Pacific Journal of Mathematics

THE PRINCIPLE OF SUBORDINATION APPLIED TO FUNCTIONS OF SEVERAL VARIABLES

TED JOE SUFFRIDGE

Vol. 33, No. 1 March 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

THE PRINCIPLE OF SUBORDINATION APPLIED TO FUNCTIONS OF SEVERAL VARIABLES

T. J. SUFFRIDGE

In this paper we consider univalent maps of domains in $C^n (n \geqq 2)$. Let P be a polydisk in C^n. We find necessary and sufficient conditions that a function $f: P \to C^n$ be univalent and map the polydisk P onto a starlike or a convex domain. We also consider maps from

$$(1)\qquad \begin{aligned} D_p &= \{z: |z|_p < 1\} \subset C^n \\ |z|_p &= |(z_1, z_2, \cdots, z_n)|_p = \left[\sum_{j=1}^{n} |z_j|^p\right]^{1/p}, \quad p \geqq 1 \end{aligned}$$

into C^n and give necessary and sufficient conditions that such a map have starlike or convex image.

In [4] Matsuno has considered a similar problem for the hypersphere $D_2 \subset C^n$. His definition of starlikeness is different from that used in this paper, but the results show that the two definitions are equivalent. However, his definition of convex-like is not equivalent to geometrically convex.

1. **Preliminary lemmas.** For $(z_1, z_2, \cdots, z_n) = z \in C^n$, define $|z| = \max_{1 \leqq j \leqq n} |z_j|$. Let $E_r = \{z \in C^n: |z| < r\}$ and $E = E_1$. Let $\mathscr{P}$ be the class of mappings $w: E \to C^n$ which are holomorphic and which satisfy $w(0) = 0$, $\operatorname{Re}[w_j(z)/z_j] \geqq 0$ when $|z| = |z_j| > 0$, $(1 \leqq j \leqq n)$ where $w = (w_1, w_2, \cdots, w_n)$. The following lemmas are generalizations of Theorems A and B of Robertson [5, p. 315–317].

LEMMA 1. *Let $v(z; t): E \times I \to C^n$ be holomorphic for each $t \in I = [0, 1]$, $v(z; 0) = z$, $v(0, t) = 0$ and $|v(z; t)| < 1$ when $z \in E$. If*

$$(2)\qquad \lim_{t \to 0^+} [(z - v(z; t))/t^\rho] = w(z)$$

exists and is holomorphic in E for some $\rho > 0$, then $w \in \mathscr{P}$.

Proof. The hypothesis (2) implies that $\lim_{t \to 0^+} v_j(z; t) = z_j$ (here $v(z; t) = (v_1(z; t), v_2(z; t), \cdots, v_n(z; t))$ so

$$\frac{2z_j(z_j - v_j(z; t))}{z_j + v_j(z; t)} \equiv \psi_j(z; t)$$

is holomorphic for $z \in E$, $z_j \neq 0$ $(1 \leqq j \leqq n)$. By Schwarz lemma, $|v(z; t)| \leqq |z|$ and hence $\operatorname{Re}[\psi_j(z; t)/z_j] \geqq 0$ when $|z| = |z_j| > 0$. Setting $\psi(z; t) = (\psi_1, \psi_2, \cdots, \psi_n)$, $(z \in E, z_1 z_2 \cdots z_n \neq 0)$ we observe that

$$\lim_{t\to 0^+} \psi(z; t)/t^\rho = w(z)$$

for these values of z and using continuity of w we conclude $w \in \mathscr{P}$.

LEMMA 2. *Let $f: E \to C^n$ be holomorphic and univalent and satisfy $f(0) = 0$. Let $F(z; t): E \times I \to C^n$ be a holomorphic function of z for each $t \in I = [0, 1]$, $F(z; 0) = f(z)$, $F(0, t) = 0$ and suppose $F(z; t) \prec f$ for each $t \in I$ (i.e., $F(E; t) \subset f(E)$ for each $t \in I$). Let $\rho > 0$ be such that $\lim_{t\to 0^+} F(z; 0) - F(z; t)/t^\rho = F(z)$ exists and is holomorphic. Then $F(z) = Jw$ where $w \in \mathscr{P}$. Here F and w are written as column vectors and J is the complex Jacobian matrix for the mapping f.*

Proof. Since $F(z; t) \prec f$ for each $t \in I$, there exists $v: E \times I \to E$ such that $f(v(z; t)) = F(z; t)$ where $|v(z; t)| \leqq |z|$. Writing f as a column vector we have $f(v(z; t)) = f(z) + J(v(z; t) - z) + R(v(z; t), z)$ where $|R(\zeta, z)|/|\zeta - z| \to 0$ as $|\zeta - z| \to 0$. Hence

$$\frac{F(z; 0) - F(z; t)}{t^\rho} = J\left(\frac{z - v(z; t)}{t^\rho}\right) - \frac{R(v(z; t), z)}{t^\rho}$$

and the lemma follows from Lemma 1.

2. Starlike and convex mappings of the polydisk.

DEFINITION. A holomorphic mapping $f: E \to C^n$ is starlike if f is univalent, $f(0) = 0$ and $(1 - t)f \prec f$ for all $t \in I$.

THEOREM 1. *Suppose $f: E \to C^n$ is starlike and that J is the complex Jacobian matrix of f. There exists $w \in \mathscr{P}$ such that $f = Jw$ where f and w are written as column vectors.*

Proof. Apply Lemma 2 with $F(z; t) = (1 - t)f(z)$. Then

$$f(z) = \lim_{t\to 0^+} \frac{f(z) - (1 - t)f(z)}{t} = \lim_{t\to 0^+} \frac{F(z; 0) - F(z; t)}{t}$$

and the theorem follows from Lemma 2.

We now consider the conclusion of Theorem 1 in component form. Let J_j be the matrix obtained by replacing the jth column in J by the column vector f, $1 \leqq j \leqq n$. Then the jth component w_j of w is $\det(J_j)/\det J$. Theorem 1 therefore says that if f is starlike then $\operatorname{Re}[\det(J_j)/z_j \det J] \geqq 0$ when $|z| = |z_j| > 0$. Also,

$$(3) \qquad f_j = \frac{\partial f_j}{\partial z_1} w_1 + \frac{\partial f_j}{\partial z_2} w_2 + \cdots + \frac{\partial f_j}{\partial z_n} w_n\,, \qquad 1 \leqq j \leqq n$$

and equating coefficients in the power series using (3) we find

$$w_j(z) = z_j + \text{terms of total degree 2 or greater} .$$

Now suppose $|z^{(0)}| = |z_j^{(0)}| > 0$ and let α_k, $(1 \leqq k \leqq n)$ be such that $z_k^{(0)} = \alpha_k z_j^{(0)}$. Then $|\alpha_k| \leqq 1$, $(1 \leqq k \leqq n)$. Consider $w_j(z)/z_j = u(z_j)$ where z is restricted to the set,

$$z = (\alpha_1, \alpha_2, \cdots, \alpha_n)z_j , \qquad |z_j| < 1 .$$

Then $\operatorname{Re} u(z_j) \geqq 0$, $0 < |z_j| < 1$ and $u(z_j) \to 1$ as $z_j \to 0$. Since $\operatorname{Re} u(z_j)$ is a harmonic function of z_j, we conclude $\operatorname{Re} u(z_j) > 0$, $|z_j| < 1$ and

$$(4) \qquad \operatorname{Re}[w_j(z)/z_j] > 0 \quad \text{when} \quad |z| = |z_j| > 0 .$$

We now prove the converse of Theorem 1.

THEOREM 2. *Suppose $f: E \to C^n$ is holomorphic, $f(0) = 0$, J is nonsingular and that*

$$(5) \qquad f(z) = Jw,\ w \in \mathscr{P} .$$

Then f is starlike.

Proof. Since $\det J \neq 0$ when $z = 0$, f is univalent in a neighborhood of 0. It is clear that $\{r: 0 \leqq r \leqq 1 \text{ and } f \text{ is univalent in } E_r\} = A$ is a closed subset of $[0, 1]$. We will show that A is also open and that if f is univalent in E_r then $f(E_r)$ is starlike with respect to 0.

Let $r > 0$ be such that f is univalent in E_r, $(0 < r < 1)$. Let z be fixed, $|z| \leqq r$ and let $v(z; t)$ be such that $f(v(z; t)) = (1 - t)f(z)$, $-\varepsilon < t < t_0$ where ε is small and positive and $t_0 > 0$. This is possible since $\det J \neq 0$.

Then

$$(6) \qquad \begin{aligned} v(z; t) &= v(z; 0) + J^{-1}\cdot(-f(z))\cdot t + g(t) \\ &= z - J^{-1}\cdot J\cdot w\cdot t + g(t) \\ v(z; t) &= z - tw + g(t) \end{aligned}$$

by (5). Here $|g(t)|/t \to 0$ as $t \to 0$. Using (4), we conclude $|v(z; t)|$ is a strictly decreasing function of t. Hence each point of the ray $(1 - t)f(z)$, $0 < t \leqq 1$ is the image of a point $v(z; t) \in E_r$ for each z such that $|z| \leqq r$. We conclude that $f(E_r)$ is starlike with respect to 0. We now show A is open. Observe that f is one-to-one in the closed polydisk $\bar{E}_r$ for if $|z| \leqq |\zeta| = r$, $z \neq \zeta$ and $f(z) = f(\zeta)$ then by (6) and (4) we can conclude that for t positive and sufficiently small there are functions $v(\zeta; t)$, $v(z; t)$ such that $v(\zeta; t)$, $v(z, t) \in E_r$, $v(\zeta; t) \neq v(z; t)$ and

$f(v(z; t)) = (1-t)f(z) = (1-t)f(\zeta) = f(v(\zeta, t))$ which is a contradiction.

We now define a continuous nonnegative function $\phi: E \times E \to R$ (R is the real numbers) such that $\phi(z, \zeta) = 0$ if and only if $f(z) = f(\zeta)$, $z \neq \zeta$. We show that ϕ is positive on the closed set $\bar{E}_r \times \bar{E}_r$ and hence has a positive minimum on this set. This will imply f is univalent in $E_{r+\varepsilon}$ for some $\varepsilon > 0$ and hence A is open. For $z, \zeta \in E$, define $G(z, \zeta) = \det(a_{ij})$ where

$$a_{ij} = \begin{cases} \dfrac{f_i(z_1, z_2, \cdots, z_j, \zeta_{j+1} \cdots, \zeta_n) - f_i(z_1, z_2, \cdots, z_{j-1}, \zeta_j, \cdots, \zeta_n)}{z_j - \zeta_j}, & (z_j \neq \zeta_j) \\ \dfrac{\partial f_i}{\partial z_j}(z_1, z_2, \cdots, z_j, \zeta_{j+1}, \cdots, \zeta_n), & (z_j = \zeta_j) \end{cases}$$

and $f = (f_1, f_2, \cdots, f_n)$.

Now set $\phi(z, \zeta) = |G(z, \zeta)| + \sum_{j=1}^{n} |f_j(z) - f_j(\zeta)|$. Then $\phi(z, z) = |\det(J(z)| > 0$ while

$$\phi(z, \zeta) > 0 \quad \text{when} \quad f(z) \neq f(\zeta) .$$

If $f(z) = f(\zeta)$ for some $z, \zeta \in E$, $z \neq \zeta$ then the columns of $G(z, \zeta)$ are not linearly independent so $G(z, \zeta) = 0$ and $\phi(z, \zeta) = 0$. The proof is now complete.

THEOREM 3. *Suppose $f: E \to C^n$ is holomorphic, $f(0) = 0$ and that J is nonsingular for all $z \in E$. Then f is a univalent map of E onto a convex domain if and only if there exist univalent mappings f_j $(1 \leqq j \leqq n)$ from the unit disk in the plane onto convex domains in the plane such that $f(z) = T(f_1(z_1), f_2(z_2) \cdots, f_n(z_n))$ where T is a nonsingular linear transformation.*

Proof. It is clear that if f satisfies the conditions given in the theorem, then f is univalent and $f(E)$ is convex. We will prove the converse.

Suppose f is a univalent map of E onto a convex domain. Let $A = (A_1, A_2, \cdots, A_n)$ where $A_j \geqq 0$ $(1 \leqq j \leqq n)$ and let

$$A_t(z) = (z_1 e^{iA_1 t}, z_2 e^{iA_2 t}, \cdots, z_n e^{iA_n t})$$

where $-1 \leqq t \leqq 1$. Then

$$F(z; t) = 1/2[f(A_t(z)) + f(A_{-t}(z))] \prec f \qquad 0 \leqq t \leqq 1$$

and $F(z; t)$ satisfies the hypotheses of Lemma 2 with $\rho = 2$. Using the same notation as in Lemma 2, we have

$$
(7) \qquad
\begin{aligned}
F(z) &= (F_1, F_2, \cdots, F_n) \\
2F_j &= \sum_{k=1}^{n} A_k^2\left(z_k^2 \frac{\partial^2 f_j}{\partial z_k^2} + z_k \frac{\partial f_j}{\partial z_k}\right) \\
&\quad + 2\sum_{k=2}^{n}\sum_{l=1}^{k-1} A_k A_l z_k z_l \frac{\partial^2 f_j}{\partial z_l \partial z_k}
\end{aligned}
$$

and also $F = Jw$, $w \in \mathscr{P}$. Hence we find that $w_j = \det J^{(j)}/\det J$ where $J^{(j)}$ is obtained from J by replacing the jth column by F written as a column vector. Fix k, $1 \leqq k \leqq n$ and choose $A_k = 1$, $A_l = 0$, $l \neq k$, $1 \leqq l \leqq n$. Suppose $|z| = |z_j| > 0$, $j \neq k$ and $z_k = 0$. Then $w_j/z_j = 0$ and since $\operatorname{Re}(w_j/z_j) \geqq 0$ when $|z| = |z_j| > 0$ we must have $w_j \equiv 0$. We have therefore shown that for $1 \leqq j \leqq n$ and $1 \leqq k \leqq n$ we have

$$
(8) \qquad z_k^2 \frac{\partial^2 f_j}{\partial z_k^2} + z_k \frac{\partial f_j}{\partial z_k} = \frac{\partial f_j}{\partial z_k}\psi_k
$$

where $\operatorname{Re}[\psi_k(z)/z_k] \geqq 0$ when $|z| = |z_k| > 0$. With k as before, fix l, $1 \leqq l \leqq n$, $l \neq k$ and choose $A_k = 1$, $A_l = \varepsilon > 0$ and $A_m = 0, 1 \leqq m \leqq n$, $m \neq k, l$.

Using (8) we conclude

$$
w_j = \varepsilon \frac{z_k z_l G_j}{\det J} + O(\varepsilon^2) \qquad (j \neq k)
$$

where G_j is obtained from $\det J$ by replacing the jth column by the column $\partial^2 f_m/\partial z_l \partial z_k (1 \leqq m \leqq n)$. Hence $\operatorname{Re}[z_k z_l/z_j \cdot G_j/\det J] \geqq 0$ when $|z| = |z_j| > 0$. Since $\operatorname{Re}[z_k z_l/z_j \cdot G_j/\det J] = 0$ when $z_k z_l = 0$ we see that $G_j \equiv 0$ for each $j, 1 \leqq j \leqq n$.

Since the system of equations

$$
\sum_{j=1}^{n} \frac{\partial f_m}{\partial z_j}\varphi_j = \frac{\partial^2 f_m}{\partial z_l \partial z_k} \qquad 1 \leqq m \leqq n
$$

has solution

$$
\varphi_j = \frac{G_j}{\det J} = 0 \qquad 1 \leqq j \leqq n
$$

we conclude

$$
\frac{\partial^2 f_m}{\partial z_l \partial z_k} = 0 \qquad 1 \leqq m \leqq n .
$$

This implies

$$
(9) \qquad f_m(z) = \sum_{j=1}^{n} a_{j,m}\phi_{j,m}(z_j) \qquad 1 \leqq m \leqq n
$$

where $\phi_{j,m}$ is analytic on the unit disk in the complex plane. Using

(8) we conclude $\phi_{j,m} = \phi_{j,k}$ $(1 \leqq m, k \leqq n)$ provided the constants $a_{j,m}$ in (9) are appropriately chosen. The theorem now follows readily from (8).

EXAMPLE 1. Let $f: E \to C^2$ be given by $f(z) = (z_1 + az_2^2, z_2)$ where a is a complex number, $a \neq 0$. Clearly f is univalent. Letting $f = Jw$, we find $w_1 = z_1 - az_2^2$, $w_2 = z_2$ so f is starlike provided $|a| < 1$. Note that Theorem 3 implies the suprising result that none of the sets $f(E_r)$ is convex $(1 > r > 0)$.

EXAMPLE 2. Let $f: E \to C^2$ be given by $f(z) = (z_1 g(z), z_2 g(z))$, $g: E \to C$ where g is holomorphic, $0 \notin g(E)$. Then $f = Jw$ implies

$$w_1/z_1 = w_2/z_2 = 1 + \left[z_1 \frac{\partial g}{\partial z_1} + z_2 \frac{\partial g}{\partial z_2}\right] \Big/ g \tag{10}$$

and f is starlike if and only if $\operatorname{Re}(w_1(z)/z_1) \geqq 0$, $z \in E$. Conversely, one can show that if $f: E \to C^2$ is holomorphic, $f = Jw$ where $w \in \mathscr{P}$ and $w_1/z_1 = w_2/z_2$ then there exists $g: E \to C$, g holomorphic, $0 \notin g(E)$ such that (10) holds and $f = ((a_1 z_1 + a_2 z_2)g, (b_1 z_1 + b_2 z_2)g)$, $(a_1 b_2 \neq a_2 b_1)$. In these cases the intersection of the polydisk E with an analytic plane $\alpha z_1 + \beta z_2 = 0$ maps into an analytic plane $\delta f_1 + \gamma f_2 = 0$. Interesting choices of g are $g(z) = (1 - z_1 z_2)^{-1}$ and $g(z) = [(1 - z_1)(1 - z_2)]^{-1}$.

3. Extension to convex and starlike maps of D_p. Since the details of the proofs for the results in this section are similar to those in §'s 2 and 3, we omit the details. We wish to find lemmas which apply to D_p (D_p is defined in equation (1)) in the same way that Lemmas 1 and 2 apply to the polydisk. The crucial point is that given equation (6) with $0 \neq z \in D_p$ we wish to conclude

$$|v(z; t)|_p \leqq |z|_p \quad \text{when} \quad 0 < t < \varepsilon$$

for some $\varepsilon > 0$. This will be true provided $\sum_{j=1}^n |z_j - tw_j|^p < \sum_{j=1}^n |z_j|^p$ for t sufficiently small. That is

$$\sum_{\substack{j=1 \\ z_j \neq 0}}^{n} |z_j|^p (1 - 2t \operatorname{Re} w_j/z_j + t^2 |w_j/z_j|^2)^{p/2} + \sum_{z_j = 0} t^p |w_j|^p < \sum_{j=1}^{n} |z_j|^p$$

or

$$t\Big(\sum_{\substack{j=1 \\ z_j \neq 0}}^{n} -p \operatorname{Re} |z_j|^p \operatorname{Re}(w_j/z_j) + \sum_{z_j=0} t^{p-1} |w_j|\Big) < 0$$

when t is sufficiently small, $t > 0$. Hence we define $\mathscr{P}_p$ for $p \geqq 1$ by $w \in \mathscr{P}_p$ if $w: D_p \subset C^n \to C^n$, $w(0) = 0$, w holomorphic and

$$
(11) \qquad \begin{aligned} &\operatorname{Re} \sum_{j=1}^{n} w_j \cdot |z_j|^p / z_j \geqq 0 \qquad \text{if } p > 1 \\ &\operatorname{Re} \sum_{\substack{j=1 \\ z_j \neq 0}}^{n} w_j \cdot |z_j| / z_j - \sum_{z_j = 0} |w_j| \geqq 0 \qquad \text{if } p = 1 \,, \end{aligned}
$$

$z \in D_p$, $w = (w_1, w_2, \cdots, w_n)$.

We now have the following lemmas and theorems which correspond to the lemmas and theorems of §§ 2 and 3.

LEMMA 3. *Let $v(z; t): D_p \times I \to C^n$ be holomorphic for each $t \in I$, $v(z, 0) = z$, $v(0, t) = 0$ and $|v(z; t)|_p < 1$ when $z \in D_p$. If*

$$\lim_{t \to 0^+} [(z - v(z; t))/t^\rho] = w(z)$$

exists and is holomorphic in D_p for some $\rho > 0$, then $w \in \mathscr{P}_p$.

LEMMA 4. *Let $f: D_p \to C^n$ be holomorphic and univalent and satisfy $f(0) = 0$. Let $F(z; t): D_p \times I \to C^n$ be a holomorphic function of z for each $t \in I$, $F(z, 0) = f(z)$, $F(0; t) = 0$ and suppose $F(z; t) \prec f$ for each $t \in I$. Let $\rho > 0$ be such that $\lim_{t \to 0^+}(F(z; 0) - F(z; t))/t^\rho = F(z)$ exists and is holomorphic. Then $F(z) = Jw$ where $w \in \mathscr{P}_p$.*

THEOREM 4. *If $f: D_p \to C^n$ is starlike then there exists $w \in \mathscr{P}_p$ such that $f = Jw$. Conversely, if $f: D_p \to C^n$, $f(0) = 0$, J is nonsingular and $f = Jw$, $w \in \mathscr{P}_p$ then f is starlike.*

THEOREM 5. *Let $f: D_p \to C^n$, $f(0) = 0$ and suppose J is nonsingular. Then $f(D_p)$ is convex if and only if $F = Jw$ where $w \in \mathscr{P}_p$ for each choice of $A = (A_1, A_2, \cdots, A_n)$, $A_j \geqq 0$ $(1 \leqq j \leqq n)$ and F is given by* (7) *with $z \in D_p$.*

Now set $p = 2$. It is easy to see that Theorem 4 above is equivalent to Matsuno's Theorem 1 [4, p. 91]. Consider $f: D_2 \to C^2$ given by $f(z) = (z_1 + az_2^2, z_2)$. Theorem 5 shows that $f(D_2)$ is convex if and only if $|a| \leqq 1/2$ while Matsuno's Lemma 3 [4, p. 94] implies f is convex-like if and only if $|a| \leqq 3\sqrt{3}/4$. This shows that convex-like is not equivalent to geometrically convex.

REFERENCES

1. S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, Vol. V., Amer. Math. Soc., New York, 1950.

2. S. Bochner and W. T. Martin, *Several complex variables*, Princeton Univ. Press, 1948.

3. R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
4. Takeshi Matsuno, *Star-like theorems and convex-like theorems in the complex vector space*, Sci. Rep. Tokyo, Kyoiku Daigaku, Sect. A **5** (1955), 88-95.
5. M. S. Robertson, *Applications of the subordination principle to univalent functions*, Pacific J. Math. **11** (1961), 315-324.

Received May 19, 1969. This research was supported by the National Science Foundation, Grant GP 8225.

UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 33, No. 1 March, 1970