THE PRINCIPLE OF SUBORDINATION APPLIED TO FUNCTIONS OF SEVERAL VARIABLES

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In this paper we consider univalent maps of domains in \( \mathbb{C}^n \), where \( n \geq 2 \). Let \( P \) be a polydisk in \( \mathbb{C}^n \). We find necessary and sufficient conditions that a function \( f: P \rightarrow \mathbb{C}^n \) be univalent and map the polydisk \( P \) onto a starlike or a convex domain. We also consider maps from

\[
D_p = \{ z : |z|_p < 1 \} \subset \mathbb{C}^n
\]

\[
|z|_p = \left( |z_1|_p^p + |z_2|_p^p + \cdots + |z_n|_p^p \right)^{1/p}, \quad p \geq 1
\]

into \( \mathbb{C}^n \) and give necessary and sufficient conditions that such a map have starlike or convex image.

In [4] Matsuno has considered a similar problem for the hypersphere \( D_\mathbb{C} \subset \mathbb{C}^n \). His definition of starlikeness is different from that used in this paper, but the results show that the two definitions are equivalent. However, his definition of convex-like is not equivalent to geometrically convex.

1. Preliminary lemmas. For \( (z_1, z_2, \ldots, z_n) = z \in \mathbb{C}^n \), define \( |z| = \max_{1 \leq j \leq n} |z_j| \). Let \( E_r = \{ z \in \mathbb{C}^n : |z| < r \} \) and \( E = E_1 \). Let \( \mathcal{P} \) be the class of mappings \( w: E \rightarrow \mathbb{C}^n \) which are holomorphic and which satisfy \( w(0) = 0 \), \( \Re \frac{w_j(z)}{|z_j|} \geq 0 \) when \( |z| = |z_j| > 0, (1 \leq j \leq n) \) where \( w = (w_1, w_2, \ldots, w_n) \). The following lemmas are generalizations of Theorems A and B of Robertson [5, p. 315–317].

**Lemma 1.** Let \( v(z; t): E \times I \rightarrow \mathbb{C}^n \) be holomorphic for each \( t \in I = [0, 1] \), \( v(z; 0) = z, v(0, t) = 0 \) and \( |v(z; t)| < 1 \) when \( z \in E \). If

\[
\lim_{t \to 0^+} \left[ (z - v(z; t))/t^p \right] = w(z)
\]

exists and is holomorphic in \( E \) for some \( \rho > 0 \), then \( w \in \mathcal{P} \).

**Proof.** The hypothesis (2) implies that \( \lim_{t \to 0^+} v_j(z; t) = z_j \) (here \( v(z; t) = (v_1(z; t), v_2(z; t), \ldots, v_n(z; t)) \) so

\[
\frac{2z_j(z_j - v_j(z; t))}{z_j + v_j(z; t)} = \phi_j(z; t)
\]

is holomorphic for \( z \in E, z_j \neq 0 (1 \leq j \leq n) \). By Schwarz lemma, \( |v(z; t)| \leq |z| \) and hence \( \Re [\phi_j(z; t)/z_j] \geq 0 \) when \( |z| = |z_j| > 0 \). Setting \( \psi(z; t) = (\psi_1, \psi_2, \ldots, \psi_n), (z \in E, z_1z_2 \cdots z_n \neq 0) \) we observe that
\[
\lim_{t \to 0^+} \psi(z; t)/t^\rho = w(z)
\]
for these values of \( z \) and using continuity of \( w \) we conclude \( w \in \mathcal{P} \).

**Lemma 2.** Let \( f: E \to \mathbb{C}^n \) be holomorphic and univalent and satisfy \( f(0) = 0 \). Let \( F(z; t): E \times I \to \mathbb{C}^n \) be a holomorphic function of \( z \) for each \( t \in I = [0, 1] \), \( F(z; 0) = f(z) \), \( F(0, t) = 0 \) and suppose \( F(z; t) < f \) for each \( t \in I \) (i.e., \( F(E; t) \subset f(E) \) for each \( t \in I \)). Let \( \rho > 0 \) be such that \( \lim_{t \to 0^+} F(z; 0) - F(z; t)/t^\rho = F(z) \) exists and is holomorphic. Then \( F(z) = Jw \) where \( w \in \mathcal{P} \). Here \( F \) and \( w \) are written as column vectors and \( J \) is the complex Jacobian matrix for the mapping \( f \).

**Proof.** Since \( F(z; t) < f \) for each \( t \in I \), there exists \( v: E \times I \to E \) such that \( f(v(z; t)) = F(z; t) \) where \( |v(z; t)| \leq |z| \). Writing \( f \) as a column vector we have \( f(v(z; t)) = f(z) + J(v(z; t) - z) + R(v(z; t), z) \) where \( |R(\zeta, z)|/|\zeta - z| \to 0 \) as \( |\zeta - z| \to 0 \). Hence
\[
\frac{F(z; 0) - F(z; t)}{t^\rho} = J\left(\frac{z - v(z; t)}{t^\rho}\right) - \frac{R(v(z; t), z)}{t^\rho}
\]
and the lemma follows from Lemma 1.

2. Starlike and convex mappings of the polydisk.

**Definition.** A holomorphic mapping \( f: E \to \mathbb{C}^n \) is starlike if \( f \) is univalent, \( f(0) = 0 \) and \( (1 - t)f < f \) for all \( t \in I \).

**Theorem 1.** Suppose \( f: E \to \mathbb{C}^n \) is starlike and that \( J \) is the complex Jacobian matrix of \( f \). There exists \( w \in \mathcal{P} \) such that \( f = Jw \) where \( f \) and \( w \) are written as column vectors.

**Proof.** Apply Lemma 2 with \( F(z; t) = (1 - t)f(z) \). Then
\[
f(z) = \lim_{t \to 0^+} f(z) - (1 - t)f(z) = \lim_{t \to 0^+} \frac{F(z; 0) - F(z; t)}{t}
\]
and the theorem follows from Lemma 2.

We now consider the conclusion of Theorem 1 in component form. Let \( J_j \) be the matrix obtained by replacing the \( j \)th column in \( J \) by the column vector \( f_j \), \( 1 \leq j \leq n \). Then the \( j \)th component \( w_j \) of \( w \) is \( \det (J_j)/\det J \). Theorem 1 therefore says that if \( f \) is starlike then \( \Re \{ \det (J_j)/z_j \det J \} \geq 0 \) when \( |z| = |z_j| > 0 \). Also,
\[
f_j = \frac{\partial f_j}{\partial z_1}w_1 + \frac{\partial f_j}{\partial z_2}w_2 + \cdots + \frac{\partial f_j}{\partial z_n}w_n \quad 1 \leq j \leq n
\]
and equating coefficients in the power series using (3) we find
\[ w_j(z) = z_j + \text{terms of total degree 2 or greater}. \]

Now suppose \(|z^{(0)}| = |z_j^{(0)}| > 0\) and let \(\alpha_k, (1 \leq k \leq n)\) be such that \(z_k^{(0)} = \alpha_k z_j^{(0)}\). Then \(|\alpha_k| \leq 1, (1 \leq k \leq n)\). Consider \(w_j(z)/z_j = u(z_j)\) where \(z\) is restricted to the set,
\[ z = (\alpha_1, \alpha_2, \ldots, \alpha_n)z_j, \quad |z_j| < 1. \]

Then \(\Re u(z_j) \geq 0, 0 < |z_j| < 1\) and \(u(z_j) \rightarrow 1\) as \(z_j \rightarrow 0\). Since \(\Re u(z_j)\) is a harmonic function of \(z_j\), we conclude \(\Re u(z_j) > 0, |z_j| < 1\) and
\[
(4) \quad \Re \left[ w_j(z)/z_j \right] > 0 \quad \text{when} \quad |z| = |z_j| > 0.
\]

We now prove the converse of Theorem 1.

**Theorem 2.** Suppose \(f: E \rightarrow \mathbb{C}^n\) is holomorphic, \(f(0) = 0, J\) is nonsingular and that
\[
(5) \quad f(z) = Jw, w \in \mathcal{H}.
\]

Then \(f\) is starlike.

**Proof.** Since \(\det J \neq 0\) when \(z = 0\), \(f\) is univalent in a neighborhood of 0. It is clear that \(\{r: 0 \leq r \leq 1\text{ and } f\text{ is univalent in } E_r\} = A\) is a closed subset of \([0, 1]\). We will show that \(A\) is also open and that if \(f\) is univalent in \(E_r\) then \(f(E_r)\) is starlike with respect to 0.

Let \(r > 0\) be such that \(f\) is univalent in \(E_r, (0 < r < 1)\). Let \(z\) be fixed, \(|z| \leq r\) and let \(v(z; t)\) be such that \(f(v(z; t)) = (1 - t)f(z), -\varepsilon < t < t_0\) where \(\varepsilon\) is small and positive and \(t_0 > 0\). This is possible since \(\det J \neq 0\).

Then
\[
v(z; t) = v(z; 0) + J^{-1} \cdot (-f(z)) \cdot t + g(t)
\]
\[
(6) \quad v(z; t) = z - J^{-1} \cdot J \cdot w \cdot t + g(t)
\]
by (5). Here \(|g(t)|/t \rightarrow 0\) as \(t \rightarrow 0\). Using (4), we conclude \(|v(z; t)|\) is a strictly decreasing function of \(t\). Hence each point of the ray \((1 - t)f(z), 0 < t \leq 1\) is the image of a point \(v(z; t) \in E_r\) for each \(z\) such that \(|z| \leq r\). We conclude that \(f(E_r)\) is starlike with respect to 0.

We now show \(A\) is open. Observe that \(f\) is one-to-one in the closed polydisk \(\bar{E}_r\) for if \(|z| \leq |\zeta| = r, z \neq \zeta\) and \(f(z) = f(\zeta)\) then by (6) and (4) we can conclude that for \(t\) positive and sufficiently small there are functions \(v(\zeta; t), v(z; t)\) such that \(v(\zeta; t), v(z, t) \in E_r, v(\zeta; t) \neq v(z; t)\) and
\( f(v(z; t)) = (1 - t)f(z) = (1 - t)f(\zeta) = f(v(\zeta, t)) \) which is a contradiction.

We now define a continuous nonnegative function \( \phi: E \times E \to \mathbb{R} \) (\( \mathbb{R} \) is the real numbers) such that \( \phi(z, \zeta) = 0 \) if and only if \( f(z) = f(\zeta) \), \( z \neq \zeta \). We show that \( \phi \) is positive on the closed set \( E_r \times E_r \) and hence has a positive minimum on this set. This will imply \( f \) is univalent in \( E_{r+\varepsilon} \) for some \( \varepsilon > 0 \) and hence \( A \) is open. For \( z, \zeta \in E \), define \( G(z, \zeta) = \det (a_{ij}) \) where

\[
a_{ij} = \begin{cases} f_j(z_1, z_2, \ldots, z_j, \zeta_j, \ldots, \zeta_n) - f_j(z_1, z_2, \ldots, z_{j-1}, \zeta_j, \ldots, \zeta_n), & (z_j \neq \zeta_j) \\
\frac{\partial f_j}{\partial z_j}(z_1, z_2, \ldots, z_j, \zeta_j, \ldots, \zeta_n), & (z_j = \zeta_j)
\end{cases}
\]

and \( f = (f_1, f_2, \ldots, f_n) \).

Now set \( \phi(z, \zeta) = |G(z, \zeta)| + \sum_{j=1}^{n} |f_j(z) - f_j(\zeta)| \). Then \( \phi(z, z) = |\det (J(z))| > 0 \) while

\[
\phi(z, \zeta) > 0 \quad \text{when} \quad f(z) \neq f(\zeta).
\]

If \( f(z) = f(\zeta) \) for some \( z, \zeta \in E \), \( z \neq \zeta \) then the columns of \( G(z, \zeta) \) are not linearly independent so \( G(z, \zeta) = 0 \) and \( \phi(z, \zeta) = 0 \). The proof is now complete.

**Theorem 3.** Suppose \( f: E \to \mathbb{C}^n \) is holomorphic, \( f(0) = 0 \) and that \( J \) is nonsingular for all \( z \in E \). Then \( f \) is a univalent map of \( E \) onto a convex domain if and only if there exist univalent mappings \( f_j \) (\( 1 \leq j \leq n \)) from the unit disk in the plane onto convex domains in the plane such that \( f(z) = T(f_1(z_1), f_2(z_2) \ldots, f_n(z_n)) \) where \( T \) is a nonsingular linear transformation.

**Proof.** It is clear that if \( f \) satisfies the conditions given in the theorem, then \( f \) is univalent and \( f(E) \) is convex. We will prove the converse.

Suppose \( f \) is a univalent map of \( E \) onto a convex domain. Let \( A = (A_1, A_2, \ldots, A_n) \) where \( A_j \geq 0 \) (\( 1 \leq j \leq n \)) and let

\[
A_t(z) = (z^1 e^{it_1}, z^2 e^{it_2}, \ldots, z^n e^{it_n})
\]

where \( -1 \leq t \leq 1 \). Then

\[
F(z; t) = 1/2[f(A_t(z)) + f(A_{-t}(z))] \quad f < f \quad 0 \leq t \leq 1
\]

and \( F(z; t) \) satisfies the hypotheses of Lemma 2 with \( \rho = 2 \). Using the same notation as in Lemma 2, we have
\[ F(z) = (F_1, F_2, \ldots, F_n) \]
\[ 2F_j = \sum_{k=1}^{n} A_{jk} \left( z_k^2 \frac{\partial^2 f_j}{\partial z_k^2} + z_k \frac{\partial f_j}{\partial z_k} \right) \]
\[ + 2 \sum_{k=2}^{n} \sum_{l=1}^{k-1} A_{kl} z_k z_l \frac{\partial^2 f_j}{\partial z_k \partial z_l} \]

and also \( F = Jw, w \in \mathcal{O} \). Hence we find that \( w_j = \det J^{(j)}/\det J \) where \( J^{(j)} \) is obtained from \( J \) by replacing the \( j \)th column by \( F \) written as a column vector. Fix \( k, 1 \leq k \leq n \) and choose \( A_k = 1, A_l = 0, l \neq k, 1 \leq l \leq n \). Suppose \(|z| = |z_j| > 0, j \neq k \) and \( z_k = 0 \). Then \( w_j/z_j = 0 \) and since \( \text{Re}(w_j/z_j) \geq 0 \) when \(|z| = |z_j| > 0 \) we must have \( w_j \equiv 0 \). We have therefore shown that for \( 1 \leq j \leq n \) and \( 1 \leq k \leq n \) we have
\[ z_k^2 \frac{\partial^2 f_j}{\partial z_k^2} + z_k \frac{\partial f_j}{\partial z_k} = \frac{\partial f_j}{\partial z_k} \psi_k \]

where \( \text{Re}[\psi_k(z)/z_k] \geq 0 \) when \(|z| = |z_j| > 0 \). With \( k \) as before, fix \( l, 1 \leq l \leq n, l \neq k \) and choose \( A_k = 1, A_l = \varepsilon > 0 \) and \( A_m = 0, 1 \leq m \leq n, m \neq k, l \).

Using (8) we conclude
\[ w_j = \varepsilon z_k z_l G_j / \det J + O(\varepsilon) \quad (j \neq k) \]

where \( G_j \) is obtained from \( \det J \) by replacing the \( j \)th column by the column \( \partial^2 f_m/\partial z_l \partial z_k (1 \leq m \leq n) \). Hence \( \text{Re}[z_k z_l / z_j G_j / \det J] \geq 0 \) when \(|z| = |z_j| > 0 \). Since \( \text{Re}[z_k z_l / z_j G_j / \det J] = 0 \) when \( z_k z_l = 0 \) we see that \( G_j = 0 \) for each \( j, 1 \leq j \leq n \).

Since the system of equations
\[ \sum_{j=1}^{n} \frac{\partial f_m}{\partial z_j} \phi_j = \frac{\partial^2 f_m}{\partial z_l \partial z_k} \]

has solution
\[ \phi_j = \frac{G_j}{\det J} = 0 \]

we conclude
\[ \frac{\partial^2 f_m}{\partial z_l \partial z_k} = 0 \]

where \( \phi_{j,m} \) is analytic on the unit disk in the complex plane. Using
we conclude \( \phi_{j,m} = \phi_{j,k} \) \((1 \leq m, k \leq n)\) provided the constants \( a_{j,m} \) in (9) are appropriately chosen. The theorem now follows readily from (8).

EXAMPLE 1. Let \( f: E \to C^2 \) be given by \( f(z) = (z_1 + az_2, z_2) \) where \( a \) is a complex number, \( a \neq 0 \). Clearly \( f \) is univalent. Letting \( f = Jw \), we find \( w_1 = z_1 - az_2, w_2 = z_2 \) so \( f \) is starlike provided \( |a| < 1 \). Note that Theorem 3 implies the suprising result that none of the sets \( f(E_r) \) is convex \((1 > r > 0)\).

EXAMPLE 2. Let \( f: E \to C^2 \) be given by \( f(z) = (z,g(z), z_2 g(z)) \), \( g: E \to C \) where \( g \) is holomorphic, \( 0 \in g(E) \). Then \( f = Jw \) implies

\[
\frac{w_1}{z_1} = \frac{w_2}{z_2} = 1 + \left[ z_1 \frac{\partial g}{\partial z_1} + z_2 \frac{\partial g}{\partial z_2} \right] / g
\]

(10) and \( f \) is starlike if and only if \( \text{Re} (w_i(z)/z_i) \geq 0, z \in E \). Conversely, one can show that if \( f: E \to C^2 \) is holomorphic, \( f = Jw \) where \( w \in \mathcal{P} \) and \( w_1/z_1 = w_2/z_2 \) then there exists \( g: E \to C, g \) holomorphic, \( 0 \in g(E) \) such that (10) holds and

\[ f = ((a_1 z_1 + a_2 z_2)g, (b_1 z_1 + b_2 z_2)g), (a_1 b_2 \neq a_2 b_1). \]

In these cases the intersection of the polydisk \( E \) with an analytic plane \( \alpha z_1 + \beta z_2 = 0 \) maps into an analytic plane \( \partial f_1 + \gamma f_2 = 0 \). Interesting choices of \( g \) are \( g(z) = (1 - z_1 z_2)^{-1} \) and \( g(z) = \left[ (1 - z_1)(1 - z_2) \right]^{-1} \).

3. Extension to convex and starlike maps of \( D_p \). Since the details of the proofs for the results in this section are similar to those in §'s 2 and 3, we omit the details. We wish to find lemmas which apply to \( D_p \) \((D_p \) is defined in equation (1)) in the same way that Lemmas 1 and 2 apply to the polydisk. The crucial point is that given equation (6) with \( 0 \neq z \in D_p \) we wish to conclude

\[ |v(z; t)|_p \leq |z|_p \quad \text{when} \quad 0 < t < \varepsilon \]

for some \( \varepsilon > 0 \). This will be true provided \( \sum_{j=1}^{n} |z_j - tw_j|^p < \sum_{j=1}^{n} |z_j|^p \) for \( t \) sufficiently small. That is

\[
\sum_{j=1}^{n} |z_j|^p (1 - 2t \text{Re} w_j/|z_j| + t^2 |w_j/|z_j|^2|^{p/2}) + \sum_{j=0}^{t} t^p |w_j|^p < \sum_{j=1}^{n} |z_j|^p
\]

or

\[
t \left( \sum_{j=1}^{n} -p \text{Re} |z_j|^p \text{Re} \left( w_j/|z_j| \right) + \sum_{j=0}^{t^{p-1}} t^p |w_j| \right) < 0
\]

when \( t \) is sufficiently small, \( t > 0 \). Hence we define \( \mathcal{P}_p \) for \( p \geq 1 \) by \( w \in \mathcal{P}_p \) if \( w: D_p \subset C^n \to C^n, w(0) = 0, w \) holomorphic and
\[
\text{Re} \sum_{j=1}^{n} w_j \cdot \frac{z_j}{|z_j|^p} / z_j \geq 0 \quad \text{if } p > 1
\]
\[
\text{Re} \sum_{j \neq 1}^{n} w_j \cdot \frac{z_j}{|z_j|} - \sum_{j=1}^{n} |w_j| \geq 0 \quad \text{if } p = 1,
\]
\[z \in D_p, \quad w = (w_1, w_2, \ldots, w_n).
\]

We now have the following lemmas and theorems which correspond to the lemmas and theorems of §§ 2 and 3.

**Lemma 3.** Let \(v(z; t): D_p \times I \rightarrow \mathbb{C}^n\) be holomorphic for each \(t \in I, v(z, 0) = z, v(0, t) = 0\) and \(|v(z; t)|_p < 1\) when \(z \in D_p\). If
\[
\lim_{t \rightarrow 0^+} [(z - v(z; t))/t^p] = w(z)
\]
exists and is holomorphic in \(D_p\) for some \(\rho > 0\), then \(w \in \mathcal{P}_p\).

**Lemma 4.** Let \(f: D_p \rightarrow \mathbb{C}^n\) be holomorphic and univalent and satisfy \(f(0) = 0\). Let \(F(z; t): D_p \times I \rightarrow \mathbb{C}^n\) be a holomorphic function of \(z\) for each \(t \in I, F(z, 0) = f(z), F(0; t) = 0\) and suppose \(F(z; t) < f\) for each \(t \in I\). Let \(\rho > 0\) be such that \(\lim_{t \rightarrow 0^+} (F(z; 0) - F(z; t))/t^\rho = F(z)\) exists and is holomorphic. Then \(F(z) = Jw\) where \(w \in \mathcal{P}_p\).

**Theorem 4.** If \(f: D_p \rightarrow \mathbb{C}^n\) is starlike then there exists \(w \in \mathcal{P}_p\) such that \(f = Jw\). Conversely, if \(f: D_p \rightarrow \mathbb{C}^n, f(0) = 0, J\) is nonsingular and \(f = Jw, w \in \mathcal{P}_p\) then \(f\) is starlike.

**Theorem 5.** Let \(f: D_p \rightarrow \mathbb{C}^n, f(0) = 0\) and suppose \(J\) is nonsingular. Then \(f(D_p)\) is convex if and only if \(F = Jw\) where \(w \in \mathcal{P}_p\) for each choice of \(A = (A_1, A_2, \ldots, A_n), A_j \geq 0 (1 \leq j \leq n)\) and \(F\) is given by (7) with \(z \in D_p\).

Now set \(p = 2\). It is easy to see that Theorem 4 above is equivalent to Matsuno's Theorem 1 [4, p. 91]. Consider \(f: D_2 \rightarrow \mathbb{C}^2\) given by \(f(z) = (z_1 + az_2^2, z_2)\). Theorem 5 shows that \(f(D_2)\) is convex if and only if \(|a| \leq 1/2\) while Matsuno's Lemma 3 [4, p. 94] implies \(f\) is convex-like if and only if \(|a| \leq 3\sqrt{3}/4\). This shows that convex-like is not equivalent to geometrically convex.

**References**


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