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If F is an open Riemann surface and $A(F)$ is the set of all analytic functions on F, then $A(F)$ is a ring under pointwise addition and multiplication. This paper is concerned with proper subrings R of $A(F)$ which are isomorphic images of $A(G)$, the ring of all analytic functions on an open Riemann surface G, under a homomorphism Φ which maps constant functions onto themselves. The ring R has the form $\{g \circ \phi$: $g \in A(G)$, ϕ an analytic map from F into $G\}$, and will be denoted R_ϕ. Relations between ϕ, R_ϕ and the spectrum of R_ϕ are given as necessary and sufficient conditions for the existence of a Riemann surface G such that R is isomorphic to $A(G)$.

Open Riemann surfaces will be denoted by F and G, the rings of all analytic functions on F and G with pointwise addition and multiplication will be denoted by $A(F)$ and $A(G)$, and Φ will denote a homomorphism from $A(G)$ into $A(F)$ which maps constant functions onto themselves. Let Φ be such a homomorphism. In [5, pp. 272–273] H. L. Royden shows there is an analytic mapping ϕ of F into G such that $\Phi(g) = g \circ \phi$, and that if Φ is an isomorphism onto $A(F)$ then ϕ is a one-to-one, onto analytic mapping. If ϕ is an analytic mapping of F into G, then Φ defined by $\Phi(g) = g \circ \phi$, $g \in A(G)$, is a homomorphism from $A(G)$ into $A(F)$ which preserves constant functions. When ϕ is one-to-one and onto, Φ is an isomorphism.

The image of $A(G)$ under Φ is the set $\{g \circ \phi$: $g \in A(G)$, ϕ is an analytic map of F into $G\}$ denoted by R_ϕ. R_ϕ is a subring of $A(F)$ and contains the constant functions, since $\Phi(\lambda) = \lambda$ for λ a constant function. The following conditions are equivalent: R_ϕ properly contains the constant functions, Φ is an isomorphism, ϕ is not a constant function. Theorems 1 and 2 give other relations between ϕ and R_ϕ.

THEOREM 1. *If R_ϕ properly contains the constant functions, then R_ϕ contains $1/f$ whenever $f \in R_\phi$, $f(z) \neq 0$ on F, if and only if ϕ maps F onto G.*

Proof. Let ϕ map F onto G, $f \in R_\phi$, $f(z) \neq 0$ on F. Then $f = \Phi h$ for some $h \in A(G)$ and $1/h \in A(G)$ if $h(y) \neq 0$ for $y \in G$. Suppose $h(a) = 0$. Since $a = \phi(z)$ for some $z \in F$, $0 = h(a) = h(\phi(z)) = \Phi h(z) = f(z)$. This contradicts $f(z) \neq 0$ on F. Thus $h(a) \neq 0$ for $a \in G$, $1/h \in A(G)$, and $1/f = \Phi(1/h) \in R_\phi$.

Suppose R_ϕ contains $1/f$ when $f \in R_\phi$, $f(z) \neq 0$ on F. Let $a \in G$.

There is $g \in A(G)$ such that $g(a) = 0$ and $g(w) \neq 0$ for $w \neq a$ [1, pp. 591–592]. The function $\Phi g \in R_\phi$. If $\Phi g(z) = g \circ \phi(z) \neq 0$ for $z \in F$, then there is $h \in R_\phi$ such that $(\Phi g)(h) = 1$. There is $k \in A(G)$ such that $h = \Phi k$. Then $(\Phi g)(\Phi k) = 1$ and $\Phi(gk) = 1$ but Φ is an isomorphism implies $gk = 1$ and $g(a)k(a) = 1$. This contradicts $g(a) = 0$. Therefore $g(\phi(z)) = 0$ and $\phi(z) = a$ for some $z \in F$.

A straightforward argument shows

THEOREM 2. *If R_ϕ properly contains the constant functions, then R_ϕ separates the points of F if and only if ϕ is one-to-one.*

Let R be a ring of analytic functions defined on F. The spectrum of R, ΣR, is the set of nonzero homomorphisms π from R into the complex numbers such that $\pi(\lambda) = \lambda$ for λ a constant function. For $x \in F$ the point evaluation mapping $\pi_x = \{(f, f(x)) \colon f \in R\}$ is a homomorphism from R into the complex numbers, and $\pi_x(\lambda) = \lambda$ for λ a constant function. Therefore ΣR always contains the point evaluation mappings defined on R. In [5, p. 272] H. L. Royden shows that the spectrum of $A(F)$ is the set of point evaluation mappings π_x defined on $A(F)$, $x \in F$. For $f \in R$ let $\hat{f} = \{(\pi, \pi f) \colon \pi \in \Sigma R\}$; $\hat{f}$ is a function from ΣR into the complex numbers. Let $\hat{R}$ denote $\{\hat{f} \colon f \in R\}$. With pointwise addition and multiplication $\hat{R}$ is a ring containing the constant functions and is isomorphic to R under $f \to \hat{f}$.

For $y \in G$, let ψ_y denote an element of $\Sigma A(G)$. The mapping $P = \{(y, \psi_y) \colon y \in G\}$ is a one-to-one function from G onto $\Sigma A(G)$. If $R = \Phi(A(G))$ and Φ is an isomorphism, $L = \{(\pi, \pi \circ \Phi) \colon \pi \in \Sigma R\}$ is a one-to-one function from ΣR onto $\Sigma A(G)$. The mapping $\pi \to \pi \circ \Phi = \psi_y \to y$ which is $P^{-1} \circ L$ defines a one-to-one correspondence between ΣR and G when Φ is an isomorphism.

THEOREM 3. *Let $R_\phi = \Phi(A(G))$, Φ be an isomorphism from $A(G)$ into $A(F)$ which preserves constant functions. Let M be the function from $\Sigma A(F)$ into ΣR_ϕ defined by $M(\pi_x) = \pi_x|_{R_\phi}$. Then M is onto if and only if ϕ is onto, and M is one-to-one if and only if ϕ is one-to-one.*

Proof. The proof that M is one-to-one if and only if ϕ is one-to-one follows from Theorem 2 and the fact that $A(F)$ separates the points of F.

Let $\pi \in \Sigma R_\phi$. Then $\pi \circ \Phi \in \Sigma A(G)$ implies there is $y \in G$ such that $\pi \circ \Phi = \psi_y$, where $\psi_y(g) = g(y)$ for $g \in A(G)$. There are two cases: $y \in \phi(F)$, $y \notin \phi(F)$. If $y \in \phi(F)$, then $y = \phi(x)$ for some $x \in F$ and $\pi(\Phi g) = g(y) = g(\phi(x)) = \Phi g(x)$ for every $g \in A(G)$, $\pi(\Phi g) = \Phi g(x)$ for every $f = \Phi g \in R_\phi$. This implies $\pi = M(\pi_x)$. If $y \notin \phi(F)$, then $y \neq \phi(x)$ for $x \in F$, and it may be shown that for every $x \in F$ there is $f \in R_\phi$ such that $\pi(f) \neq f(x)$. Let $x \in F$. Then $\phi(x) \in G$. $y \in G$, $y \neq \phi(x)$, and $A(G)$ separates the points of G implies there is a $g \in A(G)$ such that $g(y) \neq g(\phi(x))$. From $\Phi(g) \in R_\phi$ and $\pi(\Phi g) = g(y) \neq g(\phi(x)) = \Phi g(x)$ it follows that $\pi \neq M(\pi_x) = \pi_x|_{R_\phi}$.

For $\pi \in \Sigma R_\phi$, $\pi \circ \Phi = \psi_y \in \Sigma A(G)$, and it has been shown $\pi \in M(\Sigma A(F))$ if and only if $y \in \phi(F)$.

From Theorem 3 and since ΣR_ϕ and G are in one-to-one correspondence, it follows that the point evaluation maps in ΣR_ϕ are in one-to-one correspondence with the points $\phi(x) \in \phi(F)$, and the elements of ΣR_ϕ which are not point evaluation maps are in one-to-one correspondence with the points in $G - \phi(F)$.

Theorem 4 contains a necessary condition which a subring R of $A(F)$ must satisfy if R is to be $\Phi(A(G))$, the isomorphic image of $A(G)$ under Φ for some open Riemann surface G. The corollary to Theorem 5 gives a set of sufficient conditions on R in order that R be $\Phi(A(G))$ when $\Phi g = g \circ \phi$ and ϕ: $F \rightarrow G$ is an onto mapping.

Suppose F is an open Riemann surface, $p \in F$, f is analytic at p and τ is a local uniformizer which maps a neighborhood of p onto $\{z: |z| < g\}$ for some $\rho > 0$, $\tau(p) = 0$. There is a number $r > 0$ such that $f \circ \tau^{-1}(z) = \sum_{i=0}^{\infty} a_i z^i$ for $|z| < r$. The multiplicity of f at p is defined as $\inf\{k: k \neq 0 \text{ and } a_k \neq 0\}$, denoted $n(p; f)$. The multiplicity $n(p; f)$ of f at p does not depend on τ. If R contains functions other than constants, $m = \inf\{n(p; f): \mathrm{f} \in R\}$ is defined, and $n(p; f) = m$ for some $f \in R$.

THEOREM 4. *Let $p \in F$, R_ϕ contain functions other than constants and let $m = \{\inf n(p; f): f \in R_\phi\}$. There is a local uniformizer τ at p with the properties: $\tau(0) = p$, for some $\rho > 0$, τ maps $\{z: |z| < \rho\}$ onto a neighborhood of p, and if $f \in R_\phi$, $f \circ \tau(z) = \sum_{i=0}^{\infty} a_i (z^m)^i$ for $|z| < \rho$.*

The proof of Theorem 4 is based on two lemmas:

LEMMA 1. *If $p \in F$, $m = \inf\{n(p; f): f \in R_\phi\}$ and $f \in R_\phi$, then $n(p;f) = km$, where k is a positive integer.*

LEMMA 2. *Given $\sum_{i=m}^{\infty} c_i z^i$ convergent for $|z| < \rho$, $c_m \neq 0$, $m \neq 0$, there is $\sum_{i=1}^{\infty} b_i z^i$ convergent for $|z| < \rho$, $b_1 \neq 0$, such that $(\sum_{i=1}^{\infty} b_i z^i)^m = \sum_{i=m}^{\infty} c_i z^i$.*

Lemma 1 follows from the two relations: For $f \in R_\phi$, $f = g \circ \phi$ for some $g \in A(G)$, which implies $n(p; f) = (n(p; \phi))(n(\phi(p); g))$, and if $m = \inf\{n(p; f): f \in R_\phi\}$ then $n(p; \phi) = m$. Lemma 2 is proved by defining W a subset of the natural numbers N as $W = \{n \in N: b_1, b_2, \cdots, b_n$ can be defined in such a way that the coefficients of z^i for $1 \leqq m \leqq i \leqq m + n - 1$ of $(\sum_{i=1}^\infty b_i z^i)^m$ and $\sum_{i=m}^\infty c_i z^i$ are equal$\}$ and using induction to show $W = N$.

Proof of Theorem 4. Let τ_p be a local uniformizer about p such that $\tau_p(0) = p$. If $m = \inf\{n(p; f): f \in R_\phi\}$, there is $f_p \in R_\phi$ and $\rho > 0$ such that $f_p \circ \tau_p(z) = \sum_{i=m}^\infty c_i z^i$ for $|z| < \rho, c_m \neq 0$, and the range of $\sum_{i=m}^\infty c_i z^i$ contains $|z| < \rho^m$.

There is a power series $\sum_{i=1}^\infty b_i z^i$, $b_1^m = c_m$, such that $\sum_{i=m}^\infty c_i z^i = (\sum_{i=1}^\infty b_i z^i)^m$ for $|z| < \rho$ as stated in Lemma 2. $k(z) = \sum_{i=1}^\infty b_i z^i$ is defined for $|z| < \rho$, is one-to-one, and its range contains $|z| < \rho$. Thus $k^{-1}(y)$ is defined for $|y| < \rho$ and $f_p \circ \tau_p \circ k^{-1}(z) = (\sum_{i=1}^\infty b_i(k^{-1}(z))^i)^m = z^m$ for $|z| < \rho$, $\tau_p \circ k^{-1}(0) = p$. The function $\tau = \tau_r \circ k^{-1}$ is a local uniformizer about p and there is $f_p \in R_\phi$ such that $f_p \circ \tau(z) = z^m$ for $|z| < \rho$.

Let $f \in R_\phi$, f not a constant function. Then $f \circ \tau(z) = \sum_{i=0}^\infty a_i z^i$ for $|z| < \rho$. Let N denote the natural numbers and define $W = \{n \in N: f \circ \tau(z) = \sum_{i=0}^n a_{mj_i} z^{mj_i} + z^{mj_n} h_n(z)$, where $h_n(z) = \sum_{i=1}^\infty b_{n,i} z^i$ and j_i are nonnegative integers, $0 = j_0 < j_1 < \cdots < j_n\}$.

It follows from Lemma 1 that for $|z| < \rho, f \circ \tau(z) = \sum_{i=0}^\infty a_i z^i = a_0 + a_{mj_1} z^{mj_1} + z^{mj_1} h_1(z)$, where $h_1(0) = 0$. If $k \in W$, then $f \circ \tau(z) = \sum_{i=0}^k a_{mj_i} z^{mj_i} + z^{mj_k} h_k(z)$, $h_k(0) = 0$. Since $f \in R_\phi$, $z^m \in R_\phi$ and constants are contained in R_ϕ, $z^{mj_k} h_k(z) = f(z) - \sum_{i=0}^k a_{mj_i} z^{mj_i} \in R_\phi$. If $h_k \neq 0$, $n(p; z^{mj_k} h_k) = mj_{k+1}$ and $f \circ \tau(z) = \sum_{i=0}^{k+1} a_{mj_i} z^{mj_i} + z^{mj_{k+1}} h_{k+1}(z)$, where $h_{k+1}(z) = \sum_{i=1}^\infty b_{k+1,i} z^i$ on $|z| < \rho$ and $j_{k+1} > j_k$. If $h_k = 0$, then the above statement is true with $a_{mj_{k+1}} = 0$, $h_{k+1} = 0$. By induction $W = N$ and $f \circ \tau(z) = \sum_{i=0}^\infty a_{mi} z^{mi}$ on $|z| < \rho$.

If R, a subring of $A(F)$, has the property that for every $a \in F$, $f \in R$, for some local uniformizer τ about a, $f \circ \tau(z) = \sum_{i=0}^\infty a_i (z^{m(a)})^i$ for $m(a) = \inf\{n(a; f): f \in R\}$, then R has property (ξ). If R contains functions other than constants and has property (ξ), then for $a \in F$, $m(a) = \inf\{n(a; f): f \in R\} = 1$ if R separates the points of F.

THEOREM 5. *If R is a subring of $A(F)$ which contains functions other than constants and has property (ξ), then there is an open Riemann surface G, an analytic mapping ϕ of F onto G, and a separating subring S of $A(G)$ such that S is isomorphic to R under $\hat{f} \to \hat{f} \circ \phi$, $\hat{f} \in S$.*[1]

Proof. Let $G = \{\pi_p: p \in F\}$ where $\pi_p = \{(f, f(p)): f \in R\}$ and $\phi =$

$\{(p, \pi_p): p \in F\}$. The topology on G will be that which makes ϕ continuous and open. If N_p is an open neighborhood of $p \in F$, then $N_{\pi_p} = \{\pi_q: q \in N_p\}$ is an open neighborhood of π_p. The set G with this topology is a connected Hausdorff space.

Let $p \in F$, $\pi_p \in G$ and $m = \inf\{n(p; f): f \in R\}$. By the same argument used in the beginning of the proof of Theorem 4, there is a function $f_p \in R$ and a local uniformizer τ about p such that $\tau(0) = p$ and $f_p \circ \tau(z) = z^m$ for $|z| < \rho^{1/m}$ for some $\rho > 0$. Then for $f \in R$, $f \circ \tau(z) = \sum_{i=0}^{\infty} a_i(z^m)^i = g_f(z^m)$ for $|z| < \rho^{1/m}$, g_f analytic on $|z| < \rho$.

It will be shown that $\sigma_\tau = \{(z^m, \pi_{\tau(z)}): |z| < \rho^{1/m}\}$ is a local uniformizer about π_p. If $z_1^m = z_2^m$, then $f \circ \tau(z_1) = g_f(z_1^m) = g_f(z_2^m) = f \circ \tau(z_2)$, for $f \in R$ implies $\pi_{\tau(z_1)} = \pi_{\tau(z_2)}$, which implies that σ_τ is a function. If $\pi_{\tau(z_1)} = \pi_{\tau(z_2)}$ then in particular $f_p \circ \tau(z_1) = f_p \circ \tau(z_2)$, which implies $z_1^m = z_2^m$, and σ_τ is one-to-one. Since the relations $z^m \to z \to \tau(z) \to \phi(\tau(z)) = \pi_{\tau(z)}$ are open and continuous, σ_τ is open and continuous. Thus σ_τ is a homeomorphism from $\{w: |w| < \rho\}$ onto $\phi \circ \tau(\{z: |z| < \rho^{1/m}\}) = N_{\pi_p}$.

If $\pi \in W = \sigma_{\tau_2}(|z| < \rho_2) \cap \sigma_{\tau_1}(|z| < \rho_1)$, there are points z_1, z_2 such that $\pi_{\tau_1}(z_1) = \pi_{\tau_2}(z_2)$. Then $f \circ \tau_1(z_1) = f \circ \tau_2(z_2)$ for every $f \in R$, and $z_1^{m_1} = f_1(\tau_1(z_1)) = f_1(\tau_2(z_2)) = g_{f_1}(z_2^{m_2})$, so g_{f_1} is analytic on $\{w: |w| < \rho_2\}$, which contains $\sigma_{\tau_2}^{-1}(W)$. This shows that $z_1^{m_1} = \sigma_{\tau_1}^{-1} \circ \sigma_{\tau_2}(z_2^{m_2})$ is analytic on $\sigma_{\tau_2}^{-1}(W)$ to $\sigma_{\tau_1}^{-1}(W)$. The function σ_τ is a local uniformizer of a neighborhood of π_p, and G is a Riemann surface.

For $f \in R$, let $\hat{f} = \{(\pi_p, f(p)): p \in F\}$, $S = \{\hat{f}: f \in R\}$. Since f is continuous and ϕ is open, $\hat{f}$ is continuous. The function $\hat{f}$ is analytic at π_p, because if $|w| < \rho$, $w = z^m$, then $\hat{f} \circ \sigma_\tau(w) = \hat{f}(\pi_{\tau(z)}) = f(\tau(z)) = \sum_{i=0}^{\infty} a_i(z^m)^i = \sum_{i=0}^{\infty} a_i w^i$. The mapping ϕ is analytic at p, because $\sigma_\tau^{-1} \circ \phi \circ \tau(z) = \sigma_\tau^{-1}(\pi_{\tau(z)}) = z^m$ for $|z| < \rho^{1/m}$. With pointwise addition and multiplication, S is a ring and is isomorphic to R under the mapping $\hat{f} \to \hat{f} \circ \phi = f$. The ring S separates the points of G. Since S contains functions which are not constant and are analytic on G, G is an open Riemann surface.

If S is to be $A(G)$, then by Theorem 3 the mapping $M(\pi_p) = \pi_p|_R$ from $\Sigma A(F)$ to ΣR must be onto, since ϕ is an onto mapping of F to G. Thus ΣR may contain only point evaluation mappings and $\Sigma R = G$.

COROLLARY TO THEOREM 5. *If R is a subring of $A(F)$ which properly contains the constant functions and has property (ξ), if ΣR contains only point evaluation mappings, and R contains all $f \in A(F)$ such that $f \circ \tau_p(z) = \sum_{i=0}^{\infty} a_i(z^m)^i$ for $|z| < \rho^{1/m}$, $p \in F$, $m = \inf\{n(p: f): f \in R\}$, then $\Sigma R = G$ is an open Riemann surface, and R is isomorphic to $S = A(G)$.*

[1] This result and proof are similar to one given by M. Heins for a subfield of the field of all meromorphic functions on a Riemann surface [**2**, pp. 268-269].

Proof. Everything except $S = A(G)$ was shown in the proof of Theorem 5. The function $\hat{f} \in A(G)$ if and only if for every $\pi_p \in G$, $\hat{f} \circ \sigma_{\tau_p}(w) = \sum_{i=0}^{\infty} a_i w^i$ for $|w| < \rho$. Let $\hat{f} \in A(G)$, $p \in F$, $\pi_p \in G$, and $f = \hat{f} \circ \phi$. Then $f \in A(F)$ and $f \in R$, because for $|z| < \rho^{1/m}$, $f \circ \tau_p(z) = \hat{f} \circ \phi(\tau_p(z)) = \hat{f}(\pi_{\tau_p(z)}) = \hat{f} \circ \sigma_{\tau_p}(z^m) = \sum_{i=0}^{\infty} a_i(z^m)^i$.

If $R = \{\hat{f} \circ \phi : \hat{f} \in S\}$ and S separates the points of G, then R separates the points of F if and only if ϕ is a one-to-one function. If S separates the points of G, and $S = A(G)$, then R may not separate the points of F, because if it did ϕ would be a one-to-one, onto analytic function from F to G, and $R = A(F)$. If $S \neq A(G)$ there may be a surface H, a mapping ϕ_1 and a separating subring T of $A(H)$ such that ϕ_1 is analytic and one-to-one but not onto, and $T = A(H)$.

In this part of the paper it is noted that if $R = \Phi(A(G))$, then ΣR with the Gelfand topology is an open Riemann surface, and $\hat{R}$ which is isomorphic to R, is the ring of all analytic functions on ΣR. Theorem 8 gives sufficient conditions on a subring R of $A(F)$ and on $\hat{R}$ in order that ΣR be an open Riemann surface and $\hat{R}$ be a ring of analytic functions on ΣR. In conclusion sufficient conditions for $\hat{R}$ to be $A(\Sigma R)$ are given.

If R is a ring of complex valued functions on F, then the Gelfand topology on ΣR is the weakest topology on ΣR which makes each element of $\hat{R}$ continuous, where $\hat{R} = \{\hat{f} : f \in R\}$, $\hat{f} = \{(\pi, \pi f) : \pi \in \Sigma R\}$. Let $\pi_0 \in \Sigma R$, K be a finite subset of $\hat{R}$, $\varepsilon > 0$. An open neighborhood of π_0 will be $\{\pi \in \Sigma R : |\hat{f}(\pi) - \hat{f}(\pi_0)| < \varepsilon \text{ for } \hat{f} \in K\}$. If $R = \Phi(A(G))$ and Φ is an isomorphism, then ΣR and $\Sigma A(G)$ with the Gelfand topology are homeomorphic under the mapping $L(\pi) = \pi \circ \Phi$ from ΣR onto $\Sigma A(G)$. The mapping $P(y) = \psi_y$ from G onto $\Sigma A(G)$ with the Gelfand topology is one-to-one, onto and continuous. The mapping P is also open. As Royden observes [4, pp. 287–288], this is a consequence of a theorem of Remmert that an open Riemann surface can be mapped one-to-one and holomorphically into C^3 [3, p. 118]. Thus $P^{-1} \circ L$ is a homeomorphism from ΣR with the Gelfand topology onto G.

THEOREM 6. *If R is a subring of $A(F)$ such that $R = \Phi(A(G))$, and if Φ is an isomorphism which preserves constant functions, then ΣR with the Gelfand topology is an open Riemann surface, and $\hat{R}$ is the ring of all analytic functions on ΣR. Moreover $\hat{R}$ is isomorphic to R.*

Proof. The spectrum of R with the Gelfand topology is a Hausdorff space. It is homeomorphic to G under the mapping $L^{-1} \circ P$, and is connected. Let $\pi_q \in \Sigma R$ where $q \in G$, $\psi_q \in \Sigma A(G)$, and $L^{-1} \circ P$ maps $q \to \psi_q \to \pi_q$. If N_q is a neighborhood of q then $N_{\pi_q} = L^{-1} \circ P(N_q)$ is a neighborhood of π_q. There exists $h_q \in A(G)$ which has a simple zero at q [1, pp. 591–592]. h_q is a local uniformizer on a neighborhood of q, $N_q = h_q^{-1}(|z| < \rho)$ for some $\rho > 0$. If $\sigma_q = h_q|_{N_q}$, then $h_q \circ \sigma_q^{-1}(z) = z$ for $|z| < \rho$. For $h \in A(G)$, $y \in N_q$, $h(y) = \sum_{i=0}^{\infty} a_i(h_q(y))^i$.

If $f_q = \Phi h_q$ then $\hat{f}_q$ is a local uniformizer on $N_{\pi_q} = L^{-1} \circ P(N_q)$. From $\hat{f}_q(\pi_y) = h_q(y)$ follows $\hat{f}_q(\pi_y) = h_q \circ P^{-1} \circ L(\pi_y)$, $\pi_y \in N_{\pi_q}$, which implies $\hat{f}_q$ is a homeomorphism of N_{π_q} onto $|z| < \rho$. If $\pi_y \in N_{\pi_{q_1}} \cap N_{\pi_{q_2}}$, then $\hat{f}_{q_1}(\pi_y) = h_{q_1}(y) = \sum_{i=0}^{\infty} a_i(h_{q_2}(y))^i = \sum_{i=0}^{\infty} a_i(\hat{f}_{q_2}(\pi_y))^i$ since $\pi_y \in N_{\pi_{q_2}}$ or $y \in N_{q_2}$. The function $\hat{f}_q$ is a local uniformizer on N_{π_q} and ΣR is a Riemann surface.

The ring $\hat{R}$ is contained in $A(\Sigma R)$, because if $\hat{f} \in \hat{R}$, $\pi_y \in N_{\pi_q}$, $z = \hat{f}_q(\pi_y)$, then $\hat{f} \circ \hat{f}_q^{-1}(z) = \hat{f}(\pi_y) = h(y) = \sum_{i=0}^{\infty} a_i(h_q(y))^i = \sum_{i=0}^{\infty} a_i(\hat{f}_q(\pi_y))^i = \sum_{i=0}^{\infty} a_i z^i$. The function $T(q) = \pi_q$ is an analytic map of G onto ΣR. If θ is analytic on ΣR, then $\theta \circ T \in A(G)$ and $\theta \in \hat{R}$ because $\theta(\pi_q) = \theta \circ T(q) = \psi_q(\theta \circ T) = \pi_q(f)$ for $f = \Phi(\theta \circ T)$. This implies $\theta = \hat{f}$. Thus $\hat{R} = A(\Sigma R)$. Since $\hat{R}$ contains functions which are analytic and are not constant on ΣR, ΣR is an open Riemann surface.

THEOREM 7. *Let $R = \Phi(A(G))$. If $\hat{\pi} \in \Sigma R$, then $\hat{\pi}^{-1}(0)$ is a principal maximal ideal of R, and every principal maximal ideal of R is the kernel of $\pi \in \Sigma R$. If $\hat{\pi}^{-1}(0)$ is generated by f, then $\hat{f}$ is a local homeomorphism on a neighborhood $N_{\hat{\pi}}$ of $\hat{\pi}$ and if $\pi \in N_{\hat{\pi}}$, $\hat{k} \in \hat{R}$, then $\hat{k}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$.*

Proof. If $\hat{\pi} \in \Sigma R$, then $\hat{\pi} \circ \Phi = \psi_q \in \Sigma A(G)$ and $\hat{\pi}^{-1}(0) = \Phi(\psi_q^{-1}(0))$. The kernel of ψ_q, $M_q = \psi_q^{-1}(0)$, is a principal maximal ideal of $A(G)$, and every principal maximal ideal of $A(G)$ is a kernel of $\psi \in \Sigma A(G)$ [5, pp. 271–272]. If h generates M_q, then h has a single zero and it is a simple zero at q [5]. Thus h is a homeomorphism on a neighborhood of q, N_q. If $f = \Phi h$, then $\hat{\pi}^{-1}(0)$ is the ideal generated by f. Also $\hat{f}$ is a uniformizer on $N_{\hat{\pi}} = L^{-1} \circ P(N_q)$, and if $\pi \in N_{\hat{\pi}}$, $\hat{k} \in \hat{R}$, then $\hat{k}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$ as shown in the proof of Theorem 6.

LEMMA. *Let S be a ring of continuous functions on X with identity. Then X is not connected if and only if S is contained in a ring Q of continuous functions on X, where $Q = I_1 + I_2$, I_1, I_2 proper ideals of Q, $I_1 \cap I_2 = \{0\}$.*

THEOREM 8. *Let R be a subring of $A(F)$ which properly contains the constant functions, and suppose $\hat{R}$ is not contained in a ring Q of continuous functions on ΣR where $Q = I_1 + I_2$, I_1, I_2 proper ideals of Q, $I_1 \cap I_2 = \{0\}$. If for $\hat{\pi} \in \Sigma R$, $\hat{\pi}^{-1}(0)$ is a principal ideal of R*

generated by f and $\hat{f}$, the function in $\hat{R}$ which corresponds to f in R, is a homeomorphism on a neighborhood of $\hat{\pi}$, and for π in this neighborhood, $g \in R$, $\pi g = \sum_{i=0}^{\infty} a_i(\pi f)^i$, then ΣR is an open Riemann surface and $\hat{R}$ is a ring of analytic functions on ΣR.

Proof. The spectrum of R with the Gelfand topology is a Hausdorff space. By the lemma ΣR is connected. Let $\hat{\pi} \in \Sigma R$. There is $\hat{f}$ a homeomorphism of $N_{\hat{\pi}}$ onto $|z| < \rho$ for some $\rho > 0$. If $\pi \in N_{\hat{\pi}}$, $g \in R$, then $\hat{g}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$. If $\pi \in N_{\pi_1} \cap N_{\pi_2} = W$ then $\hat{f}_1 \circ \hat{f}_2^{-1}(\hat{f}_2(\pi)) = \hat{f}_1(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}_2(\pi))^i$ implies $\hat{f}_1 \circ \hat{f}_2^{-1}$ is analytic on $\hat{f}_2(W)$. $\{(N_\pi, \hat{f}_\pi)$: $\pi \in \Sigma R\}$ defines an analytic structure on ΣR. It is immediate that $\hat{R} \subset A(\Sigma R)$. Since $\hat{R}$ contains functions which are not constant and are analytic on ΣR, ΣR is an open Riemann surface.

If $\{R_n\}$ is a sequence of subrings of $A(F)$ such that R_n satisfies the conditions of Theorem 8, $\Sigma R_n|_{R_1} = \Sigma R_1$, $R_{n-1} \subset R_n$, then the chain has a maximal element, $\{\hat{f} \circ \phi$: $\hat{f} \in A(\Sigma R_1)$ and $\phi(x) = \pi_x$, $x \in F\}$. Let $\hat{\pi} \in \Sigma R_1$ and $\hat{f}$ be a local homeomorphism at $\hat{\pi}$. If R_1 satisfies the conditions of Theorem 8 and contains all functions g in $A(F)$ such that $\hat{g}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$ for $\pi \in N_{\hat{\pi}}$, π and $\hat{\pi}$ elements of ΣR_1, then $\hat{R}_1 = A(\Sigma R_1)$, because if $\hat{g} \notin \hat{R}_1$, then there is $\hat{\pi} \in \Sigma R_1$ such that $\hat{g} \circ \hat{f}^{-1}$ is not analytic on $\{z$: $|z| < \rho\}$ which implies $\hat{g} \notin A(\Sigma R_1)$.

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SAM HOUSTON STATE UNIVERSITY
HUNTSVILLE, TEXAS

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