1 - e_2}{2i}\right) \\
 &= -2^{-m}i\binom{m-1}{r}(2\bar{\gamma} - \gamma - \gamma^3)\nu + \gamma^2 2^{-m+1} \\
 &\quad \varphi(\gamma^2 e_1 + \bar{\gamma}^2 e_2, \dots, \gamma^2 e_1 + \bar{\gamma}^2 e_2, e_3) .
 \end{aligned}$$

Continuing this procedure, we obtain for any positive integer s

$$\begin{aligned}
 (15) \quad \mu &= -2^{-m}i\binom{m-1}{r}\left(s\bar{\gamma} - \sum_{j=0}^{s-1} \gamma^{2j+1}\right)\nu + \gamma^s 2^{-m+1} \\
 &\quad \varphi(\gamma^s e_1 + \bar{\gamma}^s e_2, \dots, \gamma^s e_1 + \bar{\gamma}^s e_2, e_3) .
 \end{aligned}$$

Let

$$f(z) = z\varphi(ze_1 + \bar{z}e_2, \dots, ze_1 + \bar{z}e_2, e_3) ,$$

where z is a complex variable. Then f is a continuous function of z on the complex plane, and hence f is bounded on the unit circle. Moreover, since γ is not a root of unity (in particular, $\gamma \neq \pm 1$),

$$\sum_{j=0}^{s-1} \gamma^{2j-1}$$

is also bounded as s becomes large. Thus, letting s approach infinity in (15) forces μ to become infinite, a contradiction. This proves Theorem 3 in one direction.

Now suppose all eigenvalues of T are 1 in absolute value and all elementary divisors of T are linear over C . Let 1 (p times), -1 (q times) and $\gamma_j, \bar{\gamma}_j = a_j \pm ib_j, |\gamma_j| = 1, j = 1, \dots, t$, be the eigenvalues of T . Then there is a basis of $V, v_1, \dots, v_p, u_1, \dots, u_q, x_1, y_1, \dots, x_t, y_t$ such that

$$\begin{aligned}
 (16) \quad &Tv_j = v_j, j = 1, \dots, p \\
 &Tu_j = -u_j, j = 1, \dots, q \\
 &Tx_j = a_j x_j - b_j y_j, j = 1, \dots, t \\
 &Ty_j = b_j x_j + a_j y_j, j = 1, \dots, t .
 \end{aligned}$$

Set

$$\begin{aligned}
 w_j &= x_j + iy_j \\
 \bar{w}_j &= x_j - iy_j, j = 1, \dots, t .
 \end{aligned}$$

Then $v_1, \dots, v_p, u_1, \dots, u_q, w_1, \bar{w}_1, \dots, w_t, \bar{w}_t$ form a basis of \bar{V} of eigenvectors of T . Let $f_1, \dots, f_p, g_1, \dots, g_q, h_1, k_1, \dots, h_t, k_t$ be the corresponding dual basis. If l_1, \dots, l_m are linear functionals on a space V , then $l_1 \dots l_m$ is the m -linear functional on $\times_1^m V$ such that

$$l_1 \cdots l_m(x_1, \dots, x_m) = \prod_{i=1}^m l_i(x_i).$$

Define φ as follows:

$$(17) \quad \varphi = \sum_{j=1}^p f_j^m + \sum_{j=1}^q g_j^m + \sum_{j=1}^t [(h_j k_j)^r + (\bar{h}_j \bar{k}_j)^r],$$

where $r = m/2$ and $\bar{f}(v) = \overline{f(v)}$. Now \bar{h}_j and \bar{k}_j are not linear on the complex space \bar{V} , but they are complex valued linear functionals on V , i.e., they are linear functionals on V but are not in the dual space of V . Thus φ is a real multilinear functional on V . Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma.$$

We assert that $\psi \in P_m(H, T)$. Clearly ψ is symmetric with respect to S_m , and thus with respect to any subgroup H of S_m . It remains to show that ψ is positive definite and that T is an isometry with respect to ψ . It suffices to prove these last two properties for φ . Let

$$x = \sum_{j=1}^p \alpha_j v_j + \sum_{j=1}^q \beta_j u_j + \sum_{j=1}^t (\delta_j x_j + \lambda_j y_j)$$

be an arbitrary vector of V . Then from (17),

$$\varphi(x, \dots, x) = \sum_{j=1}^p \alpha_j^m + \sum_{j=1}^q \beta_j^m + 2 \sum_{j=1}^t \left[\left(\frac{\delta_j}{2} \right)^2 + \left(\frac{\lambda_j}{2} \right)^2 \right]^r.$$

Since m is even and $\alpha_j, \beta_j, \delta_j, \lambda_j$ are all real, φ is positive definite. Now let $z_k, k = 1, \dots, m$, be arbitrary vectors in V , with

$$(18) \quad z_k = \sum_{j=1}^p a_{kj} v_j + \sum_{j=1}^q b_{kj} u_j + \sum_{j=1}^t (c_{kj} x_j + d_{kj} y_j).$$

Then

$$(19) \quad \begin{aligned} \varphi(z_1, \dots, z_m) &= \sum_{j=1}^p \prod_{k=1}^m a_{kj} + \sum_{j=1}^q \prod_{k=1}^m b_{kj} \\ &+ \sum_{j=1}^t \prod_{k=1}^r \left(\frac{c_{2k-1,j}}{2} + \frac{d_{2k-1,j}}{2i} \right) \left(\frac{c_{2k,j}}{2} - \frac{d_{2k,j}}{2i} \right) \\ &+ \sum_{j=1}^t \prod_{k=1}^r \left(\frac{c_{2k-1,j}}{2} - \frac{d_{2k-1,j}}{2i} \right) \left(\frac{c_{2k,j}}{2} + \frac{d_{2k,j}}{2i} \right). \end{aligned}$$

From (16)

$$(20) \quad \begin{aligned} Tz_k &= \sum_{j=1}^p a_{kj} v_j + \sum_{j=1}^q (-b_{kj}) u_j \\ &+ \sum_{j=1}^t (a_j c_{kj} + b_j d_{kj}) x_j + (a_j d_{kj} - b_j c_{kj}) y_j, \end{aligned}$$

$k = 1, \dots, m$. Let

$$\begin{aligned} e_{kj} &= a_j c_{kj} + b_j d_{kj} \\ f_{kj} &= a_j d_{kj} - b_j c_{kj} . \end{aligned}$$

Then from (19) and (20)

$$\begin{aligned} \varphi(Tz_1, \dots, Tz_m) &= \sum_{j=1}^p \prod_{k=1}^m a_{kj} + \sum_{j=1}^q \prod_{k=1}^m (-b_{kj}) \\ (21) \quad &+ \sum_{j=1}^t \prod_{k=1}^m \left(\frac{e_{2k-1,j}}{2} + \frac{f_{2k-1,j}}{2i} \right) \left(\frac{e_{2k,j}}{2} - \frac{f_{2k,j}}{2i} \right) \\ &+ \sum_{j=1}^t \prod_{k=1}^m \left(\frac{e_{2k-1,j}}{2} - \frac{f_{2k-1,j}}{2} \right) \left(\frac{e_{2k,j}}{2} + \frac{f_{2k,j}}{2i} \right) . \end{aligned}$$

It is easily verified that

$$\begin{aligned} (22) \quad \frac{e_{kj}}{2} + \frac{f_{kj}}{2i} &= \bar{\gamma}_j \left(\frac{c_{kj}}{2} + \frac{d_{kj}}{2i} \right) \\ \frac{e_{kj}}{2} - \frac{f_{kj}}{2i} &= \gamma_j \left(\frac{c_{kj}}{2} - \frac{d_{kj}}{2i} \right) . \end{aligned}$$

Using (22) in (21) and the fact that $|\gamma_j| = 1$, we obtain

$$\varphi(Tz_1, \dots, Tz_m) = \varphi(z_1, \dots, z_m) .$$

This completes the proof of Theorem 3.

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Mir Maswood Ali, <i>On some extremal simplexes</i>	1
Silvio Aurora, <i>On normed rings with monotone multiplication</i>	15
Silvio Aurora, <i>Normed fields which extend normed rings of integers</i>	21
John Kelly Beem, <i>Indefinite Minkowski spaces</i>	29
T. F. Bridgland, <i>Trajectory integrals of set valued functions</i>	43
Robert Jay Buck, <i>A generalized Hausdorff dimension for functions and sets</i>	69
Vlastimil B. Dlab, <i>A characterization of perfect rings</i>	79
Edward Richard Fadell, <i>Some examples in fixed point theory</i>	89
Michael Benton Freeman, <i>Tangential Cauchy-Riemann equations and uniform approximation</i>	101
Barry J. Gardner, <i>Torsion classes and pure subgroups</i>	109
Vinod B. Goyal, <i>Bounds for the solution of a certain class of nonlinear partial differential equations</i>	117
Fu Cheng Hsiang, <i>On $C, 1$ summability factors of Fourier series at a given point</i>	139
Lawrence Stanislaus Husch, Jr., <i>Homotopy groups of PL-embedding spaces</i>	149
Daniel Ralph Lewis, <i>Integration with respect to vector measures</i>	157
Marion-Josephine Lim, <i>$\mathcal{L} - 2$ subspaces of Grassmann product spaces</i>	167
Stephen J. Pierce, <i>Orthogonal groups of positive definite multilinear functionals</i>	183
W. J. Pugh and S. M. Shah, <i>On the growth of entire functions of bounded index</i>	191
Siddani Bhaskara Rao and Ayyagari Ramachandra Rao, <i>Existence of triconnected graphs with prescribed degrees</i>	203
Ralph Tyrrell Rockafellar, <i>On the maximal monotonicity of subdifferential mappings</i>	209
R. Shantaram, <i>Convergence of a sequence of transformations of distribution functions. II</i>	217
Julianne Soucek, <i>Rings of analytic functions</i>	233
Ted Joe Suffridge, <i>The principle of subordination applied to functions of several variables</i>	241
Wei-lung Ting, <i>On secondary characteristic classes in cobordism theory</i>	249
Pak-Ken Wong, <i>Continuous complementors on B^*-algebras</i>	255
Miyuki Yamada, <i>On a regular semigroup in which the idempotents form a band</i>	261