THE COMMUTATOR AND SOLVABILITY IN A GENERALIZED ORTHOMODULAR LATTICE

Edwin Leroy Marsden, Jr.
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In this paper we prove in a generalized orthomodular lattice the analog of the following theorem from group theory. For \( a \) and \( b \) members of a group \( G \), let \( aba^{-1}b^{-1} \) be the commutator of \( a \) and \( b \). The set of commutators in \( G \) generates a normal subgroup \( H \) of \( G \) possessing these properties: \( G/H \) is Abelian. Moreover, if \( K \) is any normal subgroup of \( G \) for which \( G/K \) is Abelian, then \( K \supseteq H \). Continuing the analogy with group theory, we determine a solvability condition on generalized orthomodular lattices.

An orthomodular lattice is a lattice \( L \) with 0 and 1 and with an orthocomplementation \( ' : L \to L \) satisfying the orthomodular identity: for \( e \leq f \) in \( L \), \( f = e \lor (f \land e') \). Throughout this paper \( L \) shall denote an orthomodular lattice. For \( f \in L \) the Sasaki projection determined by \( f \phi_f : L \to L \) by \( e\phi_f = (e \lor f') \land f \). We say \( e \) commutes with \( f \), \( e cf \), when \( e\phi_f = e \land f \). Basic properties of orthomodular lattices and of their coordinatizing Baer \(*\)-semigroups are contained in [1, 2].

A lattice ideal \( I \) in \( L \) is called a p-ideal if and only if \( e \in I \) and \( f \in L \) imply \( e\phi_f \in I \). Theorem 6, which concerns p-ideals in generalized orthomodular lattices, indicates the significance of p-ideals in orthomodular lattices.

2. The commutator. For elements \( e \) and \( f \) of the orthomodular lattice \( L \), we define the commutator of \( e \) and \( f \) by

\[
[e, f] = (e \lor f) \land (e \lor f') \land (e' \lor f) \land (e' \lor f') .
\]

It is easily shown that \( e cf \) if and only if \( [e, f] = 0 \), and that \( [e, f] = [e, f'] = [e', f] = [e', f'] \).

**Theorem 1.** Let \( R \) be a Baer \(*\)-ring, and let \( P'(R) \) denote the orthomodular lattice of closed projections in \( R \). Then for

\[
e, f \in P'(R), (ef-fe)'' = [e, f] .
\]

In proving the theorem, we shall use the following computation.

**Lemma 2.** \([e, f] = (f'e)f')'' \lor (e'fe)''.

**Proof.** \((f'e)f')'' = ((f'e)f')'' = f'\phi_e \phi_f = \{(f' \lor e') \land e \lor f' \} \land f =
$(f' \lor e') \land (e \lor f') \land f$, where the last equality holds by the Foulis-Holland theorem [2]—observe that $(f' \lor e')ce$, and $(f' \lor e')cf'$. Similarly, $(e'fe)'' = (f' \lor e') \land (e' \lor f) \land e$. The following expression is simplified by repeated applications of the Foulis-Holland theorem. We have

$$
(f'ef)'' \lor (e'ef)''
= [(f' \lor e') \land (e \lor f') \land f] \lor [(f' \lor e') \land (e' \lor f) \land e]
= (f' \lor e')c(e \lor f') \land f, (e' \lor f) \land e
= (f' \lor e') \land [(e \lor f') \land f] \lor [(e' \lor f) \land e]
= (e' \lor f)c(e \lor f') \land f, e
= (f' \lor e') \land (e' \lor f) \land (e \lor f') \land (f \lor e) = [e, f].
$$

**Proof of theorem.** The element $(ef-fe)''$ is the smallest closed projection serving as a right identity for $(ef-fe)$. Equivalently, $(ef-fe)'$ is the greatest closed projection which serves as a right annihilator for $ef-fe$. Thus for $k \in P'(R)$, $k \leq (ef-fe)'$ if and only if $efk = fek$.

Suppose that for some $k \in P'(R)$, $efk = fek$. Then $f'efk = f'fek = 0$ implies that $k = (f'ef)'k$, or $k \leq (f'ef)'$. Similarly $k \leq (e'fe)'$, and hence $k \leq (e'fe)' \land (f'ef)' = [e, f]'$. Also, $(ef)[e, f]' = e(f[e, f]) = e(f \land [e, f]) = e(f \land [(e \lor f) \lor (e' \lor f) \lor (e' \lor f')]) = e[(e \land f) \lor (e' \land f)] = e \land f = fe[e, f]'$. Moreover, for $k \leq [e, f]'$, then $k = [e, f]'k$ and $efk = ef[e, f]'k = f[e, f]'k = fek$. Thus we have shown that $efk = fek$ if and only if $k \leq [e, f]'$. Therefore $(ef-fe)' = [e, f]'$ and $(ef-fe)'' = [e, f]$.

**Lemma 3.** For $e, f \in L$, $f\phi_e \leq f \lor [f, e]$.

**Proof.** By the Foulis-Holland theorem,

$$
f \lor [(f \lor e) \land (f \lor e') \land (f' \lor e) \land (f' \lor e')] = (f \lor e) \land (f \lor e').
$$

**Lemma 4.** Let $L$ and $X$ be orthomodular lattices.

(i) For an ortho-homomorphism $\phi : L \to X$ and $c$ a commutator in $L$, $c\phi$ is a commutator in $X$.

(ii) For an ortho-epimorphism $\phi : L \to X$ and $x$ a commutator in $X$, $x = c\phi$ where $c$ is a commutator in $L$.

(iii) $X$ is Boolean if and only if 0 is the only commutator in $X$.

**Proof.** Ortho-homomorphisms preserve suprema, infima, and orthocomplements.
THEOREM 5. Let $L$ be an orthomodular lattice, and let $J$ be the ideal generated by the commutators in $L$. Then $J$ is a $p$-ideal, and $L/J$ is Boolean. Moreover, if $I$ is any $p$-ideal for which $L/I$ is Boolean, then $I \supseteq J$.

Proof. Let $J$ be the ideal generated by the commutators in $L$, i.e.,

$$J = \left\{ y \in L \mid \text{for some commutators } c_1, \ldots, c_n \in L, y \leq \bigvee_{i=1}^n c_i \right\}.$$  

We claim that $J$ is a $p$-ideal. Take any $x \in L$ and $y \leq \bigvee_{i=1}^n c_i$ a finite join of commutators in $L$. Then by Lemma 3, $y \phi_x \leq (\bigvee_{i=1}^n c_i) \phi_x = \bigvee_{i=1}^n (c_i \phi_x) \leq \bigvee_{i=1}^n (c_i \lor [c_i, x])$, and hence $y \phi_x \in J$.

To show that $L/J$ is Boolean, use the natural ortho-epimorphism $\phi: L \to L/J$, and apply Lemma 4 (ii). A second application of Lemma 4 completes the proof of the theorem.

3. Solvability in a generalized orthomodular lattice. At this point it is impossible to mimic the solvability conditions of group theory [4]. The difficulty is that the $p$-ideals in orthomodular lattices need not be orthomodular lattices. In fact, a $p$-ideal $I$ of $L$ contains a greatest element $d$ if and only if $I = L(0, d)$, where $d$ is central in $L$. In order to generalize both orthomodular lattices and $p$-ideals we make the following

DEFINITION. $G$ is a generalized orthomodular lattice (GOML) if and only if

(i) $0 \in G$,

(ii) for every nonzero $a \in G$, $G(0, a) = \{ x \in G \mid 0 \leq x \leq a \}$ is an orthomodular lattice, and

(iii) for $x \leq a \leq b$ in $G$, and for $a-x$ and $b-x$ the orthocomplements of $x$ in $G(0, a)$ and $G(0, b)$ respectively, $a-x = (b-x) \land a$.

M. F. Janowitz [5] has shown that every GOML $G$ can be embedded as a $p$-ideal in an orthomodular lattice $L$. If $G$ is not already an orthomodular lattice then $G$ is embedded as a prime ideal in $L$, i.e., for $a \in L$ either $a \in G$ or $a' \in G$. Let $G$ be a GOML, and let $G \subseteq L$ be the Janowitz embedding. For any $e, f \in L$, since $G$ is prime in $L$, then $[e, f] \in G$. Thus the $p$-ideal generated by the cummutators in $L$ is a subset of $G$. The following theorem clarifies this. For elements $e, f \in G$ we define the generalized Sasaki projection by $e \mathcal{T}_f = \{ e \lor [(e \lor f) - f] \} \land f$, the Sasaki projection in $G(0, e \lor f)$. An ideal $I$ of $G$ is called a $p$-ideal of $G$ when $I$ is closed under all generalized Sasaki projections. For elements $e, f \in G$ we say that $e$ is perspective to $f$ via $t$, written $e \sim t f$, if and only if for some
\( t \in G, \ e \lor t = f \lor t \) and \( e \land t = f \land t = 0. \)

**Theorem 6.** Let \( I \) be an ideal of \( G \), and let \( G \leq L \) be the Janowitz embedding. These conditions are equivalent:

(i) For \( e \in I, f \in G \) and \( e \sim_p f \), then \( f \in I \).

(ii) \( I \) is a \( p \)-ideal of \( G \).

(iii) \( I \) is a \( p \)-ideal of \( L \).

(iv) For \( e \in I, f \in L \) and \( e \sim_p f \), then \( f \in I \).

(v) \( I \) is the kernel of a (unique) congruence on \( L \).

(vi) \( I \) is the kernel of a (unique) congruence on \( G \).

**Proof.** (i) \( \implies \) (ii). Let \( e \in I \) and \( f \in G \). A computation shows that \( e\Psi_f \sim_p f\Psi_e \) via \((e \lor f) - e\Psi_f \). Since \( f\Psi_e \leq e \), then \( f\Psi_e \in I \), and by (i) \( e\Psi_f \in I \).

(ii) \( \implies \) (iii). Let \( e \in I \) and \( f \in L \). If \( f \in G \), we are finished. Otherwise, \( f' \in G \) and it follows that \( e \lor f' \in G \) and \( e\phi_f = (e \lor f') \land f \in G \). By (ii), \( e\Psi_{e\phi_f} \in I \). But

\[
e\Psi_{e\phi_f} = [e \lor [(e \lor e\phi_f) - e\phi_f]] \land e\phi_f = [e \lor [(e\phi_f') \land (e \lor e\phi_f)]] \land e\phi_f =
\]

\[
= [e \lor (e\phi_f') \land (e \lor e\phi_f) \land e\phi_f = \]

\[
= [e \lor (e' \lor f) \lor f'] \land e\phi_f = e\phi_f.
\]

(iii) \( \iff \) (iv) \( \iff \) (v) are well known [3].

(v) \( \implies \) (vi). The restriction of the congruence on \( L \) to \( G \) is a congruence. Notice that the congruence preserves relative orthocomplements. The uniqueness stems the fact in any relatively complemented lattice with 0, every ideal is the kernel of at most one congruence [3].

(vi) \( \implies \) (i). Suppose that \( \theta \) is a congruence on \( G \) with \( \ker \theta = I \). Let \( e \in I \) and \( f \in G \) with \( e \sim_p f \) via \( t \in G \). The \( e\theta_0 \) implies \( e \lor t\theta t \), or \( f \lor t\theta t \). It follows that \( f = (f \lor t) \land t\theta t \land f = 0 \). Hence \( f \in I \).

The Janowitz embedding and Theorem 6 furnish an immediate generalization of Theorem 5.

**Theorem 7.** Let \( G \) be a GOML, and let \( J \) be the commutator \( p \)-ideal in \( G \). Then \( G/J \) is distributive. Moreover, if \( I \) is a \( p \)-ideal of \( G \) for which \( G/I \) is distributive, then \( I \supseteq J \).

We are now in a position to discuss solvability of GOML. Let \( G \) be a GOML, let \( G_i \) be the \( p \)-ideal generated by the commutators in \( G \), and for \( n > 1 \) let \( G_n \) be the \( p \)-ideal generated by the commutators in \( G_{i-1} \). A GOML \( G \) will be called **solvable** if and only if for some \( n \) \( G_n = \{0\} \).

**Lemma 8.** Let \( J \) be a \( p \)-ideal in a GOML \( G \), and let \( I \) be a \( p \)-ideal in \( J \). Then \( I \) is a \( p \)-ideal in \( G \).
Proof. We shall show for $e \in I$, $f \in G$ that $e \Psi f \in I$. Since $e \in J$, a $p$-ideal in $G$, then $e \Psi f \in J$. Therefore $e \Psi e \Psi f \in I$. A computation shows that $e \Psi e \Psi f = e \Psi f$.

THEOREM 9. Let $G$ be a GOML. For $G$ to be solvable it is a necessary and sufficient condition that $G$ be distributive.

Proof. Theorem 7 proves the sufficiency. We shall prove the necessity by showing that $G = G_1$ and hence that $G_n = G_1$ for all positive integers $n$.

Let $G \leq L$ be the Janowitz embedding, and let $'$ be the orthocomplementation of $L$. For elements $e, f \in G$, set $c = (e' \vee f') \land (e' \vee f) \land e$ and $d = (f' \vee e') \land (f' \vee e) \land f$. Then $c \lor d = [e, f]$ by the computation of Lemma 2. Moreover,

\[
c \lor d' = [(e' \lor f') \land (e' \lor f) \land e] \lor (e \land f) \lor (f \land e') \lor f'
\]
\[
= [(e' \lor f) \land e] \lor (f \land e') \lor f'
\]
\[
= (e' \lor f) c e, f'
\]
\[
= (e \lor f') \lor (f \land e') = 1.
\]

Similarly $c' \lor d = 1$. Also $c' \lor d' \geq (e \land f) \lor e' \lor f' = 1$.

We have shown for any $e, f \in G$ and for $c, d$ as above that $[e, f] = [c, d] = c \lor d$. Here $c, d \leq [c, d]$ imply that $c, d \in G_1$, and thus $[e, f] = [c, d] \in G_2$. This completes the proof that $G_1 = G_2$.

REFERENCES

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