

Pacific Journal of Mathematics

THEOREMS ON CESÀRO SUMMABILITY OF SERIES

S. MUKHOTI

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1.1. We consider the Cesàro summability, for integral orders, of the series

$$(1.1) \quad \sum_{\nu=0}^{\infty} a_{\nu} d_{\nu} .$$

In this paper we establish equivalence theorems for the series (1.1) which are valid for a substantial class of sequences d_{ν} including $e^{-\nu}$ and $\nu^{-\delta}$. Results of this character, but not overlapping with those in this paper, were given by Hardy and Littlewood and by Andersen. Andersen's result was extended by Bosanquet and Chow, and further extended by Bosanquet.

Notation. 1.2. We write $A_n^0 = A_n = a_0 + a_1 + \dots + a_n$,

$$A_n^k = A_0^{k-1} + A_1^{k-1} + \dots + A_n^{k-1}$$

and we get the identities: See Hardy [8].

$$(1.2) \quad A_n^k = \sum_{\nu=0}^n B_{n-\nu}^{k-1} A_{\nu} ,$$

$$(1.3) \quad A_n^k = \sum_{\nu=0}^n B_{n-\nu}^k a_{\nu} ,$$

where

$$(1.4) \quad B_{n-\nu}^k = \binom{n - \nu + k}{k} ;$$

$E_n^k = A_n^k$ when $a_0 = 1, a_n = 0$, for $n > 0$, i.e., when $A_n = 1$, for all n .

Hence

$$(1.5) \quad E_n^k = \binom{n + k}{k} \sim \frac{n^k}{k!} .$$

If

$$(1.6) \quad \frac{A_n^k}{E_n^k} \rightarrow A, \text{ when } n \rightarrow \infty ,$$

or equivalently if

$$(1.7) \quad \frac{k! A_n^k}{n^k} \rightarrow A, \text{ when } n \rightarrow \infty ,$$

then we say that $\sum_{n=0}^{\infty} a_n$ is summable (C, k) to sum A and we write

$$(1.8) \quad \sum_{n=0}^{\infty} a_n = A(C, k).$$

1.3. *Statement of lemma and identity.* We write

$$\Delta d_n = d_n - d_{n-1}, \Delta^k u_n = \Delta \Delta^{k-1} u_n \quad (k \geq 2)$$

and $\Delta^0 u_n = u_n$.

We shall use the following well-known identity:

$$(1.9) \quad \Delta^k(u_n v_n) = \sum_{\nu=0}^k \binom{k}{\nu} \Delta^\nu u_n \Delta^{k-\nu} v_{n-\nu}.$$

LEMMA A. *In order that*

$$(1.10) \quad t_m = \sum C_{m,n} S_n \rightarrow S \quad (m \rightarrow \infty), \quad (m = 0, 1, 2, \dots)$$

whenever

$$(1.11) \quad S_n \rightarrow S \quad (n \rightarrow \infty),$$

it is necessary and sufficient that

$$(1.12) \quad (i) \quad \sum |C_{m,n}| < H,$$

where H is independent of m ;

$$(1.13) \quad (ii) \quad C_{m,n} \rightarrow 0,$$

for each n , when $m \rightarrow \infty$;

$$(1.14) \quad (iii) \quad \sum C_{m,n} \rightarrow 1, \text{ when } m \rightarrow \infty.$$

Lemma A is mentioned by Hardy [8, Th. 2], which is due to Toeplitz [12]. Toeplitz considers only *triangular* transformations, in which $C_{m,n} = 0$ for $n > m$. The extension to general transformations was made by Steinhaus [11].

2. Statement and proof of the theorem.

THEOREM (*the cases $k = 1, 2, \dots$). Suppose that $d_n > 0$ for $n \geq 0$, and*

$$(2.1) \quad (i) \quad d_{n+1}^k = o(n^k) \text{ as } n \rightarrow \infty,$$

$$(2.2) \quad (ii) \quad (1/B_n^k) \sum_{m=0}^n B_m^k \left| \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^n B_{n-\nu}^{k-1} d_{\nu+1} \right\} \right| = O(1),$$

(Δ operating on m).

Then necessary and sufficient conditions for

$$(2.3) \quad (I) \quad \sum_{\nu=0}^{\infty} a_{\nu} d_{\nu} \text{ to be summable } (C, k) \text{ to } S$$

are that

$$(2.4) \quad (II) \quad -\sum_{\nu=0}^{\infty} S_{\nu} \Delta d_{\nu+1} \text{ should be summable } (C, k) \text{ to } S$$

and

$$(2.5) \quad (III) \quad S_n d_{n+1} = o(1) (C, k) \text{ as } n \rightarrow \infty ,$$

where

$$(2.6) \quad S_n = \sum_{\nu=0}^n a_{\nu} .$$

Proof. We have

$$(2.7) \quad \sum_{\nu=0}^n a_{\nu} d_{\nu} = S_n d_{n+1} - \sum_{\nu=0}^n S_{\nu} \Delta d_{\nu+1}$$

$$\text{i.e.,} \quad C_n = F_n - G_n ,$$

and hence

$$(2.8) \quad C_n^k = F_n^k - G_n^k .$$

The *sufficiency* follows immediately from (2.8).

Necessity. We are given that

$$(2.9) \quad C_n^k / B_n^k \rightarrow S \text{ as } n \rightarrow \infty ,$$

and it will be enough to prove that

$$(2.10) \quad -G_n^k / B_n^k \rightarrow S \text{ as } n \rightarrow \infty .$$

From (2.7) we have

$$(2.11) \quad \begin{aligned} \frac{C_n \Delta d_{n+1}}{d_{n+1}} &= S_n \Delta d_{n+1} - \frac{\Delta d_{n+1}}{d_{n+1}} G_n , \\ &= \frac{d_n (d_{n+1} \Delta G_n - G_n \Delta d_{n+1})}{d_n d_{n+1}} = d_n \Delta \left(\frac{G_n}{d_{n+1}} \right) . \end{aligned}$$

Thus

$$(2.12) \quad \frac{G_n}{d_{n+1}} = \sum_{\nu=0}^n \frac{C_{\nu}}{d_{\nu}} \frac{\Delta d_{\nu+1}}{d_{\nu+1}} ,$$

so

$$-G_n = d_{n+1} \sum_{\nu=0}^n C_\nu \Delta(1/d_{\nu+1}) ,$$

and hence

$$\begin{aligned} -G_n^k &= \sum_{\nu=0}^n B_{n-\nu}^{k-1} d_{\nu+1} \sum_{m=0}^{\nu} C_m \Delta(1/d_{m+1}) \\ (2.13) \quad &= \sum_{m=0}^n C_m \Delta(1/d_{m+1}) \sum_{\nu=m}^n B_{n-\nu}^{k-1} d_{\nu+1} \\ &= \sum_{m=0}^n (-1)^k C_m^k \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^n B_{n-\nu}^{k-1} d_{\nu+1} \right\} . \end{aligned}$$

It follows that

$$\begin{aligned} -\frac{G_n^k}{B_n^k} &= \frac{1}{B_n^k} \sum_{m=0}^n (-1)^k \frac{C_m^k}{B_m^k} \cdot B_m^k \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^n B_{n-\nu}^{k-1} d_{\nu+1} \right\} \\ (2.14) \quad &= \sum_{m=0}^n T_m \gamma_{n,m} , \end{aligned}$$

where

$$(2.15) \quad T_m = C_m^k / B_m^k ,$$

and

$$(2.16) \quad \gamma_{n,m} = (-1)^k \frac{B_m^k}{B_n^k} \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^n B_{n-\nu}^{k-1} d_{\nu+1} \right\} .$$

Hence

$$(2.17) \quad (i) \quad \sum_{m=0}^n |\gamma_{n,m}| = (1/B_n^k) \sum_{m=0}^n B_m^k \left| \Delta^k \left\{ \Delta(1/d_{m+k+1}) \sum_{\nu=m+k}^n B_{n-\nu}^{k-1} d_{\nu+1} \right\} \right| < H ,$$

by hypothesis (ii).

Now, from (2.16), we have, for each m

$$\begin{aligned} \gamma_{n,m} &= (-1)^k (B_m^k / B_n^k) \left[\Delta^{k+1}(1/d_{m+k+1}) \sum_{\nu=m+k}^n B_{n-\nu}^{k-1} d_{\nu+1} \right. \\ &\quad + \alpha_k \Delta^k(1/d_{m+k}) \binom{n-m}{k-1} d_{m+k} + \dots \\ (2.18) \quad &\quad + \alpha_1^1 \Delta(1/d_{m+1}) \binom{n-m}{k-1} \Delta^{k-1} d_{m+k} \\ &\quad \left. + \dots + \alpha_{k-1}^1 \Delta(1/d_{m+1}) \Delta^{k-1} \binom{n-m}{k-1} d_{m+1} \right] , \end{aligned}$$

(α various constants)

using the identity (1.9).

Then from (2.18) it follows that for each m

$$(2.19) \quad \gamma_{n,m} = A_{n,m} + O\left(\frac{1}{n}\right) = A_{n,m} + o(1), \text{ as } n \rightarrow \infty,$$

where

$$(2.20) \quad |A_{n,m}| < \frac{k!}{n^k} B_m^k |\Delta^{k+1}(1/d_{m+k+1})| \sum_{\nu=0}^n B_{n-\nu}^{k-1} d_{\nu+1} < \frac{K}{n^k} d_{n+1}^k = o(1)$$

for each m , as $n \rightarrow \infty$, by hypothesis (i).

Hence it follows from (2.19)–(2.20) that

$$(2.21) \quad (ii) \quad \gamma_{n,m} \rightarrow 0 \text{ for each } m, \text{ as } n \rightarrow \infty.$$

Let us take

$$a_0 = 1, a_\nu = 0, \text{ for } \nu \geq 1, \text{ and } d_0 = 1$$

in

$$(2.22) \quad C_n = \sum_{\nu=0}^n a_\nu d_\nu.$$

Then we have, for $n \geq 0$, $C_n = 1$, and hence

$$(2.23) \quad C_n^k / B_n^k = 1.$$

Next, since $C_\nu = 1$, $d_0 = 1$, we obtain from (2.12)

$$-G_n = d_{n+1} \sum_{\nu=0}^n \Delta(1/d_{\nu+1}) = 1 - d_{n+1},$$

and hence

$$(2.24) \quad -G_n^k / B_n^k = 1 - d_{n+1}^k / B_n^k \rightarrow 1 \text{ as } n \rightarrow \infty,$$

by hypothesis (i).

But this implies, from (2.14)–(2.15), that

$$(2.25) \quad (iii) \quad -G_n^k / B_n^k = \sum_{m=0}^n \gamma_{n,m} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

It follows that conditions (i), (ii) and (iii) of Lemma A are satisfied, and hence

$$(2.26) \quad -G_n^k / B_n^k \rightarrow S \text{ as } n \rightarrow \infty.$$

Note. Hypotheses (i) and (ii) of the Theorem are necessary. For suppose that $-G_n^k / B_n^k \rightarrow S$ as $n \rightarrow \infty$, whenever $C_n^k / B_n^k \rightarrow S$ as $n \rightarrow \infty$.

Then from (2.14)–(2.16), condition (i) of Lemma A must hold, but this implies (2.17) and hence hypothesis (ii) of Theorem 2.

Next, let us choose C_n so that (2.23) holds. Then (2.3) holds, with $S = 1$, and hence (2.24) holds. Hence it follows that $d_{n+1}^k / B_n^k =$

$o(1)$ as $n \rightarrow \infty$, and this implies hypothesis (i) of the theorem.

Further the summability (C, k) of (2.4) can be improved to the summability $(C, k - 1)$, by the following Lemma.

LEMMA B. *If d_n is monotonically decreasing and*

$$(2.27) \quad (i) \quad n^j \Delta^j d_{n+1} = O(d_{n+1}),$$

$$(2.28) \quad (ii) \quad n^j \Delta^j t_{n+1} = O(t_{n+1}),$$

for $j = 1, 2, \dots, k + 1$, where

$$(2.29) \quad t_n = 1/d_n,$$

then

$$(2.30) \quad (iii) \quad S_n d_{n+1} = o(1) (C, k) \Rightarrow n S_n \Delta d_{n+1} = o(1) (C, k).$$

Proof. We have

$$(2.31) \quad H_n = S_n d_{n+1} = o(1) (C, k).$$

We will prove that

$$(2.32) \quad H_n g_n = o(1) (C, k),$$

where

$$(2.33) \quad g_n = \frac{n \Delta d_{n+1}}{d_{n+1}} = n \Delta d_{n+1} t_{n+1}.$$

By a theorem of Bosanquet [6, Th. 1], which is an extension of another theorem of Bosanquet [4, Lemma 1], it will be enough to prove that

$$(2.34) \quad \Delta^j g_n = O(n^{-j}), \quad j = 0, 1, \dots, k - 1,$$

and

$$(2.35) \quad \sum_{\nu=1}^n \nu^k |\Delta^k g_\nu| = O(n).$$

Now

$$(2.36) \quad g_n = n \Delta d_{n+1} t_{n+1} \leq K d_{n+1} t_{n+1} \leq K,$$

by (2.44).

Next, using the identity (1.9), we have

$$(2.37) \quad \begin{aligned} |\Delta^{k-1} g_n| &= \alpha_1 n |\Delta d_{n+1}| |\Delta^{k-1} t_{n+1}| + \dots + \alpha_{k-1} n |\Delta^k d_{n+1}| t_{n+1-k+1} \\ &\quad + \alpha^1 |\Delta d_{n+1}| |\Delta^{k-2} t_{n+1}| + \dots + \alpha^{k-2} |\Delta^{k-1} d_{n+1}| t_{n-1-k+2} \\ &\leq K/n^{k-1}. \end{aligned} \quad (\alpha \text{ various constant})$$

The other conditions in (2.34) are easily obtained similarly, but it is well known that the inequalities for $j = 0$ and $k - 1$ imply those for $j = 1, 2, \dots, k - 2$: See Hardy and Littlewood [9].

Next we have

$$(2.38) \quad \begin{aligned} |\Delta^k g_n| &\leq \alpha_1 n |\Delta d_{n+1}| |\Delta^k t_{n+1}| + \dots + \alpha_k n |\Delta^{k+1} d_{n+1}| t_{n+1-k} \\ &\quad + \alpha^1 |\Delta d_{n+1}| |\Delta^{k-1} t_{n+1}| + \dots + \alpha^{k-1} |\Delta^k d_{n+1}| t_{n+1-k+1} \\ &\leq K/n^k. \end{aligned}$$

Hence

$$(2.39) \quad \sum_{\nu=1}^n \nu^k |\Delta^k g_\nu| \leq \sum_{\nu=1}^n \nu^k \frac{K}{\nu^k} < Kn,$$

and this completes the proof the lemma.

Next we will consider the case $k = 0$.

Now we have

$$nS_n \Delta d_{n+1} \leq Kd_{n+1} S_n$$

by (2.27), and since

$$(2.40) \quad S_n d_{n+1} = o(1) \text{ as } n \rightarrow \infty,$$

it follows that

$$(2.41) \quad nS_n \Delta d_{n+1} = o(1) \text{ as } n \rightarrow \infty.$$

Next since

$$(2.42) \quad \sum_{\nu=0}^{\infty} S_\nu \Delta d_{\nu+1}$$

is convergent, it follows from the definition that (2.42) is summable $(C, -1)$.

In conclusion I wish to acknowledge my debts of gratitude to Prof. L. S. Bosanquet for suggesting the problem to me and for his valuable guidance and comments throughout the course of my work. I also appreciate several comments made by Mr. M. C. Austin and wish to record my appreciation of several suggestions for improvement made by Prof. D. Borwein.

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