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CONTINUOUS COMPLEMENTORS ON B^* -ALGEBRAS

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This paper is concerned with continuous and uniformly continuous complementors on a B^* -algebra. Let A be a B^* -algebra with a complementor p and E_p the set of all p -projections of A . We show that if A has no minimal left ideals of dimension less than three, then p is uniformly continuous if and only if E_p is a closed and bounded subset of A . We also give a characterization of the boundedness of E_p .

Let A be a complex Banach algebra and let L_r be the set of all closed right ideals of A . Following [4], we shall say that A is a right complemented Banach algebra if there exists a mapping $p: R \rightarrow R^p$ of L_r into itself having the following properties:

- (C₁) $R \cap R^p = (0) \quad (R \in L_r) ;$
- (C₂) $R + R^p = A \quad (R \in L_r) ;$
- (C₃) $(R^p)^p = R \quad (R \in L_r) ;$
- (C₄) if $R_1 \subset R_2$, then $R_2^p \subset R_1^p \quad (R_1, R_2 \in L_r) .$

The mapping p is called a right complementor on A . In this paper a complemented Banach algebra will always mean a right complemented Banach algebra. We also use $p(R)$ for R^p .

For any set S in a Banach algebra A , let S_l and S_r denote the left and right annihilators of S in A , respectively. Then A is called an annihilator algebra if, for every closed left ideal I and for every closed right ideal R , we have $I_r = (0)$ if and only if $I = A$ and $R_l = (0)$ if and only if $R = A$. If $I_{r_l} = I$ and $R_{l_r} = R$, then A is called a dual algebra.

We say that a Banach algebra A has an approximate identity if there exists a net $\{e_\alpha\}$ in A such that $\|e_\alpha\| \leq 1$, for all α , and $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$, for all $x \in A$. Every B^* -algebra has an approximate identity.

A minimal idempotent f in a complemented Banach algebra A is called a p -projection if $(fA)^p = (1 - f)A$. If A is a semi-simple annihilator complemented Banach algebra, then every nonzero right ideal, no matter whether closed or not (see [4; p. 653]), contains a p -projection. Let A be a complemented B^* -algebra with a complementor p . Since, by [4; p. 655, Lemma 5], the socle of A is dense in A , A is dual (see [3; p. 222, Th. 2.1]). Let E (resp. E_p) be the set of all self-adjoint minimal idempotents (resp. p -projections) in A . Then, for each $e \in E$, there exists a unique $P(e) \in E_p$ such that $P(e)A = eA$. It

can be shown that P is a one-to-one mapping of E onto E_p . We call P the p -derived mapping of p . The complementor p is said to be continuous if P is continuous in the relative topologies of E and E_p induced by the given norm on A (see [1; p. 463, Definition 3.7]).

Let A be a dual B^* -algebra. It has been shown in [1; p. 463, Th. 3.6] that the mapping $p: R \rightarrow (R_l)^*$ is a complementor on $A(R \in L_r)$. In this case $E_p = E$, P is the identity map, and therefore p is uniformly continuous.

The concept " p is continuous" can be defined for any semi-simple annihilator complemented Banach $*$ -algebra in which $xx^* = 0$ implies $x = 0$. In fact, let A be such an algebra and p a given complementor on A . By [2; p. 155, Th. 1], every maximal closed right ideal of A is modular. Therefore [1, p. 462, Corollary 3.4] holds for A . Hence the mapping P exists as in the case of B^* -algebra and so the concept of continuity of p can be defined.

In this paper, all algebras and spaces under consideration are over the complex field C .

2. Lemmas. In this section, unless otherwise stated, H will denote a complex Hilbert space and $A = LC(H)$, the set of all compact operators on H . There exist many complementors on A . If H is infinite dimensional, then all complementors on A are continuous ([1; p. 471, Th. 6.8]). However if $\dim H$ is finite, this is not true in general as is shown in [1; p. 475]. If $\dim H \geq 3$, then every continuous complementor on A is uniformly continuous (see [1; p. 471, Corollary 6.6]).

If u and v are elements of H , $u \otimes v$ will denote the operator on H defined by the relation $(u \otimes v)(h) = (h, v)u$, for all $h \in H$.

LEMMA 1. *Let A be any C^* -subalgebra of bounded operators on H and $E \subset A$ the set of all self-adjoint minimal idempotents. The E is a closed subset of $L(H)$, all bounded operators on H .*

Proof. Let $\{e_n\} \subset E$ be a sequence converging to some $e \in A$. Clearly $e^2 = e$ and $e^* = e$. In order that $e \in E$, it suffices to show that $e(H)$ is one dimensional. Since $(u \otimes v)^* = v \otimes u$ and since each e_n is a self-adjoint minimal idempotent, we can write $e_n = u_n \otimes u_n$, where $u_n \in H$ and $\|u_n\| = 1$ ($n = 1, 2, \dots$). Let $v, w \in H$ be such that $e(v) \neq 0$, $e(w) \neq 0$. Since $\{(v, u_n)\}$ is bounded, there exists a subsequence $\{v, u_k\}$ of $\{(v, u_n)\}$ and a nonzero constant $a \in C$ such that $(v, u_k) \rightarrow a$. Since

$$\|au_k - e(v)\| \leq |a - (v, u_k)| \|u_k\| + \|e_k - e\| \|v\|,$$

we have $au_k \rightarrow e(v)$. Similarly we can show that there exist a subsequence $\{u_i\}$ of $\{u_k\}$ and a nonzero constant $b \in C$ such that $bu_i \rightarrow e(w)$.

It follows now that $be(v) = ae(w)$, which shows that $e(H)$ is one dimensional. This completes the proof.

LEMMA 2. *Let H be finite dimensional, p a complementor on A and E_p the set of all p -projections in A . If E_p is a closed and bounded subset of A , then p is continuous.*

Proof. Let $e \in E$ and let $\{e_n\}$ be a sequence in E such that $e_n \rightarrow e$. Write $e_n = u_n \otimes u_n$, $e = u \otimes u$, where $u_n, u \in H$ and $\|u_n\| = \|u\| = 1$ ($n = 1, 2, \dots$). Since H is finite dimensional, there exists a subsequence $\{u_k\}$ of $\{u_n\}$ such that $u_k \rightarrow u'$ for some $u' \in H$; clearly $\|u'\| = 1$ and $u' \otimes u' = u \otimes u$. Thus $u = au'$, where $a = (u, u')$ and $|a| = 1$. Let $u'_k = au_k$. Then $e_k = u'_k \otimes u'_k$. Let P be the p -derived mapping of p . Since $P(e_k)$ is a minimal idempotent and since $P(e_k)A = e_kA$, we can write $P(e_k) = u'_k \otimes v'_k$, where $v'_k \in H$ ($k = 1, 2, \dots$). Similarly $P(e) = u \otimes v$ with $v \in H$. Since E_p is bounded and since $\|u'_k\| = 1$, $\{v'_k\}$ is bounded. Since H is finite dimensional, there exists a subsequence $\{v'_i\}$ of $\{v'_k\}$ such that $v'_i \rightarrow v'$ for some $v' \in H$. As $\|P(e_i)\| \geq 1$, $v' \neq 0$. Since $P(e_i) = u'_i \otimes v'_i \rightarrow u \otimes v'$ and since E_p is closed, it follows that also $u \otimes v' \in E_p$. Then both $u \otimes v', u \otimes v \in E_p$. However, by [1, p. 466, Lemma 5.1] for any $u \in H$, there exists a unique such v . Thus $v = v'$. Hence $P(e_i) \rightarrow P(e)$. Therefore P is continuous and so is p . This completes the proof.

3. Main theorem. Throughout this section A will be a B^* -algebra with a complementor p . Then A is dual (see § 1). Let $\{I_t; t \in T\}$ be the family of all minimal closed two-sided ideals of A . Then, by [3; p. 221, Lemma 2.3], $A = (\sum_t I_t)_0$, the $B^*(\infty)$ -sum of I_t . Since each I_t is a simple dual B^* -algebra, $I_t = LC(H_t)$ for some Hilbert space H_t ($t \in T$). It has been shown in [4; p. 652, Lemma 1] that p induces a complementor p_t on I_t , which is given by $p_t(R) = p(R) \cap I_t$ for all closed right ideals R of I_t ($t \in T$).

Let E (resp. E_t) be the set of all self-adjoint minimal idempotents in A (resp. in I_t) and let E_p (resp. E_p^t) be the set of all p -projections in A (resp. in I_t). Clearly $E_t = E \cap I_t$ and $E_p^t = E_p \cap I_t$ ($t \in T$). It can be shown that, if $u \neq v$ ($u, v \in T$), then $\|e_u - e_v\| = 1$, for all $e_u \in E_u$, and $e_v \in E_v$. Since each $e \in E$ belongs to some I_t , $E = \bigcup_t E_t$. Similarly, if $u \neq v$ ($u, v \in T$), then $\|f_u - f_v\| = \text{maximum}(\|f_u\|, \|f_v\|) \geq 1$, for all $f_u \in E_p^u$ and $f_v \in E_p^v$; $E_p = \bigcup_t E_p^t$. Thus p is continuous if and only if p_t is continuous for all $t \in T$ (see [1; p. 464]).

THEOREM 3. *Let A be a B^* -algebra which has no minimal left ideals of dimension less than three and p a complementor on A . Then the following statements are equivalent:*

(i) p is uniformly continuous.

(ii) There exists an involution $^{*'}$ on A for which $R^p = (R_i)^{*'}$, for every closed right ideal R of A (and hence there exists an equivalent norm $\|\cdot\|'$ on A which satisfies the B^* -condition for $^{*'}$).

(iii) The set E_p of all p -projections in A is a closed and bounded subset of A .

Proof. (i) \rightarrow (ii). This is [1; p. 477, Th. 7.4].

(ii) \Rightarrow (iii). Suppose (ii) holds. Let E_p^t be the set of all p -projections in $I_t(t \in T)$. By [1; p. 465, Corollary 4.4], each $f_t \in E_p^t$ is self-adjoint in $^{*'}$. Hence $\|f_t\|' = 1$. Since each E_p^t is the set of all self-adjoint (in $^{*'}$) minimal idempotents in I_t , by Lemma 1, E_p^t is closed in $\|\cdot\|'$. It is now easy to show that E_p is closed and bounded in $\|\cdot\|'$. This proves (iii).

(iii) \Rightarrow (i). Suppose (iii) holds. If H_t is finite dimensional, then since $I_t = LC(H_t)$, it follows from Lemma 2 that p_t is continuous. If H_t is infinite dimensional, then by [1; p. 471, Th. 6.8], p_t is continuous. Therefore each p_t is continuous and so p is continuous. We now show that p is uniformly continuous. For each $t \in T$, let Q_t be a p_t -representing operator of H_t onto itself (see [1; p. 467, Definition 5.4]). By [1; p. 470, Th. 6.4], Q_t is a continuous positive linear operator with continuous inverse Q_t^{-1} . We may assume that $\|Q_t^{-1}\| = 1$, where $\|Q_t^{-1}\|$ denotes the operator bound of Q_t^{-1} on $H_t(t \in T)$ (see [1; p. 472, Corollary 6.10]). We claim that $\{\|Q_t\|\}$ is bounded above. On the contrary, we assume that there exists a sequence $\{Q_n\} \subset \{Q_t\}$ such that $\|Q_n^{1/2}\| \geq 5n$, where $Q_n^{1/2}$ denotes the square root of Q_n ($n = 1, 2, \dots$). Since $\|Q_n^{-1}\| = 1$, we can choose $u_n \in H_n$ such that $\|u_n\| = 1$ and $\|Q_n u_n\| \leq 2$. Since $\|Q_n^{1/2}\| \geq 5n$, we can choose $v_n \in H_n$ such that $\|v_n\| = 1$, $(u_n, v_n) = 0$ and $\|Q_n^{1/2} v_n\| \geq 5n$. Let $a_n = \|Q_n^{1/2} v_n\|^{-1}$ and $h_n = a_n v_n + u_n$. Then

$$\begin{aligned} (h_n, Q_n h_n) - (u_n, Q_n u_n) &= a_n^2 (v_n, Q_n v_n) + a_n (Q_n u_n, v_n) \\ &\quad + a_n (v_n, Q_n u_n) \\ &\geq 1 - 2a_n \|Q_n u_n\| \\ &\geq 1 - 4a_n. \end{aligned}$$

Since $a_n \leq 1/5n$, we have

$$(h_n, Q_n h_n) - (u_n, Q_n u_n) \geq 1 - \frac{4}{5n} \geq \frac{1}{5}.$$

Therefore

$$\begin{aligned} \frac{1}{5} &\leq (h_n, Q_n h_n) - (u_n, Q_n u_n) = a_n (v_n, Q_n h_n) + a_n (u_n, Q_n v_n) \\ &\leq a_n |(v_n, Q_n h_n)| + 2a_n. \end{aligned}$$

Hence we get

$$(\#) \quad |(v_n, Q_n h_n)| \geq \frac{1}{5a_n} - 2 \geq n - 2.$$

Now let

$$f_n = \frac{h_n \otimes Q_n h_n}{(h_n, Q_n h_n)}.$$

By the definition of $Q_n, f_n \in E_p$. Since $\|h_n\| \geq \|u_n\| = 1$ and since

$$\begin{aligned} (h_n, Q_n h_n) &= a_n^2(v_n, Q_n v_n) + a_n(Q_n u_n, v_n) \\ &\quad + a_n(v_n, Q_n u_n) + (u_n, Q_n u_n) \\ &< 1 + 1 + 1 + 2 = 5, \end{aligned}$$

it follows from (#) that

$$\|f_n(v_n)\| = \frac{|(v_n, Q_n h_n)| \|h_n\|}{(h_n, Q_n h_n)} > \frac{n - 2}{5}.$$

Since $\|v_n\| = 1, \|f_n\| > (n - 2)/5$, contradicting the boundedness of E_p . Therefore $\{\|Q_t\|\}$ and $\{\|Q_t^{-1}\|\}$ are bounded. By using the argument in [1; p. 479], it is easy to show that p is uniformly continuous. This completes the proof of the theorem.

Finally we give a characterization of the boundedness of E_p .

Let R be a closed right ideal of A and let P_R be the projection on R along R^p , i.e., $P_R(x + y) = x$ for all $x \in R, y \in R^p$. Since $R^p = \{x \in A : P_R(x) = 0\}$, P_R is continuous. Now let $\{J_\lambda : \lambda \in \Lambda\}$ be the set of all minimal right ideals of A . Since A is dual, each J_λ is automatically closed. For every $\lambda \in \Lambda$, let P_λ be the projection on J_λ along $p(J_\lambda)$.

THEOREM 4. *Let A be a B^* -algebra with a complementor p . Then the following statements are equivalent:*

- (i) *The set E_p of all p -projections in A is a bounded subset of A .*
- (ii) *$\{\|P_\lambda\| : \lambda \in \Lambda\}$ is bounded, where $\|P_\lambda\|$ denotes the operator bounded of P_λ .*
- (iii) *There exists a constant k such that*

$$k \|x_1 + x_2\| \geq \|x_i\| \quad (i = 1, 2),$$

for all $x_1 \in J_\lambda, x_2 \in p(J_\lambda)$ ($\lambda \in \Lambda$).

Proof. (i) \Rightarrow (ii). Suppose $\sup\{\|f\| : f \in E_p\} \leq c$, where c is a constant. Let J be a minimal right ideal of A . Then there exists an $f \in E_p$ such that $J = fA$ and $J^p = (1 - f)A$. Let $x \in A$. Since

$$\|P_\lambda(x)\| = \|fx\| \leq c \|x\|,$$

$|P_\lambda| \leq c$. This proves (ii).

(ii) \Rightarrow (iii). Suppose that $\sup\{|P_\lambda|: \lambda \in A\} \leq k - 1$, where k is a constant. Then, for all $x_1 \in J_\lambda, x_2 \in p(J_\lambda)$ ($\lambda \in A$), we have

$$\|x_1\| \leq (k - 1) \|x_1 + x_2\| \leq k \|x_1 + x_2\|.$$

It now follows from $\|x_2\| - \|x_1\| \leq \|x_1 + x_2\|$ that $\|x_2\| \leq k \|x_1 + x_2\|$.

(iii) \Rightarrow (i). Suppose (iii) holds. Let $f \in E_p$ and $x \in A$. Since $x = (1 - f)x + fx$, by (iii), $k \|x\| \geq \|fx\|$. As a B^* -algebra, A has an approximate identity $\{e_\alpha\}$. Since $\|e_\alpha\| \leq 1, \|fe_\alpha\| \leq k \|e_\alpha\| \leq k$. It now follows from $\|fe_\alpha\| \rightarrow \|f\|$ that $\|f\| \leq k$. This completes the proof of the theorem.

It is Professor B. J. Tomiuk who aroused my interest in this topic. I wish to express my hearty thanks to him. I also wish to thank the referee for discovering an error in my previous demonstration of Theorem 3.

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