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This paper is concerned with continuous and uniformly continuous complementors on a B^*-algebra. Let A be a B^*-algebra with a complementor p and E_p the set of all p-projections of A. We show that if A has no minimal left ideals of dimension less than three, then p is uniformly continuous if and only if E_p is a closed and bounded subset of A. We also give a characterization of the boundedness of E_p.

Let A be a complex Banach algebra and let L_r be the set of all closed right ideals of A. Following [4], we shall say that A is a right complemented Banach algebra if there exists a mapping $p: R \to R^p$ of L_r into itself having the following properties:

$$(C_1) \qquad R \cap R^p = (0) \qquad (R \in L_r) ;$$

$$(C_2) \qquad R + R^p = A \qquad (R \in L_r) ;$$

$$(C_3) \qquad (R^p)^p = R \qquad (R \in L_r) ;$$

$$(C_4) \qquad \text{if } R_1 \subset R_2, \text{ then } R_2^p \subset R_1^p \ (R_1, R_2 \in L_r) .$$

The mapping p is called a right complementor on A. In this paper a complemented Banach algebra will always mean a right complemented Banach algebra. We also use $p(R)$ for R^p.

For any set S in a Banach algebra A, let S_l and S_r denote the left and right annihilators of S in A, respectively. Then A is called an annihilator algebra if, for every closed left ideal I and for every closed right ideal R, we have $I_r = (0)$ if and only if $I = A$ and $R_l = (0)$ if and only if $R = A$. If $I_{rl} = I$ and $R_{lr} = R$, then A is called a dual algebra.

We say that a Banach algebra A has an approximate identity if there exists a net $\{e_\alpha\}$ in A such that $\|e_\alpha\| \leqq 1$, for all α, and $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$, for all $x \in A$. Every B^*-algebra has an approximate identity.

A minimal idempotent f in a complemented Banach algebra A is called a p-projection if $(fA)^p = (1 - f)A$. If A is a semi-simple annihilator complemented Banach algebra, then every nonzero right ideal, no matter whether closed or not (see [4; p. 653]), contains a p-projection. Let A be a complemented B^*-algebra with a complementor p. Since, by [4; p. 655, Lemma 5], the socle of A is dense in A, A is dual (see [3; p. 222, Th. 2.1]). Let E (resp. E_p) be the set of all self-adjoint minimal idempotents (resp. p-projections) in A. Then, for each $e \in E$, there exists a unique $P(e) \in E_p$ such that $P(e)A = eA$. It can be shown that P is a one-to-one mapping of E onto E_p. We call P the p-derived mapping of p. The complementor p is said to be continuous if P is continuous in the relative topologies of E and E_p induced by the given norm on A (see [1; p. 463, Definition 3.7]).

Let A be a dual B^*-algebra. It has been shown in [1; p. 463, Th. 3.6] that the mapping $p: R \to (R_l)^*$ is a complementor on $A(R \in L_r)$. In this case $E_p = E$, P is the identity map, and therefore p is uniformly continuous.

The concept "p is continuous" can be defined for any semi-simple annihilator complemented Banach $*$-algebra in which $xx^* = 0$ implies $x = 0$. In fact, let A be such an algebra and p a given complementor on A. By [2; p. 155, Th. 1], every maximal closed right ideal of A is modular. Therefore [1, p. 462, Corollary 3.4] holds for A. Hence the mapping P exists as in the case of B^*-algebra and so the concept of continuity of p can be defined.

In this paper, all algebras and spaces under consideration are over the complex field C.

2. **Lemmas.** In this section, unless otherwise stated, H will denote a complex Hilbert space and $A = LC(H)$, the set of all compact operators on H. There exist many complementors on A. If H is infinite dimensional, then all complementors on A are continuous ([1; p. 471, Th. 6.8]). However if $\dim H$ is finite, this is not true in general as is shown in [1; p. 475]. If $\dim H \geqq 3$, then every continuous complementor on A is uniformly continuous (see [1; p. 471, Corollary 6.6]).

If u and v are elements of H, $u \otimes v$ will denote the operator on H defined by the relation $(u \otimes v)(h) = (h, v)u$, for all $h \in H$.

LEMMA 1. *Let A be any C^*-subalgebra of bounded operators on H and $E \subset A$ the set of all self-adjoint minimal idempotents. The E is a closed subset of $L(H)$, all bounded operators on H.*

Proof. Let $\{e_n\} \subset E$ be a sequence converging to some $e \in A$. Clearly $e^2 = e$ and $e^* = e$. In order that $e \in E$, it suffices to show that $e(H)$ is one dimensional. Since $(u \otimes v)^* = v \otimes u$ and since each e_n is a self-adjoint minimal idempotent, we can write $e_n = u_n \otimes u_n$, where $u_n \in H$ and $\|u_n\| = 1$ $(n = 1, 2, \cdots)$. Let $v, w \in H$ be such that $e(v) \neq 0$, $e(w) \neq 0$. Since $\{(v, u_n)\}$ is bounded, there exists a subsequence $\{v, u_k)\}$ of $\{(v, u_n)\}$ and a nonzero constant $a \in C$ such that $(v, u_k) \to a$. Since

$$\|au_k - e(v)\| \leqq |a - (v, u_k)| \, \|u_k\| + \|e_k - e\| \, \|v\| ,$$

we have $au_k \to e(v)$. Similarly we can show that there exist a subsequence $\{u_t\}$ of $\{u_k\}$ and a nonzero constant $b \in C$ such that $bu_t \to e(w)$.

It follows now that $be(v) = ae(w)$, which shows that $e(H)$ is one dimensional. This completes the proof.

LEMMA 2. *Let H be finite dimensional, p a complementor on A and E_p the set of all p-projections in A. If E_p is a closed and bounded subset of A, then p is continuous.*

Proof. Let $e \in E$ and let $\{e_n\}$ be a sequence in E such that $e_n \to e$. Write $e_n = u_n \otimes u_n$, $e = u \otimes u$, where $u_n, u \in H$ and $||u_n|| = ||u|| = 1$ $(n = 1, 2, \cdots)$. Since H is finite dimensional, there exists a subsequence $\{u_k\}$ of $\{u_n\}$ such that $u_k \to u'$ for some $u' \in H$; clearly $||u'|| = 1$ and $u' \otimes u' = u \otimes u$. Thus $u = au'$, where $a = (u, u')$ and $|a| = 1$. Let $u'_k = au_k$. Then $e_k = u'_k \otimes u'_k$. Let P be the p-derived mapping of p. Since $P(e_k)$ is a minimal idempotent and since $P(e_k)A = e_kA$, we can write $P(e_k) = u'_k \otimes v'_k$, where $v'_k \in H$ $(k = 1, 2, \cdots)$. Similarly $P(e) = u \otimes v$ with $v \in H$. Since E_p is bounded and since $||u'_k|| = 1$, $\{v'_k\}$ is bounded. Since H is finite dimensional, there exists a subsequence $\{v'_t\}$ of $\{v'_k\}$ such that $v'_t \to v'$ for some $v' \in H$. As $||P(e_t)|| \geqq 1$, $v' \neq 0$. Since $P(e_t) = u'_t \otimes v'_t \to u \otimes v'$ and since E_p is closed, it follows that also $u \otimes v' \in E_p$. Then both $u \otimes v', u \otimes v \in E_p$. However, by [1, p. 466, Lemma 5.1] for any $u \in H$, there exists a unique such v. Thus $v = v'$. Hence $P(e_t) \to P(e)$. Therefore P is continuous and so is p. This completes the proof.

3. Main theorem.

Throughout this section A will be a B^*-algebra with a complementor p. Then A is dual (see §1). Let $\{I_t: t \in T\}$ be the family of all minimal closed two-sided ideals of A. Then, by [3; p. 221, Lemma 2.3], $A = (\sum_t I_t)_0$, the $B^*(\infty)$-sum of I_t. Since each I_t is a simple dual B^*-algebra, $I_t = LC(H_t)$ for some Hilbert space $H_t (t \in T)$. It has been shown in [4; p. 652, Lemma 1] that p induces a complementor p_t on I_t, which is given by $p_t(R) = p(R) \cap I_t$ for all closed right ideals R of $I_t (t \in T)$.

Let E (resp. E_t) be the set of all self-adjoint minimal idempotents in A (resp. in I_t) and let E_p (resp. E_p^t) be the set of all p-projections in A (resp. in I_t). Clearly $E_t = E \cap I_t$ and $E_p^t = E_p \cap I_t (t \in T)$. It can be shown that, if $u \neq v (u, v \in T)$, then $||e_u - e_v|| = 1$, for all $e_u \in E_u$, and $e_v \in E_v$. Since each $e \in E$ belongs to some I_t, $E = \bigcup_t E_t$. Similarly, if $u \neq v (u, v \in T)$, then $||f_u - f_v|| = \text{maximum}\,(||f_u||, ||f_v||) \geqq 1$, for all $f_u \in E_p^u$ and $f_v \in E_p^v$; $E_p = \bigcup_t E_p^t$. Thus p is continuous if and only if p_t is continuous for all $t \in T$ (see [1; p. 464]).

THEOREM 3. *Let A be a B^*-algebra which has no minimal left ideals of dimension less than three and p a complementor on A. Then the following statements are equivalent:*

(i) *p is uniformly continuous.*

(ii) *There exists an involution* $*'$ *on* A *for which* $R^p = (R_l)^{*'}$, *for every closed right ideal* R *of* A (*and hence there exists an equivalent norm* $||\cdot||'$ *on* A *which satisfies the* B^*-*condition for* $*'$).

(iii) *The set* E_p *of all p-projections in* A *is a closed and bounded subset of* A.

Proof. (i) $\rightarrow$ (ii). This is [1; p. 477, Th. 7.4].

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let E_p^t be the set of all p-projections in $I_t (t \in T)$. By [1; p. 465, Corollary 4.4], each $f_t \in E_p^t$ is self-adjoint in $*'$. Hence $||f_t||' = 1$. Since each E_p^t is the set of all self-adjoint (in $*'$) minimal idempotents in I_t, by Lemma 1, E_p^t is closed in $||\cdot||'$. It is now easy to show that E_p is closed and bounded in $||\cdot||$. This proves (iii).

(iii) $\Rightarrow$ (i). Suppose (iii) holds. If H_t is finite dimensional, then since $I_t = LC(H_t)$, it follows from Lemma 2 that p_t is continuous. If H_t is infinite dimensional, then by [1; p. 471, Th. 6.8], p_t is continuous. Therefore each p_t is continuous and so p is continuous. We now show that p is uniformly continuous. For each $t \in T$, let Q_t be a p_t-representing operator of H_t onto itself (see [1; p. 467, Definition 5.4]). By [1; p. 470, Th. 6.4], Q_t is a continuous positive linear operator with continuous inverse Q_t^{-1}. We may assume that $||Q_t^{-1}|| = 1$, where $||Q_t^{-1}||$ denotes the operator bound of Q_t^{-1} on $H_t (t \in T)$ (see [1; p. 472, Corollary 6.10]). We claim that $\{||Q_t||\}$ is bounded above. On the contrary, we assume that there exists a sequence $\{Q_n\} \subset \{Q_t\}$ such that $||Q_n^{1/2}|| \geqq 5n$, where $Q_n^{1/2}$ denotes the square root of Q_n $(n = 1, 2, \cdots)$. Since $||Q_n^{-1}|| = 1$, we can choose $u_n \in H_n$ such that $||u_n|| = 1$ and $||Q_n u_n|| \leqq 2$. Since $||Q_n^{1/2}|| \geqq 5n$, we can choose $v_n \in H_n$ such that $||v_n|| = 1$, $(u_n, v_n) = 0$ and $||Q_n^{1/2} v_n|| \geqq 5n$. Let $a_n = ||Q_n^{1/2} v_n||^{-1}$ and $h_n = a_n v_n + u_n$. Then

$$\begin{aligned}(h_n, Q_n h_n) - (u_n, Q_n u_n) &= a_n^2 (v_n, Q_n v_n) + a_n (Q_n u_n, v_n) \\ &\quad + a_n (v_n, Q_n u_n) \\ &\geqq 1 - 2a_n \, ||Q_n u_n|| \\ &\geqq 1 - 4a_n \, .\end{aligned}$$

Since $a_n \leqq 1/5n$, we have

$$(h_n, Q_n h_n) - (u_n, Q_n u_n) \geqq 1 - \frac{4}{5n} \geqq \frac{1}{5} \, .$$

Therefore

$$\begin{aligned}\frac{1}{5} \leqq (h_n, Q_n h_n) - (u_n, Q_n u_n) &= a_n (v_n, Q_n h_n) + a_n (u_n, Q_n v_n) \\ &\leqq a_n \, |(v_n, Q_n h_n)| + 2a_n \, .\end{aligned}$$

Hence we get

$$(\#) \qquad |(v_n, Q_n h_n)| \geqq \frac{1}{5a_n} - 2 \geqq n - 2 .$$

Now let

$$f_n = \frac{h_n \otimes Q_n h_n}{(h_n, Q_n h_n)} .$$

By the definition of Q_n, $f_n \in E_p$. Since $||h_n|| \geqq ||u_n|| = 1$ and since

$$\begin{aligned}(h_n, Q_n h_n) &= a_n^2(v_n, Q_n v_n) + a_n(Q_n u_n, v_n) \\ &\quad + a_n(v_n, Q_n u_n) + (u_n, Q_n u_n) \\ &\quad < 1 + 1 + 1 + 2 = 5 ,\end{aligned}$$

it follows from ($\#$) that

$$||f_n(v_n)|| = \frac{|(v_n, Q_n h_n)|\, ||h_n||}{(h_n, Q_n h_n)} > \frac{n-2}{5} .$$

Since $||v_n|| = 1$, $||f_n|| > (n-2)/5$, contradicting the boundedness of E_p. Therefore $\{||Q_t||\}$ and $\{||Q_t^{-1}||\}$ are bounded. By using the argument in [1; p. 479], it is easy to show that p is uniformly continuous. This completes the proof of the theorem.

Finally we give a characterization of the boundedness of E_p.

Let R be a closed right ideal of A and let P_R be the projection on R along R^p, i.e., $P_R(x + y) = x$ for all $x \in R$, $y \in R^p$. Since $R^p = \{x \in A: P_R(x) = 0\}$, P_R is continuous. Now let $\{J_\lambda: \lambda \in \Lambda\}$ be the set of all minimal right ideals of A. Since A is dual, each J_λ is automatically closed. For every $\lambda \in \Lambda$, let P_λ be the projection on J_λ along $p(J_\lambda)$.

THEOREM 4. *Let A be a B^*-algebra with a complementor p. Then the following statements are equivalent*:

(i) *The set E_p of all p-projections in A is a bounded subset of A.*

(ii) *$\{|P_\lambda|: \lambda \in \Lambda\}$ is bounded, where $|P_\lambda|$ denotes the operator bounded of P_λ.*

(iii) *There exists a constant k such that*

$$k\, ||x_1 + x_2|| \geqq ||x_i|| \qquad (i = 1, 2) ,$$

for all $x_1 \in J_\lambda$, $x_2 \in p(J_\lambda)$ $(\lambda \in \Lambda)$.

Proof. (i) $\Rightarrow$ (ii). Suppose $\sup\{||f||: f \in E_p\} \leqq c$, where c is a constant. Let J be a minimal right ideal of A. Then there exists an $f \in E_p$ such that $J = fA$ and $J^p = (1 - f)A$. Let $x \in A$. Since

$$||P_\lambda(x)|| = ||fx|| \leqq c\, ||x|| ,$$

$|P_\lambda| \leqq c$. *This proves* (ii).

(ii) $\Rightarrow$ (iii). Suppose that $\sup\{|P_\lambda|: \lambda \in \Lambda\} \leqq k - 1$, where k is a constant. Then, for all $x_1 \in J_\lambda$, $x_2 \in p(J_\lambda)$ $(\lambda \in \Lambda)$, we have

$$||x_1|| \leqq (k-1)\,||x_1 + x_2|| \leqq k\,||x_1 + x_2|| \,.$$

It now follows from $||x_2|| - ||x_1|| \leqq ||x_1 + x_2||$ that $||x_2|| \leqq k\,||x_1 + x_2||$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $f \in E_p$ and $x \in A$. Since $x = (1-f)x + fx$, by (iii), $k\,||x|| \geqq ||fx||$. As a B^*-algebra, A has an approximate identity $\{e_\alpha\}$. Since $||e_\alpha|| \leqq 1$, $||fe_\alpha|| \leqq k\,||e_\alpha|| \leqq k$. It now follows from $||fe_\alpha|| \rightarrow ||f||$ that $||f|| \leqq k$. This completes the proof of the theorem.

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