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**THE COMMUTATOR AND SOLVABILITY IN A GENERALIZED
ORTHOMODULAR LATTICE**

EDWIN LEROY MARSDEN, JR.

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In this paper we prove in a generalized orthomodular lattice the analog of the following theorem from group theory. For a and b members of a group G , let $aba^{-1}b^{-1}$ be the commutator of a and b . The set of commutators in G generates a normal subgroup H of G possessing these properties: G/H is Abelian. Moreover, if K is any normal subgroup of G for which G/K is Abelian, then $K \supseteq H$. Continuing the analogy with group theory, we determine a solvability condition on generalized orthomodular lattices.

An *orthomodular lattice* is a lattice L with 0 and 1 and with an orthocomplementation $\prime: L \rightarrow L$ satisfying the *orthomodular identity*: for $e \leq f$ in L , $f = e \vee (f \wedge e')$. Throughout this paper L shall denote an orthomodular lattice. For $f \in L$ the *Sasaki projection determined by f* $\phi_f: L \rightarrow L$ by $e\phi_f = (e \vee f') \wedge f$. We say e *commutes with f* , ecf , when $e\phi_f = e \wedge f$. Basic properties of orthomodular lattices and of their coordinatizing Baer $*$ -semigroups are contained in [1, 2].

A lattice ideal I in L is called a *p -ideal* if and only if $e \in I$ and $f \in L$ imply $e\phi_f \in I$. Theorem 6, which concerns p -ideals in generalized orthomodular lattices, indicates the significance of p -ideals in orthomodular lattices.

2. **The commutator.** For elements e and f of the orthomodular lattice L , we define the *commutator* of e and f by

$$[e, f] = (e \vee f) \wedge (e \vee f') \wedge (e' \vee f) \wedge (e' \vee f').$$

It is easily shown that ecf if and only if $[e, f] = 0$, and that $[e, f] = [e, f'] = [e', f] = [e', f']$.

THEOREM 1. *Let R be a Baer $*$ -ring, and let $P'(R)$ denote the orthomodular lattice of closed projections in R . Then for*

$$e, f \in P'(R), (ef-fe)'' = [e, f].$$

In proving the theorem, we shall use the following computation.

LEMMA 2. $[e, f] = (f'ef)'' \vee (e'fe)''$.

Proof. $(f'ef)'' = ((f'e)''f)'' = f'\phi_e\phi_f = \{[(f' \vee e') \wedge e] \vee f'\} \wedge f =$

$(f' \vee e') \wedge (e \vee f') \wedge f$, where the last equality holds by the Foulis-Holland theorem [2]—observe that $(f' \vee e')ce$, and $(f' \vee e')cf'$. Similarly, $(e'fe)'' = (f' \vee e') \wedge (e' \vee f) \wedge e$. The following expression is simplified by repeated applications of the Foulis-Holland theorem. We have

$$\begin{aligned}
 & (f'ef)'' \vee (e'fe)'' \\
 &= [(f' \vee e') \wedge (e \vee f') \wedge f] \vee [(f' \vee e') \wedge (e' \vee f) \wedge e] \\
 &\quad (f' \vee e')c(e \vee f') \wedge f, (e' \vee f) \wedge e \\
 &= (f' \vee e') \wedge \{[(e \vee f') \wedge f] \vee [(e' \vee f) \wedge e]\} \\
 &\quad (e' \vee f)c(e \vee f') \wedge f, e \\
 &= (f' \vee e') \wedge [(e' \vee f) \wedge \{[(e \vee f') \wedge f] \vee e\}] \\
 &\quad (e \vee f')cf, e \\
 &= (f' \vee e') \wedge (e' \vee f) \wedge (e \vee f') \wedge (f \vee e) = [e, f].
 \end{aligned}$$

Proof of theorem. The element $(ef-fe)''$ is the smallest closed projection serving as a right identity for $(ef-fe)$. Equivalently, $(ef-fe)'$ is the greatest closed projection which serves as a right annihilator for $ef-fe$. Thus for $k \in P'(R)$, $k \leq (ef-fe)'$ if and only if $efk = fek$.

Suppose that for some $k \in P'(R)$, $efk = fek$. Then $f'efk = f'fek = 0$ implies that $k = (f'ef)'k$, or $k \leq (f'ef)'$. Similarly $k \leq (e'fe)'$, and hence $k \leq (e'fe)' \wedge (f'ef)' = [e, f]'$. Also, $(ef)[e, f]' = e(f[e, f]') = e(f \wedge [e, f]') = e(f \wedge [(e \wedge f) \vee (e \wedge f') \vee (e' \wedge f) \vee (e' \wedge f')]) = e[(e \wedge f) \vee (e' \wedge f)] = e \wedge [(e \wedge f) \vee (e' \wedge f)] = e \wedge f = fe[e, f]'$. Moreover, for $k \leq [e, f]'$, then $k = [e, f]'k$ and $efk = ef[e, f]'k = fe[e, f]'k = fek$. Thus we have shown that $efk = fek$ if and only if $k \leq [e, f]'$. Therefore $(ef-fe)' = [e, f]'$ and $(ef-fe)'' = [e, f]$.

LEMMA 3. For $e, f \in L$, $f\phi_e \leq f \vee [f, e]$.

Proof. By the Foulis-Holland theorem,

$$f \vee [(f \vee e) \wedge (f \vee e') \wedge (f' \vee e) \wedge (f' \vee e')] = (f \vee e) \wedge (f \vee e').$$

LEMMA 4. Let L and X be orthomodular lattices.

- (i) For an ortho-homomorphism $\phi: L \rightarrow X$ and c a commutator in L , $c\phi$ is a commutator in X .
- (ii) For an ortho-epimorphism $\phi: L \rightarrow X$ and x a commutator in X , $x = c\phi$ where c is a commutator in L .
- (iii) X is Boolean if and only if 0 is the only commutator in X .

Proof. Ortho-homomorphisms preserve suprema, infima, and ortho-complements.

THEOREM 5. *Let L be an orthomodular lattice, and let J be the ideal generated by the commutators in L . Then J is a p -ideal, and L/J is Boolean. Moreover, if I is any p -ideal for which L/I is Boolean, then $I \supseteq J$.*

Proof. Let J be the ideal generated by the commutators in L , i.e.,

$$J = \left\{ y \in L \mid \text{for some commutators } c_1, \dots, c_n \text{ in } L, y \leq \bigvee_{i=1}^n c_i \right\}.$$

We claim that J is a p -ideal. Take any $x \in L$ and $y \leq \bigvee_{i=1}^n c_i$ a finite join of commutators in L . Then by Lemma 3, $y\phi_x \leq (\bigvee_{i=1}^n c_i)\phi_x = \bigvee_{i=1}^n (c_i\phi_x) \leq \bigvee_{i=1}^n (c_i \vee [c_i, x])$, and hence $y\phi_x \in J$.

To show that L/J is Boolean, use the natural ortho-epimorphism $\phi: L \rightarrow L/J$, and apply Lemma 4 (ii). A second application of Lemma 4 completes the proof of the theorem.

3. Solvability in a generalized orthomodular lattice. At this point it is impossible to mimic the solvability conditions of group theory [4]. The difficulty is that the p -ideals in orthomodular lattices need not be orthomodular lattices. In fact, a p -ideal I of L contains a greatest element d if and only if $I = L(0, d)$, where d is central in L . In order to generalize both orthomodular lattices and p -ideals we make the following

DEFINITION. G is a *generalized orthomodular lattice* (GOML) if and only if

- (i) $0 \in G$,
- (ii) for every nonzero $a \in G$, $G(0, a) = \{x \in G \mid 0 \leq x \leq a\}$ is an orthomodular lattice, and
- (iii) for $x \leq a \leq b$ in G , and for $a-x$ and $b-x$ the orthocomplements of x in $G(0, a)$ and $G(0, b)$ respectively, $a-x = (b-x) \wedge a$.

M. F. Janowitz [5] has shown that every GOML G can be embedded as a p -ideal in an orthomodular lattice L . If G is not already an orthomodular lattice then G is embedded as a prime ideal in L , i.e., for $a \in L$ either $a \in G$ or $a' \in G$. Let G be a GOML, and let $G \leq L$ be the Janowitz embedding. For any $e, f \in L$, since G is prime in L , then $[e, f] \in G$. Thus the p -ideal generated by the commutators in L is a subset of G . The following theorem clarifies this. For elements $e, f \in G$ we define the *generalized Sasaki projection* by $e\mathcal{P}_f = \{e \vee [(e \vee f) - f]\} \wedge f$, the Sasaki projection in $G(0, e \vee f)$. An ideal I of G is called a p -ideal of G when I is closed under all generalized Sasaki projections. For elements $e, f \in G$ we say that e is *perspective to f via t* , written $e \sim_t f$, if and only if for some

$t \in G$, $e \vee t = f \vee t$ and $e \wedge t = f \wedge t = 0$.

THEOREM 6. *Let I be an ideal of G , and let $G \leq L$ be the Janowitz embedding. These conditions are equivalent.*

- (i) *For $e \in I$, $f \in G$ and $e \sim_p f$, then $f \in I$.*
- (ii) *I is a p -ideal of G .*
- (iii) *I is a p -ideal of L .*
- (iv) *For $e \in I$, $f \in L$ and $e \sim_p f$, then $f \in I$.*
- (v) *I is the kernel of a (unique) congruence on L .*
- (vi) *I is the kernel of a (unique) congruence on G .*

Proof. (i) \Rightarrow (ii). Let $e \in I$ and $f \in G$. A computation shows that $e\Psi_f \sim_p f\Psi_e$ via $(e \vee f) - e\Psi_f$. Since $f\Psi_e \leq e$, then $f\Psi_e \in I$, and by (i) $e\Psi_f \in I$.

(ii) \Rightarrow (iii). Let $e \in I$ and $f \in L$. If $f \in G$, we are finished. Otherwise, $f' \in G$ and it follows that $e \vee f' \in G$ and $e\phi_f = (e \vee f') \wedge f \in G$. By (ii), $e\Psi_{e\phi_f} \in I$. But

$$\begin{aligned} e\Psi_{e\phi_f} &= [e \vee [(e \vee e\phi_f) - e\phi_f]] \wedge e\phi_f = \{e \vee [(e\phi_f)' \wedge (e \vee e\phi_f)]\} \wedge e\phi_f \\ &= [e \vee (e\phi_f)'] \wedge [e \vee e\phi_f] \wedge e\phi_f \\ &= [e \vee (e' \wedge f) \vee f'] \wedge e\phi_f = e\phi_f. \end{aligned}$$

(iii) \Leftrightarrow (iv) \Leftrightarrow (v) are well known [3].

(v) \Rightarrow (vi). The restriction of the congruence on L to G is a congruence. Notice that the congruence preserves relative orthocomplements. The uniqueness stems the fact in any relatively complemented lattice with 0, every ideal is the kernel of at most one congruence [3].

(vi) \Rightarrow (i). Suppose that θ is a congruence on G with $\ker \theta = I$. Let $e \in I$ and $f \in G$ with $e \sim_p f$ via $t \in G$. The $e\theta 0$ implies $e \vee t\theta t$, or $f \vee t\theta t$. It follows that $f = (f \vee t) \wedge f\theta t \wedge f = 0$. Hence $f \in I$.

The Janowitz embedding and Theorem 6 furnish an immediate generalization of Theorem 5.

THEOREM 7. *Let G be a GOML, and let J be the commutator p -ideal in G . Then G/J is distributive. Moreover, if I is a p -ideal of G for which G/I is distributive, then $I \supseteq J$.*

We are now in a position to discuss solvability of GOML. Let G be a GOML, let G_1 be the p -ideal generated by the commutators in G , and for $n > 1$ let G_n be the p -ideal generated by the commutators in G_{n-1} . A GOML G will be called *solvable* if and only if for some n $G_n = \{0\}$.

LEMMA 8. *Let J be a p -ideal in a GOML G , and let I be a p -ideal in J . Then I is a p -ideal in G .*

Proof. We shall show for $e \in I$, $f \in G$ that $e\Psi_f \in I$. Since $e \in J$, a p -ideal in G , then $e\Psi_f \in J$. Therefore $e\Psi_{e\Psi_f} \in I$. A computation shows that $e\Psi_{e\Psi_f} = e\Psi_f$.

THEOREM 9. *Let G be a GOML. For G to be solvable it is a necessary and sufficient condition that G be distributive.*

Proof. Theorem 7 proves the sufficiency. We shall prove the necessity by showing that $G_2 = G_1$ and hence that $G_n = G_1$ for all positive integers n .

Let $G \leq L$ be the Janowitz embedding, and let $'$ be the orthocomplementation of L . For elements $e, f \in G$, set $c = (e' \vee f') \wedge (e' \vee f) \wedge e$ and $d = (f' \vee e') \wedge (f' \vee e) \wedge f$. Then $c \vee d = [e, f]$ by the computation of Lemma 2. Moreover,

$$\begin{aligned} & c \vee d' \\ &= [(e' \vee f') \wedge (e' \vee f) \wedge e] \vee (e \wedge f) \vee (f \wedge e') \vee f' \\ &\quad (e \wedge f)c(e' \vee f'), (e' \vee f), e \\ &= [(e' \vee f) \wedge e] \vee (f \wedge e') \vee f' \\ &\quad (e' \vee f)ce, f' \\ &= (e \vee f') \vee (f \wedge e') = 1. \end{aligned}$$

Similarly $c' \vee d = 1$. Also $c' \vee d' \geq (e \wedge f) \vee e' \vee f' = 1$.

We have shown for any $e, f \in G$ and for c, d as above that $[e, f] = [c, d] = c \vee d$. Here $c, d \leq [c, d]$ imply that $c, d \in G_1$, and thus $[e, f] = [c, d] \in G_2$. This completes the proof that $G_1 = G_2$.

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