ON THE MAXIMAL MONOTONICITY OF SUBDIFFERENTIAL MAPPINGS

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The subdifferential of a lower semicontinuous proper convex function on a Banach space is a maximal monotone operator, as well as a maximal cyclically monotone operator. This result was announced by the author in a previous paper, but the argument given there was incomplete; the result is proved here by a different method, which is simpler in the case of reflexive Banach spaces. At the same time, a new fact is established about the relationship between the subdifferential of a convex function and the subdifferential of its conjugate in the nonreflexive case.

Let $E$ be a real Banach space with dual $E^*$. A proper convex function on $E$ is a function $f$ from $E$ to $(-\infty, +\infty]$, not identically $+\infty$, such that

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

whenever $x \in E$, $y \in E$ and $0 < \lambda < 1$. The subdifferential of such a function $f$ is the (generally multivalued) mapping $\partial f : E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \forall y \in E\} ,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $E$ and $E^*$.

A multivalued mapping $T : E \to E^*$ is said to be a monotone operator if

$$\langle x_0 - x_i, x_0^* - x_i^* \rangle \geq 0 \text{ whenever } x_0^* \in T(x_0), x_i^* \in T(x_i) .$$

It is said to be a cyclically monotone operator if

$$\langle x_0 - x_i, x_0^* \rangle + \cdots + \langle x_{n-1} - x_n, x_{n-1}^* \rangle + \langle x_n - x_0, x_n^* \rangle \geq 0$$

whenever $x_i^* \in T(x_i), i = 0, \cdots, n$.

It is called a maximal monotone operator (resp. maximal cyclically monotone operator) if, in addition, its graph

$$G(T) = \{(x, x^*) \mid x^* \in T(x)\} \subset E \times E^*$$

is not properly contained in the graph of any other monotone (resp. cyclically monotone) operator $T' : E \to E^*$.

This note is concerned with proving the following theorems.
**THEOREM A.** If $f$ is a lower semicontinuous proper convex function on $E$, then $\partial f$ is a maximal monotone operator from $E$ to $E^*$.

**THEOREM B.** Let $T : E \to E^*$ be a multivalued mapping. In order that there exist a lower semicontinuous proper convex function $f$ on $E$ such that $T = \partial f$, it is necessary and sufficient that $T$ be a maximal cyclically monotone operator. Moreover, in this case $T$ determines $f$ uniquely up to an additive constant.

These theorems have previously been stated by us in [4] as Theorem 4 and Theorem 3, respectively. However, a gap occurs in the proofs in [4], as has kindly been brought to our attention recently by H. Brézis. (It is not clear whether formula (4.7) in the proof of Theorem 3 of [4] will hold for $\varepsilon$ sufficiently small, because $x^*_\varepsilon$ depends on $\varepsilon$ and could conceivably increase unboundedly in norm as $\varepsilon$ decreases to 0. The same oversight appears in the penultimate sentence of the proof of Theorem 4 of [4]). In view of this oversight, the proofs in [4] are incomplete; further arguments must be given before the maximality in Theorem A, the maximality in the necessary condition in Theorem B, and the uniqueness in Theorem B can be regarded as established. Such arguments will be given here.

2. **Preliminary result.** Let $f$ be a lower semicontinuous proper convex function on $E$. (For proper convex functions, lower semicontinuity in the strong topology of $E$ is the same as lower semicontinuity in the weak topology.) The conjugate of $f$ is the function $f^*$ on $E^*$ defined by

\[
 f^*(x^*) = \sup \{\langle x, x^* \rangle - f(x) \mid x \in E \}. 
\]

It is known that $f^*$ is a weak* lower semicontinuous (and hence strongly lower semicontinuous) proper convex function on $E^*$, and that

\[
 f(x) + f^*(x^*) = \langle x, x^* \rangle \geq 0, \forall x \in E, \forall x^* \in E^*, 
\]

with equality if and only if $x^* \in \partial f(x)$

(see Moreau [3, § 6]). The subdifferential $\partial f^*$, which is a multivalued mapping from $E^*$ to the bidual $E^{**}$, can be compared with the subdifferential $\partial f$ from $E$ to $E^*$, when $E$ is regarded in the canonical way as a weak** dense subspace of $E^{**}$ (the weak** topology being the weak topology induced on $E^{**}$ by $E^*$). Facts about the relationship between $\partial f^*$ and $\partial f$ will be used below in proving Theorems A and B.

In terms of the conjugate $f^{**}$ of $f^*$, which is the weak** lower semicontinuous proper convex function on $E^{**}$ defined by
(2.3) \[ f^{**}(x^{**}) = \sup \left\{ \langle x^{**}, x^* \rangle - f^*(x^*) \mid x^* \in E^* \right\}, \]
we have, as in (2.2),
\[ (2.4) \quad f^{**}(x^{**}) + f^*(x^*) - \langle x^{**}, x^* \rangle \geq 0, \forall x^{**} \in E^{**}, \forall x^* \in E^*, \]
with equality if and only if \( x^{**} \in \partial f^*(x^*) \).

Moreover, the restriction of \( f^{**} \) to \( E \) is \( f \) (see [3, §6]). Thus, if \( E \) is reflexive, we can identify \( f^{**} \) with \( f \), and it follows from (2.2) and (2.4) that \( \partial f^* \) is just the “inverse” of \( \partial f \), in other words one has \( x \in \partial f^*(x^*) \) if and only if \( x^* \in \partial f(x) \). If \( E \) is not reflexive, the relationship between \( \partial f^* \) and \( \partial f \) is more complicated, but \( \partial f^* \) and \( \partial f \) still completely determine each other, according to the following result.

**Proposition 1.** Let \( f \) be a lower semicontinuous proper convex function on \( E \), and let \( x^* \in E^* \) and \( x^{**} \in E^{**} \). Then \( x^{**} \in \partial f^*(x^*) \) if and only if there exists a net \( \{ x_i^* \mid i \in I \} \) in \( E^* \) converging to \( x^* \) in the strong topology and a bounded net \( \{ x_i \mid i \in I \} \) in \( E \) (with the same partially ordered index set \( I \)) converging to \( x^{**} \) in the weak** topology, such that \( x_i^* \in \partial f(x_i) \) for every \( i \in I \).

**Proof.** The sufficiency of the condition is easy to prove. Given nets as described, we have
\[ f(x_i) + f^*(x_i^*) = \langle x_i, x_i^* \rangle, \forall i \in I \]
by (2.2), where \( f(x_i) = f^*(x_i) \). Then by the lower semicontinuity of \( f^* \) and \( f^{**} \) we have
\[ f^{**}(x^{**}) + f^*(x^*) \leq \lim \inf \{ f^{**}(x_i) + f^*(x_i^*) \} \]
\[ = \lim \langle x_i, x_i^* \rangle = \langle x^{**}, x^* \rangle. \]
(The last equality makes use of the boundedness of the norms \( \| x_i \|, i \in I \).) Thus \( x^{**} \in \partial f^*(x^*) \) by (2.4).

To prove the necessity of the condition, we demonstrate first that, given any \( x^{**} \in E^{**} \), there exists a bounded net \( \{ y_i \mid i \in I \} \) in \( E \) such that \( y_i \) converges to \( x^{**} \) in the weak** topology and
\[ \lim f(y_i) = f^{**}(x^{**}). \]

Consider \( f + h_\alpha \), where \( \alpha \) is a positive real number and \( h_\alpha \) is the lower semicontinuous proper convex function on \( E \) defined by
\[ h_\alpha(x) = 0 \quad \text{if} \quad \| x \| \leq \alpha, \quad h_\alpha(x) = +\infty \quad \text{if} \quad \| x \| > \alpha. \]

Assuming that \( \alpha \) is sufficiently large, there exist points \( x \) at which \( f \) and \( h_\alpha \) are both finite and \( h_\alpha \) is continuous (i.e., points \( x \) such that \( f(x) < +\infty \) and \( \| x \| < \alpha \)). Then, by the formulas for conjugates of
sums of convex functions (see Moreau [3, pp. 38, 56, 57] or Rockafellar [5, Th. 3]), we have \((f + h_\alpha)^* = f^* \square h_\alpha^*\) (infimal convolution), and consequently

\[(2.7) \quad (f + h_\alpha)^* = (f^* \square h_\alpha^*)^* = f^{**} + h_\alpha^{**}.
\]

Moreover \(h_\alpha^*(x^*) = \alpha \|x^*\|\) for ever \(x^* \in E^*\), so that

\[
h_\alpha^*(x^{**}) = \sup \{\langle x^{**}, x^*\rangle - \alpha \|x^*\| \mid x^* \in E^*\} \]

\[
= \begin{cases} 
0 & \text{if } \|x^{**}\| \leq \alpha, \\
+\infty & \text{if } \|x^{**}\| > \alpha.
\end{cases}
\]

Hence by (2.7), given any \(x^{**} \in E^{**}\), we have

\[(2.8) \quad f^{**}(x^{**}) = (f + h_\alpha)^**(x^{**})
\]

for sufficiently large \(\alpha > 0\). On the other hand, it is known that, for any lower semicontinuous proper convex function \(g\) on \(E\), \(g^{**}\) is the greatest weak** lower semicontinuous function on \(E^{**}\) majorized by \(g\) on \(E\) (see [3, § 6]), so that

\[(2.9) \quad g^{**}(x^{**}) = \liminf_{y \to x^{**}} g(y),
\]

where the “lim inf” is taken over all nets in \(E\) converging to \(x^{**}\) in the weak** topology. Taking \(g = f + h_\alpha\), we see from (2.8) and (2.9) that

\[
f^{**}(x^{**}) = \liminf_{y \to x^{**}} [f(y) + h_\alpha(y)],
\]

implying that (2.5) holds as desired for some net \(\{y_i \mid i \in I\}\) in \(E\) such that \(y_i\) converges to \(x^{**}\) in the weak** topology and \(\|y_i\| \leq \alpha\) for every \(i \in I\).

Now, given any \(x^* \in E^*\) and \(x^{**} \in \partial f^*(x^*)\), let \(\{y_i \mid i \in I\}\) be a bounded net in \(E\) such that \(y_i\) converges to \(x^{**}\) in the weak** topology and (2.5) holds. Define \(\varepsilon_i \geq 0\) by

\[
\varepsilon_i = f(y_i) + f^*(x^*) - \langle y_i, x^*\rangle.
\]

Note that \(\lim \varepsilon_i = 0\) by (2.5) and (2.4). According to a lemma of Brøndsted and Rockafellar [1, p. 608], there exist for each \(i \in I\) an \(x_i \in E\) and an \(x_i^* \in E^*\) such that

\[
x_i^* \in \partial f(x_i), \quad \|x_i - y_i\| \leq \varepsilon_i, \quad \|x_i^* - x^*\| \leq \varepsilon_i.
\]

The latter two conditions imply that the net \(\{x_i^* \mid i \in I\}\) converges to \(x^*\) in the strong topology of \(E^*\), while the net \(\{x_i \mid i \in I\}\) is bounded and converges to \(x^{**}\) in the weak** topology of \(E^{**}\). This completes the proof of Proposition 1.
3. Proofs of Theorems A and B. In the sequel, \( f \) denotes a lower semicontinuous proper convex function on \( E \), and \( j \) denotes the continuous convex function \( E \) defined by \( j(x) = (1/2)\|x\|^2 \). We shall make use of the fact that, for each \( x \in E \), \( \partial f(x) \) is by definition a certain (possibly empty, possibly unbounded) weak* closed convex subset of \( E^* \), whereas \( \partial j(x) \) is (by the finiteness and continuity of \( j \), see \([3, \text{p. 60}]\)) a certain nonempty weak* compact convex subset of \( E^* \). Furthermore

\[
\partial(f + j) = \partial f(x) + \partial j(x), \forall x \in E
\]

(see \([3, \text{p. 62}]\) or \([5, \text{Th. 3}]\)). The conjugate of \( j \) is given by \( j^*(x^*) = (1/2)\|x^*\|^2 \), and since

\[
(f + j)^*(x^*) = (f^* - j^*)(x^*) = \min_{y^* \in E^*} \{f^*(y^*) + j^*(x^* - y^*)\}
\]

([3, §9] or \([5, \text{Th. 3}]\)) the conjugate function \((f + j)^*\) is finite and continuous throughout \( E^* \).

Proof of Theorem A. Theorem A has already been established by Minty \([2]\) in the case of convex functions which, like \( j \), are everywhere finite and continuous. Applying Minty's result to the function \((f + j)^*\), we may conclude that \( \partial(f + j)^* \) is a maximal monotone operator from \( E^* \) to \( E^{**} \). We shall show this implies that \( \partial f \) is a maximal monotone operator from \( E \) to \( E^* \).

Let \( T \) be a monotone operator from \( E \) to \( E^* \) such that the graph of \( T \) includes the graph of \( \partial f \), i.e.,

\[
T(x) \supseteq \partial f(x), \forall x \in E
\]

(3.2)

We must show that equality necessarily holds in (3.2).

The mapping \( T + \partial j \) defined by

\[
(T + \partial j)(x) = T(x) + \partial j(x)
\]

\[
= \{x^*_1 + x^*_2 \mid x^*_1 \in T(x), x^*_2 \in \partial j(x)\}
\]

is a monotone operator from \( E \) to \( E^* \), since \( T \) and \( \partial j \) are, and by (3.1) and (3.2) we have

\[
(T + \partial j)(x) \supseteq \partial(f + j)(x), \forall x \in E
\]

(3.3)

Let \( S \) be the multivalued mapping from \( E^* \) to \( E^{**} \) defined as follows: \( x^{**} \in S(x^*) \) if and only if there exists a net \( \{x^*_i \mid i \in I\} \) in \( E^* \) converging to \( x^* \) in the strong topology, and a bounded net \( \{x_i \mid i \in I\} \) in \( E \) (with the same partially ordered index set \( I \)) converging to \( x^{**} \) in the weak** topology, such that

\[
x^*_i \in (T + \partial j)(x_i), \forall i \in I.
\]
It is readily verified that $S$ is a monotone operator. (The boundedness of the nets $\{x_i | i \in I\}$ enters in here.) Moreover

$$(3.4) \quad S(x^*) \supseteq \partial(f + j)^*(x^*), \forall x^* \in E^*,$$

by (3.3) and Proposition 1. Since $\partial(f + j)^*$ is a maximal monotone operator, equality must actually hold in (3.4). This shows that one has $x \in \partial(f + j)^*(x^*)$ whenever $x \in E$ and $x \in S(x^*)$, hence in particular whenever $x^* \in (T + \partial j)(x)$. On the other hand, one always has $x^* \in \partial(f + j)(x)$ if $x \in \partial(f + j)^*(x^*)$ and $x \in E$. (This follows from applying (2.2) and (2.4) to $f + j$ in place of $f$.) Thus one has $x^* \in \partial(f + j)(x)$ if $x^* \in (T + \partial j)(x)$, implying by (3.3) and (3.1) that

$$(3.5) \quad T(x) + \partial j(x) = \partial f(x) + \partial j(x), \forall x \in E.$$

We shall show now from (3.5) that actually

$$(3.6) \quad T(x) = \partial f(x), \forall x \in E,$$

so that $\partial f$ must be a maximal monotone operator as claimed. Suppose that $x \in E$ is such that the inclusion in (3.2) is proper. This will lead to a contradiction. Since $\partial f(x)$ is a weak* closed convex subset of $E^*$, there must exist some point of $T(x)$ which can be separated strictly from $\partial f(x)$ be a weak* closed hyperplane. Thus, for a certain $y \in E$, we have

$$\sup \{\langle y, x^* \rangle | x^* \in T(x) \} > \sup \{\langle y, x^* \rangle | x^* \in \partial f(x) \}.$$

But then

$$\sup \{\langle y, z^* \rangle | z^* \in T(x) + \partial j(x) \} = \sup \{\langle y, x^* \rangle | x^* \in T(x) \} + \sup \{\langle y, y^* \rangle | y^* \in \partial j(x) \} > \sup \{\langle y, x^* \rangle | x^* \in \partial f(x) \} + \sup \{\langle y, y^* \rangle | y^* \in \partial j(x) \} = \sup \{\langle y, z^* \rangle | z^* \in \partial f(x) + \partial j(x) \},$$

inasmuch as $\partial j(x)$ is a nonempty bounded set, and this inequality is incompatible with (3.5).

**Proof of Theorem B.** Let $g$ be a lower semicontinuous proper convex function on $E$ such that

$$(3.6) \quad \partial g(x) \supseteq \partial f(x), \forall x \in E.$$

As noted at the beginning of the proof Theorem 3 of [4], to prove Theorem B it suffices, in view of Theorem 1 of [4] and its Corollary 2, to demonstrate that $g = f + \text{const}$.

We consider first the case where $f$ and $g$ are everywhere finite and continuous. Then, for each $x \in E$, $\partial f(x)$ is a nonempty weak*
compact set, and

\[(3.7) \quad f'(x; u) = \max \{\langle u, x^* \rangle \mid x^* \in \partial f(x)\}, \forall u \in E,\]

where

\[f'(x; u) = \lim_{\lambda \to 0} \frac{f(x + \lambda u) - f(x)}{\lambda}\]

[3, p. 65]. Similarly, \(\partial g(x)\) is a nonempty weak* compact set, and

\[(3.8) \quad g'(x; u) = \max \{\langle u, x^* \rangle \mid x^* \in \partial g(x)\}, \forall u \in E.\]

It follows from (3.6), (3.7) and (3.8) that

\[(3.9) \quad f'(x; u) \geq g'(x; u), \forall x \in E, \forall u \in E.\]

On the other hand, for any \(x \in E\) and \(y \in E\), we have

\[f(y) - f(x) = \int_0^1 f'((1 - \lambda)x + \lambda y; y - x) d\lambda,\]

\[g(y) - g(x) = \int_0^1 g'((1 - \lambda)x + \lambda y; y - x) d\lambda\]

(see [6, § 24]), so that by (3.9) we have

\[f(y) - f(x) \leq g(y) - g(x), \forall x \in E, \forall y \in E.\]

Of course, the latter can hold only if \(g = f + \text{const}\).

In the general case, we observe from (3.6) that

\[\partial g(x) + \partial j(x) \supset \partial f(x) + \partial j(x), \forall x \in E,\]

and consequently

\[\partial (g + j)(x) \supset \partial (f + j)(x), \forall x \in E,\]

by (3.1) (and its counterpart for \(g\)). This implies by Proposition 1 that

\[(3.10) \quad \partial (g + j)^*(x^*) \supset \partial (f + j)^*(x^*), \forall x^* \in E^*.\]

The functions \((f + j)^*\) and \((g + j)^*\) are finite and continuous on \(E^*\), so we may conclude from (3.10) and the case already considered that

\[(g + j)^* = (f + j)^* + \alpha\]

for a certain real constant \(\alpha\). Taking conjugates, we then have

\[(3.11) \quad (g + j)^{**} = (f + j)^{**} - \alpha.\]

Since \((g + j)^{**}\) and \((f + j)^{**}\) agree on \(E\) with \(g + j\) and \(f + j\), respectively, (3.11) implies that

\[g + j = f + j - \alpha,\]
and hence that $g = f + \text{const.}$

**REMARK.** The preceding proofs become much simpler if $E$ is reflexive, since then $\partial f^*$ and $\partial(f + j)^*$ are just the "inverses" of $\partial f$ and $\partial(f + j)$, respectively, and Proposition 1 is superfluous. In this case, $S$ may be replaced by the inverse of $T + \partial j$ in the proof of Theorem A.

**REFERENCES**


Received July 17, 1969. This work was supported in part by the Air Force Office of Scientific Research under grant AF-AFOSR-1202-67.

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