ORTHOGONAL GROUPS OF POSITIVE DEFINITE MULTILINEAR FUNCTIONALS

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Let $V$ be a finite dimensional vector space over the real numbers $R$ and let $T: V \to V$ be a linear transformation. If $\varphi: \times_1^m V \to R$ is a real multilinear functional and

$$\varphi(Tx_1, \cdots, T x_m) = \varphi(x_1, \cdots, x_m),$$

$x_1, \cdots, x_m \in V$, $T$ is called an isometry with respect to $\varphi$. We say $\varphi$ is positive definite if $\varphi(x, \cdots, x) > 0$ for all nonzero $x \in V$. In this paper we prove that if $\varphi$ is positive definite and $T$ is an isometry with respect to $\varphi$, then all eigenvalues of $T$ have modulus one and all elementary divisors of $T$ over the complex numbers are linear.

Let $V$ be an $n$-dimensional vector space over the real numbers $R$. Let $T: V \to V$ be a linear transformation of $V$. The following theorem [1, Th. 3] is easy to prove:

**THEOREM 1.** There exists a positive definite symmetric quadratic form $\varphi: V \times V \to R$ such that

(1) $\varphi(Tx, Ty) = \varphi(x, y), x, y \in V$

if and only if

1. all eigenvalues of $T$ have modulus 1;
2. all elementary divisors of $T$ over the complex numbers $C$ are linear.

Moreover, if $T$ satisfies (2), then there is a positive definite symmetric $\varphi$ such that (1) holds.

Theorem 1 can also be expressed in matrix theoretic terms. If $A$ is a real $n \times n$ positive definite symmetric matrix and $X$ is any automorph of $A$;

(3) $X^T AX = A$,

then $X$ satisfies (2); moreover, if an $n \times n$ matrix $X$ satisfies (2), then there is a positive definite symmetric $A$ such that (3) holds.

Let $\varphi: \times_1^m V \to R$ be a real multilinear functional. Let $H$ be a subgroup of the symmetric group $S_m$. If

(4) $\varphi(x_{a(1)}, \cdots, x_{a(m)}) = \varphi(x_1, \cdots, x_m)$
for all \( \sigma \in H \) and all \( x_i \in V, i = 1, \ldots, m \), then \( \varphi \) is said to be symmetric with respect to \( H \). If

\[
\varphi(Tx_1, \ldots, Tx_m) = \varphi(x_1, \ldots, x_m)
\]

for all \( x_i \in V \), \( T \) is called an isometry of \( V \) with respect to \( \varphi \). (Note that if \( m > 2 \), (5) has no matrix analogue). Let \( \Omega_m(H, T) \) be the set of all \( \varphi \) satisfying (4) and (5). Clearly \( \Omega_m(H, T) \) is a subspace of the vector space of all multilinear functionals symmetric with respect to \( H \). We say \( \varphi \) is positive definite if

\[
\varphi(x, \ldots, x) > 0
\]

for all nonzero \( x \) in \( V \). The set of all positive definite \( \varphi \) in \( \Omega_m(H, T) \) is denoted by \( P_m(H, T) \). It is clear that \( P_m(H, T) \) is a (possibly empty) convex cone in \( \Omega_m(H, T) \).

The following result [1] was proved as a partial generalization of Theorem 1.

**Theorem 2.** Let \( T: V \to V \) be linear. If \( P_m(H, T) \) is nonempty, then

- (a) \( m \) is even
- (b) every eigenvalue \( \gamma \) of \( T \) has modulus 1
- (c) elementary divisors of \( T \) corresponding to \( \gamma = \pm 1 \) are linear.

Conversely, if \( m \) is even, all eigenvalues of \( T \) are \( \pm 1 \), and all elementary divisors of \( T \) are linear, then \( P_m(H, T) \) is nonempty.

We conjectured that if \( P_m(H, T) \) is nonempty, then (c) can be replaced by (c') “all elementary divisors of \( T \) over the complex field are linear.” This would provide a complete generalization of Theorem 2, and thus justify (6) as a definition of a positive definite multilinear functional. The purpose of this paper is to prove this conjecture.

**Theorem 3.** If \( P_m(H, T) \) is nonempty, then

- (a) \( m \) is even
- (b) all eigenvalues of \( T \) have modulus 1
- (c') all elementary divisors of \( T \) over \( C \) are linear.

Conversely, if (a), (b), and (c') hold, then \( P_m(H, T) \) is nonempty.

**2. Proof of Theorem 3.** Assume that \( P_m(H, T) \) is nonempty. Parts (a) and (b) follow from Theorem 2. We now prove two lemmas.

**Lemma 1.** If \( \gamma \) is an eigenvalue of \( T \) and \( (x - \gamma)^k, k > 1, \) is a nonlinear elementary divisor of \( T \) corresponding to \( \gamma \), then \( \gamma^m \neq 1 \) for any integer \( m \).
Proof. Since $T$ is a real transformation, it has a real elementary divisor

$$[(x - \gamma)(x - \overline{\gamma})]^k.$$  

(By Theorem 2, $\gamma$ cannot be real in this case.) Let $W$ be the invariant subspace of $T$ determined by (7), and let $S$ be the restriction of $T$ to $W$. Then $S$ is an isometry of $W$ with respect to $\varphi$, and hence $S^r$ is also an isometry for any integer $r$. Now if $\gamma^r = 1$, then all eigenvalues of $S^r$ are 1, and hence Theorem 2 implies that all elementary divisors of $S^r$ are linear. Therefore, $S^r$ is the identity on $W$, and thus, the elementary divisors of $S$ are linear, a contradiction.

**LEMMA 2.** If Theorem 3 is true for the case $H = S_m$, then it is true for any subgroup $H$ of $S_m$.

Proof. Let $H$ be a subgroup of $S_m$ and let $\varphi \in P_m(H, T)$. For each $\sigma \in S_m$, define

$$\varphi_\sigma(x_1, \ldots, x_m) = \varphi(x_{\sigma(1)}, \ldots, x_{\sigma(m)}),$$

$x_1, \ldots, x_m \in V$. In general, $\varphi_\sigma$ is not symmetric with respect to $H$, but $\varphi_\sigma$ is positive definite and $T$ is an isometry with respect to $\varphi_\sigma$. Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma.$$  

Clearly $\psi$ is positive definite, and $T$ is an isometry with respect to $\psi$. Moreover, for any $\tau \in S_m$, and $x_1, \ldots, x_m \in V$,

$$\psi(x_{\tau(1)}, \ldots, x_{\tau(m)}) = \sum_{\sigma \in S_m} \varphi_\sigma(x_{\tau(1)}, \ldots, x_{\tau(m)})$$

$$= \sum_{\sigma \in S_m} \varphi(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

$$= \sum_{\rho \in S_m} \varphi(x_{\rho(1)}, \ldots, x_{\rho(m)})$$

$$= \sum_{\rho \in S_m} \varphi_\rho(x_1, \ldots, x_m)$$

$$= \psi(x_1, \ldots, x_m).$$

Thus $\psi \in P_m(S_m, T)$, and hence the elementary divisors of $T$ are linear. This proves Lemma 2.

We may assume henceforth that $H = S_m$ and abbreviate $P_m(S_m, T)$ to $P_m$. If $P_m$ is nonempty, and $T$ has a nonlinear elementary divisor over $C$ corresponding to the eigenvalue $\gamma = a + ib$ ($b \neq 0$), then there exist four linearly independent vectors $v_1, \ldots, v_4$ in $V$ such that
(10)

Let $\bar{V}$ be the extension of $V$ to an $n$-dimensional space over $C$, i.e., $\bar{V}$ consists of all vectors of the form $x + iy$, $x, y \in V$. By linear extension, we regard $T$ as a linear transformation of $\bar{V}$, and by multilinear extension, $\varphi$ becomes a complex valued multilinear functional on $\times_1^n \bar{V}$. Equation (5) still holds in $\bar{V}$, but $\varphi$ is no longer positive definite. Set

$$e_1 = v_1 + iv_2, \quad e_2 = v_1 - iv_2$$
$$e_3 = v_3 + iv_4, \quad e_4 = v_3 - iv_4.$$

From (10) and (11),

(12)

$$Te_1 = \gamma e_1, \quad Te_2 = \bar{\gamma} e_2$$
$$Te_3 = \gamma e_3 + v_2, \quad Te_4 = \bar{\gamma} e_4 + v_2.$$

By Lemma 1, $\gamma$ is not a root of unity; thus,

$$\varphi(e_1, \cdots, e_1, e_2, \cdots, e_2) = \varphi(Te_1, \cdots, Te_1, Te_2, \cdots, Te_2)$$
$$= \gamma^k \bar{\gamma}^{m-k} \varphi(e_1, \cdots, e_1, e_2, \cdots, e_2)$$
$$= 0,$$

unless $k = m - k$, where $k$ is the number of times $e_1$ occurs in (13). With $r = m/2$, we set

$$\varphi(e_1, \cdots, e_1, e_2, \cdots, e_2) = \nu.$$

Now $\nu \neq 0$; otherwise

$$\varphi(v_1, \cdots, v_1) = 2^{-m} \varphi(e_1 + e_2, \cdots, e_1 + e_2)$$
$$= 0,$$

contradicting (6). (Note that we are using the assumption that $\varphi$ is symmetric with respect to $S_m$; this gives us a convenient way of sorting expressions such as those on the right side of (14).)

Let $\mu = \varphi(v_1, \cdots, v_1, e_2)$. Using (13) and (14), we compute,

$$\mu = 2^{-m+1} \varphi(e_1 + e_2, \cdots, e_1 + e_2, e_2)$$
$$= 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots, \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + v_2)$$
$$= 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots, \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + \frac{e_1 - e_2}{2i})$$
$$= -2^{-m} i \binom{m-1}{r} (\gamma - \bar{\gamma}) \nu + \gamma 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots, \gamma e_1 + \bar{\gamma} e_2, e_3)$$
Continuing this procedure, we obtain for any positive integer \( s \)

\[
\mu = -2^{-m}i \left( \frac{m-1}{r} \right) (2\gamma - \gamma - \gamma^2) \nu + \gamma^2 2^{-m+1}
\]

\[\varphi(\gamma^e_1 + \bar{\gamma}^e_2, \cdots, \gamma^e_1 + \bar{\gamma}^e_2, e_3).\]

Let

\[ f(z) = z \varphi(ze_1 + \bar{z}e_2, \cdots, ze_1 + \bar{z}e_2, e_3), \]

where \( z \) is a complex variable. Then \( f \) is a continuous function of \( z \) on the complex plane, and hence \( f \) is bounded on the unit circle. Moreover, since \( \gamma \) is not a root of unity (in particular, \( \gamma \neq \pm 1 \)),

\[
\sum_{j=0}^{s-1} \gamma^{2j-1}
\]

is also bounded as \( s \) becomes large. Thus, letting \( s \) approach infinity in (15) forces \( \mu \) to become infinite, a contradiction. This proves Theorem 3 in one direction.

Now suppose all eigenvalues of \( T \) are 1 in absolute value and all elementary divisors of \( T \) are linear over \( C \). Let 1 (\( p \) times), \(-1 \) (\( q \) times) and \( \gamma_j, \bar{\gamma}_j = a_j \pm ib_j, |\gamma_j| = 1, j = 1, \cdots, t \), be the eigenvalues of \( T \). Then there is a basis of \( V, v_1, \cdots, v_p, u_1, \cdots, u_q, x_1, y_1, \cdots x_t, y_t \) such that

\[
Tv_j = v_j, j = 1, \cdots, p
\]
\[
Tu_j = -u_j, j = 1, \cdots, q
\]
\[
Tx_j = a_j x_j - b_j y_j, j = 1, \cdots, t
\]
\[
Ty_j = b_j x_j + a_j y_j, j = 1, \cdots, t.
\]

Set

\[
w_j = x_j + iy_j
\]
\[
\bar{w}_j = x_j - iy_j, j = 1, \cdots, t.
\]

Then \( v_1, \cdots, v_p, u_1, \cdots, u_q, w_1, \bar{w}_1, \cdots, w_t, \bar{w}_t \) form a basis of \( \bar{V} \) of eigenvectors of \( T \). Let \( f_1, \cdots, f_p, g_1, \cdots, g_q, h_1, k_1, \cdots, h_t, k_t \) be the corresponding dual basis. If \( l_1, \cdots, l_m \) are linear functionals on a space \( V \), then \( l_1 \cdots l_m \) is the \( m \)-linear functional on \( \times_1^m V \) such that
Define $\varphi$ as follows:

$$\varphi = \sum_{j=1}^{2} f_j^m + \sum_{j=1}^{q} g_j^m + \sum_{j=1}^{t} [(h_j k_{2j})^r + (\bar{h}_j \bar{k}_{2j})^r],$$

where $r = m/2$ and $f(v) = f(\overline{v})$. Now $h_j$ and $k_j$ are not linear on the complex space $\mathbb{V}$, but they are complex valued linear functionals on $V$, i.e., they are linear functionals on $V$ but are not in the dual space of $V$. Thus $\varphi$ is a real multilinear functional on $V$. Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma.$$

We assert that $\psi \in P_m(H, T)$. Clearly $\psi$ is symmetric with respect to $S_m$, and thus with respect to any subgroup $H$ of $S_m$. It remains to show that $\psi$ is positive definite and that $T$ is an isometry with respect to $\psi$. It suffices to prove these last two properties for $\varphi$. Let

$$x = \sum_{j=1}^{p} \alpha_j v_j + \sum_{j=1}^{q} \beta_j u_j + \sum_{j=1}^{t} (\delta_j x_j + \lambda_j y_j)$$

be an arbitrary vector of $V$. Then from (17),

$$\varphi(x, \cdots, x) = \sum_{j=1}^{p} \alpha_j^m + \sum_{j=1}^{q} \beta_j^m + 2 \sum_{j=1}^{t} \left(\frac{\delta_j^2}{2} + \frac{\lambda_j^2}{2}\right)^r.$$

Since $m$ is even and $\alpha_j$, $\beta_j$, $\delta_j$, $\lambda_j$ are all real, $\varphi$ is positive definite. Now let $z_k$, $k = 1, \cdots, m$, be arbitrary vectors in $V$, with

$$z_k = \sum_{j=1}^{p} a_{kj} v_j + \sum_{j=1}^{q} b_{kj} u_j + \sum_{j=1}^{t} (c_{kj} x_j + d_{kj} y_j).$$

Then

$$\varphi(z_1, \cdots, z_m) = \sum_{j=1}^{p} \prod_{k=1}^{m} a_{kj} + \sum_{j=1}^{q} \prod_{k=1}^{m} b_{kj}$$

$$\quad + \sum_{j=1}^{t} \prod_{k=1}^{r} \left(\frac{c_{kj} - d_{kj}}{2} - \frac{d_{kj} - c_{kj}}{2i}\right) \left(\frac{c_{kj} + d_{kj}}{2} + \frac{d_{kj} + c_{kj}}{2i}\right).$$

From (16)

$$Tz_k = \sum_{j=1}^{p} a_{kj} v_j + \sum_{j=1}^{q} (-b_{kj}) u_j$$

$$\quad + \sum_{j=1}^{t} (a_j c_{kj} + b_j d_{kj}) x_j + (a_j d_{kj} - b_j c_{kj}) y_j.$$
\( k = 1, \ldots, m \). Let
\[
e_{kj} = a_jc_{kj} + b_jd_{kj}
\]
\[
f_{kj} = a_jd_{kj} - b_jc_{kj}.
\]

Then from (19) and (20)
\[
\phi(T_{z_1}, \ldots, T_{z_m}) = \sum_{j=1}^{m} \prod_{k=1}^{n} a_{kj} + \sum_{j=1}^{m} \prod_{k=1}^{n} (-b_{kj})
\]
\[
+ \sum_{j=1}^{m} \prod_{k=1}^{n} \left( \frac{e_{2k-1,j}}{2} + \frac{f_{2k-1,j}}{2i} \right) \left( \frac{e_{2k,j}}{2} - \frac{f_{2k,j}}{2i} \right)
\]
\[
+ \sum_{j=1}^{m} \prod_{k=1}^{n} \left( \frac{e_{2k-1,j}}{2} - \frac{f_{2k-1,j}}{2i} \right) \left( \frac{e_{2k,j}}{2} + \frac{f_{2k,j}}{2i} \right).
\]

It is easily verified that
\[
\frac{e_{kj}}{2} + \frac{f_{kj}}{2i} = \gamma_j \left( \frac{e_{kj}}{2} + \frac{d_{kj}}{2i} \right)
\]
\[
\frac{e_{kj}}{2} - \frac{f_{kj}}{2i} = \gamma_j \left( \frac{e_{kj}}{2} - \frac{d_{kj}}{2i} \right).
\]

Using (22) in (21) and the fact that \( |\gamma_j| = 1 \), we obtain
\[
\phi(T_{z_1}, \ldots, T_{z_m}) = \phi(z_1, \ldots, z_m).
\]

This completes the proof of Theorem 3.

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Silvio Aurora, *On normed rings with monotone multiplication* .................... 15
Silvio Aurora, *Normed fields which extend normed rings of integers* .......... 21
John Kelly Beem, *Indefinite Minkowski spaces* ........................................ 29
T. F. Bridgland, *Trajectory integrals of set valued functions* ....................... 43
Robert Jay Buck, *A generalized Hausdorff dimension for functions and sets* . 69
Vlastimil B. Dlab, *A characterization of perfect rings* ................................. 79
Edward Richard Fadell, *Some examples in fixed point theory* ...................... 89
Michael Benton Freeman, *Tangential Cauchy-Riemann equations and uniform approximation* ................................................................. 101
Barry J. Gardner, *Torsion classes and pure subgroups* ................................. 109
Vinod B. Goyal, *Bounds for the solution of a certain class of nonlinear partial differential equations* .......................................................... 117
Fu Cheng Hsiang, *On $C_1$ summability factors of Fourier series at a given point* ................................................................. 139
Lawrence Stanislaus Husch, Jr., *Homotopy groups of PL-embedding spaces* .... 149
Daniel Ralph Lewis, *Integration with respect to vector measures* .................... 157
Marion-Josephine Lim, *$L_2$ subspaces of Grassmann product spaces* ......... 167
Stephen J. Pierce, *Orthogonal groups of positive definite multilinear functionals* ................................................................. 183
W. J. Pugh and S. M. Shah, *On the growth of entire functions of bounded index* .................................................................................. 191
Siddani Bhaskara Rao and Ayyagari Ramachandra Rao, *Existence of triconnected graphs with prescribed degrees* ................................. 203
Ralph Tyrrell Rockafellar, *On the maximal monotonicity of subdifferential mappings* ........................................................................... 209
R. Shantaram, *Convergence of a sequence of transformations of distribution functions. II* ................................................................. 217
Julianne Souchek, *Rings of analytic functions* ............................................ 233
Ted Joe Suffridge, *The principle of subordination applied to functions of several variables* .......................................................... 241
Wei-lung Ting, *On secondary characteristic classes in cobordism theory* ....... 249
Pak-Ken Wong, *Continuous complementors on $B^*$-algebras* ...................... 255
Miyuki Yamada, *On a regular semigroup in which the idempotents form a band* ............................................................................. 261