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**ON THE GROWTH OF ENTIRE FUNCTIONS OF BOUNDED  
INDEX**

W. J. PUGH AND S. M. SHAH

## ON THE GROWTH OF ENTIRE FUNCTIONS OF BOUNDED INDEX

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A class  $E$  of entire functions of zero order and with widely spaced zeros has been defined and it is proved that if  $f \in E$  then  $f', f'', \dots \in E$ . Furthermore  $f$  is of index one. This class includes many functions which are both of bounded index and arbitrarily slow growth. If  $f$  is any transcendental entire function then there is an entire function  $g$  of unbounded index with the same asymptotic behavior. When  $f$  is of infinite order then it is of unbounded index and we simply take  $g = f$ . When  $f$  is of finite order we give the construction for  $g$ .

DEFINITION 1. An entire function  $f(z)$  is said to be of bounded index if there exists an integer  $M$ , independent of  $z$ , such that

$$\left| \frac{f^{(n)}(z)}{n!} \right| \leq \max_{0 \leq s \leq M} \left\{ \left| \frac{f^{(s)}(z)}{s!} \right| \right\}$$

for all  $n$  and all  $z$ . The least such integer  $M$  is called the index of  $f(z)$ .

Although functions of bounded index have been the object of a number of recent investigations (cf: [3], [5], [6], [7]–[9]), little is known about their properties, and most of the following natural questions seem to require further study.

I. *What are the growth properties of functions of bounded index:*

- (a) can they increase arbitrarily rapidly,
- (b) can they increase arbitrarily slowly,
- (c) is it possible to derive the boundedness (or the unboundedness) of the index from the asymptotic properties of the logarithm of the maximum modulus of  $f(z)$ , i.e.,  $\log M(r, f)$ ?

II. *Classes of functions of bounded index:*

- (a) find classes of functions of bounded index,
- (b) is the sum (or product) of two functions of bounded index also of bounded index?

Question I(a) was settled by Shah [8] who proved that the growth of functions of bounded index is at most of the exponential type of order one. (See also Lepson [6].) Shah [8] and Lepson [6] have constructed functions of arbitrarily slow growth and of unbounded index.

In the present note we derive a simple answer to Question I(b) from the consideration of

*Functions with widely spaced zeros.* Let  $f(z)$  be an entire function of genus zero, and let  $\{a_j\}_{j=1}^{\infty}$  be the sequence of its zeros. We say that  $f(z)$  has widely spaced zeros if the zeros  $\{a_j\}$  are all simple and

$$|a_1| \geq a = 5, |a_{n+1}| \geq a^n |a_n| \quad (n = 1, 2, 3, \dots).$$

Using this definition we prove

**THEOREM 1.** *Let  $f(z)$  have widely spaced zeros. Then, for all  $z$ ,*

$$|f^{(n)}(z)| < \max \{|f(z)|, |f'(z)|\} \quad (n = 2, 3, 4, \dots).$$

**COROLLARY 1.1.** *Functions with widely spaced zeros are of bounded index.*

**COROLLARY 1.2.** *There exist functions of bounded index and of arbitrarily slow growth.*

Corollary 1.1 may also be considered as a contribution to Question II(a). Corollary 1.2 answers Question I(b). Other contributions, due to separate efforts of the present authors, will be found elsewhere. In [9] Shah proves that all solutions of certain classes of linear differential equations are of bounded index. In his doctoral dissertation, Pugh shows that the functions

$$F_{\sigma}(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{j^{\sigma}}\right) \quad (\sigma > 8),$$

and

$$f_q(z) = \prod_{j=0}^{\infty} (1 - q^j z) \quad \left(0 < q < \frac{1}{16}\right),$$

are of bounded index. As a contribution to II(b), Pugh [7] has shown that the sum of two functions of bounded index need not be of bounded index.

Our second result clarifies one aspect of Question I(c). We prove

**THEOREM 2.** *Let  $f(z)$  be any transcendental entire function of finite order. It is always possible to find an entire function  $g(z)$ , of unbounded index such that*

$$\log M(r, f) \sim \log M(r, g) \quad (r \rightarrow \infty).$$

Choosing  $f(z)$  to be of bounded index, we see that it is always possible to find *functions of unbounded index with the same asymptotic behavior as  $f(z)$* .

The authors gratefully acknowledge the help of Professor Albert Edrei who suggested the class of functions with widely spaced zeros, and indicated the connection between Theorem 2 and the results of [2].

### 1. Successive derivatives of functions with widely spaced zeros.

LEMMA 1. *Let  $f(z)$  be an entire function with widely spaced zeros  $\{a_j\}_{j=1}^{\infty}$ . Let  $\{b_j\}_{j=1}^{\infty}$  ( $|b_j| \leq |b_{j+1}|$ ), be the zeros of  $f'(z)$ .*

*Then*

$$(1.1) \quad \frac{|a_{n+1}|}{b} < |b_n| \leq |a_{n+1}|, \quad (n \geq 2, b = 1.6),$$

and

$$(1.2) \quad \left(1 + \frac{2R + d}{a}\right) |a_1| < |b_1| \leq |a_2|, \quad (R = 2.4, d = 10^{-3}, |a_1| \geq a = 5).$$

*Proof.* In §§ 1-3, we shall write  $1.6 = b$ ,  $2.4 = R$ ,  $10^{-3} = d$ ,  $1 + (2R + d)/a = 1.9602 = c$ . Put

$$g_n(z) = \sum_{j=1}^n \frac{1}{z - a_j}, \quad (n \geq 1),$$

and

$$(1.3) \quad h_n(z) = \frac{f'(z)}{f(z)} - g_n(z) = \sum_{j=n+1}^{\infty} \frac{1}{z - a_j}.$$

Our proof of the lemma depends on obvious applications of Rouché's theorem [4, p. 254].

Let  $z = re^{i\theta}$  and

$$(1.4) \quad |a_n| < r < |a_{n+1}|, \quad (n \geq 1).$$

Clearly

$$\begin{aligned} \operatorname{Re}(zg_n(z)) &= \sum_{j=1}^n \frac{\operatorname{Re}(r^2 - z\bar{a}_j)}{|z - a_j|^2} \\ &\geq \sum_{j=1}^n \frac{r}{r + |a_j|} \end{aligned}$$

and hence

$$|g_n(z)| \geq \sum_{j=1}^n \frac{1}{r + |a_j|}.$$

In particular by the definition of widely spaced zeros we have

$$(1.5) \quad |g_n(z)| \geq \frac{n}{|a_{n+1}| + |a_n|} \geq \frac{n}{|a_{n+1}|} \frac{25}{26}, \quad (n \geq 2),$$

$$(1.6) \quad \left| g_n \left( \frac{|a_{n+1}|}{b} e^{i\theta} \right) \right| \geq 2 \left( \frac{|a_{n+1}|}{b} + |a_2| \right)^{-1} > \frac{3}{|a_{n+1}|}, \quad (n \geq 2).$$

For  $h_n(z)$  we have

$$(1.7) \quad \begin{aligned} \left| h_n \left( \frac{|a_{n+1}|}{b} e^{i\theta} \right) \right| &\leq \left( |a_{n+1}| - \frac{|a_{n+1}|}{b} \right)^{-1} + \left( |a_{n+2}| - \frac{|a_{n+1}|}{b} \right)^{-1} + \dots \\ &< \frac{b}{b-1} \frac{1}{|a_{n+1}|} + \frac{1.25}{|a_{n+2}| - (|a_{n+1}|/b)} \\ &< \frac{2.8}{|a_{n+1}|} \quad (n \geq 2). \end{aligned}$$

Now in the disc

$$(1.8) \quad |z| \leq \frac{|a_{n+1}|}{b},$$

$g_n(z)$  has  $n$  poles, and, by the theorem of Gauss-Lucas [10, p. 6], exactly  $(n-1)$  zeros. The function  $h_n(z)$  is regular in the disc (1.8), and by (1.6) and (1.7)

$$|g_n(z)| > |h_n(z)|, \quad \left( n \geq 2, |z| = \frac{|a_{n+1}|}{b} \right).$$

Hence, by Rouché's theorem

$$g_n(z) + h_n(z) = \frac{f'(z)}{f(z)}$$

has exactly  $(n-1)$  zeros in the disc (1.8).

We have thus proved

$$(1.9) \quad \frac{|a_{n+1}|}{b} < |b_n|, \quad (n \geq 2).$$

Similarly, for

$$(1.10) \quad r = |z| = \gamma |a_n|, \quad (1 < \gamma < 1.01, n \geq 2)$$

we have

$$\begin{aligned} |h_n(z)| &< (|a_{n+1}| - \gamma |a_n|)^{-1} + (|a_{n+2}| - \gamma |a_n|)^{-1} + \dots \\ &\leq (|a_{n+1}| - \gamma |a_n|)^{-1} + (1.1)(|a_{n+2}| - \gamma |a_n|)^{-1} \\ &\leq (\gamma |a_n| + |a_1|)^{-1} < |g_n(z)|. \end{aligned}$$

Again by Rouché's theorem  $f'(z)/f(z)$  has exactly  $(n - 1)$  zeros in any disc with center at the origin and a radius  $r$  satisfying (1.10). Hence

$$|b_{n-1}| < \gamma |a_n| \quad (n \geq 2),$$

and letting  $\gamma \rightarrow 1+$ , we obtain

$$(1.11) \quad |b_{n-1}| \leq |a_n| \quad (n \geq 2).$$

The second of the inequalities (1.2) also follows from (1.11).

We complete the proof of the lemma by showing that

$$(1.12) \quad |z| \leq c |a_1|$$

implies

$$(1.13) \quad \left| \frac{f'(z)}{f(z)} \right| > 0.$$

Thus  $f'(z)$  will have no zeros in the disc (1.12) and, therefore

$$c |a_1| < |b_1|,$$

which is the first of the inequalities (1.2).

In order to verify (1.13) notice that (1.12) and the definition of widely spaced zeros imply

$$\begin{aligned} \left| \frac{f'(z)}{f(z)} \right| &\geq \frac{1}{|a_1|} \left\{ \frac{1}{1+c} - \sum_2^\infty \frac{1}{a^{j(j-1)/2} - c} \right\} \\ &> 0. \end{aligned}$$

This completes the proof of Lemma 1.

LEMMA 2. *If  $f(z)$  has widely spaced zeros all the derivatives*

$$f'(z), f''(z), \dots$$

*have the same property.*

*Proof.* It is sufficient to prove that if  $f(z)$  has widely spaced zeros, the zeros of  $f'(z)$  are also widely spaced. By (1.2)

$$(1.14) \quad 9.801 \leq c |a_1| < |b_1|.$$

By (1.1) and (1.2)

$$\begin{aligned} |b_n| &\leq |a_{n+1}|, & (n \geq 1) \\ \frac{1}{b} |a_{n+2}| &< |b_{n+1}|, & (n \geq 1). \end{aligned}$$

Hence

$$(1.15) \quad \left| \frac{b_{n+1}}{b_n} \right| > \frac{|a_{n+2}|}{b |a_{n+1}|} \geq \frac{a^{n+1}}{b} > a^n \quad (n \geq 1).$$

The relations (1.14) and (1.15) show that the  $b$ 's are widely spaced.

2. **Minimum distance between a zero of  $f(z)$  and a zero of  $f'(z)$ .** The inequalities (1.1) do not preclude the possibility that  $|a_{n+1} - b_n|$  be very small. In this section we show that

$$(2.1) \quad \inf_{\substack{1 \leq j < \infty \\ 1 \leq k < \infty}} |a_j - b_k| > 2R + d.$$

I. From now on, we denote the zeros of  $f^{(k)}(z)$ , in order of ascending moduli by  $\{a_j^{(k)}\}_{j=1}^\infty$ . By definition  $a_n^{(0)} = a_n$  and  $f^{(0)} \equiv f$ .

II. We consider systematically the sets

$$D_k(\rho) = \bigcup_{j=1}^\infty \{z: |z - a_j^{(k)}| \leq \rho\} \quad (\rho > 0, k = 0, 1, \dots).$$

LEMMA 3. *If  $f(z)$  has widely spaced zeros, and if  $z \in D_0(R)$ , then*

$$(2.2) \quad \left| \frac{f'(z)}{f(z)} \right| < 1, \quad \left| \frac{f''(z)}{f(z)} \right| < 1.$$

*Proof.* The identities

$$\frac{d}{dz} \left( \frac{f'(z)}{f(z)} \right) = - \sum_{j=1}^\infty \frac{1}{(z - a_j)^2} = \frac{f''(z)}{f(z)} - \left( \frac{f'(z)}{f(z)} \right)^2$$

imply

$$\left| \frac{f''(z)}{f(z)} \right| \leq \sum_{j=1}^\infty \frac{1}{|z - a_j|^2} + \left( \sum_{j=1}^\infty \frac{1}{|z - a_j|} \right)^2 \leq 2 \left( \sum_{j=1}^\infty \frac{1}{|z - a_j|} \right)^2.$$

Hence, the inequalities (2.2) follow from the single inequality

$$(2.3) \quad \sum_{j=1}^\infty \frac{1}{|z - a_j|} < \frac{\sqrt{2}}{2}.$$

If  $z \in D_0(R)$ , and  $|z| < |a_1|$ , then

$$(2.4) \quad |z - a_1| > R$$

and

$$(2.5) \quad |z - a_j| \geq |a_j| - |z| > |a_j| - |a_1| > |a_1|(a^{j-1} - 1) > \frac{a^j}{2}, \quad (j \geq 2).$$

Hence

$$\sum_{j=1}^{\infty} \frac{1}{|z - a_j|} < \frac{1}{R} + 2 \sum_{j=2}^{\infty} \frac{1}{a^j} < \frac{\sqrt{2}}{2},$$

so that (2.3) holds if  $|z| < |a_1|$ .

In general, the relations

$$|a_n| \leq |z| < |a_{n+1}| \quad (n \geq 1), \quad z \in D_0(R)$$

imply

$$(2.6) \quad |z - a_j| \geq |z| - |a_j| \geq |a_n| - |a_{n-1}| > \frac{a^n}{2}$$

provided

$$(2.7) \quad n \geq 2, \quad j < n.$$

Similarly, for  $j > n + 1$

$$(2.8) \quad |z - a_j| \geq |a_j| - |a_{n+1}| > (a^{j-1} - 1) |a_{n+1}| > \frac{a^{j-1}}{2} |a_{n+1}|.$$

Finally,

$$(2.9) \quad \frac{1}{|z - a_n|} + \frac{1}{|z - a_{n+1}|} \leq \frac{1}{R} + (\max\{|z - a_n|, |z - a_{n+1}|\})^{-1}$$

with

$$(2.10) \quad \max\{|z - a_n|, |z - a_{n+1}|\} \geq \frac{|a_{n+1}| - |a_n|}{2} > \frac{(a^n - 1)|a_n|}{2}.$$

Combining (2.6), (2.8), (2.9) and (2.10), we find, for  $n \geq 2$ ,

$$(2.11) \quad \sum_{j=1}^{\infty} \frac{1}{|z - a_j|} < \frac{2(n-1)}{a^n} + \frac{1}{R} + \frac{2}{(a^n - 1)|a_n|} + \frac{2a}{|a_{n+1}|} \sum_{j=n+2}^{\infty} \frac{1}{a^j} < \frac{2(n-1)}{a^n} + \frac{1}{R} + \frac{2}{(a^n - 1)a} + \frac{2}{(a-1)a^{n+2}}.$$

It is easily seen that (2.11) holds for  $n = 1$  also and that (2.11) implies (2.3). Hence the lemma is proved.



LEMMA 4. If  $z \in D_0(2R + d)$ , then  $f'(z) \neq 0$ .

*Proof.* If  $z \in D_0(2R + d)$ , then for some  $n$ ,

$$(2.12) \quad |z - a_n| \leq 2R + d = 4.801.$$

Hence, if  $j < n$  and  $n \geq 2$ ,

$$(2.13) \quad \begin{aligned} |z - a_j| &\geq |z| - |a_{n-1}| \geq |a_n| - |a_{n-1}| - (2R + d) \\ &\geq |a_n| \left(1 - \frac{1}{a} - \frac{2R + d}{a^2}\right) > \frac{6}{10} |a_n|. \end{aligned}$$

If  $j > n$ , then

$$(2.14) \quad \begin{aligned} |z - a_j| &\geq |a_j| - |a_n| - (2R + d) \\ &> |a_j| \left(1 - \frac{1}{a} - \frac{2R + d}{a^2}\right) > \frac{6}{10} |a_j|. \end{aligned}$$

By (2.12), (2.13), and (2.14) we have, for  $n \geq 2$ ,

$$(2.15) \quad \begin{aligned} \left| \frac{f'(z)}{f(z)} \right| &\geq \frac{1}{4.801} - \frac{10(n-1)}{6|a_n|} - \frac{10}{6} \sum_{j=n+1}^{\infty} \frac{1}{|a_j|}, \\ &\geq \frac{1}{4.801} - \frac{1}{3} \frac{(n-1)}{a^{n(n-1)/2}} - \frac{5}{12} \frac{1}{a^{(n+1)n/2}}. \end{aligned}$$

Again, it is easily seen that (2.15) holds for  $n = 1$  also. The expression on the right of (2.15) is positive and consequently in  $D_0(2R + d)$ ,  $f'(z) \neq 0$  unless  $f(z) = 0$ . On the other hand  $f'(z) \neq 0$  if  $f(z) = 0$  because all the zeros of  $f(z)$  are simple. This completes the proof of Lemma 4.

**3. Proof of Theorem 1.** Because all the derivatives of  $f(z)$  have widely spaced zeros, Lemmas 1 to 4 apply to all of the functions  $f^{(k)}(z)$ , ( $k = 0, 1, 2, 3, \dots$ ). In particular Lemma 4 shows that the sets  $D_{n-2}(R)$  and  $D_{n-1}(R)$  are disjoint for  $n \geq 2$ .

Hence, by Lemma 3, at least one of the two inequalities

$$(3.1) \quad \left| \frac{f^{(n)}(z)}{f^{(n-2)}(z)} \right| < 1, \quad \left| \frac{f^{(n)}(z)}{f^{(n-1)}(z)} \right| < 1 \quad (n \geq 2)$$

must hold.

Thus, for all  $z$

$$(3.2) \quad |f^{(n)}(z)| < \max\{|f^{(n-1)}(z)|, |f^{(n-2)}(z)|\} \quad (n = 2, 3, 4, \dots).$$

Theorem 1 follows from (3.2) by an obvious induction over  $n$ .

4. **Proof of Theorem 2.** In this section we assume familiarity with the most elementary results and notations of Nevanlinna's theory of meromorphic functions.

Let  $f(z)$  be a given entire, nonrational function of finite order. A theorem of Edrei and Fuchs [2; p. 384 and p. 390, formula (3.5)] asserts the existence of an entire function  $h(z)$  such that  $h(0) = 1$  and

$$(4.1) \quad N\left(r, \frac{1}{h}\right) \sim \log M(r, h) \sim \log M(r, f) \quad (r \rightarrow +\infty).$$

We take  $g(z)$  to be of the form

$$(4.2) \quad g(z) = h(z)P(z),$$

where

$$(4.3) \quad P(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{d_j}\right)^j.$$

The quantities  $d_j$  are positive and satisfy the following conditions:

- (i)  $d_1 > e^2, d_{j+1} > d_j^2$  ( $j = 1, 2, 3, \dots$ );
- (ii) for  $t \geq d_j$ ,

$$\frac{j(j+1)}{2} < \left\{ \frac{\log M(t, f)}{\log t} \right\}^{1/2}.$$

Since  $f(z)$  is not rational

$$(4.4) \quad \frac{\log M(t, f)}{\log t} \longrightarrow +\infty \quad (t \rightarrow +\infty)$$

and hence it is possible to satisfy condition (ii).

Putting

$$n(t) = n\left(t, \frac{1}{P}\right),$$

we see that

$$(4.5) \quad n(t) = 0 \quad (0 \leq t < d_1), \quad n(t) = \frac{k(k+1)}{2} \quad (d_k \leq t < d_{k+1}).$$

Hence, if

$$(4.6) \quad d_k \leq t < d_{k+1} \quad (k \geq 1)$$

(4.5) and condition (i) imply

$$(4.7) \quad n(t) < 2^k < \log d_k \leq \log t < t^{1/2} \quad (k \geq 1).$$

By (4.6), (i) and (4.5)

$$t^2 < d_{k+1}^2 < d_{k+2},$$

$$(4.8) \quad \frac{n(t^2)}{n(t)} \leq 1 + \frac{2}{k}.$$

By (4.6), (ii), (4.5) and (4.4)

$$(4.9) \quad n(t) \log t < \log M(t, f) \left\{ \frac{\log t}{\log M(t, f)} \right\}^{1/2} = o(\log M(t, f)) \quad (t \rightarrow \infty).$$

By (4.1), (4.2) and the elements of Nevanlinna's theory

$$\begin{aligned} (1 + o(1)) \log M(r, f) &= N\left(r, \frac{1}{h}\right) \leq N\left(r, \frac{1}{g}\right) \\ &\leq \log M(r, g) \leq \log M(r, h) + \log M(r, P) \\ &= \log M(r, f) \left\{ 1 + o(1) + \frac{\log M(r, P)}{\log M(r, f)} \right\} \quad (r \rightarrow +\infty). \end{aligned}$$

Hence, in order to obtain Theorem 2 it is sufficient to show that

$$(4.10) \quad \frac{\log M(r, P)}{\log M(r, f)} \longrightarrow 0 \quad (r \rightarrow +\infty)$$

and to remark that  $g(z)$  cannot be of bounded index because it has zeros of arbitrarily high multiplicity.

The relation (4.10) follows readily from the identity [1, p. 48]

$$\log M(r, P) = r \int_0^\infty \frac{n(t)}{t(t+r)} dt,$$

which, in view of (4.7), (4.8) and (4.9), leads to

$$\begin{aligned} \log M(r, P) &< n(r) \log r + r \int_r^{r^2} \frac{n(r^2)}{t^2} dt + r \int_{r^2}^\infty t^{-3/2} dt \\ &= o(\log M(r, f)) \quad (r \rightarrow +\infty). \end{aligned}$$

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