CONTINUOUS COMPLEMENTORS ON $B^*$-ALGEBRAS

Pak-Ken Wong
CONTINUOUS COMPLEMENTORS ON $B^*$-ALGEBRAS

Pak-Ken Wong

This paper is concerned with continuous and uniformly continuous complementors on a $B^*$-algebra. Let $A$ be a $B^*$-algebra with a complementor $p$ and $E_p$ the set of all $p$-projections of $A$. We show that if $A$ has no minimal left ideals of dimension less than three, then $p$ is uniformly continuous if and only if $E_p$ is a closed and bounded subset of $A$. We also give a characterization of the boundedness of $E_p$.

Let $A$ be a complex Banach algebra and let $L_r$ be the set of all closed right ideals of $A$. Following [4], we shall say that $A$ is a right complemented Banach algebra if there exists a mapping $p: R \to R^p$ of $L_r$ into itself having the following properties:

(C₁) $R \cap R^p = (0)$ (for $R \in L_r$);
(C₂) $R + R^p = A$ (for $R \in L_r$);
(C₃) $(R^p)^p = R$ (for $R \in L_r$);
(C₄) if $R_1 \subseteq R_2$, then $R_1^p \subseteq R_2^p$ ($R_1, R_2 \in L_r$).

The mapping $p$ is called a right complementor on $A$. In this paper a complemented Banach algebra will always mean a right complemented Banach algebra. We also use $p(R)$ for $R^p$.

For any set $S$ in a Banach algebra $A$, let $S_i$ and $S_r$ denote the left and right annihilators of $S$ in $A$, respectively. Then $A$ is called an annihilator algebra if, for every closed left ideal $I$ and for every closed right ideal $R$, we have $I_r = (0)$ if and only if $I = A$ and $R_l = (0)$ if and only if $R = A$. If $I_{r_l} = I$ and $R_{l_r} = R$, then $A$ is called a dual algebra.

We say that a Banach algebra $A$ has an approximate identity if there exists a net $\{e_\alpha\}$ in $A$ such that $\|e_\alpha\| \leq 1$, for all $\alpha$, and $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$, for all $x \in A$. Every $B^*$-algebra has an approximate identity.

A minimal idempotent $f$ in a complemented Banach algebra $A$ is called a $p$-projection if $(fA)^p = (1 - f)A$. If $A$ is a semi-simple annihilator complemented Banach algebra, then every nonzero right ideal, no matter whether closed or not (see [4; p. 653]), contains a $p$-projection. Let $A$ be a complemented $B^*$-algebra with a complementor $p$. Since, by [4; p. 655, Lemma 5], the socle of $A$ is dense in $A$, $A$ is dual (see [3; p. 222, Th. 2.1]). Let $E$ (resp. $E_p$) be the set of all self-adjoint minimal idempotents (resp. $p$-projections) in $A$. Then, for each $e \in E$, there exists a unique $P(e) \in E_p$ such that $P(e)A = eA$. It
can be shown that $P$ is a one-to-one mapping of $E$ onto $E_p$. We call $P$ the $p$-derived mapping of $p$. The complementor $p$ is said to be continuous if $P$ is continuous in the relative topologies of $E$ and $E_p$ induced by the given norm on $A$ (see [1; p. 463, Definition 3.7]).

Let $A$ be a dual $B^*$-algebra. It has been shown in [1; p. 463, Th. 3.6] that the mapping $p: R \rightarrow (R^1)^*$ is a complementor on $A(R \in L_r)$. In this case $E_p = E$, $P$ is the identity map, and therefore $p$ is uniformly continuous.

The concept "$p$ is continuous" can be defined for any semi-simple annihilator complemented Banach *-algebra in which $xx^* = 0$ implies $x = 0$. In fact, let $A$ be such an algebra and $p$ a given complementor on $A$. By [2; p. 155, Th. 1], every maximal closed right ideal of $A$ is modular. Therefore [1, p. 462, Corollary 3.4] holds for $A$. Hence the mapping $P$ exists as in the case of $B^*$-algebra and so the concept of continuity of $p$ can be defined.

In this paper, all algebras and spaces under consideration are over the complex field $C$.

2. Lemmas. In this section, unless otherwise stated, $H$ will denote a complex Hilbert space and $A = LC(H)$, the set of all compact operators on $H$. There exist many complementors on $A$. If $H$ is infinite dimensional, then all complementors on $A$ are continuous ([1; p. 471, Th. 6.8]). However if dim $H$ is finite, this is not true in general as is shown in [1; p. 475]. If dim $H \geq 3$, then every continuous complementor on $A$ is uniformly continuous (see [1; p. 471, Corollary 6.6]).

If $u$ and $v$ are elements of $H$, $u \otimes v$ will denote the operator on $H$ defined by the relation $(u \otimes v)(h) = (h, v)u$, for all $h \in H$.

**Lemma 1.** Let $A$ be any $C^*$-subalgebra of bounded operators on $H$ and $E \subset A$ the set of all self-adjoint minimal idempotents. The $E$ is a closed subset of $L(H)$, all bounded operators on $H$.

**Proof.** Let $\{e_n\} \subset E$ be a sequence converging to some $e \in A$. Clearly $e^2 = e$ and $e^* = e$. In order that $e \in E$, it suffices to show that $e(H)$ is one dimensional. Since $(u \otimes v)^* = v \otimes u$ and since each $e_n$ is a self-adjoint minimal idempotent, we can write $e_n = u_n \otimes u_n$, where $u_n \in H$ and $\|u_n\| = 1$ ($n = 1, 2, \cdots$). Let $v, w \in H$ be such that $e(v) \neq 0$, $e(w) \neq 0$. Since $\{(v, u_n)\}$ is bounded, there exists a subsequence $\{v, u_{n_k}\}$ of $\{(v, u_n)\}$ and a nonzero constant $a \in C$ such that $(v, u_{n_k}) \rightarrow a$. Since
\[
\|au_{n_k} - e(v)\| \leq |a - (v, u_{n_k})| \|u_{n_k}\| + \|e_{n_k} - e\| \|v\|,
\]
we have $au_{n_k} \rightarrow e(v)$. Similarly we can show that there exist a subsequence $\{u_i\}$ of $\{u_k\}$ and a nonzero constant $b \in C$ such that $bu_i \rightarrow e(v)$. 


It follows now that \( be(v) = ae(w) \), which shows that \( e(H) \) is one dimensional. This completes the proof.

**Lemma 2.** Let \( H \) be finite dimensional, \( p \) a complementor on \( A \) and \( E_p \) the set of all \( p \)-projections in \( A \). If \( E_p \) is a closed and bounded subset of \( A \), then \( p \) is continuous.

**Proof.** Let \( e \in E \) and let \( \{e_n\} \) be a sequence in \( E \) such that \( e_n \to e \). Write \( e_n = u_n \otimes u_n \), \( e = u \otimes u \), where \( u_n, u \in H \) and \( \|u_n\| = \|u\| = 1 \) \((n = 1, 2, \cdots)\). Since \( H \) is finite dimensional, there exists a subsequence \( \{u_k\} \) of \( \{u_n\} \) such that \( u_k \to u' \) for some \( u' \in H \); clearly \( \|u'\| = 1 \) and \( u' \otimes u' = u \otimes u \). Thus \( u = a u' \), where \( a = (u, u') \) and \( |a| = 1 \). Let \( u_k' = a u_k \). Then \( e_k = u_k' \otimes u_k' \). Let \( P \) be the \( p \)-derived mapping of \( p \). Since \( P(e_k) \) is a minimal idempotent and since \( P(e_k) A = e_k A \), we can write \( P(e_k) = u_k' \otimes v_k', \) where \( v_k' \in H \) \((k = 1, 2, \cdots)\). Similarly \( P(e) = u \otimes v \) with \( v \in H \). Since \( E_p \) is bounded and since \( \|u_k'\| = 1 \), \( \{v_k'\} \) is bounded. Since \( H \) is finite dimensional, there exists a subsequence \( \{v_l'\} \) of \( \{v_k'\} \) such that \( v_l' \to v' \) for some \( v' \in H \). As \( \|P(e_l)\| \geq 1 \), \( v' \neq 0 \). Since \( P(e_l) = u_l' \otimes v_l' \to u \otimes v' \) and since \( E_p \) is closed, it follows that also \( u \otimes v' \in E_p \). Then both \( u \otimes v', u \otimes v \in E_p \). However, by [1, p. 466, Lemma 5.1] for any \( u \in H \), there exists a unique such \( v \). Thus \( v = v' \). Hence \( P(e_l) \to P(e) \). Therefore \( P \) is continuous and so is \( p \). This completes the proof.

3. Main theorem. Throughout this section \( A \) will be a \( B^* \)-algebra with a complementor \( p \). Then \( A \) is dual (see § 1). Let \( \{I_t : t \in T\} \) be the family of all minimal closed two-sided ideals of \( A \). Then, by [3; p. 221, Lemma 2.3], \( A = (\bigcup_t I_t) \), the \( B^*(\infty) \)-sum of \( I_t \). Since each \( I_t \) is a simple dual \( B^* \)-algebra, \( I_t = LC(H_t) \) for some Hilbert space \( H_t(t \in T) \). It has been shown in [4; p. 652, Lemma 1] that \( p \) induces a complementor \( p_t \) on \( I_t \), which is given by \( p_t(R) = p(R) \cap I_t \) for all closed right ideals \( R \) of \( I_t(t \in T) \).

Let \( E \) (resp. \( E_t \)) be the set of all self-adjoint minimal idempotents in \( A \) (resp. in \( I_t \)) and let \( E_p \) (resp. \( E_p^t \)) be the set of all \( p \)-projections in \( A \) (resp. in \( I_t \)). Clearly \( E_t = E \cap I_t \) and \( E_p^t = E_p \cap I_t(t \in T) \). It can be shown that, if \( u \neq v(u, v \in T) \), then \( \|e_u - e_v\| = 1 \), for all \( e_u \in E_u \), and \( e_v \in E_v \). Since each \( e \in E \) belongs to some \( I_t \), \( E = \bigcup_t E_t \). Similarly, if \( u \neq v(u, v \in T) \), then \( \|f_u - f_v\| = \max(\|f_u\|, \|f_v\|) \geq 1 \), for all \( f_u \in E_p^u \) and \( f_v \in E_p^v \); \( E_p = \bigcup_t E_p^t \). Thus \( p \) is continuous if and only if \( p_t \) is continuous for all \( t \in T \) (see [1; p. 464]).

**Theorem 3.** Let \( A \) be a \( B^* \)-algebra which has no minimal left ideals of dimension less than three and \( p \) a complementor on \( A \). Then the following statements are equivalent:
(i) \( p \) is uniformly continuous.

(ii) There exists an involution \( *' \) on \( A \) for which \( R^p = (R_i)^* \), for every closed right ideal \( R \) of \( A \) (and hence there exists an equivalent norm \( || \cdot ||' \) on \( A \) which satisfies the \( B^* \)-condition for \( *' \)).

(iii) The set \( E^p \) of all \( p \)-projections in \( A \) is a closed and bounded subset of \( A \).

Proof. (i) \( \rightarrow \) (ii). This is [1; p. 477, Th. 7.4].

(ii) \( \Rightarrow \) (iii). Suppose (ii) holds. Let \( E^p_i \) be the set of all \( p \)-projections in \( I_i(t \in T) \). By [1; p. 465, Corollary 4.4], each \( f_i \in E^p_i \) is self-adjoint in \( *' \). Hence \( ||f_i||' = 1 \). Since each \( E^p_i \) is the set of all self-adjoint (in \( *' \)) minimal idempotents in \( I_i \), by Lemma 1, \( E^p_i \) is closed in \( || \cdot ||' \). It is now easy to show that \( E^p \) is closed and bounded in \( || \cdot || \). This proves (iii).

(iii) \( \Rightarrow \) (i). Suppose (iii) holds. If \( H_t \) is finite dimensional, then since \( I_i = LC(H_t) \), it follows from Lemma 2 that \( p_t \) is continuous. If \( H_t \) is infinite dimensional, then by [1; p. 471, Th. 6.8], \( p_t \) is continuous. Therefore each \( p_t \) is continuous and so \( p \) is continuous. We now show that \( p \) is uniformly continuous. For each \( t \in T \), let \( Q_t \) be a \( p_t \)-representing operator of \( H_t \) onto itself (see [1; p. 467, Definition 5.4]). By [1; p. 470, Th. 6.4], \( Q_t \) is a continuous positive linear operator with continuous inverse \( Q_t^{-1} \). We may assume that \( ||Q_t^{-1}|| = 1 \), where \( ||Q_t^{-1}|| \) denotes the operator bound of \( Q_t^{-1} \) on \( H_t(t \in T) \) (see [1; p. 472, Corollary 6.10]). We claim that \( \{||Q_i||\} \) is bounded above. On the contrary, we assume that there exists a sequence \( \{Q_n\} \subset \{Q_i\} \) such that \( ||Q_n^{1/2}|| \geq 5n \), where \( Q_n^{1/2} \) denotes the square root of \( Q_n \) \( (n = 1, 2, \ldots) \). Since \( ||Q_n^{-1}|| = 1 \), we can choose \( u_n \in H_n \) such that \( ||u_n|| = 1 \) and \( ||Q_n u_n|| \leq 2 \). Since \( ||Q_n^{1/2}|| \geq 5n \), we can choose \( v_n \in H_n \) such that \( ||v_n|| = 1 \), \( (u_n, v_n) = 0 \) and \( ||Q_n^{1/2}v_n|| \geq 5n \). Let \( a_n = ||Q_n^{1/2}v_n||^{-1} \) and \( h_n = a_n v_n + u_n \). Then

\[
(h_n, Q_n h_n) - (u_n, Q_n u_n) = a_n^2 (v_n, Q_n v_n) + a_n (Q_n u_n, v_n) + a_n^2 (v_n, Q_n v_n)
\geq 1 - 2a_n ||Q_n u_n||
\geq 1 - 4a_n.
\]

Since \( a_n \leq 1/5n \), we have

\[
(h_n, Q_n h_n) - (u_n, Q_n u_n) \geq 1 - \frac{4}{5n} \geq \frac{1}{5}.
\]

Therefore

\[
\frac{1}{5} \leq (h_n, Q_n h_n) - (u_n, Q_n u_n) = a_n (v_n, Q_n h_n) + a_n (u_n, Q_n v_n)
\leq a_n |(v_n, Q_n h_n)| + 2a_n.
\]
Hence we get

\[(\#) \quad |(v_n, Q_n h_n)| \geq \frac{1}{5a_n} - 2 \geq n - 2.\]

Now let

\[f_n = \frac{h_n \otimes Q_n h_n}{(h_n, Q_n h_n)}.\]

By the definition of $Q_n$, $f_n \in E_p$. Since $\|h_n\| \geq \|u_n\| = 1$ and since

\[
(h_n, Q_n h_n) = a_n^2(v_n, Q_n v_n) + a_n(Q_n u_n, v_n) + a_n(v_n, Q_n u_n) + (u_n, Q_n u_n) < 1 + 1 + 1 + 2 = 5,
\]

it follows from $(\#)$ that

\[\|f_n(v_n)\| = \frac{|(v_n, Q_n h_n)| \|h_n\|}{(h_n, Q_n h_n)} > \frac{n - 2}{5}.\]

Since $\|v_n\| = 1$, $\|f_n\| > (n - 2)/5$, contradicting the boundedness of $E_p$. Therefore $\{|Q_n|\}$ and $\{|Q_n^{-1}|\}$ are bounded. By using the argument in [1; p. 479], it is easy to show that $p$ is uniformly continuous. This completes the proof of the theorem.

Finally we give a characterization of the boundedness of $E_p$.

Let $R$ be a closed right ideal of $A$ and let $P_R$ be the projection on $R$ along $R^p$, i.e., $P_R(x + y) = x$ for all $x \in R$, $y \in R^p$. Since $R^p = \{x \in A: P_R(x) = 0\}$, $P_R$ is continuous. Now let $\{J_\lambda: \lambda \in \Lambda\}$ be the set of all minimal right ideals of $A$. Since $A$ is dual, each $J_\lambda$ is automatically closed. For every $\lambda \in \Lambda$, let $P_\lambda$ be the projection on $J_\lambda$ along $p(J_\lambda)$.

**Theorem 4.** Let $A$ be a $B^*$-algebra with a complementor $p$. Then the following statements are equivalent:

(i) The set $E_p$ of all $p$-projections in $A$ is a bounded subset of $A$.

(ii) $\{|P_\lambda|: \lambda \in \Lambda\}$ is bounded, where $|P_\lambda|$ denotes the operator bounded of $P_\lambda$.

(iii) There exists a constant $k$ such that

\[k \|x_1 + x_2\| \geq \|x_i\| \quad (i = 1, 2),\]

for all $x_1 \in J_\lambda, x_2 \in p(J_\lambda)$ ($\lambda \in \Lambda$).

**Proof.** (i) $\Rightarrow$ (ii). Suppose $\sup \{\|f\|: f \in E_p\} \leq c$, where $c$ is a constant. Let $J$ be a minimal right ideal of $A$. Then there exists an $f \in E_p$ such that $J = fA$ and $J^p = (1 - f)A$. Let $x \in A$. Since

\[\|P_\lambda(x)\| = \|fx\| \leq c \|x\|,
\]

...
This proves (ii).

(ii) $\Rightarrow$ (iii). Suppose that $\sup \{\|P\| : \lambda \in A\} \leq k - 1$, where $k$ is a constant. Then, for all $x_i \in J_i$, $x_\lambda \in p(J_i)$ ($\lambda \in A$), we have

$$\|x_i\| \leq (k - 1) \|x_i + x_\lambda\| \leq k \|x_i + x_\lambda\|.$$

It now follows from $\|x_\lambda\| - \|x_i\| \leq \|x_i + x_\lambda\|$ that $\|x_\lambda\| \leq k \|x_i + x_\lambda\|$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $f \in E$ and $x \in A$. Since $x = (1 - f)x + fx$, by (iii), $k \|x\| \geq \|fx\|$. As a $B^*$-algebra, $A$ has an approximate identity $\{e_\lambda\}$. Since $\|e_\lambda\| \leq 1$, $\|fe_\lambda\| \leq k \|e_\lambda\| \leq k$. It now follows from $\|fe_\lambda\| \rightarrow \|f\|$ that $\|f\| \leq k$. This completes the proof of the theorem.

It is Professor B. J. Tomiuk who aroused my interest in this topic. I wish to express my hearty thanks to him. I also wish to thank the referee for discovering an error in my previous demonstration of Theorem 3.

REFERENCES


Received September 5, 1969.

UNIVERSITY OF OTTAWA
OTTAWA, CANADA
Mir Maswood Ali, *On some extremal simplexes* ........................................... 1
Silvio Aurora, *On normed rings with monotone multiplication* .................. 15
Silvio Aurora, *Normed fields which extend normed rings of integers* ........ 21
John Kelly Beem, *Indefinite Minkowski spaces* ........................................ 29
T. F. Bridgland, *Trajectory integrals of set valued functions* ..................... 43
Robert Jay Buck, *A generalized Hausdorff dimension for functions and sets* .......................................................... 69
Vlastimil B. Dlab, *A characterization of perfect rings* ............................... 79
Edward Richard Fadell, *Some examples in fixed point theory* .................. 89
Michael Benton Freeman, *Tangential Cauchy-Riemann equations and uniform approximation* ............................................. 101
Barry J. Gardner, *Torsion classes and pure subgroups* ............................ 109
Vinod B. Goyal, *Bounds for the solution of a certain class of nonlinear partial differential equations* ...................................................... 117
Fu Cheng Hsiang, *On $C_1$ summability factors of Fourier series at a given point* .................................................................................................................. 139
Lawrence Stanislaus Husch, Jr., *Homotopy groups of PL-embedding spaces* .......................................................... 149
Daniel Ralph Lewis, *Integration with respect to vector measures* .......... 157
Marion-Josephine Lim, *$L_2$ subspaces of Grassmann product spaces* ........ 167
Stephen J. Pierce, *Orthogonal groups of positive definite multilinear functionals* .......................................................... 183
W. J. Pugh and S. M. Shah, *On the growth of entire functions of bounded index* .......................................................... 191
Ralph Tyrrell Rockafellar, *On the maximal monotonicity of subdifferential mappings* .......................................................... 209
R. Shantaram, *Convergence of a sequence of transformations of distribution functions. II* .......................................................... 217
Julianne Souchek, *Rings of analytic functions* ........................................ 233
Ted Joe Suffridge, *The principle of subordination applied to functions of several variables* .......................................................... 241
Wei-lung Ting, *On secondary characteristic classes in cobordism theory* .......................................................... 249
Pak-Ken Wong, *Continuous complementors on $B^*$-algebras* .................. 255
Miyuki Yamada, *On a regular semigroup in which the idempotents form a band* .......................................................... 261