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EXTREMELY AMENABLE ALGEBRAS

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Let S be a semigroup and m(S) the space of bounded real functions on S. A subalgebra of m(S) is extremely left amenable (ELA) if it is (sup) norm closed, left translation invariant, containing constants and has a multiplicative left invariant mean. S is ELA if m(S) is ELA. In this paper, we give a method in constructing all ELA subalgebras of m(S); it turns out that any such subalgebra of m(S) is contained in an ELA subalgebra which is the uniform limit of certain classes of simple functions on S.

A subset  $E \subseteq S$  is left thick if for any finite subset  $\sigma \subseteq S$ , there exists  $s \in S$  such that  $\{as; a \in \sigma\} \subseteq E$ . In §3, we strengthen a result of T. Mitchell and prove that a semigroup S is ELA if and only if for any subset  $E \subseteq S$ , either E is left thick or S - E is left thick. We also show how this result may be generalized to certain subalgebras of m(S).

ELA semigroups and subalgebras have been considered by Mitchell in [9] and [10], and Granirer in [5], [6] and [7]. ELA semigroups S are shown to be characterized by the fixed point property on compact hausdorff spaces by Mitchell [9] and by the algebraic property: "for any a, b in S, there is a c in S such that ac = bc = c" by Granirer [5]. ELA subalgebras are characterized by Mitchell [10] by a fixed point property on compacta (under certain kinds of actions of S on a compact hausdorff space).

1. Some notations and preliminaries. Let S be a semigroup. For each  $a \in S$ ,  $f \in m(S)$ , denote by the sup norm of f,  $||f|| = \sup_{s \in S} |f(s)|$ (and it is only this norm that will be used throughout this paper),  $_af(s) = f(as)$  and  $p_a(f) = f(a)$  for all  $s \in S$ . Then  $p_a$  is called the *point measure* on m(S) at a and any element in Co  $\{p_a; a \in S\}$  is called a *finite mean* on m(S) (where Co A denotes the convex hull of a subset A in a linear space).

If A is a norm closed left translation invariant subalgebra of m(S) (i.e.,  ${}_{a}f \in A$  whenever  $f \in A$  and  $a \in S$ ) containing 1, the constant one function on S, and  $\varphi \in A^*$ , then  $\varphi$  is a mean if  $\varphi(f) \geq 0$  for  $f \geq 0$ , and  $\varphi(1) = 1$ ;  $\varphi$  is multiplicative if  $\varphi(fg) = \varphi(f)\varphi(g)$  for all  $f, g \in A$ ;  $\varphi$  is left invariant if  $\varphi({}_{s}f) = \varphi(f)$  for all  $s \in S$  and  $f \in A$ ; and  $\varphi$  is a point measure [finite mean] on A if  $\varphi$  is the restriction of some point measure [finite mean] on m(S) to A. It is well-known that the set of [point measure] finite mean on A is w\*-dense (i.e.,  $\sigma(A^*, A)$  -dense) in the set of [multiplicative] means on A. Furthermore, the set of multiplicative means on m(S) is precisely  $\beta S$ , the Stone-Cech compactification of S ([3], p. 276).

A subalgebra of m(S) is [extremely] left amenable, sometimes denoted by [ELA] LA, if it is norm closed, left translation invariant, containing constants and has a [multiplicative] left invariant mean (LIM). A semigroup S is [ELA] LA if m(S) is [ELA] LA.

For any subset  $E \subseteq S$ ,  $a \in S$ , we shall denote by  $\overline{E}$  = the closure of E in  $\beta S$ ,  $a^{-1}E = \{s \in S; as \in E\}$ ,  $\mathbf{1}_E \in m(S)$  such that  $\mathbf{1}_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E \end{cases}$ and  $\varphi(E) = \varphi(\mathbf{1}_E)$  for any  $\varphi \in m(S)^*$ .

A subset  $E \subseteq S$  is left thick if for any finite subset  $\sigma \subseteq S$ , there exists  $s \in S$  such that  $\{as; a \in \sigma\} \subseteq E$ , or equivalently, the family  $\{s^{-1}E; s \in S\}$  has finite intersection property. Left thick subsets are first considered by Mitchell in [11]. Clearly, any left ideal of a semigroup is left thick. If S is left amenable, then every right ideal I is left thick, since if  $\varphi$  is a LIM on m(S), then  $\varphi(I) = 1$ ; consequently, the family  $\{s^{-1}I; s \in S\}$  has finite intersection property.

2. The class of extremely amenable subalgebras. For any semigroup S, and  $\mathscr{T}$  an algebra of subsets of S (i.e., a collection of subsets of S containing S and which are closed under complementation and finite union), we shall denote by

$$m(\mathscr{T},S)= ext{norm}$$
 closure of the linear span of the set  $\{1_E; E\in\mathscr{T}\}$  .

Then  $m(\mathcal{T}, S)$  is a norm closed subalgebra of m(S) containing constants. Furthermore, if  $\mu$  is a mean on m(S), denote by

$$\mathscr{C}_{\mu} = \{E \subseteq S; \ \mu(s^{-1}E) = \mu(E) = 1 \ ext{for all} \ s \in S\}$$
  
 $\mathscr{T}_{\mu} = ext{algebra generated by} \ \mathscr{C}_{\mu}$ .

**REMARK 1.** For any semigroup S:

- (a)  $\mathscr{C}_{\mu}$  is nonempty for all mean  $\mu$  on m(S) since  $S \in \mathscr{C}_{\mu}$ .
- (b) If  $\mu$  is a multiplicative LIM on m(S), then  $m(\mathcal{T}_{\mu}, S) = m(S)$ .

(c) If S has f.i.p.r.i. (finite intersection for right ideals),  $\mu \in \cap \{\overline{sS}; s \in S\}$  then  $aS \in \mathscr{C}_{\mu}$  for all  $a \in S$ . In particular, all right ideals of S are left thick. To see this we only have to observe that for each  $a, t \in S, t^{-1}(aS) \supseteq bS$  where b is chosen such that  $tb \in aS$ . Conversely, if all right ideals of a semigroup S are left thick, then S has f.i.p.r.i. since for any  $a, b \in S$ , there exists  $c \in S$  such that  $bc \in aS$ .

(d) If S generates a group G, S has f.i.p.r.i. and  $\mu \in \cap \{sS; s \in S\}$ , where the closure is taken in  $\beta G$ , then  $gS \in \mathscr{C}_{\mu}$  for all  $g \in G$ . In

fact, for any  $g \in G$ , gS contains a right ideal of S ([12], Lemma 5.1) and hence  $\mu(g_1^{-1}(g_2S)) = 1$  for all  $g_1, g_2 \in G$ . In particular, each gS(and therefore S) is a left thick subset of  $G, g \in G$ .

Our first main result is to show that for any semigroup S, every ELA subalgebra of m(S) is contained in an ELA subalgebra  $m(\mathscr{T}_{\mu}, S)$  for some mean  $\mu$  on m(S). We shall prove this result in a series of lemmas.

**LEMMA 1.** Let S be a semigroup,  $F \subseteq m(S)$  such that  ${}_{s}f \in F$  for all  $f \in F$  and  $s \in S$ . If A is the smallest norm closed subalgebra containing F and the constant functions, then A is left translation invariant. If  $\varphi({}_{s}f) = \varphi(f), \varphi \in \beta S$ , for all  $s \in S$  and  $f \in F$ , then  $\varphi$  is a multiplicative LIM on A.

*Proof.* A is the norm closure of H, where H consists of all functions of the form  $a_01 + a_1g_1 + \cdots + a_ng_n$  and for each  $i = 1, \dots, n$ ,  $g_i$  is a finite product of functions in F. Then as readily checked,  $sh \in H$  for all  $s \in S$  and  $h \in H$ . If  $f \in A$ , and  $h_{\alpha} \in H$  such that  $\lim_{\alpha} || h_{\alpha} - f || = 0$ , then  $\lim_{\alpha} || sh_{\alpha} - sf || \leq \lim_{\alpha} || h_{\alpha} - f || = 0$ , and hence  $sf \in A$  for all  $s \in S$ . The last assertion can be proved similarly.

**LEMMA 2.** Let S be a semigroup,  $f \in m(S)$  and  $\varphi \in \beta S$  be such that  $\varphi({}_sf) = \varphi(f)$  for all  $s \in S$ ;

(a) if  $\varphi(f) \neq 0$ , then  $\{s \in S; f(s) \neq 0\} \in \mathscr{C}_{\varphi}$ 

(b) if  $\varphi(f) = 0$ , then  $\{s \in S; f(s) < c\} \in \mathscr{C}_{\varphi}$  for all c > 0.

*Proof.* (a) If  $N = \{s \in S; f(s) \neq 0\}$ , then  $\varphi(f) = \varphi(1_N f) = \varphi(1_N)\varphi(f)$ and  $\varphi({}_sf) = \varphi({}_s(1_N f)) = \varphi(1_{s^{-1}N})\varphi({}_sf)$ . Hence  $\varphi(N) = \varphi(s^{-1}N) = 1$  for all  $s \in S$ .

(b) Let A be the smallest norm closed subalgebra containing f and all its left translates and constants. Then as well-known, A is a lattice [2]. Define  $h(s) = \max \{c - f(s), 0\}$ , then  $h \in A$ , and  $\varphi(_{s}h) = \varphi(h)$  for all  $s \in S$  (Lemma 1). Since  $\varphi(h) \ge c > 0$ , it follows from (a) that  $\{s \in S; h(s) \neq 0\} = \{s \in S; f(s) < c\} \in \mathscr{E}_{\varphi}$ .

LEMMA 3. For any semigroup S and  $E \subseteq S$ , if E is left thick, then there exists  $\varphi \in \beta S$  such that  $\varphi(s^{-1}E) = \varphi(E) = 1$  for all  $s \in S$ .

*Proof.* Let  $\psi \in \bigcap_{s \in S} \overline{s^{-1}E}$  and define  $\varphi \in \beta S$  by  $\varphi(f) = \psi(h)$  where  $h(s) = \psi(sf)$  for all  $s \in S$ .

THEOREM 1. Let S be a semigroup and A be a norm closed left translation invariant subalgebra of m(S) containing constants, then A is ELA if and only if  $A \subseteq m(\mathscr{T}_{\mu}, S)$  for some [multiplicative] mean  $\mu$  on m(S).

*Proof.* For any mean  $\mu$  on m(S),  $m(\mathscr{T}_{\mu}, S)$  is the smallest norm closed subalgebra containing  $F = \{1_E; E \in \mathscr{C}_{\mu}\}$  and constants. It follows from Lemma 1 that  $m(\mathscr{C}_{\mu}, S)$  is necessarily left translation invariant. Furthermore, any  $\varphi \in \bigcap_{E \in \mathscr{C}_{\mu}} \overline{E}$  (which is nonempty by compactness of  $\beta S$ ) is a multiplicative LIM on  $m(\mathscr{T}_{\mu}, S)$  since  $\varphi(E) = \varphi(s^{-1}E) = 1$  for all  $s \in S$  and  $E \in \mathscr{C}_{\mu}$  (Lemma 1). Consequently, the restriction of  $\varphi$  to A is a multiplicative LIM.

Conversely, if A has a multiplicative LIM  $\psi$ , and  $\{p_{a_{\alpha}}\}, a_{\alpha} \in S$ , is a net of point measure on m(S) such that  $\lim_{\alpha} p_{a_{\alpha}}(f) = \psi(f)$  for all  $f \in A$ , then any cluster point  $\mu$  of the net  $\{p_{a_{\alpha}}\}$  in  $\beta S$  is a multiplicative extension of  $\psi$  to m(S). Let  $I = \{f \in A; \mu(f) = 0\}, f \in I$  be arbitrary and  $\lambda > 0$ . For each  $n \in Z$ , the integers, define

$$K(n, \lambda) = \{s \in S; \lambda n \leq f(s) < \lambda(n+1)\}$$
 .

Then  $S - K(n, \lambda) \in \mathscr{C}_{\mu}$  for all  $n \in \mathbb{Z} - \{-1, 0\}$  (by Lemma 2b) and  $||f - \sum (\lambda n) \mathbf{1}_{K(n,\lambda)}|| \leq \lambda$ , where the sum is taken over all  $n \in \mathbb{Z} - \{-1, 0\}$ . Thus  $A = I \bigoplus C \subset m(\mathscr{T}_{\mu}, S)$ , where C is the algebra of constant functions, since  $m(\mathscr{T}_{\mu}, S)$  is closed in m(S).

REMARK 2. If S is endowed with a noncompact hausdorff topology such that for each compact subset  $\sigma \subseteq S$ ,  $s^{-1}\sigma$  is compact for all  $s \in S$ ; order  $E = \{\sigma; \sigma \text{ compact subset of } S\}$  by upward inclusion. For  $\sigma \in E$ , let  $a_{\sigma} \in S - \sigma$ . If  $\mu$  is a cluster point of the net of point measures  $\{p_{a_{\sigma}}; \sigma \in E\}$ , then for any  $\sigma \in E$ ,  $\mu(s^{-1}(S - \sigma)) = \mu(S - \sigma) = 1$  for all  $s \in S$ . Hence,  $S - \sigma$  is left thick for all compact subsets  $\sigma \subseteq S$  and the ELA subalgebra  $m(\mathscr{T}_{\mu}, S)$  includes all functions  $f \in m(S)$  which vanish at infinity. In fact for any such f (fixed but arbitrary), let  $\lambda > 0$ . For each  $n \in Z$ , the integers, define

$$K(n, \lambda) = \{s \in S; \lambda n \leq f(s) < \lambda(n+1)\}$$
.

Since each  $S - K(n, \lambda)$  is included in a compact subset of  $S, K(n, \lambda) \in \mathscr{T}_{\mu}$  for all  $n \in \mathbb{Z}$  and

$$||f - \sum_{n \in N} (\lambda n) \mathbf{1}_{K(n,\lambda)}|| < \lambda$$
 .

Theorem 1 yields the following consequence:

COROLLARY. For any semigroup S, m(S) has a nontrivial ELA subalgebra (i.e., other than the algebra of constant functions) if and only if S has a proper left thick subset.

*Proof.* If S has a proper left thick subset E, let  $\mu$  be a mul-

tiplicative mean on m(S) such that  $\mu(E) = \mu(s^{-1}E) = 1$  for all  $s \in S$ (Lemma 3), then  $E \in \mathscr{C}_{\mu}$ , and  $m(\mathscr{T}_{\mu}, S)$  is a nontrivial ELA subalgebra of m(S) (Theorem 1). Conversely, if A is a nontrivial ELA subalgebra of m(S), then  $A \subseteq m(\mathscr{T}_{\mu}, S)$  for some mean  $\mu$  on m(S). Consequently,  $m(\mathscr{T}_{\mu}, S)$  is nontrivial and hence  $\mathscr{C}_{\mu}$  contains a proper subset of S, which is necessary left thick.

REMARK 3. The class of semigroups S for which m(S) has a nontrivial ELA subalgebra is extremely big and they include semigroups S which satisfy any one of the following conditions:

- (a) S is finite and not right cancellative.
- (b) S is infinite and left cancellative.

(c) S is infinite and has finite intersection property for right ideals (note that any left amenable semigroup has the latter property).

(d) S has finite intersection property for left ideals and the factor semigroup  $S|(\mathcal{E})$  is infinite, where  $(\mathcal{E})$  is the two-sided stable equivalence relation defined by  $a(\mathcal{E})b$  if and only if ca = cb for some  $c \in S$  (an equivalence relation E on S is two-sided stable if aEb implies acEbc and caEcb for all  $c \in S$ ).

In fact, we only need to show that the semigroups listed in (a), (b), (c) and (d) have proper left thick subsets. (a) If  $a, b, c \in S$  are such that  $a \neq b$  and ac = bc, then Sc is a proper left thick subset in S. (b) It follows from Remark 2 (with the discrete topology) that for any finite subset  $\sigma \subseteq S, S - \sigma$  is left thick. (c) We may assume that S is not cancellative (for otherwise (b) shows that S has a proper left thick subset); then S has either a proper left ideal or a proper right ideal, which must be left thick (Remark 1(c)). (d) The factor semigroup  $S/(\checkmark)$  if left cancellative ([4], p. 372). It follows from (b) that  $S|(\checkmark)$  has a proper left thick subset  $\widetilde{A}$ . If  $A = \{s; \overline{s} \in \widetilde{A}\}$ , where  $\overline{s}$  denotes the homomorphic image of s in  $S/(\checkmark)$ , then A is a proper left thick subset in S.

Examples of semigroups S for which the only ELA subalgebra of m(S) is the algebra of constant functions include all semigroups of the form  $E' \times G'$  where E' is a left zero semigroup (i.e.,  $a \cdot b = a$  for all  $a, b \in E'$ ) and G' is a finite right cancellative semigroup as the following proposition shows:

PROPOSITION 1. The following conditions concerning a semigroup S are equivalent:

- (a) S is right cancellative and has no proper left thick subset.
- (b) S has an idempotent and has no proper left thick subset.

(c) S is the direct product  $E \times G$  of a finite group G and a left zero semigroup E.

(d) S is the direct product  $E' \times G'$  of the finite right cancellative semigroup G' and a left zero semigroup E'.

*Proof.* (a) implies (b) follows from theorem 1.2.7 in [13] (p. 38). If (b) holds, the same theorem in [13] shows that S is the direct product  $E \times G$  of a group G and a left zero semigroup E. G is finite, for otherwise G has a proper left thick subset T (Remark 3(b)) which implies that S has a proper left thick subset  $E \times T$ . (c) implies (d) is clear. Finally if (d) holds, then as readily checked, S is right cancellative. Finally if K is a left thick subset in  $S, t \in E'$  is arbitrary, there exists  $(t_0, g_0) \in E' \times G'$  such that  $\{(tt_0, gg_0); g \in G\} = \{(t, g); g \in G\} \subseteq K$ . Consequently, K = S.

3. A characterization theorem. Mitchell ([9], Th. 1) shows that a semigroup S is ELA if and only if for each finite collection of subsets  $E_i \subseteq S$ ,  $i = 1, \dots, n$  such that  $S = \bigcup_{i=1}^{n} E_i$ , it follows that at least one of the subsets  $E_i$  is left thick in S. We show in this section that Mitchell's result can be sharpened and generalized to certain subalgebras of m(S). Our proof is completely different from that of Mitchell [9].

THEOREM 2. For any semigroup S, and  $\mathcal{T}$  an algebra of subsets of S such that  $s^{-1}E \in \mathcal{T}$  for all  $s \in S$  and  $E \in \mathcal{T}$ , the following conditions are equivalent:

(a)  $m(\mathcal{T}, S)$  is ELA.

(b) For each finite collection  $\{E_1, \dots, E_n\}$  of disjoint sets from  $\mathcal{T}$  with union S, at least one of  $E_i$  is left thick.

*Proof.* (a)  $\Rightarrow$  (b) Let  $\varphi$  be a multiplicative LIM on  $m(\mathscr{T}, S)$ , then  $1 = \varphi(S) = \sum_{i=1}^{n} \varphi(E_i)$ . Hence  $\varphi(E_i) > 0$  for some *i*, which implies  $\varphi(s^{-1}E_i) = \varphi(E_i) = 1$  for all  $s \in S$ , since  $\varphi$  is multiplicative. Consequently, the family  $\{s^{-1}E_i; s \in S\}$  has finite intersection property, and hence  $E_i$  is left thick.

(b)  $\Rightarrow$  (a) Let  $\mathscr{P}$  be the set each of whose elements is a finite collection  $\{E_1, \dots, E_n\}$  of disjoint sets in  $\mathscr{T}$  with union S. Let  $\mathscr{P}$  be ordered by defining  $P_1 \leq P_2$  to mean that each set in  $P_1$  is the union of sets in  $P_2$ ,  $P_1$ ,  $P_2 \in \mathscr{P}$ . It is easy to see that  $\leq$  renders  $\mathscr{P}$  into a directed set. For each  $E \in \mathscr{T}$ , let  $K_E = \{\varphi \in \beta S; \varphi(s^{-1}E) = \varphi(E) \}$  for all  $s \in S$ .  $K_E$  is a nonempty and closed subset of  $\beta S$ , and the family  $\{K_E; E \in \mathscr{T}\}$  has the finite intersection property. In fact, if  $E_1, \dots, E_n \in \mathscr{T}$ , let  $P_i = \{E_i, S - E_i\} \in \mathscr{P}$ , and choose  $P_0 \in \mathscr{P}$  such that  $P_0 \geq P_i$  for each  $1 \leq i \leq n$ . By assumption, there exists F in  $P_0$  such that F is left thick. Let  $\varphi_0 \in \beta S$  such that  $\varphi_0(s^{-1}F) = \varphi_0(F) = 1$ 

for all  $s \in S$  (Lemma 3). If  $F \subseteq E_i$ , then  $s^{-1}F \subseteq s^{-1}E_i$  for all  $s \in S$ . Hence  $\varphi_0(s^{-1}E_i) = \varphi_0(E_i) = 1$  for all  $s \in S$ . If  $F_0 \subseteq S - E_i$ , then  $\varphi_0(s^{-1}(S - E_i)) = \varphi_0(S - E_i) = 1$  for all  $s \in S$ . Consequently,  $\varphi_0(s^{-1}E_i) = \varphi_0(E_i) = 0$  for all  $s \in S$ . Hence  $\varphi_0 \in \bigcap_{i=1}^n K_{E_i}$ . If  $\varphi \in \bigcap_{E \in \mathscr{S}} K_E$  (which is nonempty by compactness of  $\beta S$ ), then  $\varphi(s^{-1}E) = \varphi(E)$  for all  $s \in S$  and  $E \in \mathscr{T}$ . Consequently,  $\varphi$  is a multiplicative LIM on  $m(\mathscr{T}, S)$  (Lemma 1).

LEMMA 4. A semigroup S is ELA if and only if for each subset  $E \subseteq S$ , there exists a mean  $\mu_E$  on m(S) such that  $\mu_E(s^{-1}E) = \mu_E(E) \in \{0, 1\}$  for all  $s \in S$ .

*Proof.* If  $\varphi$  is a multiplicative LIM on m(S), then for any subset  $E \subseteq S, \varphi(E)$  is either 0 or 1. To see the converse, for each  $E \subseteq S$ , let  $K_E = \{ \varphi \in \beta S; \varphi(s^{-1}E) = \varphi(E) \text{ for all } s \in S \}$ . Then  $K_E$  is nonempty since if  $\mu_E(s^{-1}E) = \mu_E(E) = 1$  for all  $s \in S$ , then  $\mu_E(s^{-1}E \cap E) = 1$  for all  $s \in S$  and hence the family  $\{s^{-1}E \cap E : s \in S\}$  has finite intersection property. Let  $\varphi \in \bigcap_{s \in S} \overline{s^{-1}E \cap E}$ , then  $\varphi(s^{-1}E) = \varphi(E) = 1$  for all  $s \in S$ .  $\mu_{\scriptscriptstyle E}(s^{\scriptscriptstyle -1}E)=\mu_{\scriptscriptstyle E}(E)=0 \quad {
m for} \quad {
m all} \quad s\in S, \quad {
m then} \quad \mu_{\scriptscriptstyle E}(s^{\scriptscriptstyle -1}(S-E))=$  $\mathbf{If}$  $\mu_{E}(S-E) = 1$  for all  $s \in S$ . Hence as above, there exists  $\varphi \in \beta S$ such that  $\varphi(s^{-1}(S-E)) = \varphi(S-E) = 1$  for all  $s \in S$ , or  $\varphi(s^{-1}E) = 0$  $\varphi(E) = 0$  for all  $s \in S$ . In both cases,  $K_E \neq \emptyset$ . Furthermore,  $\mathcal{K} =$  $\{K_E; E \subseteq S\}$  is a family of nonempty w<sup>\*</sup>-compact subset of  $\beta S$ . If we can show that  $\mathcal{K}$  has the finite intersection property, then any  $\varphi \in \bigcap_{E \subseteq S} K_E$  satisfies  $\varphi(s^{-1}E) = \varphi(E)$  for all  $s \in S$  and  $E \subseteq S$ . By Lemma 1,  $\varphi$  is even a LIM on m(S). To this end, let  $\mathscr{C}$  be a family of subsets of S such that  $\bigcap_{E \in \mathcal{X}} K_E \neq \emptyset$ , and let  $E_0 \subseteq S$ . Pick  $\varphi \in \bigcap_{E \in \mathscr{C}} K_E$  and  $\mu \in K_F$  where  $F = \{s \in S; \varphi(s^{-1}E_0) = 1\}$ . Define  $\psi \in \beta S$ by  $\psi(f) = \mu(h)$ , where  $h(s) = \varphi(sf)$  for all  $s \in S$ . Then  $\psi \in (\bigcap_{E \in \mathscr{C}} K_E) \cap K_{E_0}$ since  $\psi(E) = \mu(h) = \mu(ah) = \psi(a^{-1}E)$  for all  $a \in S$ , where h(s) = $\varphi(s^{-1}E) = \varphi(s^{-1}(a^{-1}E)) = h(as)$  for all  $a, s \in S$ , and

$$\psi(E_0) = \mu(F) = \mu(a^{-1}F) = \psi(a^{-1}E_0)$$
 for all  $a \in S$ .

This finishes the proof.

Lemma 4 yields the following new characterization theorem for the class of ELA semigroups:

THEOREM 3. A semigroup S is ELA if and only if (\*) for each subset  $E \subseteq S$ , either E is left thick, or S - E is left thick.

*Proof.* Necessity follows from Theorem 2 (a)  $\Rightarrow$  (b). Conversely if (\*) holds, it follows from Lemma 3 that for each  $E \subseteq S$ , there exists a mean  $\mu$  on m(S) such that  $\mu(s^{-1}E) = \mu(E) = 1$  for all  $s \in S$ 

if E is left thick, or  $\mu(s^{-1}E) = \mu(E) = 0$  if S - E is left thick. Consequently, S is ELA by Lemma 4.

REMARK. Note that condition (\*) in Theorem 3 is formally weaker than condition (b) and (c) in [9], Theorem 1.

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336

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# Pacific Journal of Mathematics Vol. 33, No. 2 April, 1970

Raymond Balbes and Alfred Horn, <i>Projective distributive lattices</i>	273
John Findley Berglund, On extending almost periodic functions	281
Günter Krause, Admissible modules and a characterization of reduced left	
artinian rings	291
Edward Milton Landesman and Alan Cecil Lazer, <i>Linear eigenvalues and a</i>	
nonlinear boundary value problem	311
Anthony To-Ming Lau, <i>Extremely amenable algebras</i>	329
Aldo Joram Lazar, Sections and subsets of simplexes	337
Vincent Mancuso, Mesocompactness and related properties	345
Edwin Leroy Marsden, Jr., <i>The commutator and solvability in a generalized</i>	257
Share Metering Brown where I found on and the three times of a manifold	337
domains	363
S. Mukhoti, Theorems on Cesàro summability of series	385
Ngô Van Quê, Classes de Chern et théorème de Gauss-Bonnet	393
Ralph Tyrrell Rockafellar, Generalized Hamiltonian equations for convex	
problems of Lagrange	411
Ken iti Sato, On dispersive operators in Banach lattices	429
Charles Andrew Swanson, Comparison theorems for elliptic differential	
systems	445
John Griggs Thompson, Nonsolvable finite groups all of whose local	
subgroups are solvable. II	451
David J. Winter, <i>Cartan subalgebras of a Lie algebra and its ideals</i>	537