Pacific Journal of Mathematics

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Vol. 33, No. 2 April 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 2, 1970

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In this second paper, the bulk of the work is devoted to characterizing $E_2(3)$ and $S_4(3)$. These two groups are "almost" N-groups and it is relevant to treat them separately. The actual characterizations (Theorems 8.1 and 9.1) are very technical but the hypotheses deal with the structure and embedding in a simple group of certain $\{2, 3\}$-subgroups.

This paper is a continuation of an earlier paper.[1] The bibliographical references are to I.

7. Groups in which 1 is the only p-signalizer.

DEFINITION 7.1. $\mathscr{U}^*(p) = \{\mathfrak{B} \mid$ (i) $\mathfrak{B}$ is a subgroup of $\mathfrak{G}$ of type (p, p). (ii) $N(\mathfrak{B})$ contains a S_p-subgroup of $\mathfrak{G}.\}$.

HYPOTHESIS 7.1. (i) p is a prime and if $\mathfrak{B} \in \mathscr{U}^*(p)$, then no S_p-subgroup of $C(\mathfrak{B})$ normalizes any nonidentity p'-subgroup of $\mathfrak{G}$.

(ii) The centralizer of every nonidentity p-subgroup of $\mathfrak{G}$ is p-solvable.

Lemmas 7.1, 7.2, 7.3 are proved under Hypothesis 7.1.

LEMMA 7.1. (i) $\mathscr{U}(p) \subseteqq \mathscr{E}(p)$. (See Definitions 2.8 and 2.10 of I).

(ii) *If* $p \geqq 5$, *then* $\mathscr{U}^*(p) \subseteqq \mathscr{E}(p)$.

(iii) *If* $p = 3$ *and if no element of* $\mathscr{U}(3)$ *centralizes a quaternion subgroup of* $\mathfrak{G}$, *then* $\mathscr{U}^*(3) \subseteqq \mathscr{E}(3)$.

Proof. If p is odd, choose $\mathfrak{B} \in \mathscr{U}^*(p)$, while if $p = 2$, choose $\mathfrak{B} \in \mathscr{U}(2)$. We must show that either $\mathfrak{B}$ centralizes every element of $\text{И}(\mathfrak{B}; p')$ or $p = 3$, $\mathfrak{B} \in \mathscr{U}^*(3) - \mathscr{U}(3)$ and some element of $\mathscr{U}(3)$ centralizes a quaternion subgroup of $\mathfrak{G}$.

Let $\mathfrak{P}$ be a S_p-subgroup of $N(\mathfrak{B})$, so that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. Proceeding by way of contradiction, let $\mathfrak{Q}$ be an element of $\text{И}(\mathfrak{B}; p')$ minimal subject to $[\mathfrak{Q}, \mathfrak{B}] \neq 1$. Then $\mathfrak{Q}$ is a q-group for some prime $q \neq p$, $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{B}]$, and $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{Q})$ has order p. Let $\mathfrak{C} = C(\mathfrak{B}_0)$, $\mathfrak{C}_1 = C_{\mathfrak{P}}(\mathfrak{B}_0)$, and let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{C}$ containing $\mathfrak{C}_1$. Hypothesis 7.1 implies that $O_{p'}(\mathfrak{C}) = 1$. Let $\mathfrak{P}_0 = O_p(\mathfrak{C})$. If $[\mathfrak{P}_0, \mathfrak{B}] \subseteqq \mathfrak{B}$, then

[1] *Non-solvable finite groups all of whose local subgroups are solvable*, I, Bull. Amer. Math. Soc. **74** (1968), 383-437, which will be referred to as I.

Lemma 5.16 is violated. Hence, we have $|\mathfrak{P}^*: \mathfrak{C}_1| = |\mathfrak{P}_0: \mathfrak{P}_0 \cap \mathfrak{C}_1| = p$ and $[\mathfrak{P}_0, \mathfrak{B}] \not\subseteq \mathfrak{B}$.

Suppose $\mathfrak{B} \subseteq \mathfrak{P}_0$. Then $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{B}] \subseteq \mathfrak{P}_0$, so $\mathfrak{Q} = 1$. This is not the case, so $\mathfrak{B} \not\subseteq \mathfrak{P}_0$. By Lemma 6.1, it follows that $\mathfrak{B} \notin \mathscr{U}(p)$. Hence, by construction, p is odd. Since $\mathfrak{B} \not\subseteq \mathfrak{A}_0$, $(\mathfrak{B})$ implies that $p \leqq 3$. Thus, $p = 3$ and $\mathfrak{B} \in \mathscr{U}^*(3) - \mathscr{U}(3)$. By definition of $\mathscr{U}(3)$ and $\mathscr{U}^*(3)$, it follows that $\boldsymbol{Z}(\mathfrak{P})$ is non cyclic and $\mathfrak{B}$ is not contained in the center of any S_3-subgroup of $\mathfrak{G}$.

Since $[\mathfrak{P}_0, \mathfrak{B}, \mathfrak{B}] = 1$ and since $\mathfrak{B} \not\subseteq \mathfrak{P}_0$, it follows that $\mathfrak{B} \not\subseteq \boldsymbol{O}_3(\mathfrak{C}_{2,3})$, where $\mathfrak{C}_{2,3}$ is a $S_{2,3}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}^*$.

Since $\mathfrak{B}_0 \not\subseteq \boldsymbol{Z}(\mathfrak{P})$, we have $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$, where $\mathfrak{B}_1 \subseteq \boldsymbol{Z}(\mathfrak{P})$. Since $\mathfrak{B}_0 \subseteq \boldsymbol{Z}(\mathfrak{C})$, we have $\mathfrak{B}_0 \subseteq \boldsymbol{O}_3(\mathfrak{C}_{2,3})$, and so $\mathfrak{B}_1 \not\subseteq \boldsymbol{O}_3(\mathfrak{C}_{2,3})$.

Let $\mathfrak{H}$ be a subgroup of $\mathfrak{C}_{2,3}$ such that

(a) $\mathfrak{P}^* \subseteq \mathfrak{H}$.

(b) $\mathfrak{B}_1 \not\subseteq \boldsymbol{O}_3(\mathfrak{H})$.

(c) $\mathfrak{H}$ is minimal subject to (a) and (b).

Let $\mathfrak{H}_1 = \boldsymbol{O}_3(\mathfrak{H})$. Since the fixed point subspace of $\mathfrak{B}_1$ on $\mathfrak{H}_1/\boldsymbol{D}(\mathfrak{H}_1)$ is of codimension 1, Lemma 5.30 implies that $\mathfrak{H} = \mathfrak{P}^*\mathfrak{Q}^*$, where $\mathfrak{Q}^*$ is a quaternion group and $|\mathfrak{P}^*: \mathfrak{H}_1| = 3$, so that $\mathfrak{H}_1\mathfrak{B}_1 = \mathfrak{P}^*$. Since $\mathfrak{B}_1$ centralizes $\boldsymbol{D}(\mathfrak{H}_1)$, so does $\mathfrak{Q}^*$. Let $\boldsymbol{C}_{\mathfrak{H}_1}(\mathfrak{Q}^*) = \mathfrak{P}_1^*$. Thus, $\mathfrak{P}_1$ is a normal subgroup of $\mathfrak{H}$ and $|\mathfrak{H}_1: \mathfrak{P}_1^*| = 9$.

Let $\mathfrak{P}_2^* = [\mathfrak{H}_1, \mathfrak{Q}^*]$. Then $\mathfrak{P}_2^*$ is generated by 2 elements and $\mathfrak{P}_2^* \cap \mathfrak{P}_1^*$ is of index 9 in $\mathfrak{P}_2^*$. Hence, $\mathfrak{P}_2^*$ is either elementary of order 9 or is a nonabelian group of exponent 3 and order 27. Furthermore, $\mathfrak{H}_1 = \mathfrak{P}_1^*\mathfrak{P}_2^*$, $\mathfrak{P}_1^* \cap \mathfrak{P}_2^* = \boldsymbol{D}(\mathfrak{P}_2^*)$, and $[\mathfrak{P}_1^*, \mathfrak{P}_2^*] = 1$.

Since $\mathfrak{B}_1 \not\subseteq \mathfrak{H}_1$, it follows that $\mathfrak{C}_1 = \mathfrak{B}_1 \times (\mathfrak{C}_1 \cap \mathfrak{H}_1)$. Hence, $\boldsymbol{D}(\mathfrak{C}_1) = \boldsymbol{D}(\mathfrak{C}_1 \cap \mathfrak{H}_1) \subseteq \mathfrak{P}_1^*$. We will show that $\boldsymbol{D}(\mathfrak{C}_1) = 1$. Suppose false. Let $\mathfrak{C}^* = \boldsymbol{C}(\boldsymbol{D}(\mathfrak{C}_1))$, so that $\mathfrak{C}^*$ is 3-solvable. Since $\mathfrak{C}^* \lhd \boldsymbol{N}(\boldsymbol{D}(\mathfrak{C}_1))$, it follows that $\mathfrak{C}^*\mathfrak{P}$ is 3-solvable. Since $\mathfrak{B}_1 \subseteq \boldsymbol{Z}(\mathfrak{P})$, we have $\mathfrak{B}_1 \subseteq \boldsymbol{O}_3(\mathfrak{C}^*\mathfrak{P})$. Since $\mathfrak{Q}^*$ centralizes $\boldsymbol{D}(\mathfrak{H}_1)$, it follows that $\langle \mathfrak{B}_1, \mathfrak{Q}^* \rangle \subseteq \mathfrak{C}^*$. Thus, $\langle \mathfrak{B}_1, \mathfrak{Q}^* \rangle$ is 3-closed. This is impossible, since $\langle \mathfrak{B}_1, \mathfrak{Q}^* \rangle$ covers $\mathfrak{H}/\mathfrak{H}_1$.

If $\mathfrak{P}_2^*$ is nonabelian, then $\mathfrak{Q}^*$ centralizes $\boldsymbol{Z}(\mathfrak{P}^*)$. Since $\mathfrak{Q}^*$ is a quaternion group, we are done in this case.

We may now assume that $\mathfrak{P}_2^*$ is abelian, so elementary of order 9. Thus, $\mathfrak{H}_1 = \mathfrak{P}_1^* \times \mathfrak{P}_2^*$, $\mathfrak{P}_1^*$ and $\mathfrak{C}_1$ are elementary and $\boldsymbol{Z}(\mathfrak{P}^*) = \mathfrak{P}_1^* \times \mathfrak{V}$, where $\mathfrak{V} = \mathfrak{P}_2^* \cap \boldsymbol{Z}(\mathfrak{P}^*)$. Notice that $\mathfrak{B}_0 \subseteq \mathfrak{P}_1^*$. If $\mathfrak{P}_1^* \supset \mathfrak{B}_0$, then since every subgroup of $\mathfrak{P}_1^*$ of type (3,3) is in $\mathscr{U}(3)$ and since the quaternion group $\mathfrak{Q}^*$ centralizes $\mathfrak{P}_1^*$, we are done. We may therefore assume that $\mathfrak{P}_1^* = \mathfrak{B}_0$. Hence, $\boldsymbol{Z}(\mathfrak{P})$ has order 9, $|\mathfrak{P}^*| = 3^4$, $|\mathfrak{C}_1| = 3^3$. Also, $\boldsymbol{Z}(\mathfrak{P}^*) = \mathfrak{B}_0 \times \mathfrak{V}$. Let B be a generator for $\mathfrak{B}_0$ and let I be the involution of $\mathfrak{Q}^*$. Then I inverts $\mathfrak{V}$ and centralizes B.

Let $\mathfrak{N} = \langle \mathfrak{P}, \mathfrak{P}^* \rangle \subseteq \boldsymbol{N}(\mathfrak{C}_1)$. Since $SL(3, 3)$ is a minimal simple group, it follows that $\boldsymbol{N}(\mathfrak{C}_1)$ is solvable. As $\boldsymbol{O}_{3'}(\mathfrak{N}) = 1$, we have $\mathfrak{C}_1 = \boldsymbol{C}(\mathfrak{C}_1)$. Since $[\mathfrak{C}_1, \mathfrak{P}^*, \mathfrak{P}^*] = 1$, it follows that $\mathfrak{N}$ contains a normal subgroup

$\mathfrak{B}^*$ of order 3 and that S_2-subgroups of $\mathfrak{N}$ are quaternion. Now $\mathfrak{B}^* \not\subseteq \mathfrak{B}$, since $\mathfrak{P}$ does not normalize $\mathfrak{B}_0$ and $\mathfrak{P}^*$ centralizes no subgroup of $\mathfrak{B}$ other than 1 and $\mathfrak{B}_0$. Suppose $\mathfrak{B}^* = \mathfrak{B}$. Since $\mathfrak{B} = \mathfrak{P}^{*\prime}$, it follows that S_3-subgroups of $\mathfrak{N}/\mathfrak{B}^*$ are abelian. Thus $\mathfrak{N}$ is 3-closed. But this is impossible since $\mathfrak{P} \neq \mathfrak{P}^*$. Hence, $\mathfrak{B}^*$ is a subgroup of $\boldsymbol{Z}(\mathfrak{P}^*)$ of order 3 which is different from $\mathfrak{B}$ and from $\mathfrak{B}_0$. Since $\boldsymbol{Z}(\mathfrak{P}^*) = \mathfrak{B}_0 \times \mathfrak{B}$, there is a generator for $\mathfrak{B}^*$ of the shape BV, where V is a generator for $\mathfrak{B}$.

Let J be any involution of $\mathfrak{N}$. Then $JVJ = V^{-1}$ and $JBVJ = BV$. Now I and J both normalize $\mathfrak{P}^*$ and $\mathfrak{P}^{*\prime} = \mathfrak{B}$. Hence, $\langle I, J\rangle$ maps onto an abelian subgroup of $\boldsymbol{A}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{P}^*))$, which implies that J normalizes $\boldsymbol{Z}(\mathfrak{P}^*) \cap \boldsymbol{C}(I) = \mathfrak{B}_0$. Hence, $JBJ = B^f$ for some integer f, and the previous equations yield $V^2 = 1$, which is not the case. The proof is complete.

HYPOTHESIS 7.2. (i) If $\mathfrak{A} \in \mathscr{S}_{\mathscr{C}\mathscr{N}_3}(p)$, then $\boldsymbol{V}\!\!\boldsymbol{I}(\mathfrak{A})$ contains only 1.
(ii) If $\mathfrak{Z}$ is of order p and is in the center of some S_p-subgroup of $\mathfrak{G}$, then $\boldsymbol{O}_p(\mathfrak{M})$ is of symplectic type and width w, where $\mathfrak{M} = \boldsymbol{N}(\mathfrak{Z})$.

LEMMA 7.2. *Suppose Hypothesis 7.2 is satisfied and that if* $p = 2$, *then* $w \geqq 3$, *while if* $p = 3$, *then* $w \geqq 2$. *Let* $\mathfrak{B}$ *be a subgroup of* $\boldsymbol{O}_p(\mathfrak{M})$ *of type* (p, p) *which contains* $\mathfrak{Z}$. *Then* $\mathfrak{B} \in \mathscr{E}(p)$.

Proof. Let $\mathscr{V}$ be the set of subgroups of $\boldsymbol{O}_p(\mathfrak{M})$ which violate the lemma. Let $\mathscr{V}_0$ be the subset of those $\mathfrak{B}$ in $\mathscr{V}$ which centralize at least one element $\mathfrak{B}$ of $\mathscr{U}(p)$ which $\mathfrak{Z} \subset \mathfrak{B} \subset \boldsymbol{O}_p(\mathfrak{M})$. If $\mathscr{V}_0 \neq \emptyset$, choose $\mathfrak{B} \in \mathscr{V}_0$, while if $\mathscr{V}_0 = \emptyset$, choose $\mathfrak{B}$ in $\mathscr{V}$.

Let $\mathfrak{H} = \boldsymbol{O}_p(\mathfrak{M})$, $\mathfrak{H}_0 = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B})$. We first argue that $\boldsymbol{V}\!\!\boldsymbol{I}(\mathfrak{H}_0; p')$ is trivial. Namely, $\mathfrak{M}$ is p-solvable with $\boldsymbol{O}_{p'}(\mathfrak{M}) = 1$, so $\boldsymbol{C}_{\mathfrak{M}}(\mathfrak{H}) = \mathfrak{Z}(\mathfrak{H})$. This implies that $\boldsymbol{C}_{\mathfrak{M}}(\mathfrak{H}_0)$ is a p-group. Hence, $\boldsymbol{V}\!\!\boldsymbol{I}_{\mathfrak{M}}(\mathfrak{H}_0; p')$ is trivial. Suppose $\mathfrak{R} \in \boldsymbol{V}\!\!\boldsymbol{I}(\mathfrak{H}_0; p')$. It suffices to show that $\mathfrak{R} \subseteq \mathfrak{M}$. If $\mathfrak{H}_0$ contains an element $\mathfrak{B}$ of $\mathscr{U}(p)$ with $\mathfrak{Z} \subset \mathfrak{B}$, then by Lemma 7.1 we get that $\mathfrak{B}$ centralizes $\mathfrak{R}$. Hence, $\mathfrak{R} \subseteq \boldsymbol{C}(\mathfrak{Z}) \subseteq \mathfrak{M}$. If no such elements of $\mathscr{U}(p)$ are available, then by construction, $\mathscr{V}_0 = \emptyset$. But $\mathfrak{H} \lhd \mathfrak{M}$, so if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}$, then $\mathfrak{H}$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{P})$. Let $\mathfrak{H}_1 = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B})$ so that $|\mathfrak{H} : \mathfrak{H}_1| = p$. If $\mathfrak{H}_0 \cap \mathfrak{H}_1$ contains more than one subgroup of order p, then there is a subgroup $\mathfrak{B}^*$ of $\mathfrak{H}_0 \cap \mathfrak{H}_1$ of type (p, p) which contains $\mathfrak{Z}$. Since $\mathscr{V}_0 = \emptyset$, $\mathscr{V}^* \in \mathscr{E}(p)$, so $\mathfrak{R} \subseteq \boldsymbol{C}(\mathfrak{B}^*) \subseteq \boldsymbol{C}(\mathfrak{Z}) \subseteq \mathfrak{M}$. Suppose $\mathfrak{H}_0 \cap \mathfrak{H}_1$ contains only one subgroup of order p. Then by hypothesis, we have $p \geqq 5$, and so $\mathfrak{H}$ is of width 1 and is a S_p-subgroup of $\mathfrak{G}$. Hypothesis 7.1 guarantees in this case that $\boldsymbol{V}\!\!\boldsymbol{I}(\mathfrak{H}_0; p')$ is trivial, so $\mathfrak{R} = 1$. We have thus shown that $\boldsymbol{V}\!\!\boldsymbol{I}(\mathfrak{H}_0; p')$ is trivial.

Choose $\mathfrak{Q}$ in $\boldsymbol{V}\!\!\boldsymbol{I}(\mathfrak{B}; p')$ minimal subject to $[\mathfrak{B}, \mathfrak{Q}] \neq 1$. Then $\mathfrak{Q} = [\mathfrak{B}, \mathfrak{Q}]$ and $\mathfrak{B}_0 = \boldsymbol{C}_{\mathfrak{B}}(\mathfrak{Q})$ is of order p. Clearly, $\mathfrak{B}_0 \neq \mathfrak{Z}$. Let $\mathfrak{M}_1 = \boldsymbol{N}(\mathfrak{B}_0)$

so that $\mathfrak{M}_1$ is p-solvable. By the preceding argument, $\boldsymbol{O}_{p'}(\mathfrak{M}_1) = 1$. Hence, $\mathfrak{Z} \not\subseteq \boldsymbol{O}_p(\mathfrak{M}_1)$, so that $\mathfrak{H}_0$ contains an extra special subgroup $\mathfrak{H}^*$ of width $w-1$ with $\mathfrak{F}^* \cap \boldsymbol{O}_p(\mathfrak{M}_1) = 1$.

Let $\mathfrak{X} = \boldsymbol{R}_p(\mathfrak{M}_1)$ (see Definition 2.2), $\mathfrak{Y} = \boldsymbol{C}_{\mathfrak{M}_1}(\mathfrak{X})$. Suppose $\mathfrak{Z} \subseteq \mathfrak{Y}$. Since $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{Z}]$ and $\mathfrak{Y} \triangleleft \mathfrak{M}_1$, we have $\mathfrak{Q} \subseteq \mathfrak{Y}$. Let $\mathfrak{X}_p$ be a S_p-subgroup of $\mathfrak{M}_1$ which contains $\mathfrak{H}^*$. Then $\mathfrak{H}^*$ centralizes $\boldsymbol{Z}(\mathfrak{X}_p)$, so $\mathfrak{Y}\mathfrak{H}^*$ centralizes $\boldsymbol{Z}(\mathfrak{X}_p)$. Let $\tilde{\mathfrak{P}}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{X}_p$. Then $\boldsymbol{Z}(\tilde{\mathfrak{P}}) \subseteq \boldsymbol{Z}(\mathfrak{X}_p)$, so $\mathfrak{H}^*$ is contained in a conjugate $\widetilde{\mathfrak{M}}$ of $\mathfrak{M}$, $\widetilde{\mathfrak{M}} = \boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\tilde{\mathfrak{P}})))$. Furthermore, since $\mathfrak{Z}\mathfrak{Q} \subseteq \mathfrak{Y} \subseteq \widetilde{\mathfrak{M}}$, $\mathfrak{H}^*$ is faithfully represented on $\mathfrak{Q}_p^1(\widetilde{\mathfrak{M}})$. Let $\tilde{\mathfrak{H}} = \boldsymbol{O}_p(\widetilde{\mathfrak{M}})$, and let $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_0 \supset \tilde{\mathfrak{H}}_1 \supset \cdots \supset \tilde{\mathfrak{H}}_k = 1$ be part of a chief series for $\widetilde{\mathfrak{M}}$. Then since $\mathfrak{Z} \not\subseteq \boldsymbol{O}_p(\widetilde{\mathfrak{M}})$, it follows that $\mathfrak{Z}$ does not centralize $\tilde{\mathfrak{H}}_n/\tilde{\mathfrak{H}}_{n+1}$ for at least one value of n, $0 \leqq n < k$. Hence, $|\tilde{\mathfrak{H}}_n : \tilde{\mathfrak{H}}_{n+1}| = p^{a_n}$ where $a_n \geqq r_p(\mathfrak{Z}; \widetilde{\mathfrak{M}})$. Then by Lemma 5.4, $r_p(\mathfrak{Z}; \widetilde{\mathfrak{M}}) \geqq r_C(\mathfrak{Z}; \tilde{\mathfrak{P}})$. Clearly,

$$r_C(\mathfrak{Z}; \tilde{\mathfrak{P}}) \geqq r_C(\mathfrak{Z}; \mathfrak{H}^*) = p^{w-1} .$$

On the other hand, $\tilde{\mathfrak{H}}_n$ is a subgroup of $\tilde{\mathfrak{H}}$, so $2w \geqq a_n$. Hence $2w \geqq p^{w-1}$. If $p \geqq 5$, then $w = 1$ is forced, so every p-solvable subgroup of $\mathfrak{G}$ has p-length at most 1. This is absurd, so $p \leqq 3$. If $p = 3$, then $w = 2$, since $w \geqq 2$ by hypothesis. It is clear that this is impossible since $\mathfrak{H}^*$ is faithfully represented on $\boldsymbol{Q}_3^1(\widetilde{\mathfrak{M}})$. If $p = 2$, then $w = 3$ or $w = 4$, since by hypothesis $w \geqq 3$. This is also impossible by Lemma 5.13. We have shown that $\mathfrak{Z} \not\subseteq \mathfrak{Y}$.

Since $\mathfrak{X}$ is a p-group, $\mathfrak{X} \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Then $\mathfrak{X} \cap \mathfrak{H}^G$ is an abelian subgroup of $\mathfrak{H}^G$, so $m(\mathfrak{X} \cap \mathfrak{H}^G) \leqq w + 1 + e$, where $e = 0$ if p is odd and $e = 1$ if $p = 2$.

If p is odd, then $\mathfrak{X}/\mathfrak{X} \cap \mathfrak{H}^G$ is faithfully represented on the Frattini quotient group of $\Omega_1(\mathfrak{H}^G)$, and this latter group is generated by $2w$ elements. If $p = 2$, write $\boldsymbol{O}_{2,2'}(\mathfrak{M}^G) = \mathfrak{H}^G \cdot \mathfrak{N}$ where $|\mathfrak{N}|$ is odd. Then $[\mathfrak{N}, \mathfrak{G}^G]$ is generated by $2w$ elements and $\mathfrak{X}/\mathfrak{X} \cap \mathfrak{H}^G$ is faithfully represented on the Frattini quotient group of $[\mathfrak{N}, \mathfrak{H}^G]$. Thus, by a result of Schur [32], we have $m(\mathfrak{X}/\mathfrak{X} \cap \mathfrak{M}^G) \leqq w^2$, that is,

$$m(\mathfrak{X}) \leqq w^2 + w + 1 + e . \tag{7.1}$$

If $w = 1$, then $p \geqq 5$ implies that $\boldsymbol{O}_p(\mathfrak{M})$ is a S_p-subgroup of $\mathfrak{G}$. So every p-solvable subgroup of $\mathfrak{G}$ has p-length at most 1, a contradiction. Hence, $w \geqq 2$.

There is an elementary subgroup $\mathfrak{X}_1$ of $\mathfrak{X}$ such that $\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{X}_1)$ contains a subgroup $\tilde{\mathfrak{H}}^*\tilde{\mathfrak{Q}}$, where $\tilde{\mathfrak{Q}} \triangleleft \tilde{\mathfrak{H}}^*\tilde{\mathfrak{Q}}$ is special and $\tilde{\mathfrak{H}}^* \cong \mathfrak{H}^*$ operates faithfully and irreducibly on $\tilde{\mathfrak{Q}}/\boldsymbol{D}(\tilde{\mathfrak{Q}})$. Also, $\tilde{\mathfrak{H}}^*\tilde{\mathfrak{Q}}$ acts irreducibly on $\mathfrak{X}_1$.

Assume that p is odd.

Since $\tilde{\mathfrak{H}}^*$ is extra special of width $w-1$, it follows that $m(\tilde{\mathfrak{Q}}) \geqq p^{w-1}a$, where $a = |\boldsymbol{F}_q(\zeta) : \boldsymbol{F}_q|$. Here, $\tilde{\mathfrak{Q}}$ is a q-group and ζ is a primitive

p^{th} root of 1 in an extension field of the prime field F_q. By Lemma 5.3, $m(\mathfrak{X}) \geqq m(\tilde{\mathfrak{Q}})b$, where $b = 2/3$ if $q = 2$ and $b = |F_p(\tau): F_p|$ if q is odd. Here τ is a primitive q^{th} root of 1 in an extension field of the prime field F_p. Together with (7.1), we get $abp^{w-1} \leqq w^2 + w + 1$. Clearly, $ab > 1$. Suppose $w \geqq 4$. Then $3^{w-1} \leqq p^{w-1} < w^2 + w + 1$, a contradiction. Suppose $w = 3$. If $p \geqq 5$, then $5^2 = 5^{w-1} \leqq p^{w-1} < 3^2 + 3 + 1$, a contradiction. Thus, $p = 3$ and $q = 2$. Since $p = 3, w = 3$, it follows that $\mathfrak{M}^G/\mathfrak{H}^G$ is isomorphic to a 3-solvable subgroup of the 6 by 6 symplectic group over F_3. It follows readily that $\mathfrak{M}^G/\mathfrak{H}^G$ has no elementary subgroup of order 3^4. Thus, in this case, $m(\mathfrak{X}) \leqq 4 + m(\mathfrak{X} \cap \mathfrak{H}^G) \leqq 8$, against $(4/3)\cdot 3^2 = 12 = abp^{w-1} \leqq m(\mathfrak{X})$. Hence $w = 2$. We now get $abp^{w-1} < 7$, so $p \leqq 5$. Suppose $p = 5$ and q is odd. Then $ab \geqq 2$, so that $10 < 7$. Suppose $p = 5$ and $q = 2$. Then $a = 4$, so $ab = 8/3$. We get $(8/3)\cdot 5 < 7$. Hence, $p = 3$. Suppose q is odd. Since a is characterized as the smallest positive integer n with $3^n \equiv 1 \pmod{q}$, it follows that $a \geqq 3$, so $ab \geqq 3$. This gives $9 \leqq abp^{w-1} < 7$. Hence, $q = 2$. Since $p = 3, w = 2$, it follows that $\mathfrak{M}^G/\mathfrak{H}^G$ has no elementary subgroup of order 27. Hence, $m(\mathfrak{X}) \leqq 3 + 2 = 5$. In particular, $m(\mathfrak{X}_1) \leqq 5$. Suppose first that $\tilde{\mathfrak{Q}}$ is abelian. Since $\boldsymbol{Z}(\tilde{\mathfrak{H}}^*) = \tilde{\mathfrak{Z}}$ acts without fixed points on $\tilde{\mathfrak{Q}}$, it follows that $\boldsymbol{C}_{\tilde{\mathfrak{H}}^*}(\lambda) \cap \tilde{\mathfrak{Z}} = 1$ for every non trivial character λ of $\tilde{\mathfrak{Q}}$. So $|\tilde{\mathfrak{H}}^*: \boldsymbol{C}_{\tilde{\mathfrak{H}}^*}(\lambda)| \geqq 9$ for all $\lambda \neq 1$. Hence, $m(\mathfrak{X}_1) \geqq 9$, a contradiction. Suppose $\tilde{\mathfrak{Q}}$ is nonabelian. Let $\mathfrak{X}_2$ be a subgroup of $\mathfrak{X}_1$ on which $\tilde{\mathfrak{Q}}$ acts irreducibly. Thus $m(\mathfrak{X}_2) \geqq 2$, since $\tilde{\mathfrak{Q}}'$ does not centralize $\mathfrak{X}_2$. Since $m(\mathfrak{X}_1) \leqq 5$, and $p = 3$, it follows that $\mathfrak{X}_2 = \mathfrak{X}_1$ is an irreducible $\tilde{\mathfrak{Q}}$-group. Thus, $\tilde{\mathfrak{Q}}$ is extra special. But $m(\tilde{\mathfrak{Q}}) = 6$, since $q = 2$ and $|\tilde{\mathfrak{H}}^*| = 27$. This yields $m(\mathfrak{X}_1) \geqq 2^3$. All possibilities have led to contradictions. So $p = 2$.

Since $\tilde{\mathfrak{H}}^*$ is extra special of width $w - 1$, we get that $m(\tilde{\mathfrak{Q}}) \geqq 2^{w-1}$. Now Lemma 5.3(a) applied with $\tilde{\mathfrak{Q}}$ in the role of $\mathfrak{P}$, $\mathfrak{X}_1$ in the role of V, yields $m(\mathfrak{X}_1) \geqq 2^w$. On the other hand, $\mathfrak{V}_0$ is a normal subgroup of $\mathfrak{M}_1$ of order 2, so $m(\mathfrak{X}) \geqq 1 + 2^w$.

Let $\mathfrak{E}$ be an elementary subgroup of $\mathfrak{X}$ with $m(\mathfrak{E}) = 2^w + 1$, let $\mathfrak{E}_0 = \mathfrak{E} \cap \mathfrak{H}^G$, and let $\mathfrak{E}_1$ be a complement to $\mathfrak{E}_0$ in $\mathfrak{E}$. Since $m(\boldsymbol{E}_0) \leqq w + 2$, we get $m(\mathfrak{E}_1) = a \geqq 2^w - 1 - w$. Since $\mathfrak{E}_1$ acts faithfully on $\boldsymbol{O}_{2,2'}(\mathfrak{M}^G)/\mathfrak{H}^G$, Lemma 5.34 implies that $\boldsymbol{O}_{2,2'}(\mathfrak{M}^G)/\mathfrak{H}^G$ has a subgroup $\hat{\mathfrak{Q}}/\mathfrak{H}^G$ which admits $\mathfrak{E}_1$ and such that $\mathfrak{E}_1\hat{\mathfrak{Q}}/\mathfrak{H}^G$ is the direct product of a dihedral groups of order twice an odd prime. Let $\mathfrak{R}$ be a $S_{2'}$-subgroup of $\hat{\mathfrak{Q}}$. By Lemma 5.12, $[\mathfrak{H}^G, \mathfrak{R}] = \mathfrak{K}$ is extra special of width $w_1 \leqq w$. Since $\mathfrak{H}^G$ is the central product of $\mathfrak{K}$ and $\boldsymbol{C}_{\mathfrak{H}^G}(\mathfrak{R})$, and since $\mathfrak{E}_1\hat{\mathfrak{Q}} = \mathfrak{K}\cdot \boldsymbol{N}_{\mathfrak{E}_1\hat{\mathfrak{Q}}}(\mathfrak{R})$, it follows that if $M = \mathfrak{K}/\boldsymbol{D}(\mathfrak{K})$, then $\boldsymbol{A}_{\mathfrak{E}_1\hat{\mathfrak{Q}}}(M)$ has a subgroup which is the direct product of a dihedral groups of order twice an odd prime. Let $m(M) = m$. Then $m = 2w_1 \leqq 2w$. Since $w \geqq 3$ by hypothesis, we get $w < 2^w - 1 - w \leqq a$, and so $2w < 2a$, whence $m < 2a$.

This violates Lemma 5.8, and completes the proof.

HYPOTHESIS 7.3. (i) p is odd.

(ii) $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{A}$ is a normal elementary subgroup of $\mathfrak{P}$ with $m(\mathfrak{A}) \geqq 3$, $\boldsymbol{Z}(\mathfrak{P})$ is cyclic, and $A_{\mathfrak{G}}(\mathscr{C}) = A(\mathscr{C})$, where $\mathscr{C}$: $\mathfrak{A} \supset \mathfrak{B} \cap \boldsymbol{Z}(\mathfrak{P}) \supset 1$. Also, $\mathfrak{A} \lhd N(\boldsymbol{Z}(\mathfrak{P}) \cap \mathfrak{A})$.

LEMMA 7.3. *Suppose Hypothesis* 7.3 *is satisfied. Let* $\mathfrak{Z} = \boldsymbol{Z}(\mathfrak{P}) \cap \mathfrak{A}$. *Then each subgroup of* $\mathfrak{A}$ *of type* (p, p) *which contains* $\mathfrak{Z}$ *is in* $\mathscr{E}(p)$.

Proof. The lemma is an immediate consequence of Lemma 5.5, together with Hypothesis 7.1 (i).

HYPOTHESIS 7.4. (i) $\mathfrak{G}$ is simple.

(ii) $\{2, 3\} \subseteqq \pi_4(\mathfrak{G})$.

(iii) The centralizer of every involution of $\mathfrak{G}$ is solvable.

(iv) The normalizer of every nonidentity 3-sugroup of $\mathfrak{G}$ is solvable.

(v) If $\mathfrak{A} \in \mathscr{S}_{cn3}(2) \cup \mathscr{S}_{cn3}(3)$, then $\mathsf{M}(\mathfrak{A})$ contains only 1.

All remaining lemmas in this section are proved under Hypothesis 7.4.

DEFINITION 7.2.

$$\begin{aligned} \mathscr{N} = \{(\mathfrak{A}, \mathfrak{B}) \mid & 1.\ \ \mathfrak{A} \text{ is a 2-subgroup of } \mathfrak{G}. \\ & 2.\ \ \mathfrak{B} \text{ is a 3-subgroup of } \mathfrak{G}. \\ & 3.\ \ \langle \mathfrak{A}, \mathfrak{B} \rangle \text{ is not solvable.}\} \end{aligned}$$

We remark that in the following lemmas, Lemma 7.1 may be invoked, since Hypothesis 7.4 implies that Hypothesis 7.1 is satisfied for $p = 2$ and for $p = 3$.

LEMMA 7.4. *If* $\mathfrak{A}$ *is a four-subgroup of* $\mathfrak{G}$ *which centralizes every element of* $\mathsf{M}(\mathfrak{A}; 3)$ *and* $\mathfrak{B}$ *is a subgroup of* $\mathfrak{G}$ *of type* $(3, 3)$ *which centralizes every element of* $\mathsf{M}(\mathfrak{B}; 2)$, *then* $(\mathfrak{A}, \mathfrak{B}) \in \mathscr{N}$.

Proof. Notice that if $G, H \in \mathfrak{G}$, then the pair $(\mathfrak{A}^G, \mathfrak{B}^H)$ satisfies the hypothesis of the lemma.

Suppose the lemma is false and $\mathfrak{A}, \mathfrak{B}$ are chosen so that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is minimal. It follows as in Lemma 0.10.2 that $\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathfrak{A} \times \mathfrak{B}$. We may then choose A in $\mathfrak{A}^{\#}$ such that $\boldsymbol{E}(A)$ contains an element $\mathfrak{A}_1$ of $\mathscr{U}(2)$. Hence, $\langle \mathfrak{A}_1, \mathfrak{B} \rangle$ is solvable. Thus, we may assume that $\mathfrak{A} \in \mathscr{U}(2)$. Let $\mathfrak{N} = N(\mathfrak{A})$. Since $2 \in \pi_4$, we have $\boldsymbol{O}_{2'}(\mathfrak{N}) = 1$. This is absurd since

$\mathfrak{B}$ centralizes $O_2(\mathfrak{N})$ and $\mathfrak{N}$ is solvable. The proof is complete.

We set

$$\begin{aligned}\mathfrak{G}_1 &= \{G \mid G \in \mathfrak{G},\ C(G) \text{ is solvable.}\}\\ \mathfrak{G}_p &= \{G \mid G \in \mathfrak{G},\ C(G) \text{ contains an elementary subgroup } \mathfrak{E} \text{ of order } p^2 \text{ which centralizes every element of } \mathsf{N}(\mathfrak{E}; q)\}\ ,\\ &\quad p = 2, 3,\ q = 2, 3,\ p \neq q\ .\end{aligned}$$

We conclude from Lemma 7.4 that

$$\mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3 = \emptyset\ . \tag{7.2}$$

There are some subtle consequence of (7.2).

DEFINITION 7.3.

$\mathscr{D} = \{\mathfrak{B} \mid$ 1. $\mathfrak{B}$ is a noncyclic elementary 3-subgroup of $\mathfrak{G}$.
2. Every element of $\mathfrak{B}$ centralizes an element of $\mathscr{U}(3)$.
3. $\mathfrak{B}$ centralizes every abelian subgroup in $\mathsf{N}(\mathfrak{B}; 2)$.$\}$

LEMMA 7.5. *Suppose* $\mathfrak{A} \in \mathscr{U}(2)$, $\mathfrak{B} \in \mathscr{D}$ *and* $\mathfrak{T}$ *is a* 2, 3*-subgroup of* $\mathfrak{G}$ *which contains* $\langle \mathfrak{A}, \mathfrak{B}\rangle$. *Let* $\mathfrak{T}_2$ *be a* S_2*-subgroup of* $\mathfrak{T}$. *Then* $\mathscr{M}\mathscr{S}(\mathfrak{G})$ *(see Definition* 2.7) *contains an element* $\mathfrak{M}$ *such that*

(a) $O_{2'}(\mathfrak{M}) = 1$.

(b) $O_2(\mathfrak{M})$ *is the central product of* $[O_2(\mathfrak{M}), \mathfrak{B}]$, *which is extra special of width* $w = 2, 3$ *or* 4, *and of* $C_{O_2(\mathfrak{M})}(\mathfrak{B})$, *which is either cyclic or of maximal class* $\geqq 3$.

(c) $[O_2(\mathfrak{M}), \mathfrak{B}]$ *is the central product of* w $\mathfrak{B}$*-invariant quaternion groups* $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_w$ *whose centralizers in* $\mathfrak{B}$ *are* w *distinct subgroups of order* 3. *In particular, no element of* $\mathfrak{B}^\#$ *centralizes any four-subgroup of* $[O_2(\mathfrak{M}), \mathfrak{B}]$. *If* $w > 2$, *then* $C_{O_2(\mathfrak{M})}(\mathfrak{B}) = [O_2(\mathfrak{M}), \mathfrak{B}]'$ *is the center of* $O_2(\mathfrak{M})$.

(d) $\mathfrak{T}_2 \subset \mathfrak{M}$.

(e) $\mathfrak{B} \subset \mathfrak{M}$, *and if* $\mathfrak{Q}$ *is a quaternion subgroup of* $\mathfrak{T}_2$ *which is normalized by* $\mathfrak{B}$ *but is not centralized by* $\mathfrak{B}$, *then* $\mathfrak{Q} \subset O_2(\mathfrak{M})$.

(f) *If* J *is an involution of* $\mathfrak{M} \cap C(\mathfrak{B})$, *then* $J \in O_2(\mathfrak{M})$. *If* $\mathfrak{M}$ *contains a* S_2*-subgroup of* $C(J)$ *(e.g., if* $C(J) = \mathfrak{T}$), *then* $C(J) \subseteqq \mathfrak{M}$.

(g) $\mathfrak{M}$ *contains a* S_2*-subgroup of* $\mathfrak{G}$.

Proof. Let $\mathscr{S}$ be the set of 2, 3-subgroups of $\mathfrak{G}$ which contain $\langle \mathfrak{B}, \mathfrak{T}_2\rangle$. Choose $\mathfrak{S}$ in $\mathscr{S}$ so that $|\mathfrak{S}|_2$ is maximal. Let $\mathfrak{S}_p$ be a S_p-subgroup of $\mathfrak{S}$, $p = 2, 3$, chosen so that $\mathfrak{T}_2 \subseteqq \mathfrak{S}_2$, $\mathfrak{B} \subseteqq \mathfrak{S}_3$. Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ be all the elements of $\mathscr{U}(2)$ in $\mathfrak{S}_2$. By Lemma 7.1, each $\mathfrak{A}_i$ centralizes $O_3(\mathfrak{S})$, so by Lemma 7.4, $|O_3(\mathfrak{S})| \leqq 3$. In particular, $\mathfrak{B}$ is not contained in $O_2(\mathfrak{S})$ and $\mathfrak{B}$ centralizes $O_3(\mathfrak{S})$. Since $\mathfrak{B} \nsubseteq F(\mathfrak{S})$, $\mathfrak{B}$

does not centralize $\boldsymbol{O}_2(\mathfrak{S})$. Hence, $\boldsymbol{O}_2(\mathfrak{S})$ is nonabelian since $\mathfrak{B} \in \mathscr{D}$. Let $\mathfrak{K} = \boldsymbol{O}_2(\mathfrak{S})' \cap \mathfrak{Z}(\boldsymbol{O}_2(\mathfrak{S}))$, so that $\mathfrak{K} \neq 1$. Let $\mathfrak{N} = \boldsymbol{N}(\mathfrak{K})$, $\mathfrak{C} = \boldsymbol{C}(\mathfrak{K})$, and observe that $\mathfrak{B} \subseteq \mathfrak{C}$. Since the centralizer of every involution is solvable, $\mathfrak{C}$ is solvable. Let $\mathfrak{S}^*$ be a S_2-subgroup of $\mathfrak{N}$ which contains $\mathfrak{S}_2$. Then $\mathfrak{C}\mathfrak{S}_2^*$ is solvable. By $D_{2,3}$ in $\mathfrak{C}\mathfrak{S}_2^*$ and maximality of $|\mathfrak{S}|_2$, it follows that $\mathfrak{S}_2^* = \mathfrak{S}_2$. Hence, if $\mathfrak{S}_2^{**}$ is a S_2-subgroup of $\mathfrak{G}$ containing $\mathfrak{S}_2$, then $\mathfrak{S}_2$ contains every element of $\mathscr{U}(\mathfrak{S}_2^{**})$. We may therefore assume that $\mathfrak{A}_1 \in \mathscr{U}(\mathfrak{S}_2^{**})$.

Since $\mathfrak{A}_1$ centralizes $\boldsymbol{O}_3(\mathfrak{S})$, it follows that $\mathfrak{A}_1 \cap \boldsymbol{Z}(\mathfrak{S}_2^{**}) \subseteq \boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{S}))$. Since $\mathfrak{B}$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{S}))$, maximality of $|\mathfrak{S}|_2$ guarantees that $\mathfrak{S}_2 = \mathfrak{S}_2^{**}$ is a S_2-subgroup of $\mathfrak{G}$.

Let $\mathfrak{S} \subseteq \mathfrak{M} \in \mathscr{MS}(\mathfrak{G})$. Thus, (g) holds, as does (d). Since $\mathfrak{M}$ contains a S_2-subgroup of $\mathfrak{G}$, and since 1 is the only 2-signalizer of $\mathfrak{G}$, it follows that $\boldsymbol{O}_{2'}(\mathfrak{M}) = 1$, and (a) holds. Let $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$. Suppose $\mathfrak{H}$ contains a noncyclic characteristic abelian subgroup $\mathfrak{H}_0$. Then $\mathfrak{B}$ centralizes $\mathfrak{H}_0\boldsymbol{Z}(\mathfrak{H})$ and $\mathfrak{H}_0\boldsymbol{Z}(\mathfrak{H})$ contains an element of $\mathscr{U}(\mathfrak{S}_2)$. This violates Lemma 7.4.

Clearly, $\mathfrak{H}$ is noncyclic, since $\mathfrak{H} = \boldsymbol{F}(\mathfrak{M})$ and $\mathfrak{M}$ is solvable. Thus, $\mathfrak{H}$ is of symplectic type. The width w of $\mathfrak{H}$ is at least 2, since $\mathfrak{B}$ is faithfully represented on $\mathfrak{H}$.

Suppose $w \geqq 3$ and $B \in \mathfrak{B}^\#$ centralizes a four-subgroup $\mathfrak{V}$ of $\mathfrak{H}$ with $\Omega_1(\boldsymbol{Z}(\mathfrak{H})) \subset \mathfrak{V}$. By Lemma 7.2, $\mathfrak{V}$ centralizes every element of $\boldsymbol{\mathit{V}}(\mathfrak{V}; 2')$. Since $\boldsymbol{C}(B)$ contains an element of $\mathscr{U}(3)$, (7.2) is violated. Thus, if $w \geqq 3$, then no element of $\mathfrak{B}^\#$ centralizes any four-subgroup of $\mathfrak{H}$. This immediately implies that $\mathfrak{H}$ is extra special and $w \leqq 4$. Now (b) and (c) follow from Lemma 5.12.

We next prove the first assertion of (f). Let J be an involution of $\mathfrak{M} \cap \boldsymbol{C}(\mathfrak{B})$. If $w = 2$, then $\mathfrak{B}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$ and (f) is clear. Suppose $w \geqq 3$. In this case, $\mathfrak{H}$ is extra special, so $\mathfrak{H} \cap \boldsymbol{C}(\mathfrak{B}) = \mathfrak{H}'$ is of order 2. Let $\mathfrak{B}_0$ be any subgroup of $\mathfrak{B}$ of order 3. Since J centralizes $\mathfrak{B}_0$, it follows that J normalizes $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0)$. We will show that J centralizes $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0)$. This is clear if $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0) = \mathfrak{H}'$, so suppose $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0) \supset \mathfrak{H}'$. Since $\mathfrak{H}$ is extra special, so is $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0)$, so $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0)$ is a quaternion group on which $\mathfrak{B}/\mathfrak{B}_0$ is faithfully represented. Since a quaternion group has no automorphism of order 6, J necessarily centralizes $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0)$. Hence, J centralizes $\langle \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B}_0) \mid \mathfrak{B}_0 \subset \mathfrak{B}, |\mathfrak{B}_0| = 3 \rangle = \mathfrak{H}$, so $J \in \mathfrak{H}$. This proves the first assertion of (f).

Now for the second assertion of (f). If $w > 2$, then $\langle J \rangle = \mathfrak{H}'$, by what we have just shown, together with (c). So suppose $w = 2$ and $\langle J \rangle \ntriangleleft \mathfrak{M}$. Let $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{B}]$, $\mathfrak{H}_1 = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{B})$. Thus, $J \in \mathfrak{H}_1$, $J \neq Z$, where Z is central involution of $\mathfrak{H}_1$. Since $w = 2$, $\mathfrak{B}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$. Let $\mathfrak{T}_0$ be a S_2-subgroup of $\boldsymbol{C}(J)$ which is contained in $\mathfrak{M}$. Thus, $\boldsymbol{C}(J) \supseteq \mathfrak{T}_0\mathfrak{B}$, $\mathfrak{T}_0 \supseteq \mathfrak{H}_0 \times \langle J \rangle$, and $\mathfrak{H}_0$ is the central product of 2 quaternion groups.

Since $C(J)$ contains an element of $\mathscr{U}(2)$, it follows that $O_{2'}(C(J)) = 1$. Since Z centralizes $\mathfrak{T}_0$, a S_2-subgroup of $C(J)$, it follows that $Z \in O_2(C(J))$, and so $Z \in Z(O_2(C(J)))$. Since $\mathfrak{B} \subseteq C(J)$, it follows that $O_2(C(J)) \in И_{\mathfrak{M}}(\mathfrak{B}; 2)$. Hence, $O_2(C(J)) \subseteq O_2(\mathfrak{M}) = \mathfrak{H}$, since $\mathfrak{B}$ is a $S_{2'}$-subgroup of $\mathfrak{M}$. Hence, $O_2(C(J)) \subseteq \mathfrak{H} \cap \mathfrak{T}_0 = \mathfrak{H}_0 \times \langle J \rangle$. If $O_2(C(J))$ is not elementary, then $\langle Z \rangle$ char $O_2(C(J))$, and so $C(J) \subseteq \mathfrak{M}$. Suppose $O_2(C(J))$ is elementary. Since $\mathfrak{B}$ is faithfully represented on $O_2(C(J))$, it follows that $|O_2(C(J))| \geqq 2^4$. However, $O_2(\mathfrak{M})$ contains no elementary subgroup of order 2^4 on which $\mathfrak{B}$ is faithfully represented. This completes the proof of the second assertion of (f).

We turn to the proof of (e). Let $\mathfrak{Q}$ be a quaternion subgroup of $\mathfrak{M}$ normalized but not centralized by $\mathfrak{B}$. Let $\mathfrak{B}_0 = \mathfrak{B} \cap C(\mathfrak{Q})$, so that $\mathfrak{B}\mathfrak{Q} = \mathfrak{B}_0 \times \mathfrak{B}_1\mathfrak{Q}$, where $|\mathfrak{B}_i| = 3$ and $\mathfrak{B}_1$ is faithfully represented on $\mathfrak{Q}$. Let $\mathfrak{Q}_0 = \mathfrak{Q} \cap \mathfrak{H}$. By (f), $\mathfrak{Q}_0 \supseteqq \mathfrak{Q}'$. Since $\mathfrak{Q}/\mathfrak{Q}'$ is an irreducible $\mathfrak{B}$-group, we may assume by way of contradiction that $\mathfrak{Q}_0 = \mathfrak{Q}'$.

Let $\mathfrak{R}$ be a $\mathfrak{B}\mathfrak{Q}$-invariant subgroup of $Q_2^1(\mathfrak{M})$ minimal subject to $[\mathfrak{R}, \mathfrak{Q}] \neq 1$. Thus, $\mathfrak{R}$ may be viewed as a $\mathfrak{B}\mathfrak{Q}/\mathfrak{Q}'$-group; as such $\mathfrak{B}_1\mathfrak{Q}/\mathfrak{Q}'$ acts faithfully. Since $w \leqq 4$, it follows that $\mathfrak{R}$ is elementary of order 3^3 and is centralized by $\mathfrak{B}_0$. Thus, $w = 4$ and S_3-subgroups of $\mathfrak{M}$ are of order 3^5. Also, $\mathfrak{R}$ is incident with an elementary subgroup $\mathfrak{R}_0$ of $\mathfrak{M}$ such that $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$ is elementary of order 3^4.

Let $\mathfrak{P}$ be a S_3-subgroup of $\mathfrak{G}$ containing $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$ and choose $\mathfrak{U}$ in $\mathscr{U}(\mathfrak{P})$. Then $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$ contains an elementary subgroup $\mathfrak{E}$ of order 3^3 which centralizes $\mathfrak{U}$. Since $\mathfrak{E} \subseteq \mathfrak{M}$, there is an element E of $\mathfrak{E}^\#$ such that $\mathfrak{H} \cap C(E)$ contains a four-group. But then $E \in \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3$, against (7.2). This contradiction completes the proof of (e) and the lemma.

Throughout the remainder of this section, $\mathfrak{P}$ denotes a S_3-subgroup of $\mathfrak{G}$.

LEMMA 7.6. *Suppose* $|\mathfrak{P}| > 3^4$.

(a) *If* $\mathfrak{P}_0$ *is a subgroup of* $\mathfrak{P}$ *of index at most* 9 *and* $\mathfrak{P}_0$ *contains an element of* $\mathscr{U}^*(\mathfrak{P})$, *then* $И(\mathfrak{P}_0; 2)$ *contains only* 1.

(b) *If* $\mathfrak{A}$ *is a subgroup of* $\mathfrak{P}$ *of type* (3, 3) *and* $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{A})| \leqq 3$, *then* $\mathfrak{A}$ *centralizes every element of* $И(\mathfrak{A}; 2)$.

(c) *If* $\mathfrak{A}$ *is a subgroup of* $\mathfrak{P}$ *of type* (3, 3), *if* $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{A})| \leqq 9$, *and if* $C_{\mathfrak{P}}(\mathfrak{A})$ *contains an element of* $\mathscr{U}^*(\mathfrak{P})$, *then* $\mathfrak{A} \in \mathscr{D}$.

(d) *If* $\mathfrak{E}$ *is a normal elementary subgroup of* $\mathfrak{P}$ *of order* 27 *and* $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{E})| = 3$, *then* $\mathfrak{A} \in \mathscr{E}(3)$ *for each subgroup* $\mathfrak{A}$ *of index* 3 *in* $\mathfrak{E}$.

Proof. (a) Let $\mathfrak{B}$ be an element of $\mathscr{U}^*(\mathfrak{P})$ with $\mathfrak{B} \subseteq \mathfrak{P}_0$. We will show that $И(\mathfrak{B}; 2)$ contains only 1. To do this, we first show that if $\mathfrak{X} \in \mathscr{U}(3)$, then $|C(\mathfrak{X})|$ is odd. Suppose J is an involution of of $C(\mathfrak{X})$. By Lemma 5.38, $C(J)$ contains an element $\mathfrak{Y}$ of $\mathscr{U}(2)$. Hence, by Lemmas 7.1 and 7.4, $\langle \mathfrak{X}, \mathfrak{Y} \rangle$ is nonsolvable, against $\langle \mathfrak{X}, \mathfrak{Y} \rangle \subseteq C(J)$. In particular, $\mathfrak{X}$ does not centralize any quaternion subgroup of $\mathfrak{S}$. By Lemma 7.1(iii), it follows that $\mathfrak{B}$ centralizes every element of $\text{И}(\mathfrak{B}; 2)$. Suppose K is an involution of $C(\mathfrak{B})$. Then by Lemma 5.38, $C(\mathfrak{K})$ contains an element $\mathfrak{Z}$ of $\mathscr{U}(2)$, so by Lemmas 7.1 and 7.4, $\langle \mathfrak{B}, \mathfrak{Z} \rangle$ is nonsolvable, against $\langle \mathfrak{B}, \mathfrak{Z} \rangle \subseteq C(K)$. We conclude that $|C(\mathfrak{B})|$ is odd, and so $\text{И}(\mathfrak{B}; 2)$ contains only 1. Since $\text{И}(\mathfrak{P}_0; 2) \subseteq \text{И}(\mathfrak{B}; 2)$, (a) follows.

Suppose (b) is false. Let $\mathfrak{Q}$ be a 2-group normalized by $\mathfrak{A}$ and minimal subject to $[\mathfrak{Q}, \mathfrak{A}] \neq 1$. Then $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}]$ is either a quaternion group or a four-group, and $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$ where $|\mathfrak{A}_i| = 3$ and $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{Q})$.

Let $\mathfrak{C} = C(\mathfrak{A}_0) \supseteq \langle C_{\mathfrak{P}}(\mathfrak{A}), \mathfrak{Q} \rangle$. Since $C_{\mathfrak{P}}(\mathfrak{A})$ is of index at most 3 in $\mathfrak{P}$, it follows that $C_{\mathfrak{P}}(\mathfrak{A})$ contains an element $\mathfrak{U}$ of $\mathscr{U}(\mathfrak{P})$. We argue that $C_{\mathfrak{P}}(\mathfrak{A})$ contains an element of $\mathscr{SCN}_3(\mathfrak{P})$. Namely, let

$$\mathfrak{Z} = \Omega_1(Z(C_{\mathfrak{P}}(\mathfrak{A}))) .$$

If $m(\mathfrak{Z}) \geqq 3$, let $\mathfrak{B}$ be an element of $\mathscr{SCN}_3(\mathfrak{P})$ which contains $\mathfrak{Z}$. Since $\mathfrak{A} \subseteq \mathfrak{B}$, we get $\mathfrak{B} \subseteq C_{\mathfrak{P}}(\mathfrak{A})$. Suppose $m(\mathfrak{Z}) \leqq 2$. Then $\mathfrak{Z} = \mathfrak{A} \triangleleft \mathfrak{P}$, so by Lemma 0.8.9, $\mathfrak{A}$ is contained in some element of $\mathscr{SCN}_3(\mathfrak{P})$. So $C_{\mathfrak{P}}(\mathfrak{A})$ contains an element of $\mathscr{SCN}_3(\mathfrak{P})$. By Hypothesis 7.4(v), $O_{3'}(\mathfrak{C}) = 1$.

Let $\mathfrak{P}^*$ be a S_3-subgroup of C which contains $C_{\mathfrak{P}}(\mathfrak{A})$. Since $\mathfrak{A}_1$ does not centralize $O_3(\mathfrak{C}) = \mathfrak{H}$, it follows that $\mathfrak{P}^* = \mathfrak{H} C_{\mathfrak{P}}(\mathfrak{A})$ is a S_3-subgroup of $\mathfrak{S}$. Also, since $\mathfrak{A}_1\mathfrak{H}/\mathfrak{H} \subseteq Z(\mathfrak{P}^*/\mathfrak{H})$, it follows that $\mathfrak{A}_1 \subseteq O_{3,3',3}(\mathfrak{C})$. Hence, $\mathfrak{Q} \subseteq O_{3,3'}(\mathfrak{C})$. Since $C_{\mathfrak{H}}(\mathfrak{A}_1)$ is of index 3 in $\mathfrak{H}$, it follows that $[Q_3^1(\mathfrak{C}), \mathfrak{A}_1]$ is a quaternion group. Hence, $\tilde{\mathfrak{C}} = \mathfrak{Q}\mathfrak{P}^*$ is a group. Let $\tilde{\mathfrak{H}} = O_3(\tilde{\mathfrak{C}})$. Thus, $\mathfrak{P}^* = \tilde{\mathfrak{H}}\mathfrak{A}_1$ and $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q})$ is of index 9 in $\mathfrak{H}$, while $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q}) \triangleleft \mathfrak{P}^*$. Since $\mathfrak{Q}'$ centralizes no element of $\mathscr{U}^*(\mathfrak{P}^*)$, it follows that $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q})$ is cyclic. Since $|\mathfrak{P}| > 3^4$, so also $|\mathfrak{P}^*| > 3^4$, so $\mathfrak{A}_0$ is a proper subgroup of $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q}) = \tilde{\mathfrak{A}}_0$.

Let $\tilde{\mathfrak{H}}_0 = [\mathfrak{Q}, \tilde{\mathfrak{H}}]$. By the three subgroups lemma, $\tilde{\mathfrak{H}}_0$ and $\tilde{\mathfrak{A}}_0$ commute elementwise. Furthermore, either $\tilde{\mathfrak{H}}_0$ is elementary of order 9 and $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_0 \times \tilde{\mathfrak{A}}_0$ or $\tilde{\mathfrak{H}}_0$ is a non abelian group of order 27 and exponent 3 and $\tilde{\mathfrak{H}}$ is the central product of $\tilde{\mathfrak{H}}_0$ and $\tilde{\mathfrak{A}}_0$.

Since $C(\mathfrak{A}_1) \cap \tilde{\mathfrak{H}}$ is of index 3 in $\tilde{\mathfrak{H}}$, it follows that $\mathfrak{A}_1$ centralizes $\tilde{\mathfrak{A}}_0$. Set $\mathfrak{B} = \langle \mathfrak{A}_1, \tilde{\mathfrak{A}}_0 \rangle = \mathfrak{A}_1 \times \tilde{\mathfrak{A}}_0$, and let I be the involution of $\mathfrak{Q}$. Thus, $\mathfrak{B} \subseteq C(I)$ and $C(I)$ contains an element of $\mathscr{U}(2)$. Thus, $C(I)$ contains no element of $\mathscr{U}^*(3)$. Since $\mathfrak{B}$ is of index 9 in $\mathfrak{P}^*$, it follows that $\mathfrak{B}$ is a S_3-subgroup of $C(I)$. Let $\mathfrak{L}$ be a $S_{2,3}$-subgroup of $C(I)$ which contains $\mathfrak{B}\mathfrak{Q}$. Then $O_3(\mathfrak{L}) = 1$, so $\mathfrak{B}$ is faithfully represented on $O_2(\mathfrak{L})$. We can thus choose a subgroup $\mathfrak{B}_0$ of order 3 in $\mathfrak{B}$ such that $\tilde{\mathfrak{A}}_0$ is faithfully represented on $O_2(\mathfrak{L}) \cap C(\mathfrak{B}_0)$. Let $\mathfrak{X} = C(\mathfrak{B}_0)$. Then $O_{3'}(\mathfrak{X})$ is of odd order by (a). Thus, $O_{3',3}(\mathfrak{X}) \cap \tilde{\mathfrak{A}}_0 = 1$, so that $|O_{3',3}(\mathfrak{X})|_3 \leqq 27$. But $\tilde{\mathfrak{A}}_0$ is faithfully represented on the Frattini quotient group $\mathfrak{V}$ of $O_{3',3}(\mathfrak{X})/O_{3'}(\mathfrak{X})$. Since $|\mathfrak{V}| \leqq 27$ and $\tilde{\mathfrak{A}}_0$ is cyclic of order $\geqq 9$, we have a contradiction. The proof of (b) is complete.

Suppose (c) is false. Let $\mathfrak{B}$ be a four group in $\mathscr{N}(\mathfrak{A})$ which is not centralized by $\mathfrak{A}$. Then $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$ where $|\mathfrak{A}_i| = 3$ and $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{B})$.

Set $\mathfrak{C} = C(\mathfrak{A}_0)$ and let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$ which contains $C_{\mathfrak{P}}(\mathfrak{A})$. By (a), $O_{3'}(\mathfrak{C})$ is of odd order, so $\mathfrak{A}_1\mathfrak{B}$ is faithfully represented on $O_{3',3}(\mathfrak{C})/O_{3'}(\mathfrak{C})$. Set $\mathfrak{H} = \mathfrak{P}^* \cap O_{3',3}(\mathfrak{C})$. By (B), $|\mathfrak{P}: C_{\mathfrak{H}}(\mathfrak{A}_1)| \geqq 9$. Thus, $\mathfrak{P}^* = \mathfrak{H}C_{\mathfrak{P}}(\mathfrak{A})$ is a S_3-subgroup of $\mathfrak{G}$, so that $O_{3'}(\mathfrak{C}) = 1$.

We may now apply Lemma 5.42 with $\mathfrak{C}/\mathfrak{H}$ in the role of $\mathfrak{S}$, $\mathfrak{H}/D(\mathfrak{H})$ in the role of $\mathfrak{E}$, and $\mathfrak{A}_1$ in the role of $\mathfrak{Z}$. Let $\mathfrak{C}_1$ be the inverse image in $\mathfrak{C}$ of $[Q_3^1(\mathfrak{C}), \mathfrak{A}_1]$. Thus, $\mathfrak{C}_1 = \mathfrak{H}\mathfrak{Q}$ where $\mathfrak{Q}$ is either a four-group or is the central product of 2 quaternion groups. Since $C_{\mathfrak{P}}(\mathfrak{A}_1)$ covers $\mathfrak{P}^*/\mathfrak{H}$, it follows that $\mathfrak{C}_1$ is a minimal subgroup of the group $\mathfrak{K} = \mathfrak{B}_1\mathfrak{P}^*$. Let $\mathfrak{L} = O_3(\mathfrak{K})$, so that $\mathfrak{P}^*/\mathfrak{L}$ is elementary of order 3 or 9. Since $\mathfrak{B} \subset \mathfrak{C}_1$, we assume without loss of generality that $\mathfrak{B} \subseteqq \mathfrak{Q}$.

Since $\mathfrak{A}_1$ centralizes $D(\mathfrak{L})$, so does $\mathfrak{Q}$. Thus, $C(\mathfrak{Q}) \cap \mathfrak{L} \lhd \mathfrak{L}$. Since $N(\mathfrak{Q}) \cap \mathfrak{K}$ normalizes $C(\mathfrak{Q}) \cap \mathfrak{L}$, it follows that $C(\mathfrak{Q}) \cap \mathfrak{L} \lhd \mathfrak{P}^*$. Since $\mathfrak{Q}$ centralizes no element of $\mathscr{U}^*(3)$, it follows that $C(\mathfrak{Q}) \cap \mathfrak{L}$ is cyclic. Naturally, $\mathfrak{A}_0 \subseteqq C(\mathfrak{Q}) \cap \mathfrak{L}$.

Case 1. $\mathfrak{P}^* = \mathfrak{L}\mathfrak{A}_1$.

Since $\mathfrak{A}_1$ normalizes $\mathfrak{B}$, it follows that $\mathfrak{K}_1 = \mathfrak{P}^*\mathfrak{B}$ is a group and that $\mathfrak{L} = O_3(\mathfrak{K}_1)$. Let $\mathfrak{L}_1 = C_{\mathfrak{L}}(\mathfrak{B}) \supseteqq D(\mathfrak{L})$. Thus, $\mathfrak{L}_1 \lhd \mathfrak{K}_1$ and $\mathfrak{L}/\mathfrak{L}_1$ is elementary of order 27. Also, $\mathfrak{L}_1$ is cyclic, since no element of $\mathscr{U}^*(3)$ is centralized by $\mathfrak{B}$. Since $\mathfrak{L}/\mathfrak{L}_1$ is a chief factor of $\mathfrak{K}_1$ or order 27, it follows that $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$, where $\mathfrak{L}_2 = [\mathfrak{L}, \mathfrak{B}]$ is elementary of order 27, $\mathfrak{L}_2 \lhd \mathfrak{K}_1$. Let V be an involution of $\mathfrak{B}$. Thus, $\mathfrak{B} = \langle C(V) \cap \mathfrak{L}_2, \mathfrak{A}_0 \rangle$ is elementary of order 9 and $|\mathfrak{P}^*: C_{\mathfrak{P}^*}(\mathfrak{B})| = 3$.

By (b), $\mathfrak{B}$ centralizes every element of $\mathscr{N}(\mathfrak{B}; 2)$. Since $C(V)$ contains an element of $\mathscr{U}(2)$, (7.2) is violated.

Case 2. $\mathfrak{P}^* \supset \mathfrak{L}\mathfrak{A}_1$.

In this case, $\mathfrak{P}^*/\mathfrak{L}$ is elementary of order 9, so $\mathfrak{Q}$ is the central product of 2 quaternion groups.

Suppose $\mathfrak{L}$ is abelian. Then $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$ where $\mathfrak{L}_1 = [\mathfrak{L}, \mathfrak{Q}]$ is elementary of order 3^4 and $\mathfrak{L}_2 = C_{\mathfrak{L}}(\mathfrak{Q})$ is cyclic. Notice that $\mathfrak{A}_0 \subseteqq \mathfrak{L}_2$. Since $C_{\mathfrak{L}}(\mathfrak{B})$ is of index 27 in $\mathfrak{L}$ by (B), it follows that $\mathfrak{L} \cap C(\mathfrak{A}_1) \cap C(\mathfrak{B})$ contains a subgroup $\mathfrak{B}$ of type (3, 3). But then $|\mathfrak{P}^*: C_{\mathfrak{P}^*}(\mathfrak{B})| \leqq 3$, so by (b), $\mathfrak{B}$ centralizes every element of $\mathscr{N}(\mathfrak{B}; 2)$. Hence, $\mathfrak{B}^\# \subseteqq \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3$, against (7.2). We conclude that $\mathfrak{L}$ is non abelian.

Since $\mathfrak{A}_1$ centralizes $D(\mathfrak{L})$, so does $\mathfrak{Q}$, so $\mathfrak{L}_2 = C_{\mathfrak{L}}(\mathfrak{Q}) \lhd \mathfrak{L}$. Hence $\mathfrak{L}_2 \lhd \mathfrak{P}^*$, and $\mathfrak{L}_2$ is cyclic. Let $\mathfrak{L}_1 = [\mathfrak{L}, \mathfrak{Q}]$. Then $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is elementary of order 3^4, $\mathfrak{L}_1' = D(\mathfrak{L}_1)$ and $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is a chief factor of K. Being a chief factor, $\mathfrak{L}_1/D(\mathfrak{L}_1)$ is centralized by $\mathfrak{L}$. Hence, $[\mathfrak{L}_2, \mathfrak{L}_1] \subseteqq D(\mathfrak{L}_1) \subseteqq \mathfrak{L}_2$, so $[\mathfrak{L}_2, \mathfrak{L}_1, \mathfrak{Q}] = 1$. Since $[\mathfrak{Q}, \mathfrak{L}_2] = 1$, so also $[\mathfrak{Q}, \mathfrak{L}_2, \mathfrak{L}_1] = 1$. By the three subgroups lemma, $[\mathfrak{L}_1, \mathfrak{Q}, \mathfrak{L}_2] = 1$, that is, $[\mathfrak{L}_1, \mathfrak{L}_2] = 1$. Hence,

$\boldsymbol{D}(\mathfrak{L}_1) \subseteq \boldsymbol{Z}(\mathfrak{L}_1)$. Since $\mathfrak{L}$ is nonabelian, so is $\mathfrak{L}_1$. Since $\mathfrak{L}_1/\boldsymbol{D}(\mathfrak{L}_1)$ is a chief factor of $\mathfrak{R}$, $\boldsymbol{D}(\mathfrak{L}_1) = \boldsymbol{Z}(\mathfrak{L}_1) = \mathfrak{L}_1' \subseteq \mathfrak{L}_2$, so $\mathfrak{L}_1$ is extra special of order 3^5.

Now $\mathfrak{A}_1\mathfrak{B}$ is faithfully represented on $\mathfrak{L}_1$. Also, $|\mathfrak{L}: \mathfrak{L} \cap \boldsymbol{C}(\mathfrak{B})| = |\mathfrak{L}_1: \mathfrak{L}_1 \cap \boldsymbol{C}(\mathfrak{B})| = 3^3$, by (B). Hence, $\mathfrak{L}_1 \cap \boldsymbol{C}(\mathfrak{B}) = 3^2$. This is not the case, since $\mathfrak{L}_1 \cap \boldsymbol{C}(\mathfrak{B})$ is either extra special or is $\mathfrak{L}_1'$. The proof of (c) is complete.

Suppose (d) is false. Let $\mathfrak{Q}$ be an element of $\mathsf{N}(\mathfrak{A}; 3')$ minimal subject to $[\mathfrak{A}, \mathfrak{Q}] \neq 1$. Then $\mathfrak{Q}$ is a q-group for some prime q, $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}]$, and $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$, where $|\mathfrak{A}_i| = 3$ and $\mathfrak{A}_0 = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{Q})$. By (b), $q \neq 2$. Let $\mathfrak{C} = \boldsymbol{C}(\mathfrak{A}_0)$. Since $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{A})$ contains an element of $\mathscr{SCN}_3(\mathfrak{P})$, it follows from Hypothesis 7.4(v) that $\boldsymbol{O}_{3'}(\mathfrak{C}) = 1$. Let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$ which contains $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{A}_0)$. Since $|\mathfrak{P}: \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{A})| \leqq 3$, so also $|\mathfrak{P}^*: \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{A}_0)| \leqq 3$, and so $[\mathfrak{P}^*, \mathfrak{A}, \mathfrak{A}] = 1$.

Let $\mathfrak{H}$ be a $S_{3,q}$-subgroup of $\mathfrak{C}$ which contains $\mathfrak{P}^*$. Since q is odd, (B) implies that $\mathfrak{A} \subseteq \boldsymbol{O}_3(\mathfrak{H})$. Let $\mathfrak{H}^*$ be a $S_{3,q}$-subgroup of $\mathfrak{C}$ which contains $\mathfrak{A}\mathfrak{Q}$. By Lemma 0.7.5, we get $\mathfrak{A} \subseteq \boldsymbol{O}_3(\mathfrak{H}^*)$, so $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}] \subseteq \boldsymbol{O}_3(\mathfrak{H}^*)$. This contradiction completes the proof of (d) and the lemma.

LEMMA 7.7. *Assume the following*:

(a) $\mathfrak{W}$ *is a normal elementary subgroup of* $\mathfrak{P}$, $\mathfrak{A} = \boldsymbol{A}_{\mathfrak{G}}(\mathfrak{W})$.

(b) $\bar{\mathfrak{P}}$ *is the image of* $\mathfrak{P}$ *in* $\mathfrak{A}$ *and* $\bar{\mathfrak{P}}$ *is faithfully represented on* $\mathfrak{Q}$, $\mathfrak{Q}$ *being a non abelian special* 2-*subgroup of* $\mathfrak{A}$.

(c) $\bar{\mathfrak{P}}$ *contains a subgroup* $\bar{\mathfrak{P}}_0$ *of order* 3 *which centralizes a hyperplane of* $\mathfrak{W}$.

Then $\bar{\mathfrak{P}}$ *centralizes* $\mathfrak{Q}'$.

Proof. Let $\mathfrak{Q}_0 = [\bar{\mathfrak{P}}_0, \mathfrak{Q}]$. Thus, $\mathfrak{Q}_0$ is a quaternion group, and $\mathfrak{W} = \mathfrak{W}_0 \times \mathfrak{W}_1$, where $\mathfrak{W}_0 = [\mathfrak{Q}_0, \mathfrak{W}]$ is of order 9 and $\mathfrak{W}_1 = \boldsymbol{C}_{\mathfrak{W}}(\mathfrak{Q}_0)$. Since $|\boldsymbol{C}(\mathfrak{W})|$ is odd, some involution I of $\boldsymbol{N}(\mathfrak{W})$ maps to the involution of $\mathfrak{Q}_0$. Let $\bar{\mathfrak{P}}_1$ be the normal closure of $\bar{\mathfrak{P}}_0$ in $\bar{\mathfrak{P}}$. Thus, $\bar{\mathfrak{P}}_1$ centralizes $\mathfrak{Q}'$. Let $\mathfrak{W}_2 = \boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{P}}_1)$ so that $\mathfrak{Q}'$ is faithfully represented on $\mathfrak{W}_2$. Suppose $\bar{\mathfrak{P}}$ does not centralize $\mathfrak{Q}'$. Then by Lemma 4.4 of [17], there is an elementary subgroup $\mathfrak{W}^*$ of $\mathfrak{W}_2$ which is of order 27, normal in $\mathfrak{P}$ and with $|\mathfrak{P}: \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{W}^*)| = 3$. Since $\bar{\mathfrak{P}}_0$ centralizes $\mathfrak{W}^*$, it follows that $\mathfrak{W}^* \cap \mathfrak{W}_1$ is noncyclic. Let $\mathfrak{B}$ be a subgroup of $\mathfrak{W}^* \cap \mathfrak{W}_1$ of order 9. With $\mathfrak{W}^*$ in the role of $\mathfrak{E}$ in Lemma 7.6(d), we conclude that $\mathfrak{B} \in \mathscr{E}(3)$. But now $\boldsymbol{C}(I)$ contains an element of $\mathscr{U}(2)$ and also contains $\mathfrak{B}$, against Lemma 7.4. The proof is complete.

LEMMA 7.8. *Suppose that* $\mathfrak{P}$ *is of exponent* 3, *order* 81 *and that* $|\boldsymbol{Z}(\mathfrak{P})| = 9$. *Then* $\boldsymbol{N}(\mathfrak{P})$ *is the unique element of* $\mathscr{MS}(\mathfrak{G})$ *which contains* $\mathfrak{P}$.

Proof. Suppose false. Let $\mathfrak{S}$ be a solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{P}$ and is minimal subject to $\mathfrak{P} \not\triangleleft \mathfrak{S}$. Let $\mathfrak{P}_0 = O_3(\mathfrak{S})$. Since $Z(\mathfrak{P}) \subset \mathfrak{P}_0 \subset \mathfrak{P}$, it follows that $\mathfrak{P}_0$ is abelian of order 27. Since $\mathfrak{S}$ is not 3-closed, it follows that $\mathfrak{S} = \mathfrak{P}\mathfrak{Q}$ where $\mathfrak{Q}$ is a quaternion group.

Let $\mathfrak{P}_0 = \mathfrak{P}_1 \times \mathfrak{P}_2$ where $\mathfrak{P}_1 = C_{\mathfrak{P}_0}(\mathfrak{Q})$, $\mathfrak{P}_2 = [\mathfrak{P}_0, \mathfrak{Q}]$. Thus, $|\mathfrak{P}_i| = 3^i$, $i = 1, 2$. Let $\tilde{\mathfrak{P}}_1 = \mathfrak{P}_2 \cap Z(\mathfrak{P})$. Thus, $Z(\mathfrak{P}) = \mathfrak{P}_1 \times \tilde{\mathfrak{P}}_1$ and $\mathfrak{P}' = \tilde{\mathfrak{P}}_1$. Let $\mathfrak{Q}' = \langle I \rangle \subseteq N(\mathfrak{P})$. Write $N(\mathfrak{P}) = \mathfrak{P}\mathfrak{R}$ where $\mathfrak{R}$ is a complement to $\mathfrak{P}$ in $N(\mathfrak{P})$ which contains I. Since $\mathfrak{R}$ normalizes $\tilde{\mathfrak{P}}_1$, it follows that $A_{\mathfrak{G}}(Z(\mathfrak{P}))$ is abelian. Hence, $Z(\mathfrak{P}) \cap C(I) \triangleleft N(\mathfrak{P})$. Since $Z(\mathfrak{P}) \cap C(I) = \mathfrak{P}_1$, we get that $N(\mathfrak{P}) \subseteq N(\mathfrak{P}_1)$. Since $I \in N(\mathfrak{P})$, it follows that $\mathfrak{P}_1$ may be characterized as the only subgroup of $Z(\mathfrak{P})$ of order 3 which is normal in $N(\mathfrak{P})$ and is not contained in $\mathfrak{P}'$.

Let $\mathfrak{K}$ be any solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{P}$. We will show that $\mathfrak{K} \subseteq N(\mathfrak{P}_1)$. We may assume that $\mathfrak{P} \not\triangleleft \mathfrak{K}$. Let $\tilde{\mathfrak{P}}_0 = O_3(\mathfrak{K}) \supset Z(\mathfrak{P})$. Thus, $\tilde{\mathfrak{P}}_0$ is abelian of order 27. By our characterization of $\mathfrak{P}_1$, it follows that $\mathfrak{P}_1 \triangleleft \mathfrak{K}$, that is, $\mathfrak{K} \subseteq N(\mathfrak{P}_1)$.

Set $\mathfrak{M} = N(\mathfrak{P}_1)$, so that $\mathfrak{M}$ is the unique element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{P}$. Let $\mathfrak{A}$ be any elementary subgroup of $\mathfrak{P}$ of order 27. Then $\mathfrak{P} \subseteq N(\mathfrak{A})$, so $N(\mathfrak{A}) \subseteq \mathfrak{M}$. Now let A be any element of $\mathfrak{P}^{\#}$. We will show that $C(A) \subseteq \mathfrak{M}$. This is clear if $A \in Z(\mathfrak{P})$. Suppose $A \notin Z(\mathfrak{P})$. Then $C_{\mathfrak{P}}(A) = \mathfrak{A}$ is of order 27 and is abelian. Hence, $N(\mathfrak{A}) \subseteq \mathfrak{M}$. This implies that some S_3-subgroup of $C(A)$ is contained in $\mathfrak{M}$. If $C(A)$ contains a S_3-subgroup of $\mathfrak{G}$, then $C(A) \subseteq \mathfrak{M}$, by uniqueness of $\mathfrak{M}$. So suppose that $\mathfrak{A}$ is a S_3-subgroup of $C(A)$. Then since $\mathsf{M}(\mathfrak{A})$ is trivial, we get that $\mathfrak{A} \triangleleft C(A)$, so in any case, $C(A) \subseteq \mathfrak{M}$.

Let $\mathfrak{E}$ be any non identity subgroup of $\mathfrak{P}$. We will show that $N(\mathfrak{E}) \subseteq \mathfrak{M}$. If $|\mathfrak{E}'| = 3$, it suffices to show that $N(\mathfrak{E}') \subseteq \mathfrak{M}$. If $|\mathfrak{E}'| \neq 3$, then $\mathfrak{E}$ is abelian, since $|\mathfrak{P}'| = 3$. This, we may assume that $\mathfrak{E}$ is abelian. By the preceding paragraph, $C(\mathfrak{E}) \subseteq \mathfrak{M}$. Let $\mathfrak{E}^*$ be a S_3-subgroup of $C(\mathfrak{E})$. Then $N(\mathfrak{E}) = C(\mathfrak{E}) \cdot (N(\mathfrak{E}) \cap N(\mathfrak{E}^*))$, so it suffices to show that $N(\mathfrak{E}^*) \subseteq \mathfrak{M}$. But $|\mathfrak{E}^*| \geqq 27$, so $N(\mathfrak{E}^*) \subseteq \mathfrak{M}$.

It is a consequence of the preceding results, that if $\mathfrak{H}$ is a solvable subgroup of $\mathfrak{G}$ such that $\mathfrak{H} \cap \mathfrak{P}$ is noncyclic, then $\mathfrak{H} \subseteq \mathfrak{M}$.

Let $\mathfrak{B} = \mathfrak{P} \cap N(\mathfrak{Q}')$ so that $\mathfrak{B}$ is noncyclic. Hence, $N(\mathfrak{Q}') \subseteq \mathfrak{M}$. This is not the case since $N(\mathfrak{Q}')$ contains an element of $\mathscr{U}(2)$, while $\mathfrak{M}$ contains an element of $\mathscr{U}(3)$.

8. A characterization of $E_2(3)$.

THEOREM 8.1. *$E_2(3)$ is the only simple group $\mathfrak{G}$ with the following properties*:

(i) *1 is the only 3-signalizer of $\mathfrak{G}$.*

(ii) *The center of a S_3-subgroup of $\mathfrak{G}$ is noncyclic.*

(iii) *The normalizer of every nonidenty 3-subgroup of $\mathfrak{G}$ is solvable.*

(iv) *The centralizer of every involution of* $\mathfrak{G}$ *is solvable.*

(v) S_2*-subgroup of* $\mathfrak{G}$ *contain normal elementary subgroups of order* 8.

(vi) *If* $\mathfrak{T}$ *is a* S_2*-subgroup of* $\mathfrak{G}$ *and* $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{T})$, *then* $\mathsf{N}(\mathfrak{A})$ *is trivial.*

(vii) $2 \sim 3$.

The proof of Theorem 8.1 is elaborate. I am indebted to J. Tits for helpful discussion.

We first derive some properties of $E_2(q)$. We use the notation and calculations of Ree [30]. In addition, we let $\mathfrak{B} = \mathfrak{U}\mathfrak{H}$, $\mathfrak{N} = \langle \mathfrak{H}, \omega_a, \omega_b \rangle$. F_q is the field of $q = p^n$ elements, and if $x \in F_q$, then $tr(x) = tr_{F_q/F_p}(x) = \Sigma x^\sigma$, σ ranging over all the automorphisms of F_q. If $r \in \Sigma$, then $\mathfrak{X}_r = \langle x_r(t) \mid t \in F_q \rangle$.

We need the usual sort of omnibus lemma.

LEMMA 8.1. *Let* $\mathfrak{U}, \mathfrak{B}, \mathfrak{H}, \mathfrak{N}$ *denote the subgroups of* $E_2(q)$ *given above.*

(i) $\mathfrak{W}_0 = \langle \omega_b^2 \omega_a, \omega_a^2 \omega_b \rangle$ *is a dihedral group of order* 12 *and is a complement to* $\mathfrak{H}$ *in* $\mathfrak{N}$.

(ii) $\mathfrak{H}$ *is the direct product of two cyclic groups of order* $q-1$, *with generators* $H_1 = h(\chi_{a,z})$, $H_2 = h(\chi_{b,z})$. *Here* z *is a generator for* F_q^*. *If* $W_1 = \omega_b^2\omega_a$, $W_2 = \omega_a^2\omega_b$, *then*

$$W_1^{-1}H_1W_1 = H_1^{-1}\,, \qquad W_1^{-1}H_2W_1 = H_1H_2\,,$$
$$W_2^{-2}H_1W_2 = H_1H_2^3\,, \qquad W_2^{-1}H_2W_2 = H_2^{-1}\,,$$

(iii) *If* q *is a power of* 3 *and* ν *is a nonsquare in* F_q, *then*

$$\{x_{3a+2b}(1),\ x_{2a+b}(1),\ x_{3a+2b}(1)x_{2a+b}(1),\ x_{a+b}(1)x_{3a+b}(1),\ x_{a+b}(1)x_{3a+b}(\nu)\}$$

is a set of representatives for the conjugacy classes of $E_2(q)$ *of order* 3. *If* $c \in F_q$ *satisfies* $tr(c) = 1$, *then* $\{x_a(1)x_b(1)x_{3a+b}(ec),\ e = 0, 1, -1\}$ *is a set of representatives for the conjugacy classes of elements of* $E_2(q)$ *of order* 9. $\mathfrak{U}$ *is of exponent* 9.

(iv) *Assume that* q *is odd.*

(a) *Let* $\tilde{\mathfrak{B}} = C_{\mathfrak{B}}(\omega_a^2)$, $\tilde{\mathfrak{N}} = C_{\mathfrak{N}}(\omega_a^2)$. *Let* $\mathfrak{C} = C_{E_2(q)}(\omega_a^2)$. *Then* $\mathfrak{C} = \tilde{\mathfrak{B}}\tilde{\mathfrak{N}}\tilde{\mathfrak{B}}$. $\mathfrak{C}$ *contains a subgroup* $\mathfrak{C}_0 = \mathfrak{C}_1\mathfrak{C}_2$, *where* $\mathfrak{C}_i \cong SL(2, q)$, $i = 1, 2$, $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \boldsymbol{Z}(\mathfrak{C}_i) = \langle \omega_a^2 \rangle$, $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *commute elementwise and* $|\mathfrak{C}: \mathfrak{C}_0| = 2$. *Furthermore,* $\mathfrak{C}_i \lhd \mathfrak{C}$, $i = 1, 2$.

(b) *For* $i = 1, 2$, *let* α_i *be the isomorphism from* $\mathfrak{C}_i$ *to* $SL(2, q)$ *induced by* $x_{r_i}(t) \rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $z_{-r_i}(t) \rightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, where $r_1 = a$, $r_2 = 3a + 2b$. *Each element* X *in* $\mathfrak{C} - \mathfrak{C}_0$ *induces an automorphism* $\varphi_X^{(i)}$ *of* $\mathfrak{C}_i$ *such that* $\alpha_i\varphi_X^{(i)}\alpha_i^{-1}$ *coincides with the automorphism of* $SL(2, q)$

induced by an element of $GL(2, q)$ whose determinant is a nonsquare.

(c) *There are involutions in $\mathfrak{C} - \mathfrak{C}_0$. If X is an involution in $\mathfrak{C} - \mathfrak{C}_0$, and $q \equiv \varepsilon (\mathrm{mod}\ 4)$, $\varepsilon = \pm 1$, then $C(X) \cap \mathfrak{C}_0$ has order $2(q+\varepsilon)^2$ and*

$$C(X) \cap \mathfrak{C}_0 \cong gp\langle w, x, y, z \mid x^{(q+\varepsilon)/2} = y^{(q+\varepsilon)/2} = w,\ w^2 = 1,\ xy = yx,\ z^{-1}xz = x^{-1},\ z^{-1}yz = y^{-1},\ z^2 = 1\rangle .$$

(v) *If q is odd, then $i(E_2(q)) = 1$.*

Proof. The Weyl group of G_2 is dihedral of order 12, so $w_a w_b$ is of order 6. By (1.8) of [30], $(\omega_a\omega_b)^6 = h(\chi)$, for some $\chi \in X$. We show that $\chi = 1$. It suffices to show that $\chi(a) = \chi(b) = 1$, that is, $\eta_a = \eta_b = 1$. This follows readily from table (3.4) of [30]. Since $\omega_a^{-1}\omega_b^2\omega_a = \omega_b^2\omega_a^2$, and $\omega_b^{-1}\omega_a^2\omega_b = \omega_a^2\omega_b^2$, the elements $\omega_b^2\omega_a$ and $\omega_a^2\omega_b$ are involutions. We have $(\omega_b^2\omega_a)(\omega_a^2\omega_b) = \omega_b^2\omega_a^{-1}\omega_b \sim \omega_b^{-1}\omega_a^{-1} = (\omega_a\omega_b)^{-1}$, proving (i).

It is convenient for calculations to use the following character table:

	a	b
$\chi_{a,z}$	z^2	z^{-3}
$\chi_{b,z}$	z^{-1}	z^2

.

To determine this character table, we need to compute the values $u(r)$, u, $r \in \Sigma$ (see [30], p. 433). The relevant values of $u(r)$ are given as follows:

$u \backslash r$	a	b
a	2	-1
b	-3	2

.

Using this table, we compute the values $w_r(s)$, as follows:

$r \backslash s$	a	b
a	$-a$	$3a + b$
b	$a + b$	$-b$

.

(Using the geometric interpretation of w_r, we can read these results directly from Figure 1 of [30].)

We next compute $w_r(\chi)$ for $r = a, b$ and $\chi = \chi_{a,z}, \chi_{b,z}$. For example,

$[w_a(\chi_{a,z})](a) = \chi_{a,z}(w_a(a)) = \chi_{a,z}(-a) = \chi_{a,z}(a)^{-1} = z^{-2}$. Continuing in this fashion, we get the following table of values:

	a	b
$w_a(\chi_{a,z})$	z^{-2}	z^3
$w_a(\chi_{b,z})$	z	z^{-1}
$w_b(\chi_{a,z})$	z^{-1}	z^3
$w_b(\chi_{b,z})$	z	z^{-2}

Referring back to the character table, we have

$$w_a(\chi_{a,z}) = \chi_{a,z}^{-1},\ w_a(\chi_{b,z}) = \chi_{a,z}\chi_{b,z}\ ,$$
$$w_b(\chi_{a,z}) = \chi_{a,z}\chi_{b,z}^3,\ w_b(\chi_{b,z}) = \chi_{b,z}^{-1}\ .$$

The map from X to $\mathfrak{H}$ induced by $\chi_{r,z} \to h(\chi_{r,z})$ is an isomorphism, since X and $\mathfrak{H}$ have order $(q-1)^2$. The previous information, together with (1. 7) of [30] implies that (ii) holds.

Let $\mathfrak{U}_1 = \mathfrak{U} \cap \mathfrak{U}^{\omega_a}$, $\mathfrak{U}_2 = \mathfrak{U} \cap \mathfrak{U}^{\omega_b}$. By using (3.10) of [30] it is straightforward to verify that $\mathfrak{U}_1 \cup \mathfrak{U}_2$ is the set of elements of $\mathfrak{U}$ of order 1 or 3. This then implies easily that every element of $E_2(q)$ of order 3 is conjugate to an element of $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \langle \mathfrak{X}_{a+b}, \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+b}, \mathfrak{X}_{3a+2b}\rangle$. Since $\mathfrak{B} = N(\mathfrak{U})$, it follows from Lemma 14.3.1 of [21] that elements of $Z(\mathfrak{U})$ are conjugate in $E_2(q)$ only if they are conjugate in $\mathfrak{B}$. Since the action of $\mathfrak{H}$ on $Z(\mathfrak{U}) = \langle \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+2b}\rangle$ is determined by (1.5) of [30] and our character table, it follows that any element of $E_2(q)^{\#}$ which is conjugate to an element of $Z(\mathfrak{U})$ is conjugate to exactly one of $x_{2a+b}(1)$, $x_{3a+2b}(1)$, $x_{2a+b}(1)x_{3a+2b}(1)$. Furthermore, since the Weyl group permutes transitively the roots of a given length, and since $2a+b$ and $3a+2b$ have different lengths, it follows that every element of the shape $x_r(t)$, $r \in \Sigma$, is conjugate to an element of $Z(\mathfrak{U})$. Suppose $x \in \mathfrak{U}_1 \cap \mathfrak{U}_2$, $x = x_{a+b}(t_1)x_{2a+b}(t_2)x_{3a+b}(t_3)x_{3a+2b}(t_4)$, and that x is conjugate to no element of $Z(\mathfrak{U})$. Hence, either $t_1 \neq 0$ or $t_3 \neq 0$. Suppose $t_3 = 0$. Conjugation by $x_a(-t_1^{-1}t_2/2)$ enables us to assume that $t_2 = 0$. Conjugation by ω_a then yields that x is conjugate to an element of $Z(\mathfrak{U})$. Hence, $t_3 \neq 0$. Suppose $t_1 = 0$. Conjugation by $x_b(t_3^{-1}t_4)$ enables us to assume that $t_4 = 0$. Conjugation by ω_b yields that x is conjugate to an element of $Z(\mathfrak{U})$. Hence, $t_1t_3 \neq 0$. Conjugation by $x_a(-t_1^{-1}t_2/2)x_b(t_3^{-1}t_4)$ enables us to assume that $t_2 = t_4 = 0$. Since $h(\chi_{a,z})x_{a+b}(t_1)h(\chi_{a,z})^{-1} = x_{a+b}(z^{-1}t_1)$, we may assume that $t_1 = 1$. Since $h(\chi_{a,z}\chi_{b,z})$ centralizes $x_{a+b}(1)$ and since $h(\chi_{a,z}\chi_{b,z})x_{3a+b}(t_3)h(\chi_{a,z}\chi_{b,z})^{-1} = x_{3a+b}(t_3z^2)$, we may assume that $t_3 =$ 1 or ν. A direct calculation shows that the centralizer of $x_{a+b}(1)x_{3a+b}(u)$ does not contain a S_3-subgroup of $E_2(q)$ for any u in F_q^*, and a further calculation shows that $x_{a+b}(1)x_{3a+b}(1)$ is not conjugate to $x_{a+b}(1)x_{3a+b}(\nu)$, completing the proof of the first part of (iii).

If $tu \neq 0$, it is easy to verify that $x_a(t)x_b(u)x$ has order 9 for all x in $\mathfrak{U}_1 \cap \mathfrak{U}_2$ and that $(x_a(t)x_b(u))^3 = (x_a(t)x_b(u)x)^3$. A calculation shows that $\mathfrak{H}$ permutes transitively the elements $x_a(t)x_b(u)$, $tu \in F_q^*$, so every element of $E_2(q)$ of order 9 is conjugate to an element of the shape $x_a(1)x_b(1)x$, with x in $\mathfrak{U}_1 \cap \mathfrak{U}_2$. Let $x = x_{a+b}(t_1)x_{2a+b}(t_2)x_{3a+b}(t_3)x_{3a+2b}(t_4)$. Conjugation by $x_a(u)$ enables us to assume that $t_3 = 0$. A further conjugation by $x_{a+b}(u_1)x_{3a+b}(u_2)$ enables us to assume that $t_2 = t_4 = 0$. Thus, it suffices to show that

$$x_a(1)x_b(1)x_{a+b}(u) \text{ is conjugate to } x_a(1)x_b(1)x_{a+b}(v)$$

if and only if $tr(u) = tr(v)$. If g conjugates the first element into the second then g centralizes $(x_a(1)x_b(1))^3$. A calculation shows that the centralizer of $(x_a(1)x_b(1))^3$ is $\mathfrak{U}$, and a further calculation completes the proof of (iii).

By a direct calculation, $\widetilde{\mathfrak{B}} = \langle \mathfrak{X}_a, \mathfrak{X}_{3a+2b}, \mathfrak{H} \rangle$, $\widetilde{\mathfrak{N}} = \langle \mathfrak{H}, \omega_a, (\omega_a\omega_b)^3 \rangle$. Suppose ω_a^2 centralizes $xh\omega x'$, $x \in \mathfrak{U}$, $h \in \mathfrak{H}$, $\omega \in \mathfrak{W}_0$, $x' \in \mathfrak{U}_w$, w being the image of ω in the Weyl group. Then the normal form implies that $x, x', h, \omega \in C(\omega_a^2)$, so the first assertion of (iv) is proved.

Let $\mathfrak{C}_1 = \langle \mathfrak{X}_a, \mathfrak{X}_{-a} \rangle$, $\mathfrak{C}_2 = \langle \mathfrak{X}_{3a+2b}, \mathfrak{X}_{-(3a+2b)} \rangle$, so that $\mathfrak{C}_1 \cong \mathfrak{C}_2 \cong SL(2, q)$. Clearly, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ commute elementwise. Since $\chi_{3a+2b,-1} = \chi_{a,-1}$, it follows that $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \langle \omega_a^2 \rangle$, so that $\mathfrak{C}_0 = \mathfrak{C}_1\mathfrak{C}_2$ is the central product of $\mathfrak{C}_1$ and $\mathfrak{C}_2$. Setting $\widetilde{\mathfrak{U}} = \mathfrak{U} \cap \widetilde{\mathfrak{B}}$, we have

$$|\widetilde{\mathfrak{U}} \cap \widetilde{\mathfrak{U}}^{\omega_a}| = |\widetilde{\mathfrak{U}} \cap \widetilde{\mathfrak{U}}^{\omega_a(\omega_a\omega_b)^3}| = q ,$$

and $\widetilde{\mathfrak{U}} \cap \widetilde{\mathfrak{U}}^{(\omega_a\omega_b)^3} = 1$, it follows that $|\mathfrak{C}| = q^2(q-1)^2(1+2q+q^2)$. Hence,

$$(*) \qquad |\mathfrak{C} : \mathfrak{C}_0| = |\mathfrak{H} : \mathfrak{H} \cap \mathfrak{C}_0| = 2 .$$

Since $\mathfrak{H}$ normalizes $\mathfrak{X}_r$ for all r in Σ, (iv) (a) is proved.

We observe that by (1.5) of [30],

$$h(\chi_{b,z})x_a(t)h(\chi_{b,z})^{-1} = x_a(z^{-1}t) ,$$
$$h(\chi_{b,z})x_{-a}(t)h(\chi_{b,z})^{-1} = x_{-a}(zt) .$$

Hence, if $\eta = \varphi^{(1)}_{h(\chi_{b,z})}$ denotes the automorphism of $\mathfrak{C}_1$ induced by $h(\chi_{b,z})^{-1}$, then $\alpha_1\eta\alpha_1^{-1}$ is the automorphism of $SL(2, q)$ induced by the map

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & z^{-1}t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix}.$$

This automorphism therefore coincides with the automorphism induced by $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. A similar argument applies to $\mathfrak{C}_2$. Since $\mathfrak{C} - \mathfrak{C}_0$ coincides with the coset $\mathfrak{C}_0 h(\chi_{b,z})$ whenever z is not a square of F_q^*, the proof of (iv)(b) is complete.

Let $K = (W_1W_2)^3$. By (i), K is an involution, and by (ii), K inverts $\mathfrak{H}$. Thus, $\mathfrak{H}K$ is a set of involutions in $\mathfrak{C}$. If $\mathfrak{H}K \subseteqq \mathfrak{C}_0$, we get $\mathfrak{H} \subseteqq \langle \mathfrak{H}K \rangle \subseteqq C_0$, against (*). So $\mathfrak{C} - \mathfrak{C}_0$ contains an involution.

We will use (iv)(b) in the proof of (iv)(c). First, suppose $\varepsilon = -1$. In this case, -1 is not a square in F_q. Since $\mathfrak{C}_1$ and $\mathfrak{C}_2$ commute elementwise, we assume without loss of generality that for $i = 1, 2$, $\alpha_i \varphi_X^{(i)} \alpha_i^{-1}$ is the automorphism of $SL(2, q)$ induced by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, $\mathfrak{C}_i \cap C(X)$ is cyclic of order $q - 1$. Since the commutator of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is $-I$, and since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ inverts $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ for all $x \in F_q^*$, (iv)(c) follows in this case.

Now suppose $\varepsilon = 1$. In this case, -1 is a square in F_q. Choose $a, b \in F_q$ such that $a^2 + b^2 = c$ is a nonsquare. We may assume that for $i = 1, 2$, $\alpha_i \varphi_X^{(i)} \alpha_i^{-1}$ is the automorphism of $SL(2, q)$ induced by $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. A short calculation shows that (iv)(c) holds.

Since $|C(\omega_a^2)| = (q(q^2 - 1))^2$, it follows that $C(\omega_a^2)$ contains a S_2-subgroup of $E_2(q)$. Thus, to prove (v), it suffices to show that each involution X of $\mathfrak{C}$ is conjugate to ω_a^2 in $E_2(q)$. Since $E_2(q)$ is simple, Lemma 5.38 (a)(i) implies that X is conjugate in $E_2(q)$ to an element of $\mathfrak{C}_0$.

Thus, it suffices to show that all involutions of $\mathfrak{C}_0$ are conjugate in $E_2(q)$. Since $\mathfrak{C}_i$ has just involution for $i = 1, 2$, it follows that every involution of $\mathfrak{C}_0$ different from ω_a^2 is of the shape I_1I_2 where $I_i \in \mathfrak{C}_i$ and $I_i^2 = \omega_a^2$. Since $\mathfrak{C}_i$ has just 1 conjugacy class of elements of order 4, it follows that $\mathfrak{C}_0$ has two conjugacy classes of involutions.

Case 1. Every involution of $\mathfrak{H}$ is in $\mathfrak{C}_0$.

By (ii), all involutions of $\mathfrak{H}$ are fused in $\mathfrak{N}$. By the preceding paragraph, (v) follows.

Case 2. J is an involution of $(\mathfrak{C} - \mathfrak{C}_0) \cap \mathfrak{H}$.

Set $I = \omega_a^2$, $K = (W_1W_2)^3$, so that K inverts $\mathfrak{H}$ and so centralizes I and J. Let $\mathfrak{A} = \langle I, J, K \rangle$. By (ii), the involutions of $\mathfrak{A}$ are fused in $\mathfrak{N}$ as follows:

$$I \sim J \sim IJ, IK \sim JK \sim IJK .$$

It is clear that in $\mathfrak{C}$ all the involutions of $\mathfrak{C} - \mathfrak{C}_0$ are conjugate. Let $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{C}_0$. Thus, $\mathfrak{A}_0$ is one of $\langle I, K \rangle, \langle I, JK \rangle$. Suppose $\mathfrak{A}_0 = \langle I, K \rangle$. Then, J, JI, JK, JIK are the involutions of $\mathfrak{A} - \mathfrak{A}_0$, so are all conjugate in $\mathfrak{C}$. Since $K \in \mathfrak{C}_0$, K and KI are conjugate in $\mathfrak{C}_0$. Thus, all involutions of $\mathfrak{A}$ are conjugate in $E_2(q)$, so (v) follows. Suppose $\mathfrak{A}_0 = \langle I, JK \rangle$. Then, J, JI, K, KI are the involutions of $\mathfrak{A} - \mathfrak{A}_0$, so are all conjugate in $\mathfrak{C}$. Again all involutions of $\mathfrak{A}$ are conjugate in $E_2(q)$. The proof of (v) is complete.

LEMMA 8.2. (a) *Suppose $\mathfrak{G}$ is a finite group and $\mathfrak{V}$ is a four-subgroup of $\mathfrak{G}$. Suppose also that whenever I and J are distinct involutions of $\mathfrak{V}$, I and IJ are conjugate in $C(J)$. Then $A(\mathfrak{V}) = \mathrm{Aut}(\mathfrak{V})$.*
(b) *$A_{E_2(3)}(\mathfrak{V}) = \mathrm{Aut}(\mathfrak{V})$ for every four-subgroup $\mathfrak{V}$ of $E_2(3)$.*

Proof. Choose X in $C(J)$ such that $X^{-1}IX = IJ$. Thus,

$$X \in N(V) \cap C(J) .$$

Replacing the pair I, J by the pair J, I we choose Y in $C(I)$ with $Y^{-1}JY = IJ$. Then $\mathfrak{A} = \langle XY \rangle$ permutes I, J, IJ cyclically. Thus, $\langle X, Y \rangle$ maps onto $\mathrm{Aut}(\mathfrak{V})$, proving (a).

Let $\mathfrak{V}$ be a four-subgroup of $E_2(3)$ and let I, J be distinct involutions of $\mathfrak{V}$. We will produce X in $C(J)$ such that $X^{-1}IX = IJ$. We may assume that $J = \omega_a^2$, since $i(E_2(3)) = 1$. Since $O_2(C(\omega_a^2))$ is extra special, we are done in case $I \in O_2(C(\omega_a^2))$. If $I \notin O_2(C(\omega_a^2))$, then I induces an outer automorphism of both quaternion subgroups of $O_2(C(\omega_a^2))$, so again X is available. Now (b) follows from (a).

We omit the proof that $\mathfrak{C} = C_{E_2(3)}(\omega_a^2)$ has exactly $19 + 72$ involutions; namely, $O_2(\mathfrak{C})$ has exactly 19 involutions, while all involutions of $\mathfrak{C} - O_2(\mathfrak{C})$ are conjugate in $\mathfrak{C}$. Furthermore, it is straightforward to verify that $\mathfrak{C}$ has exactly 3 conjugacy classes of elementary subgroups of order 8. Representatives $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$ for these classes may be chosen so that if $\mathfrak{T}$ denotes a fixed S_2-subgroup of $\mathfrak{C}$, then $\mathfrak{E}_i \lhd \mathfrak{T}$, and $\mathfrak{E}_i \subseteq O_2(\mathfrak{C})$, $i = 1, 2$.

We argue that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are not conjugate in $E_2(3)$. Suppose $\mathfrak{E}_1^G = \mathfrak{E}_2$. Then $\mathfrak{T}^G$ normalizes $\mathfrak{E}_2$, as does $\mathfrak{T}$. Then $\mathfrak{T}^G = \mathfrak{T}^N$ for some N in $N(\mathfrak{E}_2)$. Hence, $GN^{-1} \in N(\mathfrak{T}) = \mathfrak{T}$, so $G \in \mathfrak{T}N \subseteq N(\mathfrak{E}_2)$. Since $\mathfrak{E}_1^G = \mathfrak{E}_2$, we get $\mathfrak{E}_1 = \mathfrak{E}_2$, a contradiction.

Set $\mathfrak{V} = \mathfrak{E}_1 \cap \mathfrak{E}_2$ so that $\mathfrak{V}$ is a four-subgroup of $\mathfrak{T}$ and $O_2(\mathfrak{T}) \cap C(\mathfrak{V}) = \mathfrak{E}_1\mathfrak{E}_2$ is the direct product of a group of order 2 and a dihehral group of order 8. Let $\mathfrak{D} = C_{E_2(3)}(\mathfrak{V}) = C_{\mathfrak{T}}(\mathfrak{V})$, a group of order 32. We omit the proof that $\mathfrak{D}$ has exactly 4 elementary subgroups of order 8, among which are $\mathfrak{E}_1$ and $\mathfrak{E}_2$. By Lemma 8.2 (b), $N(\mathfrak{V})$ has an element A of order 3 which permutes transitively the involutions of $\mathfrak{V}$. If A normalizes both $\mathfrak{E}_1$ and $\mathfrak{E}_2$, then A normalizes the derived group of $\mathfrak{E}_1\mathfrak{E}_2$, that is, A normalizes $\langle \omega_a^2 \rangle$. Since this is not the case, we can choose i in $\{1, 2\}$ so that the orbit of $\mathfrak{E}_i$ under $\langle A \rangle$ has 3 elements. Since $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are in different orbits under $\langle A \rangle$, it follows that A normalizes $\mathfrak{E}_j$, where $\{i, j\} = \{1, 2\}$.

We omit the proof that $N(\mathfrak{E}_j) \cap \mathfrak{C}$ permutes transitively the involutions of $\mathfrak{E}_j - \langle \omega_a^2 \rangle$. Since A does not centralize ω_a^2, it follows that $N(\mathfrak{E}_j)$ permutes transitively the involutions of $\mathfrak{E}_j$. Thus, $|N(\mathfrak{E}_j)| = 7 \cdot |N(\mathfrak{E}_j) \cap \mathfrak{C}| = 8 \cdot 24 \cdot 8$. Hence,

$$\mathfrak{A}_{E_2(3)}(\mathfrak{C}_j) = \operatorname{Aut}(\mathfrak{C}_j)\ .$$

We have proved (a) of the next lemma.

LEMMA 8.3. (a) *$E_2(3)$ is not an N-group.*
(b) *$E_2(3)$ satisfies the hypotheses of Theorem* 8.1.

Proof. It suffices to verify (b).

By Lemma 8.1 (iv), hypothesis (iv) of Theorem 8.1 is satisfied. By definition of $\sim$, so is hypothesis (vii), $C_{E_2(3)}(\omega_a^2)$ being the relevant solvable group. Hypothesis (ii) is clearly satisfied, since

$$Z(\mathfrak{U}) = \langle \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+2b} \rangle\ .$$

Clearly, 1 is the only 2-signalizer of $C(\omega_a^2)$, so if $\mathfrak{T}$ is a S_2-subgroup of $C(\omega_a^2)$ and $\mathfrak{L}$ is a nonidentity 2′-subgroup of $E_2(3)$ normalized by $\mathfrak{T}$, then ω_a^2 inverts $\mathfrak{L}$, so $\mathfrak{L}$ is abelian. Furthermore, $\mathfrak{L}$ is a 3-group, as every $\{2, 3\}'$-subgroup of $E_2(3)$ is cyclic. Since $\mathfrak{L}$ is a faithful $\mathfrak{T}$-module, $|\mathfrak{L}| \geqq 3^4$. Since $\mathfrak{U}$ has no abelian subgroup of order 3^5, it follows that is elementary of order 3^4. It is straightforward to verify that every elementary subgroup of $\mathfrak{U}$ of order 3^4 is conjugate to

$$\langle \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+b}, \mathfrak{X}_{a+b}, \mathfrak{X}_{3a+2b} \rangle\ ;$$

the normalizer of this last group is $\mathfrak{B}$, so does not contain a S_2-subgroup of $E_2(3)$. Thus, 1 is the only 2-signalizer of $E_2(3)$. It is trivial to verify that 1 is the only 3-signalizer of $E_2(3)$, so hypothesis (i) is verified.

Since $E_2(3)$ is of order $2^6.3^6.7.13$, and since the centralizer of every nonidentity 3-element of $E_2(3)$ is a 2, 3-group, it is easy to check that hypothesis (iii) is satisfied. Since S_2-subgroups of $E_2(3)$ are of order 64, and since (**) holds, hypothesis (v) is satisfied.

Suppose that $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ for a S_2-subgroup $\mathfrak{P}$ of $E_2(3)$, and $\mathfrak{B}$ is minimal nontrivial element of $\mathsf{N}(\mathfrak{A})$. Then $\mathfrak{A}\mathfrak{B}$ is contained in the centralizer of an involution; say $\mathfrak{A}\mathfrak{B} \subseteqq \mathfrak{C} = C(\omega_a^2)$. But, by Lemma 8.1 (iv), $\mathfrak{C}$ contains no nontrivial 2′-subgroup $\mathfrak{B}$ for which $N_{\mathfrak{C}}(\mathfrak{B})$ contains an elementary subgroup of order 2^3. This contradiction proves that $\mathsf{N}(\mathfrak{A}) = \{1\}$, which is hypothesis (vi). The proof is complete.

The remaining results in this section are proved under the hypothesis that $\mathfrak{G}$ satisfies the hypothesis of Theorem 8.1.

LEMMA 8.4. (i) *$\mathfrak{G}$ satisfies Hypothesis* 7.4.
(ii) *$\mathfrak{G}$ satisfies Hypothesis* 7.1 *for $p = 2$ and for $p = 3$.*

Proof. We first show that $\mathscr{SCN}_3(3) \neq \emptyset$. Suppose false. Let $\mathfrak{P}$ be a S_3-subgroup of $\mathfrak{G}$. Since $\mathscr{SCN}_3(\mathfrak{P}) = \emptyset$, it follows that

$\Omega_1(\mathfrak{P}) = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$ is of type (3, 3). This implies that every 3-solvable subgroup of $\mathfrak{G}$ has 3-length at most 1. Since 1 is the only 3-signalizer of $\mathfrak{G}$, it follows that $\mathfrak{P} = \boldsymbol{C}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$. Hence, 1 is the only element in $\mathsf{M}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})); 3')$. Thus, if $\mathfrak{K}$ is a 3-solvable subgroup of $\mathfrak{G}$ and S_3-subgroups of $\mathfrak{K}$ are noncyclic, then $\mathfrak{K}$ is 3-closed. This implies by definition of $\sim$ that $A_{\mathfrak{G}}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$ contains an abelian subgroup of type (4, 2) or an elementary subgroup of order 8. Neither of these possibilities holds in Aut $(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$. Hence, $\mathscr{S}_{\mathscr{C}\mathscr{N}3}(3) \neq \emptyset$. We have shown that (i), (ii), (iii), (iv) of Hypothesis 7.4 hold. If $\mathfrak{A} \in \mathscr{S}_{\mathscr{C}\mathscr{N}3}(2)$, then $\mathsf{M}(\mathfrak{A})$ contains only 1 by Hypothesis (vi) of Theorem 8.1. Suppose $\mathfrak{A} \in \mathscr{S}_{\mathscr{C}\mathscr{N}3}(3)$, and $\mathfrak{Q} \in \mathsf{M}(\mathfrak{A})$, $\mathfrak{Q} \neq 1$, $\mathfrak{Q}$ minimal with these properties. Let $\mathfrak{P}$ be a S_3-subgroup of $\boldsymbol{N}(\mathfrak{A})$. Since $\boldsymbol{Z}(\mathfrak{P})$ is noncyclic, we may choose Z in $\boldsymbol{C}(\mathfrak{Q}) \cap \boldsymbol{Z}(\mathfrak{P})^{\#}$. It follows that $\mathfrak{Q} \subseteq \boldsymbol{O}_{3'}(\boldsymbol{C}(Z))$ against Hypothesis (i) of Theorem 8.1. (i) is proved.

Hypothesis 7.1 follows from Hypothesis 7.4 since if $p = 2$ or 3 and $\mathfrak{B} \in \mathscr{U}^*(p)$, then $\boldsymbol{C}(\mathfrak{B})$ contains an element of $\mathscr{S}_{\mathscr{C}\mathscr{N}3}(p)$.

In the remainder of this section, $\mathfrak{P}$ denotes a S_3-subgroup of $\mathfrak{G}$, and $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$.

Let $\mathfrak{B}_i$, $1 \leqq i \leqq 4$, be the subgroups of $\mathfrak{B}$ of order 3. Let $\mathfrak{N}_i = \boldsymbol{N}(\mathfrak{B}_i)$, let $\mathfrak{D}_i = \mathfrak{B}^{\mathfrak{N}_i}$ and let $\mathfrak{C}_i = \boldsymbol{C}_{\mathfrak{N}_i}(\mathfrak{D}_i)$. Since $3 \in \pi_4$ and $\mathfrak{P} \subseteq \mathfrak{N}_i$, we have $\boldsymbol{O}_{3'}(\mathfrak{N}_i) = 1$. Hence, by Lemma 5.10, $\mathfrak{D}_i$ is 3-reducible in $\mathfrak{N}_i$. Finally, let $\mathfrak{L}_i = \mathfrak{N}_i/\mathfrak{C}_i$. Thus, $\mathfrak{L}_i$ may be identified with a subgroup of Aut $(\mathfrak{D}_i)$, $\mathfrak{L}_i \cong \boldsymbol{A}_{\mathfrak{N}_i}(\mathfrak{D}_i)$, and as such $\mathfrak{L}_i$ is a 3-solvable group with no nontrivial normal 3-subgroups. We let $\mathfrak{K}_i = \boldsymbol{O}^{3'}(\mathfrak{L}_i)$, so that $\mathfrak{K}_i$ is that subgroup of $\mathfrak{L}_i$ generated by the 3-elements of $\mathfrak{L}_i$.

The following lemma is important.

LEMMA 8.5. *Suppose for some i, $1 \leqq i \leqq 4$, $\mathfrak{K}_i$ contains an element of order 3 which centralizes a subgroup of $\mathfrak{D}_i$ of index 3. Then*

(a) $|\mathfrak{D}_i| = 27$.

(b) $\mathfrak{K}_i \cong SL(2, 3)$.

(c) $\mathfrak{D}_i = \mathfrak{B}_i \times \mathfrak{E}_i$, *where* $\mathfrak{E}_i \lhd \mathfrak{N}_i$.

(d) $\mathfrak{K}_i$ *is faithfully and irreducibly represented on* $\mathfrak{E}_i$.

Proof. Let $\mathfrak{U}$ be the set of 3-elements of $\mathfrak{N}_i$ which centralize some subgroup of index 3 in $\mathfrak{D}_i$. Since $\mathfrak{D}_i \lhd \mathfrak{N}_i$, $\mathfrak{U}$ is an invariant subset of $\mathfrak{N}_i$. By hypothesis, $\mathfrak{U} - \mathfrak{C}_i \neq \emptyset$.

Let $\mathfrak{U}^* = \mathfrak{U} \cap \mathfrak{P}$, and let $\mathfrak{U}_1 = \langle U \mid U \in \mathfrak{U}^* \rangle$. For any subset $\mathfrak{H}$ of $\mathfrak{N}_i$, let $\bar{\mathfrak{H}} = \mathfrak{H}\mathfrak{C}_i/\mathfrak{C}_i$. Since $\mathfrak{L}_i$ is 3-reduced, so is $\mathfrak{K}_i$. Furthermore, if $U \in \mathfrak{U} - \mathfrak{C}_i$, then $\bar{U}$ is an exceptional element in the sense of Hall-Higman [26, p. 10], or as we might say, an exceptional element, being the identity on a hyperplane of $\mathfrak{D}_i$. (In a perhaps more frequently used terminology, $\bar{U}$ is a transvection.)

Let $\mathfrak{H} = \boldsymbol{O}_{3'}(\mathfrak{N}_i \bmod \mathfrak{C}_i)$, so that $\bar{\mathfrak{U}}_1$ is faithfully represented on $\bar{\mathfrak{H}}$.

By (B), $\bar{\mathfrak{U}}_1$ centralizes some $S_{2'}$-subgroup of $\bar{\mathfrak{H}}$. Let $\bar{\mathfrak{R}} = [\bar{\mathfrak{H}}, \bar{\mathfrak{U}}_1]$. Since $\bar{\mathfrak{R}}$ is solvable, it follows that $\bar{\mathfrak{R}}$ is a $\bar{\mathfrak{H}}$-invariant 2-group on which $\mathfrak{U}_1$ is faithfully represented, and that $\bar{\mathfrak{R}} = [\bar{\mathfrak{R}}, \bar{\mathfrak{U}}_1]$. By Lemma 5.17, $\bar{\mathfrak{R}}$ is special, since by (B), $\bar{\mathfrak{U}}_1$ centralizes every abelian $\bar{\mathfrak{U}}_1$-invariant subgroup of $\bar{\mathfrak{R}}$.

It may now be verified that if $U \in \mathfrak{U}^* - \mathfrak{E}_i$ and $\mathfrak{V} = \langle U \rangle$, then $[\bar{\mathfrak{R}}, \bar{\mathfrak{V}}]$ is a quaternion group and $[\bar{\mathfrak{R}}, \mathfrak{V}]$ centralizes a subgroup $\mathfrak{F}$ of index 9 in $\mathfrak{G}_i$. Furthermore, $\mathfrak{D}_i = \mathfrak{E}_i \times \mathfrak{F}$, where $\mathfrak{E}_i = [\mathfrak{R}, \mathfrak{V}, \mathfrak{D}_i]$ is of order 9, and $\mathfrak{E}_{i0} = [\mathfrak{E}_i, \mathfrak{V}]$ is of order 3. Since $\mathfrak{E}_i$ and $\mathfrak{F}$ are $\mathfrak{U}$-invariant, and since U centralizes some hyperplane of $\mathfrak{D}_i$, it follows that $C_{\mathfrak{D}_i}(U) = \mathfrak{E}_{i0} \times \mathfrak{F}$.

Let $\bar{\mathfrak{P}}_1 = \bar{\mathfrak{P}} \cap C(\bar{\mathfrak{R}})$ and let $\bar{\mathfrak{L}} = \bar{\mathfrak{P}}\bar{\mathfrak{R}}$. Since $\bar{\mathfrak{L}}$ is faithfully represented on $\mathfrak{D}_i$, so is its subgroup $\bar{\mathfrak{P}}_1\bar{\mathfrak{R}} = \bar{\mathfrak{P}}_1 \times \bar{\mathfrak{R}}$. Hence, by Lemma 3.7 of [20], $\bar{\mathfrak{R}}$ is faithfully represented on $\tilde{\mathfrak{D}}_i = C_{\mathfrak{D}_i}(\mathfrak{P}_1)$. Since $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$ is faithfully represented on $\bar{\mathfrak{R}}$, it follows that $\bar{\mathfrak{L}}/\bar{\mathfrak{P}}_1$ is faithfully represented on $\tilde{\mathfrak{D}}_i$. By Lemma 7.7, $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$ centralizes $\bar{\mathfrak{R}}'\bar{\mathfrak{P}}_1/\bar{\mathfrak{P}}_1$. Since $\bar{\mathfrak{R}}\bar{\mathfrak{P}}_1 = \bar{\mathfrak{R}} \times \bar{\mathfrak{P}}_1$, it follows that $\bar{\mathfrak{P}}$ centralizes $\bar{\mathfrak{R}}'$.

Since $\bar{\mathfrak{U}}_1$ is faithfully represented on $\bar{\mathfrak{R}}/\bar{\mathfrak{R}}'$, and since each element $\bar{\mathfrak{U}}^*$ centralizes a subgroup of $\bar{\mathfrak{R}}/\bar{\mathfrak{R}}'$ of index 4, it is straightforward to verify that $\bar{\mathfrak{U}}_1$ is elementary. It then follows that every element of $\bar{\mathfrak{U}}_1^{\#}$ is exceptional (though we don't contend that every element of $\bar{\mathfrak{U}}_1$ centralizes a hyperplane of $\mathfrak{D}_i$).

The preceding paragraph, together with $[\bar{\mathfrak{R}}', \bar{\mathfrak{P}}] = 1$ and Corollary 2 of § 2.6 of [24] imply that $\bar{\mathfrak{U}}_1 \subseteq Z(\bar{\mathfrak{P}})$. Returning to $\mathfrak{V}$, we see that $[\bar{\mathfrak{R}}, \bar{\mathfrak{V}}]$ is $\bar{\mathfrak{P}}$-invariant. This in turn implies that $\mathfrak{F}$ is $\mathfrak{P}$-invariant. If $|\mathfrak{F}| \geqq 9$, then $\mathfrak{F}$ contains an element of $\mathscr{U}^*(3)$ and Lemma 7.4 is violated. Hence, $|\mathfrak{F}| < 9$. Since $\mathfrak{V}_i \subseteq \mathfrak{F}$, we see that (a) and (c) hold. By construction, (b) and (d) follow. The proof is complete.

J denotes the subset of $\{1, 2, 3, 4\}$ whose elements satisfy the hypothesis of Lemma 8.5.

LEMMA 8.6. *Let* $i \in J$ *and let* $\mathfrak{A}$ *be a subgroup of* $\mathfrak{E}_i$ *of order* 3. *Let* $\mathfrak{N} = N(\mathfrak{A})$, *let* $\tilde{\mathfrak{N}}$ *be the normal closure of* $\mathfrak{V}_i$ *in* $\mathfrak{N}$ *and* $\tilde{\mathfrak{N}}_1$ *be the normal closure of* $\mathfrak{D}_i$ *in* $\mathfrak{N}$. *Then*

(a) $[\tilde{\mathfrak{N}}, \tilde{\mathfrak{N}}_1] = 1$.

(b) $[\tilde{\mathfrak{N}}_1, \tilde{\mathfrak{N}}_1] \subseteq \mathfrak{A}$.

Proof. Let $\mathfrak{A}^* = \mathfrak{A} \times \mathfrak{V}_i$. Since the subgroups of $\mathfrak{E}_i$ of order 3 are permuted transitively in $\mathfrak{N}_i$, it follows that $C_{\mathfrak{N}_i}(\mathfrak{A}^*)$ contains a S_3-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$. Thus, $\mathfrak{V}_i$ is contained in the center of a S_3-subgroup of $\mathfrak{N}$, namely $\mathfrak{P}^*$. By Lemma 5.10, $\tilde{\mathfrak{N}}$ is 3-reducible in $\mathfrak{N}$. Since $\mathfrak{P}^* \subseteq C_{\mathfrak{G}}(\mathfrak{A})$ and $3 \in \pi_4$, we have $O_{3'}(C_{\mathfrak{G}}(\mathfrak{A})) = 1$, which implies that $O_{3'}(C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}) = 1$. Since $\mathfrak{D}_i/\mathfrak{A} \subseteq Z(\mathfrak{P}^*/\mathfrak{A})$, we conclude that

$$\mathfrak{D}_i/\mathfrak{A} \subseteq O_3(C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}) \subseteq O_3(\mathfrak{N}/\mathfrak{A}) ,$$

and so $\mathfrak{D}_i \subseteq C_{\mathfrak{N}}(\tilde{\mathfrak{N}})$, by 3-reducibility of $\tilde{\mathfrak{N}}$ in $\mathfrak{N}$. Since $C_{\mathfrak{N}}(\tilde{\mathfrak{N}}) \lhd \mathfrak{N}$, we have $[\tilde{\mathfrak{N}}, \tilde{\mathfrak{N}}_1] = 1$. Since $\mathfrak{D}_i/\mathfrak{A} \subseteq Z(O_3(\mathfrak{N}/A))$, we also have $[\tilde{\mathfrak{N}}_1, \tilde{\mathfrak{N}}_1] \subseteq \mathfrak{A}$.

We now set $\mathfrak{D} = \langle \mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4 \rangle$. Since $\mathfrak{B} \subseteq Z(\mathfrak{P})$, it is clear that $\mathfrak{B} \subseteq Z(\mathfrak{D})$ and that $\mathfrak{D} \lhd \mathfrak{P}$.

HYPOTHESIS 8.1. (i)

$$\mathfrak{P} \subseteq \mathfrak{M} \in \mathscr{M}\mathscr{S}(\mathfrak{G}),\ \mathfrak{B} = \Omega_1(Z(\mathfrak{P}))^{\mathfrak{M}} ,$$
$$\mathfrak{C} = C_{\mathfrak{M}}(\mathfrak{B}),\ \mathfrak{B}^* = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{P}) .$$

(ii) $\mathfrak{B}^* \not\subseteq \mathfrak{C}$.

The long argument to follow is carried out under Hypothesis 8.1.

Choose G in $\mathfrak{G}$ so that $\mathfrak{D}^G \subseteq \mathfrak{P}$ but $\mathfrak{D}^G \not\subseteq \mathfrak{C}$. The element G plays a passive but important role. If $\mathfrak{H}$ is any subset of $\mathfrak{G}$, we set $\mathfrak{H}^{\cdot} = \mathfrak{H}^G$, while if $\mathfrak{H}$ is any subset of $\mathfrak{M}$, we set $\bar{\mathfrak{H}} = \mathfrak{H}\mathfrak{C}/\mathfrak{C}$.

Let $\mathfrak{R}$ be any subgroup of $O_{3'}(\bar{\mathfrak{M}})$ which admits $\mathfrak{D}^{\cdot}$ and is minimal subject to $[\bar{\mathfrak{D}}^{\cdot}, \mathfrak{R}] \neq 1$. (Notice that $\mathfrak{R}$ is available.) Let $N = N(\mathfrak{R}) = \{i \mid 1 \leqq i \leqq 4, \bar{\mathfrak{D}}_i^{\cdot}$ does not centralize $C_{\mathfrak{R}}(\bar{\mathfrak{B}}_i^{\cdot})\}$. We argue that $N \neq \emptyset$. This is clear if $\bar{\mathfrak{B}}^{\cdot}$ centralizes $\mathfrak{R}$, so we may assume that $[\bar{\mathfrak{B}}^{\cdot}, \mathfrak{R}] \neq 1$. Since $\mathfrak{B}^{\cdot}$ is noncyclic, it follows that $\mathfrak{R} = \langle \mathfrak{R} \cap C(\mathfrak{B}_i^{\cdot}) \mid 1 \leqq i \leqq 4 \rangle$, so we can choose i such that $\bar{\mathfrak{D}}^{\cdot}$ does not centralize $C_{\mathfrak{R}}(\bar{\mathfrak{B}}_i^{\cdot})$. Minimality of $\mathfrak{R}$ guarantees that $\mathfrak{R} = C_{\mathfrak{R}}(\bar{\mathfrak{B}}_i^{\cdot})$. Thus, $\mathfrak{B}.$ does not centralize $C_{\mathfrak{R}}(\bar{\mathfrak{B}}_i^{\cdot})$. Since $\mathfrak{B}^{\cdot} \subseteq \mathfrak{D}_i^{\cdot}$, we have $i \in N(\mathfrak{R})$. In the following discussion, $\mathfrak{R}$ is a fixed subgroup of $O_{3'}(\bar{\mathfrak{M}})$ which admits $\mathfrak{D}^{\cdot}$ and is minimal subject to $[\bar{\mathfrak{D}}^{\cdot}, \mathfrak{R}] \neq 1$, and j is a fixed element of $N(\mathfrak{R})$. As already observed, $\bar{\mathfrak{B}}_j^{\cdot}$ centralizes $\mathfrak{R}$.

Let $\tilde{\mathfrak{Q}}$ be a $\mathfrak{D}_j^{\cdot}$-subgroup of $\mathfrak{R}$ minimal subject to $[\bar{\mathfrak{D}}_j^{\cdot}, \tilde{\mathfrak{Q}}] \neq 1$. Let $\mathfrak{D}_0^{\cdot} = \ker(\mathfrak{D}_j^{\cdot} \rightarrow \mathrm{Aut}(\tilde{\mathfrak{Q}}))$, so that $|\mathfrak{D}_j : \mathfrak{D}_0| = 3$ and $\mathfrak{B}_j \subseteq \mathfrak{D}_0$.

Since $\tilde{\mathfrak{Q}}$ is faithfully represented on $\mathfrak{B}$, Lemma 3.7 of [18] implies that $\tilde{\mathfrak{Q}}$ faithfully represented on $C_{\mathfrak{B}}(\mathfrak{D}_0^{\cdot})$. Since $\mathfrak{D}_j^{\cdot}$ does not centralize $C_{\mathfrak{B}}(\mathfrak{D}_0^{\cdot})$, we may choose V in $\mathfrak{D}_{\mathfrak{B}}(\mathfrak{D}_0^{\cdot}) - C(\mathfrak{D}_j^{\cdot})$. Then

$$V \in C(\mathfrak{B}_j^{\cdot}) \subseteq N(\mathfrak{B}_j^{\cdot}) = \mathfrak{N}_j^{\cdot} .$$

Thus, GVG^{-1} is a 3-element of $\mathfrak{N}_j - \mathfrak{C}_j$ which centralizes a subgroup of $\mathfrak{D}_j$ of index 3. By Lemma 8.5,

$$(8.1) \qquad j \in J,\ |\mathfrak{D}_j| = 27, \cdots .$$

This implies that $|C_{\mathfrak{B}}(\mathfrak{D}_0^{\cdot}) : C_{\mathfrak{B}}(\mathfrak{D}_j^{\cdot})| = 3$, which in turn implies that $\tilde{\mathfrak{Q}}$ is a quaternion group.

Since $\tilde{\mathfrak{Q}}$ is a quaternion group, the following assertions hold:

(a) $\mathfrak{D}^{\cdot}$ centralizes a $S_{2'}$-subgroup of $\boldsymbol{O}_{3'}(\bar{\mathfrak{M}})$.

(b) $\mathfrak{D}^{\cdot}$ centralizes every abelian subgroup of $\boldsymbol{O}_{3'}(\bar{\mathfrak{M}})$ which $\mathfrak{D}^{\cdot}$ normalizes.

(c) If $\mathfrak{F}$ is the normal closure in $\mathfrak{P}$ of $\mathfrak{D}^{\cdot}$, then $[\bar{\mathfrak{F}}, \boldsymbol{O}_{3'}(\bar{\mathfrak{M}})] = \bar{\mathfrak{E}}$ is a special 2-group whose derived group is centralized by $\bar{\mathfrak{F}}$. Namely, if either (a) or (b) were false, we could find $\mathfrak{R}$ such that $\mathfrak{R}$ contains no quaternion group. Since this is not the case, (a) and (b) hold. Now (c) follows from Lemma 5.17, together with the solvability of $\boldsymbol{O}_{3'}(\bar{\mathfrak{M}})$. We retain the previous notation and continue the argument.

Let $\mathfrak{A}^{\cdot} = [\boldsymbol{C}_{\mathfrak{B}}(\mathfrak{D}_0^{\cdot}), \mathfrak{D}_j^{\cdot}]$. Thus, $\mathfrak{A}^{\cdot}$ is a subgroup of $\mathfrak{E}_j^{\cdot}$ of order 3.

Let $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$, where $\mathfrak{B}_0 = \boldsymbol{C}_{\mathfrak{B}}(\tilde{\mathfrak{Q}})$, $\mathfrak{B}_1 = [\mathfrak{B}, \tilde{\mathfrak{Q}}]$. Since $\mathfrak{A}^{\cdot} \subseteq \mathfrak{B}$, we have $\mathfrak{B} \subseteq \boldsymbol{N}(\mathfrak{A}^{\cdot}) = \mathfrak{N}^{\cdot}$, so that $[\mathfrak{B}, \mathfrak{B}_j^{\cdot}] \subseteq \mathfrak{B}_j^{\cdot\mathfrak{N}^{\cdot}}$. By Lemma 8.6, $[\mathfrak{B}, \mathfrak{B}_i^{\cdot}, \mathfrak{D}_i^{\cdot}] = 1$. This implies that $\tilde{\mathfrak{Q}}$ centralizes $[\mathfrak{B}, \mathfrak{B}_j^{\cdot}]$, which in turn implies that $[\mathfrak{B}, \mathfrak{B}_j^{\cdot}] \subseteq \mathfrak{B}_0$. Hence, $\mathfrak{B}_j^{\cdot}$ centralizes $\mathfrak{B}_1$. Hence, $\mathfrak{D}_j^{\cdot}$ centralizes $[\mathfrak{B}_1, \mathfrak{D}_0^{\cdot}]$. As $\tilde{\mathfrak{Q}}$ normalizes $[\mathfrak{B}_1, \mathfrak{D}_0^{\cdot}]$, it follows that $\tilde{\mathfrak{Q}}$ centralizes $[\mathfrak{B}_1, \mathfrak{D}_0^{\cdot}]$. By definition of $\mathfrak{B}_1$, we get $[\mathfrak{B}_1, \mathfrak{D}_0^{\cdot}] = 1$. Hence, $[\mathfrak{B}_1, \mathfrak{D}_j^{\cdot}] = \mathfrak{A}^{\cdot}$ and $|\mathfrak{B}_1| = 9$.

Suppose $\bar{\mathfrak{P}}$ centralizes $\bar{\mathfrak{E}}'$. Since $\tilde{\mathfrak{Q}} \subseteq \bar{\mathfrak{E}}$, it follows that $\bar{\mathfrak{P}}$ centralizes $\tilde{\mathfrak{Q}}'$, a group of order 2. Hence, $\bar{\mathfrak{P}}$ normalizes $\mathfrak{B}_0 = \boldsymbol{C}_{\mathfrak{B}}(\tilde{\mathfrak{Q}}')$. Since the inverse image of $\tilde{\mathfrak{Q}}'$ in $\mathfrak{M}$ contains an involution, it follows that $\mathfrak{B}_0$ contains no element of $\mathscr{U}^*(3)$. But $\mathfrak{B}_0 \lhd \mathfrak{P}$, so the only possibility is that $\mathfrak{B}_0$ is cyclic. Since $\boldsymbol{Z}(\mathfrak{P})$ is non cyclic, we get $|\mathfrak{B}_0| = 3$.

Suppose $\bar{\mathfrak{P}}$ does not centralize $\bar{\mathfrak{E}}'$. Let $\bar{\mathfrak{E}}_1 = [\bar{\mathfrak{E}}', \bar{\mathfrak{P}}]$, and let $\mathfrak{W}$ be a subgroup of $\mathfrak{B}$ which admits $\bar{\mathfrak{P}}\bar{\mathfrak{E}}'$ and is minimal subject to $[\bar{\mathfrak{E}}_1, \mathfrak{W}] \neq 1$. Since $\bar{\mathfrak{F}}$ centralizes $\bar{\mathfrak{E}}'$, it follows that $\bar{\mathfrak{F}}$ centralizes $\mathfrak{W}$; so $\mathfrak{D}^{\cdot}$ centralizes $\mathfrak{W}$. Hence, $\mathfrak{W} \subseteq \mathfrak{B}_0 \times \mathfrak{A}^{\cdot}$. By Lemma 4.4 of [19], $\mathfrak{W}$ contains a subgroup $\mathfrak{W}_0$ of order 27 such that $\mathfrak{W}_0 \lhd \mathfrak{P}$, $|\mathfrak{P}: \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{W}_0)| = 3$. Since $|\mathfrak{A}^{\cdot}| = 3$, it follows that $\mathfrak{B}_0 \cap \mathfrak{W}_0$ is noncyclic. Let $\mathfrak{W}_1$ be a subgroup of $\mathfrak{B}_0 \cap \mathfrak{W}_0$ of order 9. Since $|\mathfrak{P}|$ is clearly larger than 3^4, we conclude from Lemma 7.6 (d) that $\mathfrak{W}_1 \in \mathscr{E}(3)$. Let I be an involution in the inverse image of $\tilde{\mathfrak{Q}}$ in $\mathfrak{M}$; then I centralizes $\mathfrak{B}_0$, so centralizes $\mathfrak{W}_1$. Hence, by Lemma 7.4, $C(I)$ is nonsolvable. This contradiction shows that $[\bar{\mathfrak{P}}, \bar{\mathfrak{E}}'] = 1$. Hence, $|\mathfrak{B}_0| = 3$, an important equality.

Since $\mathfrak{M} \in \mathscr{MS}(G)$, it follows that $\mathfrak{M} = \boldsymbol{N}(\mathfrak{B}_0)$, so that $\mathfrak{M} = \mathfrak{N}_i$ for some i, $1 \leqq i \leqq 4$. Thus, $i \in J$, $\mathfrak{B}_0 = \mathfrak{B}_i$, $\mathfrak{B}_1 = \mathfrak{E}_i$, $\mathfrak{B} = \mathfrak{D}_i$, $\mathfrak{E} = \mathfrak{E}_i$.

Let $\mathfrak{P}_0 = \mathfrak{P} \cap \mathfrak{E}_i$, $\mathfrak{N}_0 = N_{\mathfrak{N}_i}(\mathfrak{P}_0)$, so that $\mathfrak{N}_0\mathfrak{E}_i = \mathfrak{N}_i$. Let $\mathfrak{Q}_0$ be a S_2-subgroup of $\mathfrak{N}_0$ permutable with $\mathfrak{P}$. Let $\mathfrak{S} = \mathfrak{P}\mathfrak{Q}_0 \cap C(\mathfrak{B}_i)$, and set $\mathfrak{Q} = \mathfrak{S} \cap \mathfrak{Q}_0$. Then $\mathfrak{S} = \mathfrak{P}\mathfrak{Q}$.

Since $i \in J$, Lemma 8.5 implies that a $S_{2,3}$-subgroup of $\mathfrak{N}_i/\mathfrak{E}_i$ is not 3-closed. Since $\mathfrak{P}\mathfrak{Q}_0$ is not 3-closed, neither is $\mathfrak{S}$, since $|\mathfrak{P}\mathfrak{Q}_0: \mathfrak{S}| \leqq 2$. Let I be an involution of $\mathfrak{Q}$. Suppose $\mathfrak{D}_i \cap \boldsymbol{C}(I) \supset \mathfrak{B}_i$. Then $\mathfrak{D}_i \cap \boldsymbol{C}(I)$ contains a subgroup $\tilde{\mathfrak{D}}$ of order 9 with $\tilde{\mathfrak{D}} \supset \mathfrak{B}_i$. Since $\mathfrak{S}$ permutes transitively the subgroups of $\mathfrak{E}_i$ of order 3, it follows that $\tilde{\mathfrak{D}}$ is central in some S_3-subgroup of $\mathfrak{S}$, that is, I centralizes an element of $\mathscr{U}(3)$. This is not the case, since $\boldsymbol{C}(I)$ contains an element of $\mathscr{U}(2)$. This contradiction forces $\mathfrak{D}_i \cap \boldsymbol{C}(I) = \mathfrak{B}_i$ for all involutions I of $\mathfrak{Q}$. Since $\mathfrak{P} \ntriangleleft \mathfrak{S}$, $\mathfrak{Q}$ is not cyclic. Thus, $\mathfrak{Q}$ is a quaternion group. Also, $\mathfrak{P}_0 = \boldsymbol{O}_3(\mathfrak{S}) = \mathfrak{S} \cap \mathfrak{E}_i$ and $\mathfrak{S}/\mathfrak{P}_0 \cong SL(2, 3)$. In particular, $\mathfrak{B}_j^\cdot \subseteq \mathfrak{P}_0$, while $\mathfrak{D}_j^\cdot \nsubseteq \mathfrak{P}_0$.

Since $j \in J$, it follows that $[\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{B}_j^\cdot), \mathfrak{D}_j^\cdot] \subseteq \mathfrak{E}_j^\cdot$. Since $\mathfrak{B}_j^\cdot \subseteq \mathfrak{P}_0$ and $\mathfrak{D}_j^\cdot \nsubseteq \mathfrak{P}_0$, it follows that $\mathfrak{E}_j^\cdot \nsubseteq \mathfrak{P}_0$. Hence, $\mathfrak{E}_j^\cdot \cap \mathfrak{P}_0 = \mathfrak{A}^\cdot$. This implies that

(8.2) $$[\boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{B}_j^\cdot), \mathfrak{D}_j^\cdot] = \mathfrak{A}^\cdot .$$

For any subset $\mathfrak{X}$ of $\mathfrak{S}$, let $\bar{\mathfrak{X}} = \mathfrak{X}\mathfrak{E}_i/\mathfrak{E}_i$. It is important to show that

(8.3) $$\boldsymbol{C}_{\bar{\mathfrak{P}}_0}(\bar{\mathfrak{B}}_j^\cdot) = \boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{B}_j^\cdot)/\mathfrak{E}_i .$$

Namely, suppose P in $\mathfrak{P}_0$ satisfies $[\mathfrak{B}_j^\cdot, P] \subseteq \mathfrak{E}_i$. Now $\mathfrak{E}_i = \mathfrak{A}^\cdot \times \mathfrak{A}^*$, where $\mathfrak{A}^\cdot$ and $\mathfrak{A}^*$ are of order 3 and $\mathfrak{A}^\cdot \subseteq \mathfrak{E}_j^\cdot$. We may apply Lemma 8.6 to $\mathfrak{N}^\cdot$. Since $P \in \mathfrak{N}^\cdot$, we get $[\mathfrak{B}_j^\cdot, P, \mathfrak{D}_j^\cdot] = 1$. Hence, $[\mathfrak{B}_j^\cdot, P] \subseteq \mathfrak{E}_i \cap \boldsymbol{C}(\mathfrak{D}_j^\cdot) = \mathfrak{A}^\cdot$. Consider the group $\mathfrak{B}_j^\cdot \times \mathfrak{A}^\cdot$, which is normalized by the 3-element P. Since $\mathfrak{N}_j^\cdot$ permutes transitively the subgroups of $\mathfrak{E}_j^\cdot$ of order 3, it follows that $\mathfrak{B}_j^\cdot \times \mathfrak{A}^\cdot$ is in the center of some S_3-subgroup of $\mathfrak{G}$. Hence, $\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{B}_j^\cdot \times \mathfrak{A}^\cdot)$ is a $3'$-group, so P centralizes $\mathfrak{B}_j^\cdot \times \mathfrak{A}^\cdot$. We have proved (8.3).

Since $\mathfrak{A}^\cdot \subseteq \mathfrak{E}_i$, so also $\mathfrak{P}_0 \subseteq \mathfrak{N}^\cdot$. Hence

(8.4) $$[\mathfrak{P}_0, \mathfrak{D}_j^\cdot, \mathfrak{D}_j^\cdot] \subseteq \mathfrak{A}^\cdot ,$$

by Lemma 8.6 (b) applied to $\mathfrak{N}^\cdot$. We will use this fact several times.

We next show that

(8.5) $$\boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{Q}') = \boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{Q}) .$$

Since $\mathfrak{Q}\mathfrak{E}_i$ is a Frobenius group, it suffices to show that $\boldsymbol{C}_{\bar{\mathfrak{P}}_0}(\mathfrak{Q}') = \boldsymbol{C}_{\bar{\mathfrak{P}}_0}(\mathfrak{Q})$. Let $\mathscr{C}$ be part of a chief series of $\mathfrak{S}$ from $\mathfrak{P}_0$ to 1, one of whose terms is $\mathfrak{E}_i$. If $\mathfrak{F}$ is a chief factor of $\mathscr{C}$, it suffices to show that if $\mathfrak{Q}'$ centralizes $\mathfrak{F}$, so does $\mathfrak{Q}$. If this were not the case, then elements of $\mathfrak{D}_j^\cdot - \mathfrak{P}_0$ would have minimal polynomial $(x - 1)^3$ on $\mathfrak{F}$, against (8.4). Thus, (8.5) holds.

Suppose $\mathfrak{Q}'$ centralizes $\bar{\mathfrak{P}}_0$. Let $\mathfrak{P}_1 = \boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{Q}')$. We get $\mathfrak{P}_0 = \mathfrak{P}_1\mathfrak{E}_i$, $\mathfrak{P}_1 \cap \mathfrak{E}_i = 1$. Since $\mathfrak{E}_i$ is an irreducible $\mathfrak{Q}$-module, we have $\mathfrak{P}_0 = \mathfrak{P}_1 \times \mathfrak{E}_i$. If $\mathfrak{P}_1$ is not cyclic, then $\mathfrak{Q}'$ centralizes an element of $\mathscr{U}^*(3)$, which is not the case, since $\mathfrak{Q}'$ centralizes an element of $\mathscr{U}(2)$. Thus, $\mathfrak{P}_1$ is cyclic. Clearly, $\mathfrak{P}_1 \neq 1$, since $\mathfrak{B}_i \subseteq \mathfrak{P}_1$. If

$$|\mathfrak{P}_1| > 3 , \quad \text{then} \quad \mho^1(\mathfrak{P}_1) \lhd \langle \mathfrak{N}_i, \mathfrak{N}_j^\cdot \rangle ,$$

while it is trivial that $\langle \mathfrak{N}_i, \mathfrak{N}_j^\cdot \rangle$ is non solvable. Hence, $\mathfrak{P}_1 = \mathfrak{B}_i$. But then Lemma 7.8 is violated.

Let $\mathfrak{P}_2 = [\mathfrak{P}_0, \mathfrak{Q}]$. By the preceding paragraph,

$$(8.6) \qquad \bar{\mathfrak{P}}_2 \neq 1 .$$

We will show that

$$(8.7) \qquad \bar{\mathfrak{P}}_2 \text{ is of exponent 3 and class at most 2 .}$$

Let $\mathfrak{R}_1 = [\mathfrak{P}_2, \mathfrak{D}_j^*]$. Since $\mathfrak{P}_2 \subseteq \mathfrak{N}^*$, Lemma 8.6 (b) implies that $[\mathfrak{R}_1, \mathfrak{R}_1] \subseteq \mathfrak{A}^*$, so that $\bar{\mathfrak{R}}_1$ is abelian. Since $\mathfrak{D}_j^*$ is elementary so is $\bar{\mathfrak{R}}_1$. Thus $\bar{\mathfrak{R}}_1$ is a normal elementary subgroup of $\bar{\mathfrak{P}}_2$. Let Q be an element of $\mathfrak{Q}$ of order 4, and set $\mathfrak{R}_2 = \mathfrak{R}_1^Q$. We argue that $\bar{\mathfrak{R}}_1\bar{\mathfrak{R}}_2 = \bar{\mathfrak{P}}_2$. To see this, observe that since $\mathfrak{Q}'$ inverts $\mathfrak{P}_2/\boldsymbol{D}(\mathfrak{P}_2)$, and since the minimal polynomial of each element of $\mathfrak{D}_j^*$ on $\bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2)$ is a divisor of $(x-1)^2$, it follows that $\mathfrak{R}_1, \mathfrak{R}_2$ map onto subspaces of $\bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2)$ which generate $\bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2)$, so our assertion follows. Since $\bar{\mathfrak{R}}_1, \bar{\mathfrak{R}}_2$ are normal elementary subgroups of $\bar{\mathfrak{P}}_2$, (8.7) holds. Since we now have $\boldsymbol{D}(\bar{\mathfrak{P}}_2) = [\bar{\mathfrak{R}}_1, \bar{\mathfrak{R}}_2] \subseteq \bar{\mathfrak{R}}_1$, and since $\bar{\mathfrak{D}}_j^*$ centralizes $\bar{\mathfrak{R}}_1$, it follows that

$$(8.8) \qquad \mathfrak{Q} \text{ centralizes } \boldsymbol{D}(\bar{\mathfrak{P}}_2) .$$

Since $\mathfrak{Q}$ has no fixed points on $\bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2)$, it follows from (8.4) that

$$(8.9) \qquad \begin{array}{l}\mathfrak{Q} \text{ operates on } \bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2) \text{ as a multiple } d \text{ of the} \\ \text{faithful irreducible } \mathfrak{Q}\text{-representation .}\end{array}$$

In particular,

$$(8.10) \qquad |\bar{\mathfrak{P}}_2 : \boldsymbol{D}(\bar{\mathfrak{P}}_2)| = 3^{2d} .$$

Let B a generator for $\mathfrak{B}_j^*$, and for any element S of $\mathfrak{S}$, let $S]$ be the mapping of $\mathfrak{P}_2$ into itself which sends P to $[P, S]$. We may view $B]$ in more than one way. Since $\mathfrak{Q}$ centralizes $\mathfrak{P}_0/\mathfrak{P}_2$, we have $B = CU$, where $C \in \boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{Q})$ and $U \in \mathfrak{P}_2$. Since $U \in \mathfrak{P}_2$, $B]$ and $C]$ induce the same mapping from $\mathfrak{P}_2/\boldsymbol{D}(\mathfrak{P}_2)$ to itself. In particular, $[\mathfrak{P}_2, B]\boldsymbol{D}(\mathfrak{P}_2)$ admits $\mathfrak{Q}$. By Lemma 8.6 applied to $\mathfrak{N}^*$, we have $[\mathfrak{P}_2, B, \mathfrak{D}_j^*] = 1$. This implies that $\mathfrak{Q}$ centralizes $[\mathfrak{P}_2, B]\boldsymbol{D}(\mathfrak{P}_2)/\boldsymbol{D}(\mathfrak{P}_2)$, so by construction of $\mathfrak{P}_2$, we have

$$(8.11) \qquad [\mathfrak{P}_2, B] \subseteq \boldsymbol{D}(\mathfrak{P}_2) .$$

Hence, $[\mathfrak{P}_2, C] \subseteq \boldsymbol{D}(\mathfrak{P}_2)$. Since C centralizes $\mathfrak{Q}$ and $\mathfrak{Q}$ centralizes $\boldsymbol{D}(\bar{\mathfrak{P}}_2)$, we conclude that C centralizes $\bar{\mathfrak{P}}_2$, by the three subgroups lemma. Hence, B and U induce the same automorphism of $\bar{\mathfrak{P}}_2$.

By Lemma 8.6 applied to $\mathfrak{N}^*$, B centralizes the normal closure of $\mathfrak{D}_j^*$ in $\mathfrak{N}^*$. Hence, $\boldsymbol{C}_{\mathfrak{P}_2}(B) \supseteq [\mathfrak{P}_2, \mathfrak{D}_j^*]\mathfrak{E}_i\mathfrak{P}_2'$. We will show that

$$(8.12) \qquad \boldsymbol{C}_{\mathfrak{P}_2}(B) = [\mathfrak{P}_2, \mathfrak{D}_j^*]\mathfrak{E}_i\mathfrak{P}_2' ,$$

$$(8.13) \qquad |\overline{\boldsymbol{C}_{\mathfrak{P}_2}(B)} : \boldsymbol{D}(\bar{\mathfrak{P}}_2)| = 3^d .$$

Let $\mathfrak{W}_1$ be the set of fixed points of $\mathfrak{D}_j^{\cdot}$ on $\bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2)$, let

$$\mathfrak{W}_2 = \overline{[\mathfrak{D}_j^{\cdot}, \mathfrak{P}_2]}\boldsymbol{D}(\bar{\mathfrak{P}}_2)/\boldsymbol{D}(\bar{\mathfrak{P}}_2)\ , \quad \text{and let} \quad W_3 = \overline{\boldsymbol{C}_{\mathfrak{P}_2}(B)}/\boldsymbol{D}(\bar{\mathfrak{P}}_2)\ .$$

From (8.2) and (8.3), we get that $\mathfrak{W}_3 \subseteq \mathfrak{W}_1$. By Lemma 8.6 (a) applied to $\mathfrak{N}^{\cdot}$, we get $\mathfrak{W}_2 \subseteq \mathfrak{W}_3$. By (8.10) and Lemma 5.2, it follows that $|\mathfrak{W}_1| \leqq 3^d$. Using (8.10) once again, we get $|\mathfrak{W}_2| \geqq 3^d$. Since $\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_1$, it follows that $\mathfrak{W}_1 = \mathfrak{W}_2 = \mathfrak{W}_3$ is of order 3^d. This yields (8.12) and (8.13).

Let $\mathfrak{B}^* = \mathfrak{B}_j^{\cdot\mathfrak{P}}$. Since $\mathfrak{P}$ centralizes $\mathfrak{A}^{\cdot}$, we have $\mathfrak{B}^* \subseteq \mathfrak{B}_j^{\cdot\mathfrak{N}^{\cdot}}$. By Lemma 8.6, $\mathfrak{B}^*$ and $\mathfrak{D}_j^{\cdot\mathfrak{P}}$ commute elementwise. Set $\mathfrak{P}_3 = [\mathfrak{P}_2, \mathfrak{B}^*]\mathfrak{E}_i \lhd \mathfrak{P}$. Since B centralizes $\bar{\mathfrak{P}}_2/\boldsymbol{D}(\bar{\mathfrak{P}}_2)$, (8.8) implies that $\mathfrak{Q}$ normalizes $\mathfrak{P}_3$. Thus, $\mathfrak{P}_3 \lhd \mathfrak{S}$. Since $\mathfrak{D}_j^{\cdot\mathfrak{P}}$ and $\mathfrak{B}^*$ commute elementwise, $[\mathfrak{P}_2, \mathfrak{D}_j^{\cdot}]$ centralizes $\mathfrak{P}_3$. Hence, $[\mathfrak{P}_2, \mathfrak{D}_j^{\cdot}]^Q$ centralizes $\mathfrak{P}_3^Q = \mathfrak{P}_3$, Q being an element of $\mathfrak{Q} - \mathfrak{Q}'$. Since $\mathfrak{P}_2 = [\mathfrak{P}_2, \mathfrak{D}_j^{\cdot}][\mathfrak{P}_2, \mathfrak{D}_j^{\cdot}]^Q$, it follows that $\mathfrak{P}_2$ centralizes $\mathfrak{P}_3$.

Let $\tilde{\mathfrak{P}}_3 = \mathfrak{P}_3 \cap \boldsymbol{C}(\mathfrak{Q})$. Thus, $\mathfrak{P}_3 = \tilde{\mathfrak{P}}_3 \times \mathfrak{E}_i$. Clearly, $\boldsymbol{N}_{\mathfrak{S}}(\mathfrak{Q})$ normalizes $\tilde{\mathfrak{P}}_3$; so does $\mathfrak{P}_2$ since $\mathfrak{P}_2$ centralizes $\mathfrak{P}_3$. Since $\mathfrak{S} = \mathfrak{P}_2\boldsymbol{N}_{\mathfrak{S}}(\mathfrak{Q})$, we have $\tilde{\mathfrak{P}}_3 \lhd \mathfrak{S}$. Since $\mathfrak{Q}$ contains an involution, no subgroup of $\tilde{\mathfrak{P}}_3$ is in $\mathscr{U}^*(\mathfrak{P})$. Hence, $\tilde{\mathfrak{P}}_3$ is cyclic. Since $\mathfrak{P}_3$ is isomorphic to a subgroup of $\bar{\mathfrak{P}}_2$, it follows that $\tilde{\mathfrak{P}}_3$ is of order 1 or 3.

Suppose $[\mathfrak{P}_2, B] \subseteq \mathfrak{E}_i$. Then (8.3) forces $[\mathfrak{P}_2, B] = 1$. This violates (8.6), (8.10), (8.13). Thus, $\tilde{\mathfrak{P}}_3$ is of order 3 and $\mathfrak{P}_3 = [\mathfrak{P}_2, B]\mathfrak{E}_i$, and $[\mathfrak{P}_2, B]$ is of order 3. Now (8.10) and (8.13) yield that $d = 1$.

Suppose $\boldsymbol{D}(\bar{\mathfrak{P}}_2) = 1$. Then by (8.11), B centralizes $\bar{\mathfrak{P}}_2$. This conflicts with (8.10) and (8.13). Hence, $\boldsymbol{D}(\bar{\mathfrak{P}}_2) \neq 1$, so that

(8.14) $$|\mathfrak{P}_2| = 3^5\ .$$

Since $\mathfrak{B}_i\tilde{\mathfrak{P}}_3$ is a normal subgroup of $\mathfrak{S}$ centralized by $\mathfrak{Q}$, we get $\mathfrak{B}_i = \tilde{\mathfrak{P}}_3$, as $\mathfrak{Q}$ centralizes no element of $\mathscr{U}^*(3)$. Hence,

(8.15) $$\boldsymbol{Z}(\mathfrak{P}_2) = \mathfrak{D}_i\ .$$

Since $\mathfrak{P}_2$ is the normal closure of $[\mathfrak{P}_2, \mathfrak{D}_j^{\cdot}]$ in $\mathfrak{S}$, and since $\mathfrak{D}_j^{\cdot\mathfrak{N}^{\cdot}}$ is of exponent 3, it follows that $\mathfrak{P}_2$ is generated by elements of order 3. Since $\mathfrak{P}_2$ is of class 2, it follows that

(8.16) $$\mathfrak{P}_2 \text{ is of exponent } 3\ .$$

Since $\mathfrak{B}_i \subset \mathfrak{P}_2$, the group $\mathfrak{P}_2/\mathfrak{B}_i$ is of order 3^4 and is inverted by the involution of $\mathfrak{Q}$. Hence, $\mathfrak{P}_2' \subseteq \mathfrak{B}_i$. Since $\mathfrak{P}_2$ is non abelian it follows that

(8.17) $$\mathfrak{P}_2' = \mathfrak{B}_i\ .$$

We next show that $B \in \mathfrak{P}_2$. Namely, $\mathfrak{B} = CU$, so that $[C, U] = [C, CU] = [C, B]$. As we have already seen, C centralizes $\bar{\mathfrak{P}}_2$, that is, $[C, U] \in \mathfrak{E}_i$. Since $[C, U] = [C, B]$, (8.3) implies that $[C, B] = 1$. Since we have also shown that $[\mathfrak{P}_2, B]$ has order $3^d = 3$, it follows that U is not in $\boldsymbol{Z}(\mathfrak{P}_2)$. Since $\bar{\mathfrak{P}}_2/\bar{\mathfrak{P}}_2'$ is an irreducible $\mathfrak{Q}$-module, it follows that C centralizes $\mathfrak{P}_2$, as C centralizes an element of $\mathfrak{P}_2$ (namely, U) which does not map into $\bar{\mathfrak{P}}_2'$. Since C and U commute, and since B and U have order 1 or 3, it follows that C has order 1 or 3. If $C \notin \mathfrak{P}_2$, then $\Omega_1(C_{\mathfrak{P}_0}(\mathfrak{P}_2)) \cap \boldsymbol{C}(\mathfrak{Q})$ is noncyclic, so that $\mathfrak{Q}$ centralizes an element of $\mathscr{U}^*(3)$. Since this is not the case, we conclude that

$$B \in \mathfrak{P}_2 \,. \tag{8.18}$$

We will next show that $\mathfrak{P}_0 = \mathfrak{P}_2$.

Since $B \in \mathfrak{P}_2$, $[\mathfrak{P}_0, B]$ is a subgroup of $\mathfrak{P}_2$ centralized by $\mathfrak{D}_j^\cdot$. Suppose $[\mathfrak{P}_0, B] \not\subseteq \mathfrak{P}_2'$ $(=\mathfrak{B}_i)$. Let B_i be a generator for $\mathfrak{B}_i$, A a generator for $\mathfrak{A}^\cdot$. Choose P in $\mathfrak{P}_0$ so that $[P, B] = B_i^a A^b$ with $b \neq 0$. Clearly, $a \neq 0$, since $A_{\mathfrak{G}}(\langle B, A\rangle)$ is a $3'$-group. Since $[\mathfrak{P}_2, B] = \mathfrak{P}_2' = \mathfrak{B}_i$, we may choose P_2 in $\mathfrak{P}_2$ so that $[P_2, B] = B_i^{-a}$. Then $[PP_2, B] = A^b$, which is impossible. Hence, $[\mathfrak{P}_0, B] = \mathfrak{P}_2'$. Hence, $[\mathfrak{P}_1, \mathfrak{P}_2] = 1$, by the three subgroups lemma. Here $\mathfrak{P}_1 = \mathfrak{P}_0 \cap \boldsymbol{C}(\mathfrak{Q})$. Hence, $\mathfrak{P}_1 \lhd \mathfrak{S}$, so $\mathfrak{P}_1$ is cyclic, as $\mathfrak{Q}$ centralizes no element of $\mathscr{U}^*(3)$. If $|\mathfrak{P}_1| > 3$, it is easy to verify that $\mho^1(\mathfrak{P}_1) \lhd \langle \mathfrak{N}_i, \mathfrak{N}_j^\cdot\rangle$ against the nonsolvability of $\langle \mathfrak{N}_i, \mathfrak{N}_j^\cdot\rangle$. Hence, $\mathfrak{P}_1$ is of order 3, so that $\mathfrak{P}_1 = \mathfrak{B}_i$. Hence,

$$\mathfrak{P}_0 = \mathfrak{P}_2 \text{ is of order } 3^5 \,, \tag{8.19}$$

$$\mathfrak{P} \text{ is of order } 3^6 \,. \tag{8.20}$$

With the preceding information at our disposal, we turn our attention to $\mathfrak{M} = \mathfrak{N}_i$ once again. Let $\tilde{\mathfrak{S}}$ be a $S_{\{2,3\}'}$-subgroup of $\mathfrak{M}$ permutable with $\mathfrak{P}$. Then $\tilde{\mathfrak{S}}$ centralizes $\mathfrak{D}_i$, for otherwise $2^3 \cdot 3 \cdot 13$ divides $A_{\mathfrak{M}}(\mathfrak{D}_i)$, forcing nonsolvability of $A_{\mathfrak{M}}(D_i)$. Since $|\boldsymbol{O}_2(\mathfrak{M}) \colon \mathfrak{D}_i| \leqq 9$, $\tilde{\mathfrak{S}}$ also centralizes $\boldsymbol{O}_3(\mathfrak{M})/\mathfrak{D}_i$. Hence, $\tilde{\mathfrak{S}}$ centralizes $\boldsymbol{O}_3(\mathfrak{M})$, or equivalently,

$$\mathfrak{M} \text{ is a } 2, 3\text{-group} \,. \tag{8.21}$$

Let $\mathfrak{T}$ be a S_2-subgroup of $\mathfrak{N}_i$ containing $\mathfrak{Q}$. Since $\mathfrak{E}_i \in \mathscr{U}^*(\mathfrak{P})$, no element of $\mathfrak{T}^\#$ centralizes $\mathfrak{E}_i$. Thus,

$$\mathfrak{P}_0 = \boldsymbol{C}(\mathfrak{E}_i) \,. \tag{8.22}$$

Since $\mathfrak{B}_i \subseteq \mathfrak{B}$, it follows that $\boldsymbol{C}(\mathfrak{B}) \subseteq \mathfrak{N}_i = \mathfrak{M}$. By Lemma 7.4, $|\boldsymbol{C}(\mathfrak{B})|$ is odd. From (8.21) we conclude that

$$\mathfrak{P} = \boldsymbol{C}(\mathfrak{B}) \,. \tag{8.23}$$

By construction, $\mathfrak{D}_i \subseteq \mathfrak{P}_0$. Suppose j_0 is an index such that $\mathfrak{D}_{j_0} \not\subseteq \mathfrak{P}_0$. Then $[\mathfrak{D}_i, \mathfrak{D}_{j_0}] \neq 1$. In this case, two applications of Lemma 8.5 imply that $|\mathfrak{D}_i| = |\mathfrak{D}_{j_0}| = 27$ and that $[\mathfrak{P}, \mathfrak{D}_{j_0}]$ is of order 3. Hence, $[\mathfrak{P}, \mathfrak{D}_{j_0}] =$

$[\mathfrak{D}_i, \mathfrak{D}_{j_0}]$ so that $\mathfrak{D}_{j_0}$ centralizes $\mathfrak{P}_0/\mathfrak{D}_i$. This contradicts (8.6) and (8.19). Hence, no such j_0 exists, that is,

$$(8.24) \qquad \mathfrak{D} \subseteqq \mathfrak{P}_0 .$$

Our previous information shows that $Z(\mathfrak{P}) = \mathfrak{B}$. Hence, $N(\mathfrak{P})$ normalizes $\mathfrak{B}$, so permutes the groups $\mathfrak{B}^{N(\mathfrak{B}_k)}$ among themselves, $1 \leqq k \leqq 4$. By definition of $\mathfrak{D}$, we get

$$(8.25) \qquad N(\mathfrak{P}) \subseteqq N(D) .$$

Suppose $\mathfrak{D}^{\boldsymbol{\cdot}} \lhd \mathfrak{P}$. Since $\mathfrak{D} \lhd \mathfrak{P}$, it follows that $\mathfrak{D}^{\boldsymbol{\cdot}} \lhd \langle \mathfrak{P}, \mathfrak{P}^{\boldsymbol{\cdot}} \rangle$. We can choose N in $N(\mathfrak{D}^{\boldsymbol{\cdot}})$ so that $\mathfrak{P}^{\boldsymbol{\cdot}N} = \mathfrak{P}$. Hence, $\mathfrak{D}^{\boldsymbol{\cdot}N} = \mathfrak{D}^{\boldsymbol{\cdot}}$ and $\mathfrak{P}^{GN} = \mathfrak{P}$. Let $H = GN$. Then $\mathfrak{D}^{\boldsymbol{\cdot}} = \mathfrak{D}^H$ and $H \in N(\mathfrak{P})$. By (8.25), we get $\mathfrak{D}^{\boldsymbol{\cdot}} = \mathfrak{D}^H = \mathfrak{D}$. This conflicts with (8.24), since by construction $\mathfrak{D}^{\boldsymbol{\cdot}} \not\subseteq \mathfrak{P}_0$. Thus,

$$(8.26) \qquad \mathfrak{D}^{\boldsymbol{\cdot}} \not\lhd \mathfrak{P} .$$

Suppose $\mathfrak{P}^*$ is a S_3-subgroup of $\mathfrak{N}_j^{\boldsymbol{\cdot}}$ and that $\mathfrak{P}^* \subseteqq \mathfrak{N}_i$. Thus, $Z(\mathfrak{P}^*) \subseteqq \mathfrak{D}_i \cap \mathfrak{D}_j^{\boldsymbol{\cdot}} = \mathfrak{A}^{\boldsymbol{\cdot}}$. This is impossible since $|\mathfrak{A}^{\boldsymbol{\cdot}}| = 3$, $|Z(\mathfrak{P}^*)| = 9$. We conclude that

$$(8.27) \qquad \mathfrak{N}_i \cap \mathfrak{N}_j^{\boldsymbol{\cdot}} \text{ contains no } S_3\text{-subgroup of } \mathfrak{G} .$$

Since $\mathfrak{D}^{\boldsymbol{\cdot}} \not\lhd \mathfrak{P}$, (8.20) implies that $|\mathfrak{D}| \leqq 3^4$. Suppose $|\mathfrak{D}| \leqq 3^3$. Then $\mathfrak{D} \supseteqq \mathfrak{D}_i$ implies $|\mathfrak{D}| = 3^3$ and $\mathfrak{D} = \mathfrak{D}_i$. Thus, for each k, $1 \leqq k \leqq 4$, we have $\mathfrak{B} \subseteqq \mathfrak{D}_k \subseteqq \mathfrak{D}_i$. If $\mathfrak{B} = \mathfrak{D}_k$, then $\mathfrak{N}_k$ normalizes $C(\mathfrak{B})$. By (8.23), we have $\mathfrak{P} \lhd \mathfrak{N}_k$, so by (8.25), we have $\mathfrak{N}_k \subseteqq N(\mathfrak{D})$. Thus, if $|\mathfrak{D}| = 3^3$, then $\langle \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4 \rangle \subseteqq N(\mathfrak{D})$. Since $\mathfrak{A}^{\boldsymbol{\cdot}} \subseteqq Z(\mathfrak{P})$, it follows that $\mathfrak{A}^{\boldsymbol{\cdot}} = \mathfrak{B}_k$ for some k. Hence, $N(\mathfrak{A}^{\boldsymbol{\cdot}}) \subseteqq \mathfrak{N}_i$. But $\mathfrak{A}^{\boldsymbol{\cdot}}$ is a subgroup of $\mathfrak{D}_j^{\boldsymbol{\cdot}}$, so there is a S_3-subgroup $\mathfrak{P}^*$ of $\mathfrak{N}_j^{\boldsymbol{\cdot}}$ which contains $\mathfrak{A}^{\boldsymbol{\cdot}}$ in its center. This violates (8.27). Hence, $|\mathfrak{D}| = 3^4$. Since $\mathfrak{P}_0 \supset \mathfrak{D} \supset \mathfrak{D}_i$, we conclude that

$$(8.28) \qquad \mathfrak{D} \text{ is elementary of order } 3^4 .$$

Let $\mathfrak{C}$ be any subgroup of $\mathfrak{D}_j^{\boldsymbol{\cdot}}$ which is of order 3 and is not contained in $\mathfrak{P}_0$. Thus, $\mathfrak{D}_j^{\boldsymbol{\cdot}} = \mathfrak{C} \times \mathfrak{B}_j^{\boldsymbol{\cdot}} \times \mathfrak{A}^{\boldsymbol{\cdot}}$. Since $\mathfrak{D}^{\boldsymbol{\cdot}} \subseteqq \mathfrak{P}$, it follows that $\mathfrak{D}^{\boldsymbol{\cdot}} \subseteqq C_{\mathfrak{P}}(\mathfrak{D}_j^{\boldsymbol{\cdot}}) = \mathfrak{C} \cdot C_{\mathfrak{P}_0}(\mathfrak{D}_j^{\boldsymbol{\cdot}})$, and as we have already shown, $C_{\mathfrak{P}_0}(\mathfrak{B}_j^{\boldsymbol{\cdot}}) = \mathfrak{D}_i \mathfrak{B}_j^{\boldsymbol{\cdot}}$ (that is, $\mathfrak{B}_j^{\boldsymbol{\cdot}} \not\subseteq Z(\mathfrak{P}_0)$). Since $\mathfrak{D}_j^{\boldsymbol{\cdot}}$ does not centralize $\mathfrak{E}_i$, it follows that $C_{\mathfrak{P}_0}(\mathfrak{D}_j^{\boldsymbol{\cdot}}) = \mathfrak{B}_j^{\boldsymbol{\cdot}} \times \mathfrak{B}_i \times \mathfrak{A}^{\boldsymbol{\cdot}}$. Since $\mathfrak{D}^{\boldsymbol{\cdot}} \not\lhd \mathfrak{P}$, it follows that $N_{\mathfrak{P}}(\mathfrak{D}^{\boldsymbol{\cdot}}) = \mathfrak{D}^{\boldsymbol{\cdot}} \mathfrak{E}_i$. Choose P in $\mathfrak{P}_0 - N_{\mathfrak{P}}(\mathfrak{D}^{\boldsymbol{\cdot}})$. Since

$$[P, \mathfrak{B}_j^{\boldsymbol{\cdot}}, \mathfrak{B}_i \mathfrak{A}^{\boldsymbol{\cdot}}] \subseteqq \mathfrak{B}_i \subseteqq \mathfrak{D}^{\boldsymbol{\cdot}} ,$$

it follows that $[P, C] \notin \mathfrak{D}^{\boldsymbol{\cdot}}$, where C is a generator for $\mathfrak{C}$. Hence, $[P, C] = DE_i$ with E_i in $\mathfrak{E}_i - \mathfrak{A}^{\boldsymbol{\cdot}}$ and D in $\mathfrak{D}^{\boldsymbol{\cdot}} \cap \mathfrak{P}_0$. Hence, $[P, C, C] = [DE_i, C] = [E_i, C]$ is a generator for $\mathfrak{A}^{\boldsymbol{\cdot}}$. This is a subtle and important bit of information, since it shows that the $\mathfrak{E}$ module $\mathfrak{P}_0/\mathfrak{P}_0'$ has an indecomposable constituent of dimension 3. Thus,

(8.29) the indecomposable direct factors of $\mathfrak{P}_0/\mathfrak{P}_0'$ as $\mathfrak{E}$-modules are of dimensions 1 and 3 .

We note

$$\mathfrak{P} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{P}) \ . \tag{8.30}$$

Namely, $\mathfrak{Q}$ does not normalize $\mathfrak{D}$, $\mathfrak{D}^{\mathfrak{Q}} = \mathfrak{P}_0$. Since $\mathfrak{P} = \langle \mathfrak{P}_0, \mathfrak{D}^{\cdot} \rangle$, (8.30) holds. This fact has an important consequence. Namely, if $\mathfrak{P} \subseteq \mathfrak{M}^* \in \mathscr{MS}(\mathfrak{G})$ and $\mathfrak{P} \not\triangleleft \mathfrak{M}^*$, then $\mathfrak{M}^*$ satisfies Hypothesis 8.1. If this were not so, then $\mathfrak{P} \subseteq C_{\mathfrak{M}^*}(\Omega_1(Z(\mathfrak{P}))^{\mathfrak{M}^*})$. Now (8.23) implies that $\mathfrak{P} \triangleleft \mathfrak{M}^*$. Thus,

(8.31) if $\mathfrak{P} \subseteq \mathfrak{M}^* \in \mathscr{MS}(\mathfrak{G})$, then either $\mathfrak{P} \triangleleft \mathfrak{M}^*$ or $\mathfrak{M}^*$ satisfies Hypothesis 8.1 .

Let $\tilde{\mathfrak{M}}$ be an element of $\mathscr{MS}(\mathfrak{G})$ which contains $N(\mathfrak{A}^{\cdot})$ and let $\tilde{\mathfrak{P}}_0 = O_3(\tilde{\mathfrak{M}})$. We argue that

$$\mathfrak{P} \not\triangleleft N(\mathfrak{A}^{\cdot}) \ . \tag{8.32}$$

Namely, $\mathfrak{P} \subseteq N(\mathfrak{A}^{\cdot})$. Also, $N(\mathfrak{A}^{\cdot})$ contains a S_3-subgroup of $\mathfrak{N}_j$. By (8.27), this implies that $N(\mathfrak{A}^{\cdot})$ has more than one S_3-subgroup so (8.32) holds. By (8.31), it follows that $|\tilde{\mathfrak{P}}_0| = 3^5$ and that $\tilde{\mathfrak{M}} = N(\mathfrak{X})$, where $\mathfrak{X}$ is some subgroup of $Z(\mathfrak{P})$ of order 3. Clearly, $\mathfrak{X} \neq \mathfrak{B}_i$, since $N(\mathfrak{A}^{\cdot}) \not\subseteq \mathfrak{M}$. On the other hand, if I is the involution of $\mathfrak{Q}$, then $I \in \tilde{\mathfrak{M}}$. Since $\mathfrak{A}^{\cdot}$ and $\mathfrak{B}_i$ are the only subgroups of $Z(\mathfrak{P})$ of order 3 which are normalized by I, it follows that $\mathfrak{X} = \mathfrak{A}^{\cdot}$.

Since $\mathfrak{M} \neq \tilde{\mathfrak{M}}$, so also $\mathfrak{P}_0 \neq \tilde{\mathfrak{P}}_0$. Hence, (8.20) implies that $\mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0$ is of order 3^4. Now (8.24) implies that $\mathfrak{D} \subseteq \mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0$, so by (8.28), we have

$$\mathfrak{D} = \mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0 \ . \tag{8.33}$$

Since $\tilde{\mathfrak{M}} = N(\mathfrak{A}^{\cdot})$, we have $\mathfrak{D}^{\cdot} \subseteq O_3(\tilde{\mathfrak{M}})$. Hence,

$$\tilde{\mathfrak{P}}_0 = \langle \mathfrak{D}, \mathfrak{D}^{\cdot} \rangle \ . \tag{8.34}$$

Since I inverts $\mathfrak{A}^{\cdot}$, we have $I \in \tilde{\mathfrak{M}}$. Let $\tilde{\mathfrak{Q}}$ be a S_2-subgroup of $O^{3'}(\tilde{\mathfrak{M}})$ which is normalized by I. Thus, $\tilde{\mathfrak{Q}}$ is a quaternion group, and by (8.22) (with $\tilde{\mathfrak{M}}$ in the role of $\mathfrak{M}$), we get that

$$\tilde{\mathfrak{Q}}\langle I \rangle \text{ is a } S_2\text{-subgroup of } \tilde{\mathfrak{M}} \ . \tag{8.35}$$

Let J be the involution of $\tilde{\mathfrak{Q}}$ and set

$$\mathfrak{H} = \langle I, J \rangle \ . \tag{8.36}$$

Thus, $\mathfrak{H}$ is a four-group and $\mathfrak{H} \subseteq N(\mathfrak{P})$. Since $\mathfrak{B}_i = Z(\mathfrak{P}) \cap C(I)$, it follows that $\mathfrak{H} \subseteq \mathfrak{M}$. Hence,

$$(8.37) \qquad \mathfrak{P} \lhd \mathfrak{P}\mathfrak{H} = \mathfrak{M} \cap \widetilde{\mathfrak{M}} .$$

Notice that by (8.22), we have $\mathfrak{M} = \mathfrak{S}\langle J\rangle$. Consider $C_{\mathfrak{M}}(I) = C_{\mathfrak{S}}(I)\langle J\rangle$. By (8.5), it follows that $\mathfrak{Q} \lhd C_{\mathfrak{S}}(I)$. Thus, J normalizes $\mathfrak{Q}$ so that

$$(8.38) \qquad \mathfrak{Q}\langle J\rangle \text{ is a } S_2\text{-subgroup of } \mathfrak{M} .$$

Let Q be an element of $\mathfrak{Q}$ of order 4 which normalizes $\mathfrak{H}$ and let $\tilde{Q}$ be an element of $\tilde{\mathfrak{Q}}$ of order 4 which normalizes $\mathfrak{H}$. By (8.22), it follows that $N_{\mathfrak{M}}(\mathfrak{Q})/N_{\mathfrak{P}_0}(\mathfrak{Q}) \cong GL(2, 3)$. Similarly for $\widetilde{\mathfrak{M}}$. Hence, neither Q nor $\tilde{Q}$ centralizes $\mathfrak{H}$, that is,

$$(8.39) \qquad \langle Q, \mathfrak{H}\rangle \text{ and } \langle \tilde{Q}, \mathfrak{H}\rangle \text{ are dihedral groups of order 8} .$$

We set

$$(8.40) \qquad I_1 = JQ , \qquad I_2 = IQ .$$

Thus, I_1 and I_2 are involutions and

$$(8.41) \qquad \begin{aligned} I_1JI_1 &= JI , & I_1II_1 &= I , \\ I_2JI_2 &= J , & I_3II_2 &= IJ . \end{aligned}$$

Finally, we get

$$(8.42) \qquad \mathfrak{W}_0 = \langle I_1, I_2\rangle \subseteq N(\mathfrak{H}) .$$

We next show that $\mathfrak{P}_0$ is complemented in $\mathfrak{M}$. It is clear from the structure of $\mathfrak{M} = \mathfrak{N}_i$ that $C_{\mathfrak{M}}(I)$ covers $\mathfrak{N}_i/\mathfrak{C}_i = \mathfrak{N}_i/\mathfrak{P}_0$ and that $C_{\mathfrak{M}}(I) \cap \mathfrak{P}_0 = \mathfrak{B}_0$. Hence $\mathfrak{M}$ will split over $\mathfrak{P}_0$ if $C_{\mathfrak{M}}(I)$ splits over $\mathfrak{B}_i$. Since $\mathfrak{B}_i$ is an abelian 3-group, this occurs if and only if a S_3-subgroup of $C_{\mathfrak{M}}(I)$ splits over $\mathfrak{B}_i$, hence, if and only if $C_{\mathfrak{P}}(I)$ is elementary of order 3^2. Regarding I as an element of $\widetilde{\mathfrak{M}}$, we know from the structure of this group that $C_{\mathfrak{P}}(I) = C_{\tilde{\mathfrak{P}}_0}(I)$. But $\tilde{\mathfrak{P}}_0$ has exponent 3, by (8.16) and (8.19). Since the structure of $\mathfrak{M}$ implies that $|C_{\mathfrak{P}}(I)| = 3^2$, we have proved that

$$(8.43) \qquad \mathfrak{M} \text{ splits over } \mathfrak{P}_0; \ \widetilde{\mathfrak{M}} \text{ splits over } \tilde{\mathfrak{P}}_0 .$$

We define

$$(8.44) \qquad \mathfrak{X}_6 = \mathfrak{A}^{\boldsymbol{\cdot}}, \ \mathfrak{X}^5 = \mathfrak{B}_i, \ \mathfrak{X}_4 = \mathfrak{A}^{\boldsymbol{\cdot} Q}, \ \mathfrak{X}_3 = \mathfrak{X}_5^{\tilde{Q}} .$$

Since $\langle \mathfrak{X}_4, \mathfrak{X}_5, \mathfrak{X}_6\rangle = \mathfrak{B}^{\mathfrak{M}}$ and $\langle \mathfrak{X}_3, \mathfrak{X}_5, \mathfrak{X}_6\rangle = \mathfrak{B}^{\widetilde{\mathfrak{M}}}$, (8.28) implies that

$$(8.45) \qquad \mathfrak{D} = \langle \mathfrak{X}_3, \mathfrak{X}_4, \mathfrak{X}_5, \mathfrak{X}_6\rangle .$$

We set

(8.46) $$\mathfrak{X}_1 = \mathfrak{X}_3^{Q}, \mathfrak{X}_2 = \mathfrak{X}_4^{\tilde{Q}} .$$

It then follows that

(8.47) $$\mathfrak{X}_{i+1} \cdots \mathfrak{X}_6 \text{ is a subgroup of } \mathfrak{P} \text{ of order } 3^{6-i}, i = 0, \cdots, 5 .$$

Further, by construction,

(8.48) $$\mathfrak{H} \text{ normalizes } \mathfrak{X}_i, 1 \leqq i \leqq 6 .$$

We now set up a 6 by 2 array whose (i, j) entry is $\mathfrak{X}_i^{I_j}$, in case $\mathfrak{X}_i^{I_j} \subseteqq \mathfrak{P}$, and is - otherwise.

(8.49)

	I_1	I_2
$\mathfrak{X}_1$	$\mathfrak{X}_3$	—
$\mathfrak{X}_2$	—	$\mathfrak{X}_4$
$\mathfrak{X}_3$	$\mathfrak{X}_1$	$\mathfrak{X}_5$
$\mathfrak{X}_4$	$\mathfrak{X}_6$	$\mathfrak{X}_2$
$\mathfrak{X}_5$	$\mathfrak{X}_5$	$\mathfrak{X}_3$
$\mathfrak{X}_6$	$\mathfrak{X}_4$	$\mathfrak{X}_6$

We will eventually determine $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ in terms of generators and relations. To do this, a number of choices must be made, and some care is required to guarantee that these choices are possible. We have already chosen the groups $\mathfrak{X}_i$, $1 \leqq i \leqq 6$, each of order 3 and each normalized by $\mathfrak{H}$. Since $\mathfrak{X}_1 \not\subseteq \tilde{\mathfrak{P}}_0$, and since $\mathfrak{X}_1$ centralizes J, we have $\mathfrak{X}_1 \subseteqq C_{\mathfrak{M}}(J) = \tilde{\mathfrak{Q}}\langle I\rangle\mathfrak{X}_1\mathfrak{X}_6$. Hence, $\mathfrak{X}_1$ normalizes $\tilde{\mathfrak{Q}}$. We therefore may choose a generator X_1 of $\mathfrak{X}_1$ such that $X_1\tilde{Q}$ has order 3. Namely, let X_1 be any generator for $\mathfrak{X}_1$. Then $(X_1\tilde{Q})^3 \in \langle J\rangle$, so either $(X_1\tilde{Q})^3 = 1$ or $(X_1\tilde{Q})^3 = J$. Since $\tilde{Q}J = \tilde{Q}^{-1}$, in the second case we get $(X_1\tilde{Q}^{-1})^3 = 1$, or equivalently, $(\tilde{Q}X_1^{-1})^3 = 1$, or equivalently, $(X_1^{-1}\tilde{Q})^3 = 1$. Thus, we may assume that

(8.50) $$(X_1\tilde{Q})^3 = 1 .$$

For the same reason, we may choose a generator X_2 for $\mathfrak{X}_2$ such that

(8.51) $$(X_2Q)^3 = 1 .$$

We set $X_3 = X_1^{Q}$, $X_4 = X_2^{\tilde{Q}}$, $X_5 = X_3^{\tilde{Q}}$, $X_6 = X_4^{Q}$. Notice that

(8.52) $$\langle X_i\rangle = \mathfrak{X}_i, 1 \leqq i \leqq 6 .$$

It is now convenient to draw up a table listing the action of $\mathfrak{H}$ on each $\mathfrak{X}_i$. This information is available since we know the action of $\mathfrak{H}$ on $\mathfrak{X}_1$ and $\mathfrak{X}_2$, and we know the action of Q, $\tilde{Q}$ on $\mathfrak{H}$, and of course we know the way in which Q, $\tilde{Q}$ permute the $\mathfrak{X}_i$. The result of this calculation is given in the following self-explanatory table:

(8.53)

	I	J
X_1	-1	1
X_2	1	-1
X_3	-1	-1
X_4	-1	-1
X_5	1	-1
X_6	-1	1

Since $Q^2 = I$, $\tilde{Q}^2 = J$, we couple our two tables and determine the action of Q, $\tilde{Q}$ on $\mathfrak{P}_0$, $\mathfrak{P}_0$ respectively. The result of this calculation is summarized below:

(8.54)

	Q	$\tilde{Q}$
X_1	X_3	—
X_2	—	X_4
X_3	X_1^{-1}	X_5
X_4	X_6	X_2^{-1}
X_5	X_5	X_3^{-1}
X_6	X_4^{-1}	X_6

It remains to determine the commutation relations in $\mathfrak{P}$. Since $\mathfrak{D}$ is abelian and $\mathfrak{D}_i = \boldsymbol{Z}(\mathfrak{P}_0)$, we get

$$(8.55)\qquad \begin{aligned} [X_i, X_j] &= 1\ , \qquad 3 \leqq i, j \leqq 6\ , \\ [X_1, X_j] &= 1\ , \qquad 4 \leqq j \leqq 6\ . \end{aligned}$$

Since $\langle \mathfrak{X}_3, \mathfrak{X}_5, \mathfrak{X}_6 \rangle = \boldsymbol{Z}(\tilde{\mathfrak{P}}_0)$, we get

$$(8.56)\qquad [X_2, X_3] = [X_2, X_5] = [X_2, X_6] = 1\ .$$

The three remaining commutation relations can be written as follows:

$$(8.57)\qquad [X_1, X_3] = X_5^a\ ,$$

$$(8.58)\qquad [X_2, X_4] = X_6^b\ .$$

$$(8.59)\qquad [X_1, X_2] = X_3^c X_4^d X_5^e X_6^f\ .$$

Here $a, b, c, d, e, f \in F_3$. Since $\mathfrak{P}_0$ and $\tilde{\mathfrak{P}}_0$ are non abelian, we see that

$ab \neq 0$. It follows from (8.29) that $[X_1, X_2, X_2]$ does not lie in $\mathfrak{X}_5$, so $d \neq 0$. By symmetry, $c \neq 0$. To determine the values a through f explicitly, we make use of the following identities:

$$\begin{aligned}[AB, C] &= [A, C][A, C, B][B, C]\\ [A, BC] &= [A, C][A, B][A, B, C]\\ [A^{-1}, C] &= [A, C, A^{-1}]^{-1}[A, C]^{-1}\\ [A, B^{-1}] &= [A, B, B^{-1}]^{-1}[A, B]^{-1}\\ [A^{-1}, B^{-1}] &= [A, B^{-1}, A^{-1}]^{-1}[A, B^{-1}]^{-1} .\end{aligned}$$

Since X_2Q has order 3, we have

$$Q^{-1}X_2Q = IQX_2Q = IX_2^{-1}Q^{-1}X_2^{-1} = X_2^{-1}QX_2^{-1} .$$

Using this relation, conjugate (8.59) by Q, to obtain

$$[X_3, X_2^{-1}QX_2^{-1}] = X_1^{-c}X_6^{d}X_5^{e}X_4^{-f} .$$

Since X_2 and X_3 commute, we have

$$[X_3, X_2^{-1}QX_2^{-1}] = [X_3, QX_2^{-1}] .$$

By the preceding identities, $[X_3, QX_2^{-1}] = [X_3, Q][X_3, Q, X_2^{-1}]$. Now

$$[X_3, Q] = X_3^{-1}Q^{-1}X_3Q = X_3^{-1}X_1^{-1} ,$$

so that

$$[X_3, QX_2^{-1}] = X_3^{-1}X_1^{-1}[X_3^{-1}X_1^{-1}, X_2^{-1}] .$$

Since $\mathfrak{X}_2$ and $\mathfrak{X}_3$ commute, we have $[X_3^{-1}X_1^{-1}, X_2^{-1}] = [X_1^{-1}, X_2^{-1}]$. Now by the preceding identities, we have

$$\begin{aligned}[X_1^{-1}, X_2^{-1}] &= [X_1, X_2^{-1}, X_1^{-1}]^{-1}[X_1, X_2^{-1}]^{-1}\\ &= [[X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1}, X_1^{-1}]^{-1}\\ &\quad \times ([X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1})^{-1} .\end{aligned}$$

Since $[X_1, X_2, X_2^{-1}] \in \mathfrak{C}_i$, it follows that

$$[[X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1}, X_1^{-1}]^{-1} = [[X_1, X_2]^{-1}, X_1^{-1}]^{-1} .$$

We get that $[X_1^{-1}, X_2^{-1}] = X_5^{ac}X_3^{c}X_4^{d}X_5^{e}X_6^{f+bd}$. Since $X_3^{-1}X_1^{-1} = X_1^{-1}X_3^{-1}X_5^{-a}$, we see that $[X_3, X_2^{-1}QX_2^{-1}] = X_1^{-1}X_3^{-1}X_5^{-a+ac}X_3^{c}X_4^{d}X_5^{e}X_6^{f+bd}$. This gives us the following equations: $c = 1, d = -f, f + bd = d$. Conjugating (8.59) successively by I, J, IJ and using the fact that $d \neq 0$ yield the values $b = -1, a = e, d = -f$. No more information is forthcoming from $\mathfrak{M}$, so we conjugate (8.59) by $\tilde{Q}$ and work in $\tilde{\mathfrak{M}}$. We state the result of these calculations:

$$(8.60)\qquad a = -1, b = -1, c = 1, d = -1, e = -1, f = 1\,.$$

Let $\mathfrak{N}^* = \langle \mathfrak{X}_2, \mathfrak{X}_5 \rangle$ and note that $\mathfrak{N}^* = C_{\mathfrak{P}}(I)$. By construction, $\mathfrak{X}_2 \subseteq O_3(\widetilde{\mathfrak{M}})$ and $\mathfrak{X}_3 \subseteq Z(O_3(\widetilde{\mathfrak{M}}))$. Hence,

$$C_{\mathfrak{P}}(\mathfrak{X}_2) = C_{\mathfrak{P}}(\mathfrak{N}^*) = \langle X_2, X_3, X_5, X_6 \rangle\,,$$

so that $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{N}^*)| = 9$. With $\mathfrak{N}^*$ in the role of $\mathfrak{A}$ in Lemma 7.6 (c), it follows that $\mathfrak{N}^*$ centralizes every abelian subgroup of $\mathcal{U}(\mathfrak{N}^*; 2)$.

Since $O^{3'}(\mathfrak{M}) \cap C(I) = \mathfrak{N}^*\mathfrak{Q}$, it follows that $\mathfrak{Q}$ is normalized by $\mathfrak{N}^*$ but is not centralized by $\mathfrak{N}^*$. Let $\mathfrak{T}^*$ be a $S_{2,3}$-subgroup of $C(I)$ which contains $\mathfrak{N}^*\mathfrak{Q}$. Then $\mathfrak{T}^*$ contains an element $\mathscr{U}(2)$. Let $\mathfrak{T}_2$ be a S_2-subgroup of $\mathfrak{T}^*$ which contains $\mathfrak{Q}$. By Lemma 7.5, there is an element $\mathfrak{M}_1$ of $\mathscr{M}\mathscr{S}(\mathfrak{G})$ such that $(\mathfrak{N}^*, \mathfrak{T}^*, \mathfrak{T}_2, \mathfrak{R} = O_2(\mathfrak{M}_1), \mathfrak{M}_1)$ satisfies all parts of Lemma 7.5 with $\mathfrak{N}^*$ in the role of $\mathfrak{B}$, $\mathfrak{R}$ in the role of $\mathfrak{H}$, $\mathfrak{M}_1$ in the role of $\mathfrak{M}$. Since by (e) of Lemma 7.5, $\mathfrak{Q} \subseteq \mathfrak{R}$, it follows that $\mathfrak{M}_1 = C(I)$. Hence, $J \in \mathfrak{M}_1$.

The next task is shown that

$$(8.61)\qquad N(\mathfrak{D}) \subseteq N(\mathfrak{P})\,.$$

By our preceding results, $\mathfrak{D} \lhd \mathfrak{P}$. It is straightforward to verify that $N_{\mathfrak{M}}(\mathfrak{D}) \subseteq N_{\mathfrak{M}}(\mathfrak{P})$. Let $\mathfrak{M}^* \in \mathscr{M}\mathscr{S}(\mathfrak{G})$ with $N(\mathfrak{D}) \subseteq \mathfrak{M}^*$. If $\mathfrak{P} \lhd \mathfrak{M}^*$, we have our desired containment. Otherwise, $\mathfrak{M}^*$ satisfies Hypothesis 8.1. Hence, $N(\mathfrak{D}) = N_{\mathfrak{M}^*}(\mathfrak{D}) \subseteq N_{\mathfrak{M}^*}(\mathfrak{P}) \subseteq N(\mathfrak{P})$, as desired.

We next show that

$$(8.62)\qquad \text{if}\quad X \in \mathfrak{X}_2^\#,\ Y \in \mathfrak{X}_5^\#\,,\quad \text{then}\quad |C(XY)|_3 = 3^4\,.$$

Let $Z = XY$. Let $X^* = X^{\tilde{Q}}$, $Y^* = Y^{\tilde{Q}}$, $Z^* = X^*Y^*$. Then $X^* \in \mathfrak{X}_4$, $Y^* \in \mathfrak{X}_3$, so it suffices to show that $\mathfrak{D}$ is a S_3-subgroup of $C(Z^*)$. Suppose false. Let $\tilde{\mathfrak{D}}$ be a S_3-subgroup of $C(\mathfrak{X}^*)$ which contains $\mathfrak{D}$, and let $\mathfrak{D} \subseteq \mathfrak{D}^* \subseteq \tilde{\mathfrak{D}}$, with $|\mathfrak{D}^*: \mathfrak{D}| = 3$. Then $\mathfrak{D}^* \subseteq N(\mathfrak{D}) \subseteq N(\mathfrak{P})$, so $\mathfrak{D}^* \subseteq \mathfrak{P}$. However, $\mathfrak{D} = C_{\mathfrak{P}}(Z^*)$. Notice that we have shown that $\mathfrak{D} \triangle C(Z^*)$. Namely, $\mathfrak{D}$ is a S_3-subgroup of $C(Z^*)$, and since $\mathfrak{D} \in \mathscr{S}_{\mathscr{C}\mathscr{N}}3(P)$, we have $O_{3'}(C(Z^*)) = 1$ so that

$$(8.63)\qquad \mathfrak{D} \text{ is a normal } S_3\text{-subgroup of } C(Z^*)\,.$$

Retaining the preceding notation we will show that $\langle I \rangle = C_{\mathfrak{R}}(Z)$. Suppose false. Since $\langle I \rangle = C_{\mathfrak{R}}(\mathfrak{N}^*)$, it follows that $1 \neq [C_{\mathfrak{R}}(Z), \mathfrak{N}^*]$. This violates the fact that $C(Z)$ is 3-closed by (8.63).

We next observe that $\mathfrak{X}_2 \sim \mathfrak{X}_6$ so that $|C(\mathfrak{X}_2)|_2 = |C(\mathfrak{X}_5)|_2 = 8$. These equalities together with the preceding paragraph show that $\mathfrak{R}$ is extra special of width 2 and that

$$(8.64)\qquad \begin{array}{l}\mathfrak{R} \text{ is the central product of quaternion groups} \\ \mathfrak{Q}, \mathfrak{Q}_1, \text{ where}\quad \mathfrak{Q} = C_{\mathfrak{R}}(\mathfrak{X}_5), \mathfrak{Q}_1 = C_{\mathfrak{R}}(\mathfrak{X}_2)\,.\end{array}$$

This choice of notation conforms with our previous definition of $\mathfrak{Q}$.

Since $\mathfrak{R}^*$ maps onto a S_3-subgroup of $\mathfrak{A}_{\mathfrak{G}}(\mathfrak{K})$, it follows that $\mathfrak{R}^*\mathfrak{K} \triangleleft \mathfrak{M}_1$. Let

$$\mathfrak{L} = N_{\mathfrak{M}_1}(\mathfrak{R}^*) \, , \tag{8.65}$$

so that $\mathfrak{L} \cap \mathfrak{K} = \langle I \rangle$, and $\mathfrak{M}_1 = \mathfrak{K}\mathfrak{L}$. $\mathfrak{L}$ acts as a permutation group on the subgroups of $\mathfrak{R}^*$ of order 3. By the previous arguments, $\mathfrak{X}_2$ and $\mathfrak{X}_5$ are permuted among themselves. Let

$$\mathfrak{L}^* = N_{\mathfrak{L}}(\mathfrak{X}_2) = N_{\mathfrak{L}}(\mathfrak{X}_5) \tag{8.66}$$

so that $|\mathfrak{L}:\mathfrak{L}^*| \leqq 2$. Also, if $L \in \mathfrak{L}^*$ and L centralizes $\mathfrak{X}_5$, then $L \in \mathfrak{R}^*\langle I \rangle$. Hence, $\mathfrak{L}^* = \mathfrak{R}^*\mathfrak{H}$, and $|\mathfrak{M}_1 : \mathfrak{K}\mathfrak{L}^*| \leqq 2$. Let $\mathfrak{L}_2$ be a S_2-subgroup of $\mathfrak{L}$ which contains $\mathfrak{H}$. Thus, $|\mathfrak{L}_2| = 4$ or 8.

We must now show that

$$\mathfrak{L} = \mathfrak{L}^* \, . \tag{8.67}$$

Suppose false. Since $\mathfrak{R}^{*\tilde{Q}} = \langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle$, it follows that $\mathfrak{L}^{\tilde{Q}}$ normalizes $\langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle$. From (8.63), we conclude that $\mathfrak{D}$ char $C(\mathfrak{X}_3\mathfrak{X}_4)$. Hence, $N(\mathfrak{X}_3\mathfrak{X}_4) \subseteqq N(\mathfrak{D})$. Now by (8.61), we have $N(\mathfrak{D}) \subseteqq N(\mathfrak{P})$. Thus $\mathfrak{L}^{\tilde{Q}}$ normalizes $\mathfrak{P}$.

It is a straightforward consequence of (8.55) through (8.60) that $\mathfrak{P}_0 \cup \tilde{\mathfrak{P}}_0$ is the set of elements of $\mathfrak{P}$ of order at most 3. Hence, $\mathfrak{P}$ contains exactly $3^6 - 2 \cdot 3^5 + 3^4 = 4.3^4$ elements of order 9. Thus, some involution I_0 of $\mathfrak{L}_2^{\tilde{Q}}$ centralizes an element P of $\mathfrak{P}$ of order 9. It is clear from (8.53) that $I_0 \notin \mathfrak{H}$.

If $X \in \mathfrak{X}_5^{\#}\mathfrak{X}_6^{\#}$, we will show that $C(X) \subseteqq N(\mathfrak{P})$. Suppose false. Let $\mathfrak{M}^* \in \mathscr{M}\mathscr{S}(\mathfrak{G})$ with $C(X) \subseteqq \mathfrak{M}^*$. We may apply all the preceding results to $\mathfrak{M}^*$ in place of $\mathfrak{M}$ and conclude that $O_3(\mathfrak{M}^*)$ is of exponent 3 and order 3^5. However, $\mathfrak{P}_0$ and $\tilde{\mathfrak{P}}_0$ are the only subgroups of $\mathfrak{P}$ meeting these conditions, so $C(X) \subseteqq \mathfrak{M}$ or $C(X) \subseteqq \tilde{\mathfrak{M}}$, from which the desired containment is obvious. In particular,

$$C(P^3) \subseteqq N(\mathfrak{P}) \, . \tag{8.68}$$

Let $\mathfrak{N}_2$ be a S_2-subgroup of $N(\mathfrak{P})$ which contains $\mathfrak{L}_2^{\tilde{Q}}$. By (8.23) $\mathfrak{N}_2$ is faithfully represented on $\mathfrak{B} = Z(\mathfrak{P}) = \langle \mathfrak{X}_5, \mathfrak{X}_6 \rangle$. It is clear that Aut $(\mathfrak{P})$ is a 2, 3-group, so we conclude that

$$N(\mathfrak{P}) = \mathfrak{P}\mathfrak{L}^{\tilde{Q}} \, . \tag{8.69}$$

It now follows from (8.68), (8.69), and (8.53) that

$$C(P^3) = \mathfrak{P}\langle I_0 \rangle \, . \tag{8.70}$$

By hypothesis, $C(I_0)$ is solvable. Let $\mathfrak{F} = O_2(C(I_0))$. Suppose $\langle P \rangle$ acts faithfully on $\mathfrak{F}$. Then $m(\mathfrak{F}) \geqq 6$, since P has order 9. But $\mathfrak{M}_1$

contains a S_2-subgroup of $\mathfrak{G}$, and since S_2-subgroups of $\mathfrak{M}_1$ are extensions of $\mathfrak{R}$ by a 4 group, it follows that every 2-subgroup of $\mathfrak{G}$ is generated by 4-elements (naturally, this uses the action of the 4-group on $\mathfrak{R}$). Hence, P^3 centralizes $\mathfrak{F}$. By (8.70), we get $\mathfrak{F} = \langle I_0 \rangle$.

By Lemma 5.38 (a)(ii), $C(I_0)$ contains an element $\mathfrak{U}$ of $\mathscr{U}(2)$. Since $O_2(C(I_0)) = \langle I_0 \rangle$, we get that $O_{2,2'}(C(I_0)) = \langle I_0 \rangle \times O_{2'}(C(I_0))$, so that by Lemma 7.1, $\mathfrak{U}$ centralizes $O_{2,2'}(C(I_0))$. But $C(I_0)$ is solvable, so that $O_{2,2'}(C(I_0))$ contains its centralizer. Thus, $\mathfrak{U} \subseteq O_{2,2'}(C(I_0))$, an absurdity. This contradiction establishes (8.67). Notice that (8.67) is equivalent to

(8.71) $$\mathfrak{M}_1 = \mathfrak{R}\mathfrak{N}^* \langle J \rangle .$$

Since J inverts $\mathfrak{N}^*$, it follows that $\mathfrak{Q}\langle J \rangle$ and $\mathfrak{Q}_1\langle J \rangle$ are both isomorphic to S_2-subgroups of $GL(2, 3)$. This implies that

(8.72) $$C_{\mathfrak{M}_1}(J) \text{ is elementary of order } 8 .$$

The hard work is now completed. We may now determine the Weyl group. Recall that $I_1 = JQ$, $I_2 = I\tilde{Q}$, so that I_1 and I_2 are involutions. Let $W = I_1 I_2$. Thus W^3 centralizes $\mathfrak{H}$. Since W centralizes no element of $\mathfrak{H}^\#$, W^3 is not in $\mathfrak{H}^\#$. Since $W^3 \in O(\mathfrak{H}) \subseteq \mathfrak{M}_1$, and since the structure of $C_{\mathfrak{M}_1}(J)$ is given in (8.72), it follows that $W^6 = 1$, so that W is of order 3 or 6.

From (8.49), we get that $\mathfrak{X}_1^{W^3} = \mathfrak{X}_1^{I_2} \neq \mathfrak{X}_1$, and conclude that W is of order 6. Thus,

(8.73) $$W_0 = W^3 \text{ is an involution in the center of } \langle I_1, I_2 \rangle = \mathfrak{W}_0 .$$

We argue that

(8.74) $$\mathfrak{P} \cap \mathfrak{P}^{W_0} = 1 .$$

Since $\mathfrak{P} \cap \mathfrak{P}^{W_0}$ is normalized by $\mathfrak{H}$ and by W_0, (8.53) implies that if $\mathfrak{P}^* = \mathfrak{P} \cap \mathfrak{P}^{W_0}$, then

$$\mathfrak{P}^* = (\mathfrak{P}^* \cap \langle \mathfrak{X}_1, \mathfrak{X}_6 \rangle)(\mathfrak{P}^* \cap \langle \mathfrak{X}_2, \mathfrak{X}_5 \rangle)(\mathfrak{P}^* \cap \langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle) .$$

If $X \in \mathfrak{X}_3^\# \mathfrak{X}_4^\#$, we know that $C(X) \subseteq N(\mathfrak{D})$. This fact, coupled with (8.49) implies that $\mathfrak{P}^* = 1$, so that (8.74) holds.

Let $\mathfrak{B} = \mathfrak{P}\mathfrak{H}$. (No confusion with previous notation is to be feared.) We then get that $\mathfrak{M} = \mathfrak{B} \cup \mathfrak{B} I_1 \mathfrak{B}$, $\tilde{\mathfrak{M}} = \mathfrak{B} \cup \mathfrak{B} I_2 \mathfrak{B}$. Hence, (8.49) implies that conditions (i′) and (iv) of Théorème 1 of [40] are satisfied. Hence, $\mathfrak{B}\mathfrak{W}_0\mathfrak{B} = \mathfrak{G}_0$ is a group and if we let $\mathfrak{B}_X$ be the largest subset of $\mathfrak{B}$ such that $\mathfrak{B}_X^X \subseteq \mathfrak{P}^{W_0}$, it follows easily from (8.74) that each element of $\mathfrak{G}_0$ has a unique representation of the shape BXB_X, $B \in \mathfrak{B}$, $X \in \mathfrak{W}_0$, $B_X \in \mathfrak{B}_X$. Thus, $|\mathfrak{G}_0| = |E_2(3)|$, by an easy calculation. Hence, (8.41), (8.50), (8.51), (8.53), (8.54), (8.57), (8.58), (8.59), (8.60), (8.73) determine the multiplication table of $\mathfrak{G}_0$. Thus, if $\mathfrak{G}^*$ is any group which satisfies the hypothesis of Theorem 8.1 and also satisfies Hypothesis 8.1, it follows that $\mathfrak{G}^*$ contains a subgroup isomorphic to $\mathfrak{G}_0$. Since we may take $\mathfrak{G}^* = E_2(3)$, it follows that $\mathfrak{G}_0 \cong E_2(3)$, and so $i(\mathfrak{G}_0) = 1$. Clearly, $\mathfrak{G}_0$ contains $\mathfrak{M}_1$, so that $\mathfrak{G}_0$ contains the centralizer of each of its involutions. Hence, $i(\mathfrak{G}) = 1$, by Lemma 5.35.

Since $E_2(3)$ does not satisfy $E_{7,13}$ (by Sylow's theorem), it follows from Lemma 5.35 that $\mathfrak{G}_0 = \mathfrak{G} \cong E_2(3)$.

The remaining lemmas are proved under the following hypothesis:

HYPOTHESIS 8.2. Whenever $\mathfrak{P} \subseteq \mathfrak{M} \in \mathscr{MS}(\mathfrak{G})$ and $\mathfrak{B} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))^{\mathfrak{M}}$, then $V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{P}) \subseteq \boldsymbol{C}_{\mathfrak{M}}(\mathfrak{B})$.

We must derive a contradiction from this hypothesis. When this is done, the proof of Theorem 8.1 will be complete.

LEMMA 8.7. *If $\mathfrak{T}$ is a 2, 3-subgroup of $\mathfrak{G}$ and $\mathfrak{T}_3$ is a S_3-subgroup of $\mathfrak{T}$, then* $V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{T}_3) \lhd \mathfrak{T}$.

Proof. We assume without loss of generality that $\mathfrak{T}_3 \subseteq \mathfrak{P}$. First, suppose $\mathfrak{T}_3 = \mathfrak{P}$. Let $\mathfrak{T} \subseteq \mathfrak{M} \in \mathscr{MS}(\mathfrak{G})$, and let $\mathfrak{T}^*$ be a $S_{2,3}$-subgroup of $\mathfrak{M}$ containing $\mathfrak{T}$. Let $\mathfrak{B} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))^{\mathfrak{M}}$, $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{M}}(\mathfrak{B})$. As $\mathfrak{C} \lhd \mathfrak{M}$, $\mathfrak{C} \cap \mathfrak{T}^*$ is a $S_{2,3}$-subgroup of $\mathfrak{C}$. By Hypothesis 8.2, $\mathfrak{B}^* \subseteq \mathfrak{C} \cap \mathfrak{T}^*$, where $\mathfrak{B}^* = \mathfrak{B}(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{P})$. Since $\mathfrak{B} \subseteq \mathfrak{B}$, Lemmas 7.4 and 5.38 imply that $| \mathfrak{C} \cap \mathfrak{T}^* |$ is odd. Hence, $\mathfrak{C} \cap \mathfrak{T}^* \lhd \mathfrak{T}^*$ implies $\mathfrak{B}^* = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{C} \cap \mathfrak{T}^*) \lhd \mathfrak{T}^*$.

We may now assume that $\mathfrak{T}_3 \subset \mathfrak{P}$. We proceed by induction on $| \mathfrak{P} | / | \mathfrak{T}_3 |$. Let $\mathfrak{B}^* = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{T}_3)$. As $\mathfrak{B}^*$ is generated by conjugates of $\mathfrak{B}$, it follows that $\mathfrak{B}^*$ centralizes $\boldsymbol{O}_2(\mathfrak{T})$. Hence, if $\mathfrak{B}^* \neq 1$, then $\boldsymbol{O}_2(\mathfrak{T}) = 1$, so that $\boldsymbol{O}_{2,3}(\mathfrak{T}) = \boldsymbol{O}_3(\mathfrak{T})$. If $\mathfrak{B}^* = 1$, the lemma is trivial, so suppose $\mathfrak{B}^* \neq 1$. In particular, $\boldsymbol{O}_3(\mathfrak{T}) \neq 1$. If $\mathfrak{T}_3$ is not a S_3-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{T}))$, let $\mathfrak{T}^*$ be a $S_{2,3}$-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{T}))$ containing $\mathfrak{T}$, and let $\mathfrak{T}_3^*$ be a S_3-subgroup of $\mathfrak{T}^*$ which contains $\mathfrak{T}_3$. Then $V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{T}_3^*) \lhd \mathfrak{T}^*$. In particular, $[\mathfrak{B}^*, \mathfrak{T}]$ is a 3-group, so $\mathfrak{B}^* \lhd \mathfrak{T}$. Hence, we may assume that $\mathfrak{T}_3$ is a S_3-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{T}))$.

Let $\mathfrak{W}_0 = \Omega_1(\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{T})))$, so that $\mathfrak{B} \subseteq \mathfrak{W}_0$. Since $| \boldsymbol{C}_T(\mathfrak{W}_0) |$ is odd, it follows that $\boldsymbol{O}_3(\mathfrak{T}) = \boldsymbol{C}_{\mathfrak{T}}(\mathfrak{W}_0)$. Suppose $\mathfrak{B}^* \not\subseteq \boldsymbol{O}_2(\mathfrak{T})$. Choose G in $\mathfrak{G}$ so that $\mathfrak{D}^G \subseteq \mathfrak{B}^*$ but $\mathfrak{D}^G \not\subseteq \boldsymbol{O}_3(\mathfrak{T})$, and for any subset $\mathfrak{S}$ of $\mathfrak{G}$, let $\mathfrak{S}^\cdot = \mathfrak{S}^G$.

It is a straightforward consequence of Hypothesis 8.2 that $\mathfrak{D}' = 1$.

As $\mathfrak{D}^\cdot$ acts nontrivially on $\boldsymbol{Q}_3^1(\mathfrak{T})$, we let $\mathfrak{Q}$ be a $\mathfrak{D}^\cdot$-invariant subgroup of $\boldsymbol{Q}_3^1(\mathfrak{T})$ minimal subject to $[\mathfrak{D}^\cdot, \mathfrak{Q}] \neq 1$. Let $\mathfrak{D}_0^\cdot = \boldsymbol{C}_{\mathfrak{D}^\cdot}(\mathfrak{Q})$, so that $| \mathfrak{D}^\cdot : \mathfrak{D}_0^\cdot | = 3$. Thus, $\widetilde{\mathfrak{W}}_0 = \boldsymbol{C}_{\mathfrak{W}_0}(\mathfrak{D}_0^\cdot)$ is invariant under $\mathfrak{D}^\cdot$ and $\mathfrak{Q}$.

Let $N = \{i \mid 1 \leqq i \leqq 4, \mathfrak{B}_i^\cdot \subseteq \mathfrak{D}_0^\cdot, \mathfrak{D}_i^\cdot \not\subseteq \mathfrak{D}_0^\cdot\}$. If $\mathfrak{B}^\cdot \subseteq \mathfrak{D}_0^\cdot$, then it is obvious that $N \neq \varnothing$. If $\mathfrak{B}^\cdot \subseteq \mathfrak{D}_0^\cdot$, then no $\mathfrak{D}_i^\cdot$ is contained in $\mathfrak{D}_0^\cdot$, $1 \leqq i \leqq 4$. Since $\mathfrak{B}^\cdot \cap \mathfrak{D}_0^\cdot$ is of order 3 in this case, we again conclude that $N \neq \varnothing$. Choose $i \in N$. Thus, $\mathfrak{D}^\cdot = \langle \mathfrak{D}_0^\cdot, \mathfrak{D}_i^\cdot \rangle$ and $| \mathfrak{D}_i^\cdot : \mathfrak{D}_i^\cdot \cap \mathfrak{D}_0^\cdot | =$

3. Since $\mathfrak{Q}$ is faithfully represented on $\tilde{\mathfrak{W}}_0$, it follows that $[\mathfrak{D}_i^\cdot, \tilde{\mathfrak{W}}_0] \neq 1$. By Lemma 8.5, $[\mathfrak{Q}, \tilde{\mathfrak{W}}_0]$ is of order 9 and is not centralized by $\mathfrak{D}^\cdot$. Since $\mathfrak{B} \subseteq \mathfrak{W}_0$, so also $\mathfrak{B} \subseteq \tilde{\mathfrak{W}}_0$. Hence, $\mathfrak{Q}$ centralizes some B of $\mathfrak{B}^\#$, so if $\mathfrak{Q} = \mathfrak{Q}^*/C_{\mathfrak{X}}(\mathfrak{W}_0)$, then $\langle \mathfrak{D}^\cdot, \mathfrak{Q}^* \rangle \subseteq C(B)$. By the preceding argument, $[\mathfrak{Q}^*, \mathfrak{D}^\cdot]$ is a 3-group, violating the nontrivial action of $\mathfrak{D}^\cdot$ on $\mathfrak{Q}$. Thus, $\mathfrak{B}^* \subseteq O_3(\mathfrak{X})$, and so $\mathfrak{B}^* \lhd \mathfrak{X}$, completing the proof of this lemma.

For the remainder of this section, we let

$$\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{P}), \mathfrak{N} = N(\mathfrak{B}) .$$

LEMMA 8.8. (i) *$\mathfrak{N}$ contains no element of $\mathscr{T}(2)$.* (See Definition 2.9.)

(ii) *If $\mathfrak{X}_0$ is any 2-subgroup of $\mathfrak{N}$, then $A_{\mathfrak{N}}(\mathfrak{X}_0)$ does not contain a subgroup of type* (3, 3).

(iii) *If $\mathfrak{C}$ is any subgroup of $Z(\mathfrak{B})$ of type* (3, 3), *then* $(\mathfrak{A}, \mathfrak{C}) \in \mathscr{N}$ *for all* $\mathfrak{A} \in \mathscr{U}(2)$. (See Definition 7.2).

(iv) *If $\mathfrak{X}_1$ is an abelian 2-subgroup of $\mathfrak{N}$, the $A_{\mathfrak{N}}(\mathfrak{X}_1)$ is a $3'$-group.*

Proof. We first prove (iii). We invoke Lemma 7.4, so that (iii) will hold if we can show that $\mathfrak{C}$ centralizes every element of $\mathsf{M}(\mathfrak{C}; 2)$. Suppose $\mathfrak{Q} \in \mathsf{M}(\mathfrak{C}; 2)$ is minimal subject to $[\mathfrak{Q}, \mathfrak{C}] \neq 1$. Let $\mathfrak{C}_0 = C_{\mathfrak{C}}(\mathfrak{Q}) \neq 1$. Let $\mathfrak{X}$ be a $S_{2,3}$-subgroup of $C(\mathfrak{C}_0)$ containing $\mathfrak{C}\mathfrak{Q}$. Since $C(\mathfrak{C}_0) \supseteq \mathfrak{B}$, it follows that if $\mathfrak{X}_0$ is a $S_{2,3}$-subgroup of $C(\mathfrak{C}_0)$ containing $\mathfrak{B}$, then $\mathfrak{B} \subseteq O_3(\mathfrak{X}_0)$. By Lemma 0.7.5, we have $\mathfrak{C} \subseteq O_3(\mathfrak{X})$, so that $[\mathfrak{Q}, \mathfrak{C}] \subseteq \mathfrak{Q} \cap O_3(\mathfrak{X}) = 1$. (iii) is proved.

Let $\mathfrak{X} \in \mathscr{T}(2), \mathfrak{X} \subseteq \mathfrak{N}$. We may assume that $\mathfrak{X}$ is a noncyclic abelian group of order 8. Since $\mathfrak{B} \subseteq Z(\mathfrak{B})$, $Z(\mathfrak{B})$ is noncyclic. Hence, $\mathfrak{X}$ contains an involution I such that $C(I) \cap Z(\mathfrak{B})$ is noncyclic. Thus, $C(I)$ contains an element of $\mathscr{U}(2)$ and also a subgroup $\mathfrak{C}$ of $Z(\mathfrak{B})$ of type (3, 3). By hypothesis, $C(I)$ is solvable, in violation of (iii). (i) is proved.

(ii) is a straightforward consequence of (i).

To prove (iv), let $\mathfrak{X}_1$ be an abelian 2-subgroup of $\mathfrak{N}$ minimal subject to $3 \mid |A_{\mathfrak{N}}(\mathfrak{X}_1)|$. Thus, $\mathfrak{X}_1$ is a four-group, and the involutions of $\mathfrak{X}_1$ are all $\mathfrak{N}$-conjugate. Thus, (iii) implies that $C(I) \cap Z(\mathfrak{B})$ is cyclic for all $I \in \mathfrak{X}_1^\#$. This implies that $|\Omega_1(Z(\mathfrak{B}))| \leq 3^3$. Since the reverse inequality holds by (B), we find that $Z(\mathfrak{P}) \cap Z(\mathfrak{B})$ is cyclic. This is not the case, since $\mathfrak{B} \subseteq Z(\mathfrak{P}) \cap Z(\mathfrak{B})$. (iv) is proved.

LEMMA 8.9. *If $\mathfrak{X}$ is a 2, 3-subgroup of $\mathfrak{G}$ and $\mathfrak{X}$ contains a conjugate of $\mathfrak{B}$, then $\mathfrak{X}$ is contained in conjugate of $\mathfrak{N}$.*

Proof. We assume without loss of generality that $\mathfrak{X}$ is a maximal

2, 3-subgroup of $\mathfrak{G}$, and that $\mathfrak{B} \subseteq \mathfrak{T}$. Since $\mathfrak{B}$ centralizes $\boldsymbol{O}_2(\mathfrak{T})$, it follows that $\boldsymbol{O}_2(\mathfrak{T}) = 1$, and so $\boldsymbol{O}_3(\mathfrak{T}) \neq 1$. Let $\mathfrak{T}_3$ be a S_3-subgroup of $\mathfrak{T}$. By maximality of $\mathfrak{T}$, $\mathfrak{T}_3$ is a S_3-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{T}))$. We assume without loss of generality that $\mathfrak{T}_3 \subseteq \mathfrak{P}$. This implies that $\mathfrak{B} \subseteq \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{T}))$.

If $\mathfrak{T}$ contains a conjugate of $\mathfrak{D}$, we are done by Lemma 8.7. We therefore suppose that for each G in $\mathfrak{G}$, $\mathfrak{D}^G \not\subseteq \mathfrak{T}$.

Suppose $1 \leqq i \leqq 4$, and $\mathfrak{D}_i \cap \boldsymbol{O}_3(\mathfrak{T}) = \mathfrak{D}_i \cap \mathfrak{T}_3$. We conclude that $\mathfrak{D}_i \subseteq \boldsymbol{O}_3(\mathfrak{T})$. Since $\mathfrak{D} \not\subseteq \mathfrak{T}_3$, we may choose i with $1 \leqq i \leqq 4$ such that $\mathfrak{D}_i \cap \boldsymbol{O}_3(\mathfrak{T}) \subset \mathfrak{D}_i \cap \mathfrak{T}_3$. Set $\mathfrak{F} = \mathfrak{D}_i \cap \boldsymbol{O}_3(\mathfrak{T})$, $\mathfrak{F}^* = \mathfrak{D}_i \cap \mathfrak{T}_3$. The index i is fixed in the following discussion. We note that $\mathfrak{F}$ and $\mathfrak{F}^*$ are normal elementary subgroups of $\mathfrak{T}_3$.

Let $\mathfrak{Q}$ be a $\mathfrak{F}^*$-invariant subgroup of $\boldsymbol{Q}_3^1(\mathfrak{T})$ minimal subject to $[\mathfrak{F}^*, \mathfrak{Q}] \neq 1$. Thus, $\mathfrak{F}^*$ acts irreducibly on the Frattini quotient group of $\mathfrak{Q}$. We remark that $\mathfrak{Q}$ is available, since $\boldsymbol{O}_2(\mathfrak{T}) = 1$.

Let $\mathfrak{B}_0 = \Omega_1(\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{T})))$, so that $\mathfrak{B} \subseteq \mathfrak{B}_0$.

Choose $Q \in \mathfrak{Q} - \mathfrak{Q}'$. We will show that $\mathfrak{B}^Q \cap \boldsymbol{C}(\mathfrak{D}_i) = 1$. Suppose false, and that B in $\mathfrak{B}^\#$ satisfies $B^Q \in \boldsymbol{C}(\mathfrak{D}_i)$. Hence, for D in $\mathfrak{F}^*$ $(\subseteq \mathfrak{D}_i)$, we have $B^{QD} = B^Q$, or $B^{QDQ^{-1}} = B$. Hence, $QDQ^{-1}D^{-1}$ centralizes B for each D in $\mathfrak{F}^*$. This implies that $\mathfrak{Q}$ centralizes B. Apply Lemma 8.7 to $\boldsymbol{C}(B)$ and conclude that if $\mathfrak{Q} = \mathfrak{Q}^*/\boldsymbol{O}_3(\mathfrak{T})$, then $[\mathfrak{Q}^*, \mathfrak{F}^*]$ is a 3-group. As this violates the nontrivial action of $\mathfrak{F}^*$ on $\mathfrak{Q}$, the assertion follows.

Since $\mathfrak{B}^Q \subseteq \boldsymbol{O}_3(\mathfrak{T}) \subseteq \mathfrak{T}_3 \subseteq \mathfrak{P}$, we have $\mathfrak{B}^Q \subseteq \boldsymbol{N}(\mathfrak{D}_i)$. Since $\mathfrak{D}_i$ is 3-reducible in $\boldsymbol{N}(\mathfrak{D}_i)$, it follows that $\mathfrak{B}^Q$ is faithfully represented on $\mathfrak{L} = \boldsymbol{O}_{3'}(\boldsymbol{N}(\mathfrak{D}_i)/\boldsymbol{C}(\mathfrak{D}_i))$. On the other hand, if $B \in \mathfrak{B}^\#$, then $[\boldsymbol{C}(B)^Q, \mathfrak{B}^Q, \mathfrak{B}^Q] = 1$. This implies that $\mathfrak{B}^Q$ centralizes every $2'$-subgroup of $\mathfrak{L}$ which $\mathfrak{B}^Q$ normalizes. Thus, there is a 2-subgroup $\mathfrak{T}_0$ of $\boldsymbol{N}(\mathfrak{D}_i)$ such that $\boldsymbol{A}_{\boldsymbol{N}(\mathfrak{D}_i)}(\mathfrak{T}_0)$ contains a subgroup of type (3, 3). This violates Lemma 8.8 by $D_{2,3}$ in $\boldsymbol{N}(\mathfrak{D}_i)$. The proof is complete.

LEMMA 8.10. *If $\mathfrak{E}$ is any subgroup of $\mathfrak{G}$ of type* (3, 3), *then $\mathfrak{E}$ centralizes every abelian subgroup in $\mathscr{M}(\mathfrak{E}; 2)$.*

Proof. Suppose $\mathfrak{Q}$ is a four-group in $\mathscr{M}(\mathfrak{E}; 2)$ with $[\mathfrak{Q}, \mathfrak{E}] \neq 1$. Let $\mathfrak{E}_0 = \boldsymbol{C}_{\mathfrak{E}}(\mathfrak{Q})$. Let $\mathfrak{T}$ be a $S_{2,3}$-subgroup of $\boldsymbol{C}(\mathfrak{E}_0)$ which contains $\mathfrak{E}\mathfrak{Q}$. By Lemma 8.9, $\mathfrak{T}^G \subseteq \mathfrak{N}$ for some G in $\mathfrak{G}$. Lemma 8.8 (iv) is violated.

LEMMA 8.11. *Hypothesis* 7.2 *is satisfied with* $p = 2$. *Furthermore, $\mathfrak{M}$ has the following properties*:

(i) *S_3-subgroups of $\mathfrak{M}$ are noncyclic.*

(ii) *$\mathfrak{M}$ is a* 2, 3-*group.*

(iii) *$\mathfrak{M}$ contains no elementary subgroup of order* 27.

(iv) $m(\mathfrak{M}_0) \leqq 2$ *for every* 3-*subgroup $\mathfrak{M}_0$ of $\mathfrak{M}$.*

Proof. Let $\mathfrak{T}$ be a 2, 3-subgroup of $\mathfrak{G}$ which contains elements of $\mathscr{T}(2)$ and $\mathscr{T}(3)$; $\mathfrak{T}$ is available by hypothesis (vii) of Theorem 8.1. We assume without loss of generality that $\mathfrak{T}$ is a maximal 2, 3-subgroup of $\mathfrak{G}$. By Lemma 8.8 (i), $\mathfrak{T}$ is contained in no conjugate of $\mathfrak{N}$. By Lemma 8.9, $\mathfrak{T}$ contains no conjugate of $\mathfrak{B}$. This fact together with maximality of $\mathfrak{T}$ implies that $\boldsymbol{O}_3(\mathfrak{T}) = 1$.

Let $\mathfrak{C}$ be a subgroup of $\mathfrak{T}$ of type (3, 3) and let $\mathfrak{T}_3$ be a S_3-subgroup of $\mathfrak{T}$ containing $\mathfrak{C}$. By Lemma 8.10, $\mathfrak{C}$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T}))$. Hence, $\Omega_1(\mathfrak{T}_3)$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T}))$. By Lemma 8.9, each $S_{2,3}$-subgroup of $\boldsymbol{N}(\Omega_1(\mathfrak{T}_3))$ is contained in a conjugate of $\mathfrak{N}$. Hence, $\mathfrak{T}_2$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T}))$ by Lemma 8.8 (iv). By hypothesis (iv) of Theorem 8.1, $\mathfrak{T}\cdot\boldsymbol{C}(\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T}))$ is solvable, so by maximality of $\mathfrak{T}$, we conclude that $\mathfrak{T}$ is a $S_{2,3}$-subgroup of $\mathfrak{T}\boldsymbol{C}(\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T})))$. Hence, we can choose a S_2-subgroup $\mathfrak{P}_2$ of $\mathfrak{G}$ such that $\mathfrak{P}_2 \cap \mathfrak{T} = \mathfrak{T}_2$ is a S_2-subgroup of $\mathfrak{T}$, and be guaranteed that $\boldsymbol{Z}(\mathfrak{P}_2) \subseteq \boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T}))$. Hence, $\mathfrak{T} \subseteq \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{P}_2))$, so by maximality of $\mathfrak{T}$, we have $\mathfrak{T}_2 = \mathfrak{P}_2$.

By Lemma 7.4, $\Omega_1(\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{T}))) = \Omega_1(\boldsymbol{Z}(\mathfrak{T}_2))$ is of order 2 and

$$\boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{T}_2))) = \mathfrak{M} \in \mathscr{MS}(G) .$$

By construction, $\mathfrak{T}_3 \subseteq \mathfrak{M}$, so (i) is satisfied. By Lemma 7.5, $\boldsymbol{O}_2(\mathfrak{M}) = \mathfrak{H}$ is of symplectic type with $w \leqq 4$. (ii) is an easy consequence of this fact together with (i).

Suppose $\mathfrak{E}$ is an elementary subgroup of $\mathfrak{M}$ of order 27. Clearly, the width of $\mathfrak{H}$ is at least 3. By Lemma 7.5, no element of $\mathfrak{E}^\#$ centralizes any four-subgroup of $\mathfrak{H}$. This is obviously impossible.

It remains to prove (iv). By Lemma 7.5 (c), $\mathfrak{M}_0$ is isomorphic to a subgroup $\mathfrak{M}_1$ of $(Z_3 \wr Z_3) \times Z_3$. By Lemma 8.11 (iii), the intersection of $\mathfrak{M}_1$ with the normal abelian subgroup $\mathfrak{A}$ such that $m(\mathfrak{A}) = 4$ in $(Z_3 \sim Z_3) \times Z_3$, is of order at most 3^2. It follows that $\mathfrak{M}_0$ is either trivial, abelian of type (3), (3, 3) or $(3^2, 3)$, or non abelian of order 3^3. In all cases, $m(\mathfrak{M}_0) \leqq 2$. The proof is complete.

Let $\mathfrak{C}$ be a subgroup of $\mathfrak{M}$ of type (3, 3), let $\mathfrak{T}_3$ be a S_3-subgroup of $\mathfrak{M}$ containing $\mathfrak{C}$, and let $\mathfrak{H}_1 = [\mathfrak{H}, \mathfrak{C}]$, where $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{M})$. Let I be the involution of $\mathfrak{H}'$. Choose C in $\mathfrak{C}^\#$ so that $\boldsymbol{C}_{\mathfrak{H}_1}(C) = \mathfrak{Q}$ is not centralized by $\mathfrak{C}$. We may assume that $\boldsymbol{C}_{\mathfrak{M}}(C) \subseteq \mathfrak{N}$, since replacing $\mathfrak{M}$ by a suitable conjugate guarantees this. Let $\mathfrak{L}$ be a $S_{2,3}$-subgroup of $\mathfrak{N}$ containing $\mathfrak{C}\mathfrak{Q}$. This notation is fixed throughout the concluding argument.

LEMMA 8.12. (i) *$\mathfrak{Q}$ is a quaternion group.*

(ii) *$\mathfrak{N}$ is a 2, 3-group.*

(iii) *$\mathfrak{N} \in \mathscr{MS}(\mathfrak{G})$ and $\mathfrak{N}$ is the only element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{P}$.*

(iv) *$\mathfrak{N}$ is the only element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{C}$.*

Proof. Since $\mathfrak{H}_1$ is extra special, so is $\mathfrak{Q}$. Since $\mathfrak{Q} \subseteq \boldsymbol{O}^{3'}(\mathfrak{N})$, Lemma 8.8 and Lemma 5.27 imply (i)

By (i) and Lemma 8.8, $\mathfrak{Q}$ is a S_2-subgroup of $\boldsymbol{O}^{3'}(\mathfrak{N})$. Clearly, since $\mathfrak{M}$ is a 2, 3-group, $N_{\mathfrak{N}}(\mathfrak{Q})$ is a 2, 3-group. Since $\mathfrak{C}\mathfrak{Q} \subseteq \boldsymbol{O}^{3'}(\mathfrak{N})$, it follows that $\mathfrak{Q} \subseteq \boldsymbol{O}^{3'}(\mathfrak{N})'$. Hence, $\mathfrak{Q}$ has a normal complement $\mathfrak{K}$ in $\boldsymbol{O}^{3'}(\mathfrak{N})'$. To prove (ii), it suffices to show that $\mathfrak{K}$ is a 3-group. Let $\mathfrak{K}_0$ be a $S_{3'}$-subgroup of $\mathfrak{K}$ normalized by $\mathfrak{Q}$. Then I inverts $\mathfrak{K}_0$ since $\mathfrak{M}$ is a 2, 3-group. Choose $\mathfrak{F}$ char $\boldsymbol{O}_3(\mathfrak{K})$ with $\ker(\mathfrak{K} \rightarrow \operatorname{Aut}(\mathfrak{F}))$ a 3-group, and with $\mathfrak{F}$ of exponent 3. Such an $\mathfrak{F}$ is available by Lemma 5.18 and 0.3.6. As $\mathfrak{Q}$ is nonabelian, $\mathfrak{K}_0$ is noncyclic. It follows readily that I centralizes a subgroup of $\mathfrak{F}/\boldsymbol{D}(\mathfrak{F})$ of order 27. This implies that $\boldsymbol{C}_{\mathfrak{F}}(I)$ contains an elementary subgroup of order 27, in violation of Lemma 8.11. (ii) is proved.

Let $\mathfrak{P} \subseteq \mathfrak{N}_1 \in \mathscr{MS}(\mathfrak{G})$. By Hypothesis 8.2, it suffices to show that $\boldsymbol{C}_{\mathfrak{N}_1}(\tilde{\mathfrak{B}}) \subseteq \mathfrak{N}$, where $\tilde{\mathfrak{B}} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))^{\mathfrak{N}_1}$. Since $\boldsymbol{C}_{\mathfrak{N}_1}(\tilde{\mathfrak{B}}) \subseteq \boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$, and since $I \in \boldsymbol{N}(\mathfrak{P}) \subseteq \boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$, we may replace $\mathfrak{N}_1$ by an element of $\mathscr{MS}(\mathfrak{G})$ which contains $\boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$ and so assume that $I \in \mathfrak{N}_1$.

Let $\mathfrak{L}$ be a $S_{3'}$-subgroup of $\mathfrak{N}_1$ which contains I. By Lemma 8.8 (i) and Lemma 8.9, it follows that $\mathfrak{L}$ has a normal 2-complement $\mathfrak{K}$. Since $\mathfrak{M}$ is a 2, 3-group, I inverts $\mathfrak{K}$. Suppose by way of contradiction that $\mathfrak{K} \neq 1$. Since $\boldsymbol{O}_{3'}(\mathfrak{N}_1) = 1$, $\mathfrak{K}\langle I \rangle$ is faithfully represented as automorphisms of $\boldsymbol{O}_3(\mathfrak{N}_1)$. By Lemma 8.11 (iv), the only possibility is that $|\mathfrak{K}| = 5$, that $\boldsymbol{C}(\mathfrak{K}) \cap \boldsymbol{O}_3(\mathfrak{N}_1) \supseteq \boldsymbol{D}(\boldsymbol{O}_2(\mathfrak{N}_1))$ and that $\boldsymbol{O}_3(\mathfrak{N}_1)/\boldsymbol{C}(\mathfrak{K}) \cap \boldsymbol{O}_3(\mathfrak{N}_1)$ is elementary of order 3^4. Since $\mathfrak{K}$ is an S-subgroup of $\mathfrak{N}_1$, it follows that $\mathfrak{K}\boldsymbol{O}_{5'}(\mathfrak{N}_1)/\boldsymbol{O}_{5'}(\mathfrak{N}_1)$ is a chief factor of $\mathfrak{N}_1$. Hence, $I \notin \mathfrak{N}_1'$. This implies that $\boldsymbol{O}_3(\mathfrak{N}_1) = \mathfrak{P}$, so that $\mathfrak{N}_1 \subseteq \boldsymbol{N}(\mathfrak{B}) = \mathfrak{N}$. Hence, $\mathfrak{N}_1 = \mathfrak{N}$. This is absurd since $I \in \mathfrak{N}'$. This contradiction forces $\mathfrak{K} = 1$, that is, $\mathfrak{N}_1$ is a 2, 3-group.

Since $|\boldsymbol{C}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))|$ is odd, it follows that $\boldsymbol{C}_{\mathfrak{N}_1}(\tilde{\mathfrak{B}}) = \boldsymbol{C}_{\mathfrak{P}}(\tilde{\mathfrak{B}}) \subseteq \mathfrak{P} \subseteq \mathfrak{N}$. Thus, (iii) holds.

We turn to (iv). Let $\mathscr{P} = \{\mathfrak{P}_0 \mid$ (i) $\mathfrak{P}_0$ is a 3-subgroup of N, (ii) $\mathfrak{P}_0 \supseteq \mathfrak{B}^N$ for some N in $\mathfrak{N}$, (iii) $\mathfrak{P}_0$ is contained in a solvable subgroup of $\mathfrak{G}$ which is not contained in $\mathfrak{N}$.$\}$. Suppose by way of contradiction that $\mathscr{P} \neq \emptyset$. Choose $\mathfrak{P}_0$ in $\mathscr{P}$ with $|\mathfrak{P}_0|$ maximal. We assume without loss of generality that $\mathfrak{P}_0 \subseteq \mathfrak{P}$. Let $\mathfrak{K}$ be a solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{P}_0$ and is minimal subject to $\mathfrak{K} \not\subseteq \mathfrak{N}$. Since $\mathfrak{P} \notin \mathscr{P}$, it follows that $\mathfrak{P}_0 \subset \mathfrak{P}$, so maximality of $|\mathfrak{P}_0|$ forces $N_{\mathfrak{P}}(\mathfrak{P}_0) \notin \mathscr{P}$. In particular, $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{N}$. This implies that $\mathfrak{P}_0$ is a S_p-subgroup of $\mathfrak{K}$. Minimality of $\mathfrak{K}$ yields that $\mathfrak{K} = \mathfrak{P}_0\mathfrak{K}_1$, where $\mathfrak{K}_1$ is a q-group for some prime $q \neq 3$.

Since $\mathfrak{B}^N \subseteq \mathfrak{P}_0$ for some N in $\mathfrak{N}$, it follows that $\boldsymbol{O}_{3'}(\mathfrak{K}) \subseteq \mathfrak{N}$, as $\boldsymbol{O}_{3'}(\mathfrak{K})$ is generated by its subgroups $\boldsymbol{O}_{3'}(\mathfrak{K}) \cap \boldsymbol{C}(B)$, $B \in (\mathfrak{B}^N)^\#$.

Suppose $q = 2$. Then by Lemma 8.9, $\mathfrak{K} \subseteq \mathfrak{N}^G$ for some G in $\mathfrak{G}$. Hence $\mathfrak{P}_0 \subseteq \mathfrak{N}^G$. Let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{N} \cap \mathfrak{N}^G$ which contains $\mathfrak{P}_0$.

Maximality of $|\mathfrak{P}_0|$ forces $\mathfrak{P}_0 = \mathfrak{P}^*$. But then since $N(\mathfrak{P}_0) \subseteq \mathfrak{N}$, we get that $\mathfrak{P}_0$ is a S_3-subgroup of $\mathfrak{N}^G$. This is absurd. Hence, $q \neq 2$.

It is a consequence of [43] that $\mathfrak{K} = O_{3'}(\mathfrak{K})\mathfrak{A}_1\mathfrak{A}_2$, where

$$\mathfrak{A}_1 = C_{\mathfrak{K}}(Z(\mathfrak{P}_0)),\ \mathfrak{A}_2 = N_{\mathfrak{K}}(J(\mathfrak{P}_0))\ .$$

Maximality of $|\mathfrak{P}_0|$ forces $N(Z(\mathfrak{P}_0)) \subseteq \mathfrak{N}$, $N(J(\mathfrak{P}_0)) \subseteq \mathfrak{N}$, so $\mathfrak{K} \subseteq \mathfrak{N}$. This establishes (iv).

We may now complete the proof of Theorem 8.1. Choose C_1 in $\mathfrak{C}^\#$. Then $C(C_1) \supseteq \mathfrak{B}$, so that $C(C_1) \subseteq \mathfrak{N}$. Hence, $\mathfrak{H} \subseteq \mathfrak{N}$, in violation of Lemma 8.8 (ii).

9. A characterization of $S_4(3)$.

THEOREM 9.1. *$S_4(3)$ is the only simple group $\mathfrak{G}$ with the following properties:*

(i) *$\mathfrak{G}$ contains an elementary subgroup of order 27.*

(ii) *If $\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{S}\!\mathscr{C}\!\mathscr{N}_3(\mathfrak{P})$, then $\mathsf{M}(\mathfrak{A})$ is trivial.*

(iii) *The center of a S_3-subgroup of $\mathfrak{G}$ is cyclic.*

(iv) *The normalizer of every nonidentity 3-subgroup of $\mathfrak{G}$ is solvable.*

(v) *S_2-subgroups of $\mathfrak{G}$ contain normal elementary subgroups of order 8.*

(vi) *If $\mathfrak{T}$ is a S_2-subgroup of $\mathfrak{G}$ and $\mathfrak{B} \in \mathscr{S}\!\mathscr{C}\!\mathscr{N}_3(\mathfrak{T})$, then $\mathsf{M}(\mathfrak{B})$ is trivial.*

(vii) *The centralizer of every involution of $\mathfrak{G}$ is solvable.*

(viii) $2 \sim 3$. (See Definition 2.9.).

After careful translation, it can be shown that Dickson [12] lists several properties of $S_4(3)$. Namely, $A(4, 3)$ is Dickson's notation for $S_4(3)$ (pp. 89–100). Now in § 194 (pp. 109–191), Dickson sets $FO(m, p^n) = O_1'(m, p^n)$ (for m odd), so by § 189 (pp. 179–183), $A(4, 3) \cong FO(5, 3) \cong S_4(3)$. Thus, by § 270 (pp. 292–293), $S_4(3)$ has a subgroup of index 27 which is a split extension of an elementary group of order 16 by A_5. So $S_4(3)$ is not an N-group. That $S_4(3)$ satisfies the hypothesis of Theorem 9.1 is left as an exercise. We remark that (viii) holds for $S_4(3)$, the centralizers of suitable involutions exhibiting $2 \sim 3$.

Throughout most of this section the following notation is used: $\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{G}$,

$$\mathfrak{Z} = \Omega_1(Z(\mathfrak{P})),\ \mathfrak{N} = N(\mathfrak{Z}),\ \mathfrak{H} = O_3(\mathfrak{N})\ .$$

By hypothesis (iii), $|\mathfrak{Z}| = 3$, and by hypothesis (ii), $O_{3'}(\mathfrak{N}) = 1$. By hypothesis (iv), $\mathfrak{N}$ is solvable, so by Lemma 1.2.3 of [26], $C_{\mathfrak{N}}(\mathfrak{H}) = Z(\mathfrak{H})$.

Clearly, $C_{\mathfrak{N}}(\mathfrak{H}) = C(\mathfrak{H})$.

We remark that $\mathfrak{G}$ satisfies Hypothesis 7.4 and also satisfies Hypothesis 7.1 for $p = 2$ and for $p = 3$.

HYPOTHESIS 9.1. $\mathfrak{H}$ is the central product of a cyclic group and a nonabelian group of order 27 and exponent 3.

LEMMA 9.1. *Assume that Hypothesis* 9.1 *is satisfied. Then*
(i) $|\mathfrak{P}:\mathfrak{H}| = 3$.
(ii) $|\mathfrak{H}| = 27$.
(iii) $O^{3'}(\mathfrak{N})/\mathfrak{H} \cong SL(2, 3)$.

Proof. We remark that $GL(2, 3)$ contains no noncyclic abelian subgroup of order 8.

As $\mathfrak{N}/\mathfrak{H}$ is faithfully represented on $\Omega_1(\mathfrak{H})/D(\Omega_1(\mathfrak{H}))$, it follows that $\mathfrak{N}$ is a 2, 3-group, and $\mathfrak{N}/\mathfrak{H}$ is isomorphic to a subgroup of $GL(2, 3)$. As $\mathfrak{H}$ contains no elementary subgroup of order 27 and $\mathfrak{P}$ does, (i) holds.

Let $\mathfrak{A}$ be a normal elementary subgroup of $\mathfrak{P}$ of order 27. Since $|\mathfrak{P}:\mathfrak{H}| = 3$, it follows that $O^{3'}(\mathfrak{N})/\mathfrak{H} \cong SL(2, 3)$, yielding (iii). Let $\mathfrak{Q}$ be a S_2-subgroup of $O^{3'}(\mathfrak{N})$ so that $\mathfrak{Q}$ is a quaternion group. Let $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{H}$. Let I be the involution of $\mathfrak{Q}$. As I inverts every element of $\Omega_1(\mathfrak{H})/\mathfrak{Z}$, it folllows that I normalizes $\mathfrak{A}_0$. Since I also normalizes $\mathfrak{P}$, it follows that I normalizes $C_{\mathfrak{P}}(\mathfrak{A}_0) = \langle \mathfrak{A}, Z(\mathfrak{H}) \rangle$. Hence, I normalizes $\Omega_1(C_{\mathfrak{P}}(\mathfrak{A}_0)) = \mathfrak{A}$. Since I centralizes the factor $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'\mathfrak{H} \cong \mathfrak{P}/\mathfrak{H}$, it follows that $\mathfrak{A} - \mathfrak{A}_0$ contains an element A_1 such that $A_1^I = A_1$. Since I also centralizes $\mathfrak{Z}$, it follows that $C_{\mathfrak{A}}(I) = \langle A_1 \rangle \times \mathfrak{Z} = \mathfrak{A}_1$. Also, $C_{\mathfrak{P}}(I) = \mathfrak{A}_1\mathfrak{Z}_1$, where $\mathfrak{Z}_1 = Z(\mathfrak{H})$, and it is clear that $C_{\mathfrak{P}}(I)$ is a S_3-subgroup of $C_{\mathfrak{N}}(I)$.

Suppose $|\mathfrak{Z}_1| > 3$. Thus, $|\mathfrak{P}| > 3^4$, so Lemma 7.6 is at our disposal. If $G \in \mathfrak{G}$ and $\mathfrak{Z}_1^G \subseteq \mathfrak{P}$, then $\mho^1(\mathfrak{Z}_1^G)$ centralizes $\mathfrak{H}$, and so $\Omega_1(\mathfrak{Z}_1^G) = \Omega_1(\mathfrak{Z}_1) = \mathfrak{Z}$, so that $G \in \mathfrak{N}$, $\mathfrak{Z}_1^G = \mathfrak{Z}_1$. We may therefore apply Theorem 14.4.2 of [21] and conclude that $\mathfrak{P} \subseteq \mathfrak{N}'$. Since Aut $(\mathfrak{Z}_1)$ is abelian, this implies that $\mathfrak{Z}_1 = Z(\mathfrak{P})$. We may therefore appeal to Lemma 7.6 (d) and conclude that if $\mathfrak{A}^*$ is any subgroup of $\mathfrak{A}$ of type (3, 3), then $\mathfrak{A}^*$ centralizes every element of $\mathsf{M}(\mathfrak{A}^*; 2)$. Taking $\mathfrak{A}^* = \mathfrak{A}_1$, Lemma 7.4 is violated. This completes the proof of (ii).

LEMMA 9.2. *Assume that Hypothesis* 9.1 *is satisfied. Let* $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$ *and let I be an involution of* $\mathfrak{N}'$. *Then*
(i) *S_2-subgroups of $\mathfrak{N}$ are quaternion.*
(ii) *If* $\mathfrak{A}_0 = C_{\mathfrak{A}}(I)$, *then*
  (a) $|\mathfrak{A}_0| = 9$.
  (b) $\mathfrak{A}_0$ *contains a subgroup* $\mathfrak{A}_1$ *of order* 3 *such that* $C(\mathfrak{A}_1) \not\subseteq \mathfrak{N}$.

(iii) *If* $\mathfrak{M} = C(I)$, *then* $O_2(\mathfrak{M})$ *is extra special of width* 2, $O_{2'}(\mathfrak{M}) = 1$, *and* $|\mathfrak{M}: O_2(\mathfrak{M})|_2 = 2$.
(iv) $A_{\mathfrak{G}}(\mathfrak{A}) \cong \Sigma_4$.

Proof. Let $\mathfrak{Q}$ be a S_2-subgroup of $O^{3'}(\mathfrak{N})$. By Lemma 9.1 (i), $\mathfrak{Q}$ is a quaternion group. It clearly suffices to prove the lemma on the assumption that I is the involution of $\mathfrak{Q}$.

By Lemma 9.1 and hypothesis (i) of Theorem 9.1, the group $\mathfrak{P}$ is $Z_3 \wr Z_3$. Hence, $\mathfrak{A}$ char $\mathfrak{P}$. Since I normalizes $\mathfrak{P}$, it therefore normalizes $\mathfrak{A}$. This implies (ii)(a), since I centralizes $Z(\mathfrak{H})$ and $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$.

Clearly, $\mathfrak{M}$ contains an element of $\mathscr{U}(2)$. It is equally clear from (B) and Lemma 9.1 that

$$(*) \qquad \begin{array}{l}\text{if } \mathfrak{X} \text{ is any noncyclic subgroup of } \mathfrak{A}, \text{ then } \mathfrak{X} \\ \text{centralizes every abelian subgroup of } \mathsf{M}(\mathfrak{X}; 2)\,.\end{array}$$

Let $\mathfrak{T}$ be a $S_{2,3}$-subgroup of $\mathfrak{M}$ which contains $\langle \mathfrak{A}_0, \mathfrak{Q}\rangle$. Let $\mathfrak{T}_2$ be a S_2-subgroup of $\mathfrak{T}$ which contains $\mathfrak{Q}$. We may apply Lemma 7.5 with $\mathfrak{A}_0$ in the role of $\mathfrak{B}$. Thus, there is an element $\widetilde{\mathfrak{M}}$ of $\mathscr{MS}(\mathfrak{G})$ satisfying the conclusions of Lemma 7.5. By Lemma 7.5 (e), we get $\mathfrak{Q} \subseteq O_2(\widetilde{\mathfrak{M}})$. Hence, $\langle I\rangle \lhd \widetilde{\mathfrak{M}}$, so $\widetilde{\mathfrak{M}} = C(I) = \mathfrak{M}$. Since $\langle I\rangle$ is a S_2-subgroup of $C(\mathfrak{A}_0)$, it follows that

$$(9.1) \qquad O_2(\mathfrak{M}) \text{ is extra special of width } w = 2, 3, \text{ or } 4\,.$$

Thus, (ii)(b) holds.

Again, let $\mathfrak{X}$ be a noncyclic subgroup of $\mathfrak{A}$. Suppose that $|C(\mathfrak{X}) \cap N(\mathfrak{A})|$ is even. Then of course $|\mathfrak{X}| = 9$, as $\mathfrak{A}$ is a self-centralizing subgroup of $\mathfrak{G}$. Let J be an involution of $C(\mathfrak{X}) \cap N(\mathfrak{A})$. Then $(*)$ and Lemma 7.5 yield that J and I are conjugate in $\mathfrak{G}$. Since $\mathfrak{X}$ is faithfully represented on $O_2(C(J))$, we can choose a subgroup $\mathfrak{Y}$ of $\mathfrak{X}$ of order 3 such that

$$(9.2) \qquad \mathfrak{X} \text{ does not centralize } C(\mathfrak{Y}) \cap O_2(C(J))\,.$$

Thus, $C(\mathfrak{Y})$ is not 3-closed. Thus, $\mathfrak{A}$ is not a S_3-subgroup of $C(\mathfrak{Y})$. This implies that

$$(9.3) \qquad C(\mathfrak{Y}) \text{ contains a } S_3\text{-subgroup of } \mathfrak{G}\,.$$

Let $\widetilde{\mathfrak{P}}$ be a S_3-subgroup of $C(\mathfrak{Y})$ which contains $\mathfrak{A}$. Thus $\langle \widetilde{\mathfrak{P}}, J\rangle \subseteq C(\mathfrak{Y})$. Thus, J normalizes both $\mathfrak{A}$ and $O_3(C(\mathfrak{Y}))$, so J normalizes $\langle \mathfrak{A}, O_3(C(\mathfrak{Y}))\rangle$. Thus, Lemma 9.1 yields that

$$(9.4) \qquad \begin{array}{l}\text{if } \mathfrak{X} \text{ is any noncyclic subgroup of } \mathfrak{A}, \text{ then each involution} \\ \text{of } C(\mathfrak{X}) \cap N(\mathfrak{A}) \text{ normalizes some } S_3\text{-subgroup of } N(\mathfrak{A})\,.\end{array}$$

By (9.3) with the pair $(\mathfrak{A}_1, \mathfrak{A}_0)$ in the role of $(\mathfrak{Y}, \mathfrak{X})$, we conclude that $\boldsymbol{C}(\mathfrak{A}_1)$ contains a S_3-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$ with $\mathfrak{A} \subset \mathfrak{P}^*$. Hence, $\boldsymbol{N}(\mathfrak{A})$ is not 3-closed, since $\mathfrak{P}$ and $\mathfrak{P}^*$ are distinct S_3-subgroups of $\boldsymbol{N}(\mathfrak{A})$.

Set $\tilde{\mathfrak{N}} = \boldsymbol{N}(\mathfrak{A})$. Clearly,

$$\mathfrak{A} = \boldsymbol{O}_3(\tilde{\mathfrak{N}}) = \boldsymbol{C}(\mathfrak{A}),\ 1 = \boldsymbol{O}_{3'}(\tilde{\mathfrak{N}}),\ I \in \tilde{\mathfrak{N}} .$$

Suppose $13 \mid |\tilde{\mathfrak{N}}|$. Since $I \in \tilde{\mathfrak{N}}$, it follows that I centralizes a S_{13}-subgroup of $\tilde{\mathfrak{N}}$, since the nonidentity 13-elements of $GL(3, 3)$ are nonreal. However, $13 \nmid |\mathfrak{M}|$, since $4 \geqq w$. Hence, $\tilde{\mathfrak{N}}$ is a 2, 3-group.

Let $\mathfrak{T}_0$ be a S_2-subgroup of $\boldsymbol{O}_{3,2}(\tilde{\mathfrak{N}})$, and let $\mathfrak{K} = N_{\tilde{\mathfrak{N}}}(\mathfrak{T}_0)$. Thus, $\tilde{\mathfrak{N}} = \mathfrak{A}\mathfrak{K}$, $\mathfrak{A} \cap \mathfrak{K} = 1$, so that $\mathfrak{K} \cong \boldsymbol{A}_{\mathfrak{G}}(\mathfrak{A})$. Suppose J is an involution of $\mathfrak{T}_0$ and $\mathfrak{A} \cap \boldsymbol{C}(J) = \mathfrak{X}$ is noncyclic. Thus, $|\mathfrak{X}| = 9$. By (9.4), J normalizes some S_3-subgroup $\tilde{\mathfrak{P}}$ of $\tilde{\mathfrak{N}}$. Since $\tilde{\mathfrak{P}} \supset \mathfrak{A}$, $\tilde{\mathfrak{P}} \cap \mathfrak{K}$ is of order 3. Hence, $[\tilde{\mathfrak{P}} \cap \mathfrak{K}, J] \subseteqq \mathfrak{T}_0 \cap \tilde{\mathfrak{P}} = 1$, so that $\tilde{\mathfrak{P}} \cap \mathfrak{K}$ centralizes J. Hence, $\tilde{\mathfrak{P}} \cap \mathfrak{K}$ normalizes $\mathfrak{X}$, that is, $\mathfrak{X} \lhd \tilde{\mathfrak{P}}$. Hence, $\mathfrak{X} \in \mathscr{U}(3)$. Thus, J centralizes elements of $\mathscr{U}(3)$ and $\mathscr{U}(2)$. This violates the solvability of $\boldsymbol{C}(J)$. Hence,

(9.5) no element of $\mathfrak{T}_0^\#$ centralizes a noncyclic subgroup of $\mathfrak{A}$.

Since $|\boldsymbol{Z}(\mathfrak{P})| = 3$, $\mathfrak{P} \cap \mathfrak{K}$ is indecomposable on $\mathfrak{A}$. Hence, $\mathfrak{T}_0$ acts on $\mathfrak{A}$ as a multiple of the sum of the $\mathfrak{K}$-conjugates of a fixed F_3-irreducible representation ρ. If $\mathfrak{T}_0$ is nonabelian, then 2 divides $\deg \rho$. Hence, 2 divides $3 = m(\mathfrak{A})$, a contradiction. Hence, $\mathfrak{T}_0$ is abelian. If $\mathfrak{T}_0$ is not elementary, then $\deg \rho \neq 1$ or 3. So $\deg \rho = 2$, which again gives a contradiction. Hence, $\mathfrak{T}_0$ is elementary. Now (9.5) implies that $|\mathfrak{T}_0| \leqq 4$, so we must have equality, since $\mathfrak{T}_0 = \boldsymbol{O}_2(\mathfrak{K})$ and $|\mathfrak{P} \cap \mathfrak{K}| = 3$. Since $I \in \mathfrak{N} \cap \boldsymbol{N}(\mathfrak{P})$, it follows that $\mathfrak{K} \cong \Sigma_4$, which establishes (iv) and also (i).

It remains to show that $w = 2$ and that $|\mathfrak{M}: \boldsymbol{O}_2(\mathfrak{M})|_2 = 2$, since by (9.1) we know that as $\boldsymbol{O}_2(\mathfrak{M})$ is extra special.

Suppose $\mathfrak{A}$ is any subgroup of $\mathfrak{A}$ of order 3 which is conjugate to $\mathfrak{Z}$, $\mathfrak{A}^* \sim_{\mathfrak{G}} \mathfrak{Z}$. We contend that $\mathfrak{A}^* \sim_{\boldsymbol{N}(\mathfrak{A})} \mathfrak{Z}$. Namely, let $\mathfrak{P}^*$ be a S_3-subgroup of $\boldsymbol{C}(\mathfrak{A}^*)$ which contains $\mathfrak{A}$. Then $\mathfrak{P}$ and $\mathfrak{P}^*$ both normalize $\mathfrak{A}$, since $|\mathfrak{P}^* : \mathfrak{A}| = 3$. We may thus choose N in $\boldsymbol{N}(\mathfrak{A})$ such that $\mathfrak{P}^{*N} = \mathfrak{P}$; since $\mathfrak{Z} = \boldsymbol{Z}(\mathfrak{P})$, we necessarily have $\mathfrak{A}^{*N} = \mathfrak{Z}$, as desired.

Since $\mathfrak{P}\langle I \rangle$ normalizes $\mathfrak{Z}$, we obtain all $\boldsymbol{N}(\mathfrak{A})$-conjugates of $\mathfrak{Z}$ by transformation with elements of $\mathfrak{T}_0$. We will show that $\mathfrak{Z}$ and $\mathfrak{A}_1$ are the only $\boldsymbol{N}(\mathfrak{A})$-conjugates of $\mathfrak{Z}$ which are in $\mathfrak{A}_0$. If $K \in \mathfrak{T}_0^\#$ and $\mathfrak{Z}^K \subseteqq \mathfrak{A}_0$, then since no element of $\mathfrak{T}_0^\#$ normalizes $\mathfrak{Z}$, we conclude that K normalizes $\mathfrak{A}_0$. It is clear that $\mathfrak{T}_0$ does not normalize $\mathfrak{A}_0$, so our assertion follows.

It is an immediate consequence of the preceding paragraph that $w = 2$. That is, only $\mathfrak{Z}$ and $\mathfrak{A}_1$ centralize elements of $\boldsymbol{O}_2(\mathfrak{M}) - \langle I \rangle$. Since $\boldsymbol{N}(\mathfrak{A}_0) \subseteqq \boldsymbol{N}(\mathfrak{A})$, we have $|\mathfrak{M}: \boldsymbol{O}_2(\mathfrak{M})|_2 = 2$, and the proof is complete.

We now change notation somewhat in order to conform with more standard notation. Let $\mathfrak{B}_1 = N(\mathfrak{Z})$, $\mathfrak{B}_2 = N(\mathfrak{A})$, and let $\mathfrak{B} = \mathfrak{P}\langle I\rangle$, $\mathfrak{H} = \langle I\rangle$. Let $\mathfrak{Q}_1$ be a S_2-subgroup of $\mathfrak{B}_1$ which contains I, and let $\mathfrak{Q}_2$ be a S_2-subgroup of $\mathfrak{B}_2$ which contains I. Thus, $\mathfrak{Q}_1$ is a quaternion group and $\mathfrak{Q}_2$ is a dihedral group of order 8. Let $\mathfrak{T}_2 = \mathfrak{Q}_2 \cap \boldsymbol{O}^{3'}(\mathfrak{B}_2)$, so that $\mathfrak{T}_2$ is a four-group.

Let $\mathfrak{C}_2 = N_{\mathfrak{B}_2}(\mathfrak{T}_2)$. Thus, $\mathfrak{C}_2$ is a complement to $\mathfrak{A}$ in $\mathfrak{B}_2$ and $\mathfrak{C}_2 \cong \Sigma_4$. Let $\mathfrak{X}_1 = \mathfrak{P} \cap \mathfrak{C}_2$, so that $\mathfrak{X}_1$ is of order 3 and is inverted by I. Since $\boldsymbol{O}_3(\mathfrak{B}_1)$ contains all elements of $\mathfrak{P}$ which are inverted by I, we have $\mathfrak{X}_1 \subseteq \boldsymbol{O}_3(\mathfrak{B}_1)$. Since $\mathfrak{Q}_1$ permutes transitively the subgroups of $\boldsymbol{O}_3(\mathfrak{B}_1)/\mathfrak{Z}$ of order 3, we may choose Q in $\mathfrak{Q}_1$ so that $\mathfrak{X}_2 = \mathfrak{X}_1^Q$ lies in $\mathfrak{A}$. Thus, I inverts $\mathfrak{X}_2$, since Q centralizes I. Let $\langle J\rangle = \mathfrak{T}_2 \cap \boldsymbol{C}(I)$, that is, let J be a generator for $\boldsymbol{Z}(\mathfrak{Q}_2)$. Since $\mathfrak{A} \cap \boldsymbol{C}(I)$ is of order 9, $\mathfrak{X}_2$ is the only subgroup of $\mathfrak{A}$ of order 3 which is inverted by I, so that $\mathfrak{X}_2^J = \mathfrak{X}_2$. Let $\mathfrak{X}_4 = \mathfrak{Z}$ and set $\mathfrak{X}_3 = \mathfrak{X}_4^J$. We may now draw up the following table:

	J	Q
$\mathfrak{X}_1$	—	$\mathfrak{X}_2$
$\mathfrak{X}_2$	$\mathfrak{X}_2$	$\mathfrak{X}_1$
$\mathfrak{X}_3$	$\mathfrak{X}_4$	—
$\mathfrak{X}_4$	$\mathfrak{X}_3$	$\mathfrak{X}_4$.

Let X_i be a generator for $\mathfrak{X}_i$, so that we have the following table:

	I
X_1	X_1^{-1}
X_2	X_2^{-1}
X_3	X_3
X_4	X_4 .

Let $\mathfrak{N} = \langle J, Q\rangle$. Since $\mathfrak{N} \subseteq \mathfrak{M}$, the structure of $\mathfrak{N}$ may be easily determined. Let $\mathfrak{Q}_1, \mathfrak{Q}_1^*$ be the quaternion subgroups of $\boldsymbol{O}_3(\mathfrak{M})$, $\mathfrak{Q}_1$ being as above. As J normalizes $\langle \mathfrak{X}_3, \mathfrak{X}_4\rangle$ and $\langle \mathfrak{X}_3, \mathfrak{X}_4\rangle$ is a S_3-subgroup of $\mathfrak{M}$, it follows that $\mathfrak{Q}_1^J = \mathfrak{Q}_1^*$. Hence, $(JQ)^2 = JQJQ$ is an involution distinct from I. This means that $\mathfrak{N}/\langle I\rangle$ is a dihedral group of order 8 with involutory generators $J\langle I\rangle$, $Q\langle I\rangle$.

Finally, notice that $\mathfrak{B} = \mathfrak{P}\langle I\rangle = N(\mathfrak{P})$.

Since $(JX_1)^3 = 1$ and since $(\mathfrak{Q}X_3)^3 \in \langle I\rangle$, it is straightforward to deduce from the first table that $\mathfrak{B}\mathfrak{N}\mathfrak{B} = \mathfrak{G}_1$ is a group. We will determine the multiplication table of $\mathfrak{G}_1$. First, we assume without loss of generality that $(QX_3)^3 = 1$, since replacement of Q by $Q^{-1} = QI$ will achieve this if $(QX_3)^3 = I$. Since I inverts X_1 and centralizes X_3, it follows easily that I neither inverts nor centralizes $[X_1, X_3]$. Thus, we may choose X_2, X_4 as generators for $\mathfrak{X}_2, \mathfrak{X}_4$ respectively such that

$$[X_1, X_3] = X_2X_4 . \tag{9.6}$$

By construction, $\mathfrak{X}_4 = \mathfrak{Z} = \boldsymbol{Z}(\mathfrak{P})$, so to complete the determination of $\mathfrak{B}$, we must compute $[X_1, X_2]$. Conjugation of (9.6) by I yields $[X_1^{-1}, X_3] = X_2^{-1}X_4$, from which we find easily that $[X_1, X_2] = X_4$.

Let $X_1^Q = X_2^a$. Since $(QX_3)^3 = 1$, an easy calculation (conjugation of (9.6) by Q) shows that $a = +1$. Since $J \in \mathfrak{T}_2 \lhd \mathfrak{C}_2$, it follows that $\boldsymbol{C}_{\mathfrak{A}}(J)$ has order 3. Since J normalizes but does not invert $\langle X_3, X_4\rangle$, it follows that $\boldsymbol{C}_{\mathfrak{A}}(J) \subseteq \langle X_3, X_4\rangle$. Hence, $X_2^J = X_2^{-1}$. Let $X_3^J = X_4^b$. Since $(X_1J)^3 = 1$, an easy calculation (conjugation of (9.6) by J) shows that $b = -1$.

Set $W_0 = (JQ^2)$. We argue that

$$\mathfrak{P} \cap \mathfrak{P}^{W_0} = 1 . \tag{9.7}$$

Suppose by way contradiction that 9.7 is false. Since W_0 is an involution, $\mathfrak{D} = \mathfrak{P} \cap \mathfrak{P}^{W_0}$ is normalized by W_0. Since $W_0 \in \boldsymbol{Z}(\mathfrak{N})$, it follows that $\mathfrak{H} = \langle I\rangle$ also normalizes $\mathfrak{D}$. Since $\boldsymbol{C}_{\mathfrak{P}}(I) = \langle X_3, X_4\rangle$ and since $W_0 \in \mathfrak{M}$, it follows from the construction of $\mathfrak{N}$ that I inverts $\mathfrak{D}$. Thus, $\mathfrak{D} \subset \boldsymbol{O}_3(\mathfrak{B}_1)$, since $\boldsymbol{O}_3(\mathfrak{B}_1)$ contains all the elements of $\mathfrak{P}$ which are inverted by I. As $\mathfrak{D}$ is abelian, and as I centralizes $\mathfrak{X}_4 = \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{B}_1))$, it follows that $|\mathfrak{D}| = 3$. There are exactly 4 subgroups of $\boldsymbol{O}_3(\mathfrak{B}_1)$ of order 3 which are inverted by I; they are all of the shape $\mathfrak{X}_2^{Q^*}$ for some Q^* in $\mathfrak{Q}_1$. Since I normalizes $\mathfrak{X}_2$ and since $\mathfrak{Q}_1 = \langle Q\rangle \cup \langle Q^{X_3}\rangle \cup \langle Q^{X_3^{-1}}\rangle$, we may assume that $\mathfrak{D} = \mathfrak{X}_2^{Q^*}$, where Q^* is one of $1, Q, X_3^{-1}QX_3, X_3QX_3^{-1}$. Since W_0 normalizes $\mathfrak{D}$, we get that $Q^*W_0Q^{*-1} \in \boldsymbol{N}(\mathfrak{X}_2)$. Since $Q^* \in \mathfrak{M}$ and $W_0 \in \boldsymbol{O}_2(\mathfrak{M})$, we get that $Q^*W_0Q^{*-1} \in \boldsymbol{N}(\mathfrak{X}_2) \cap \boldsymbol{O}_2(\mathfrak{M}) = \mathfrak{L}$, say. Since I inverts X_2, it follows that $I \notin \boldsymbol{D}(\mathfrak{L})$. Thus, $\mathfrak{L}$ is elementary. But 1 and $\langle I\rangle$ are the only elementary subgroups of $\boldsymbol{O}_2(\mathfrak{M})$ which admit $\langle X_3, X_4\rangle$, so $Q^*W_0Q^{*-1} = \langle I\rangle$. This is not the case, since $Q^* \in \mathfrak{M}$, $I \in \boldsymbol{Z}(\mathfrak{M})$, and $I \neq W_0$. This proves (9.7).

Set $\mathfrak{W} = \{1, Q, J, QJ, JQ, QJQ, JQJ, W_0\}$, a set of representatives for the cosets of $\mathfrak{H}$ in $\mathfrak{N}$. For each W in $\mathfrak{W}$, let $\mathfrak{B}_W = \langle \mathfrak{X}_i \mid 1 \leqq i \leqq 4, \mathfrak{X}_i^{W^{-1}} \subseteq \mathfrak{P}^{W_0}\rangle$. It follows that condition (iii) of Théorème I of [36] is satisfied, so by that theorem, so is condition (ii), that is, if $W_1, W_2 \in \mathfrak{W}$ and $BW_1B = BW_2B$, then $W_1 = W_2$. In view of our preceding information, we conclude that each element of $\mathfrak{G}_1$ has a normal form of the shape $PHWP'$, where $P \in \mathfrak{P}$, $H \in \mathfrak{H}$, $W \in \mathfrak{W}$, $P' \in \mathfrak{B}_W$. Furthermore, it is clear that the normal forms for $PHWP'J$ and $PHWP'Q$ are determined by our information. This implies immediately that if $\mathfrak{G}^*$ is any group which satisfies the hypothesis of Theorem 9.1 and Hypothesis 9.1, then $\mathfrak{G}^*$ contains a subgroup $\mathfrak{G}_1^*$ isomorphic to $\mathfrak{G}_1$. Taking $\mathfrak{G}^* = S_4(3)$, a comparison of orders yields $\mathfrak{G}_1 \cong S_4(3)$. In particular, $i(\mathfrak{G}_1) = 2$ and I, W_0 are representatives for the two classes of involutions of $\mathfrak{G}_1$. Since $\mathfrak{M} = \langle Q_1, J, X_3, X_4\rangle$, we have $\mathfrak{M} \subseteq \mathfrak{G}_1$. We will show that $\boldsymbol{C}(W_0) \subseteq \mathfrak{G}_1$. Let $\mathfrak{R} = \boldsymbol{O}_2(\boldsymbol{C}_{\mathfrak{G}_1}(W_0))$. Then $\mathfrak{R}$ is elementary of order 2^4 and $\mathfrak{R}$ is characteristic in a S_2-subgroup of $\boldsymbol{C}_{\mathfrak{G}_1}(W_0)$. Thus, it suffices to show that $\boldsymbol{N}(\mathfrak{R}) = \boldsymbol{N}_{\mathfrak{G}_1}(\mathfrak{R})$. Since $\boldsymbol{N}_{\mathfrak{G}_1}(\mathfrak{R})$ is an extension of $\mathfrak{R}$ by A_5 and since $\mathfrak{R} = \boldsymbol{C}(\mathfrak{R})$, it follows that $\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{R})$ is a subgroup of Aut $(\mathfrak{R}) = L_4(2)$ which contains a subgroup isomorphic to A_5 and has S_2-subgroups of order 4. Hence, $\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{R}) = \boldsymbol{A}_{\mathfrak{G}_1}(\mathfrak{R}) \cong A_5$. Hence, $\mathfrak{G}_1$ contains the centralizer of each of its involutions. By Lemma 5.35, $\mathfrak{G} = \mathfrak{G}_1$. Thus, Theorem 9.1 is proved in case Hypothesis 9.1 is satisfied.

We now revert to our previous notation.

HYPOTHESIS 9.2. $\mathfrak{H}$ is of symplectic type and width $w \geqq 2$.

HYPOTHESIS 9.3. (i) $\mathfrak{H}$ is extra special of order 3^5.
(ii) $|\mathfrak{P}| = 3^6$.
(iii) $\mathfrak{Z}$ is not weakly closed in $\mathfrak{H}$.

Lemmas 9.3 through 9.10 are all proved under Hypothesis 9.2. Notice that Hypothesis 9.3 trivially implies Hypothesis 9.2.

LEMMA 9.3. (i) *$\boldsymbol{C}(\mathfrak{Z})$ does not contain a four-group.*
(ii) *If $\mathfrak{Q}$ is any abelian 2-subgroup of $\mathfrak{N}$, then $\boldsymbol{A}_{\mathfrak{N}}(\mathfrak{Q})$ is a 2-group.*
(iii) *If $\mathfrak{A}$ is any subgroup of $\mathfrak{H}$ of type* (3, 3) *which contains $\mathfrak{Z}$, then $|\boldsymbol{C}(\mathfrak{A})|$ is odd.*
(iv) *If $\mathfrak{A}$ is any subgroup of $\mathfrak{H}$ of type* (3, 3) *which contains $\mathfrak{Z}$, then $\mathfrak{A} \in \mathscr{E}(3)$.*

Proof. Clearly, (i) implies (ii), and (iii) implies (i). Suppose (iv) holds, but I is an involution in $\boldsymbol{C}(\mathfrak{A})$. By Lemma 5.37, $\boldsymbol{C}(I)$ contains an element of $\mathscr{U}(2)$. By Lemma 7.4, $\boldsymbol{C}(I)$ is nonsolvable. Hence, (iv) implies (iii). To complete the proof of the lemma, it suffices to prove (iv). However, (iv) is a consequence of Lemma 7.2.

LEMMA 9.4. *Suppose $B \in \Omega_1(\mathfrak{H}) - \mathfrak{Z}$ and $\mathfrak{H}_0 = \boldsymbol{C}_{\mathfrak{H}}(B)$. Then*

$$\boldsymbol{C}(\mathfrak{H}_0) = \boldsymbol{Z}(\mathfrak{H}_0) = \langle B\rangle \times \boldsymbol{Z}(\mathfrak{H}) .$$

Proof. Since $\mathfrak{Z} \subset \mathfrak{H}_0$, it follows that $\boldsymbol{C}(\mathfrak{H}_0) = \boldsymbol{C}_{\mathfrak{N}}(\mathfrak{H}_0)$. Since a $S_{3'}$-subgroup of $\mathfrak{N}$ is faithfully represented on $\mathfrak{H}$, it follows that $\boldsymbol{C}(\mathfrak{H}_0)$ is a 3-group. It suffices to show that $\boldsymbol{C}(\mathfrak{H}_0) \subseteq \mathfrak{H}$. Suppose false and

$C \in \boldsymbol{C}(\mathfrak{H}_0)$, $C \notin \mathfrak{H}$. We may assume that $C^3 \in \mathfrak{H}$. In this case, $\langle C\rangle/\langle C^3\rangle$ is faithfully represented on $\boldsymbol{Q}_3^1(\mathfrak{N})$ and by Lemma 5.30, it follows that $[\boldsymbol{Q}_3^1(\mathfrak{N}), \langle C\rangle] = \tilde{\mathfrak{Q}}$ is a quaternion group. Let $\mathfrak{Q}$ be a subgroup of $\mathfrak{N}$ incident with $\tilde{\mathfrak{Q}}$. Clearly, $\mathfrak{H} = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Q})[\mathfrak{Q}, \mathfrak{H}]$ and $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Q})$ commutes elementwise with $[\mathfrak{Q}, \mathfrak{H}]$. By Lemma 9.3 (iii), $\mathfrak{Q}'$ centralizes no noncyclic subgroup of $\mathfrak{H}$. It follows that $\boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Q}) = \boldsymbol{Z}(\mathfrak{H})$ is cyclic. However, $w \geqq 2$ and C centralizes $\mathfrak{H}_0$.

LEMMA 9.5. *Hypothesis* 9.3 *is not satisfied.*

Proof. Suppose false.

Let $\mathscr{X} = \{\mathfrak{Z}_1 \mid \mathfrak{Z}_1 \subseteqq \mathfrak{H}, \mathfrak{Z}_1 \sim \mathfrak{Z}, \mathfrak{Z}_1 \neq \mathfrak{Z}\}$. By Hypothesis 9.3 (iii), $\mathscr{X} \neq \emptyset$. Since $\mathfrak{H} \lhd \mathfrak{N}$, $\mathscr{X}$ is invariant in $\mathfrak{N}$. Choose $\mathfrak{Z}_1 \in \mathscr{X}$ such that $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{Z}_1)$ is a S_3-subgroup of $\boldsymbol{C}_{\mathfrak{N}}(\mathfrak{Z}_1)$. Let $\mathfrak{P}_0 = \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{Z}_1)$.

If $\mathfrak{P}_0 = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{Z}_1)$, then $\mathfrak{Z}$ char $\mathfrak{P}_0$. This is impossible since $\mathfrak{P}_0$ is not a S_3-subgroup of $\boldsymbol{C}(\mathfrak{Z}_1)$. Hence, $|\mathfrak{P}_0| = 3^5$.

Let $\mathfrak{D} = \langle \mathfrak{Z}, \mathfrak{Z}_1\rangle$, so that $\mathfrak{D} \subseteqq \boldsymbol{Z}(\mathfrak{P}_0)$. If $\mathfrak{D} \subset \boldsymbol{Z}(\mathfrak{P}_0)$, then choose $Z \in \boldsymbol{Z}(\mathfrak{P}_0) - \mathfrak{H}$, so that Z centralizes a 3-dimensional subspace of $\mathfrak{H}/\mathfrak{H}'$. This implies that some involution of $\boldsymbol{O}^{3'}(\mathfrak{N})$ has a noncyclic fixed point set on $\mathfrak{H}$, in violation of Lemma 9.3 (iii). Hence, $\mathfrak{D} = \boldsymbol{Z}(\mathfrak{P}_0)$.

Let $\mathfrak{P}^*$ be a S_3-subgroup of $\boldsymbol{C}(\mathfrak{Z}_1)$ which contains $\mathfrak{P}_0$. Thus, $\langle \mathfrak{P}, \mathfrak{P}^*\rangle \subseteqq \boldsymbol{N}(\mathfrak{P}_0) \subseteqq \boldsymbol{N}(\mathfrak{D})$, so $\boldsymbol{O}^{3'}(\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{D})) \cong SL(2, 3)$.

By Lemma 9.3, $|\boldsymbol{C}(\mathfrak{D})|$ is odd. Since $\mathfrak{P}_0$ is a S_3-subgroup of $\boldsymbol{C}(\mathfrak{D})$ and since $\boldsymbol{O}_{3'}(\boldsymbol{C}(\mathfrak{D})) = 1$, it follows that $\mathfrak{P}_0 = \boldsymbol{C}(\mathfrak{D})$. Hence, $\boldsymbol{N}(\mathfrak{P}_0) = \boldsymbol{N}(\mathfrak{D})$. Let $\mathfrak{M} = \boldsymbol{N}(\mathfrak{D})$.

Let $\mathfrak{Q}$ be a S_2-subgroup of $\boldsymbol{O}^{3'}(\mathfrak{N})$. Thus, $\mathfrak{Q}$ is a quaternion group. Let J be the involution of $\mathfrak{Q}$. Let $\mathfrak{Q}^*$ be a S_2-subgroup of $\boldsymbol{O}^{3'}(\mathfrak{M})$. Thus $\mathfrak{Q}^*$ is a quaternion group. Let I be the involution of $\mathfrak{Q}^*$. Since J inverts $\mathfrak{H}/\mathfrak{H}'$, $J \in \mathfrak{M}$. Since I inverts $\mathfrak{D}$, $I \in \mathfrak{N}$. We assume without loss of generality that I normalizes $\mathfrak{Q}$ and J normalizes $\mathfrak{Q}^*$.

Since J neither inverts nor centralizes $\mathfrak{D}$, it is clear that $\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{D}) \cong GL(2, 3)$ and so $\langle J, \mathfrak{Q}^*\rangle$ is isomorphic to a S_2-subgroup of $GL(2, 3)$. Let Q^* be an element of $\mathfrak{Q}^*$ of order 4 which is inverted by J.

We will show that $\langle I, \mathfrak{Q}\rangle$ is isomorphic to a S_2-subgroup of $GL(2, 3)$. Since $I \in \mathfrak{N}$, we need only prove that I inverts $\mathfrak{P}_0/\mathfrak{P}_0 \cap \mathfrak{H}$. Suppose false. Then I centralizes $\mathfrak{P}_0/\mathfrak{P}_0 \cap \mathfrak{H}$. We know that $\mathfrak{D} \subseteqq \mathfrak{P}_0'$, because $\mathfrak{M}$ operates irreducibly on $\mathfrak{D}$ and $\mathfrak{D}$ contains $\mathfrak{Z} = (\mathfrak{P}_0 \cap \mathfrak{H})'$. Since $\mathfrak{Q}^*$ is faithfully represented on $\mathfrak{P}_0$, there must be a 2-dimensional subspace of $\mathfrak{P}_0/\boldsymbol{D}(\mathfrak{P}_0)$ which I inverts and $\mathfrak{Q}^*$ leaves invariant. Since I centralizes $\mathfrak{P}_0/\mathfrak{P}_0 \cap \mathfrak{H}$ and $|\mathfrak{P}_0| = 3^5$, we conclude that $\mathfrak{D} = \mathfrak{P}_0' = \boldsymbol{D}(\mathfrak{P}_0)$, and that $\mathfrak{P}_0 \cap \mathfrak{H}/\mathfrak{D}$ is the subspace of $\mathfrak{P}_0/\mathfrak{P}_0'$ inverted by I. This forces I to invert both $\mathfrak{P}_0 \cap \mathfrak{H}/\mathfrak{D}$ and $\mathfrak{D}$, that is, to invert $\mathfrak{P}_0 \cap \mathfrak{H}$. So $\mathfrak{P}_0 \cap \mathfrak{H}$ is abelian, which is false.

Let Q be an element of $\mathfrak{Q}$ of order 4 which is inverted by I.

Set $\mathfrak{X}_6 = \mathfrak{Z}$. Since J normalizes $\mathfrak{D}$ and centralizes $\mathfrak{X}_6$, we can choose an element X_5 of $\mathfrak{D}$ of order 3 such that $X_5^J = X_5^{-1}$. Let $\mathfrak{X}_5 = \langle X_5 \rangle$, $\mathfrak{X}_4 = \mathfrak{X}_5^Q$, $X_4 = X_5^Q$. Then we have relations $X_5^I = X_5^{-1}, X_5^J = X_5^{-1}$, $X_4^I = X_4$, $X_4^J = \mathfrak{X}_4^{-1}$. Suppose $[X_4, X_5] \neq 1$. The following argument is designed to exclude this possibility.

Let $[X_4, X_5] = X_6$ so that X_6 is a generator for $\mathfrak{X}_6$. Since $X_4 \notin \mathfrak{P}_0$, it follows that $\mathfrak{P}_0 \cap C(I)$ is of order 3 with generator X_3, say. Thus, $\langle X_3, X_4 \rangle = C_{\mathfrak{P}}(I)$ is of order 9, so that $[X_3, X_4] = 1$. As $\mathfrak{H}$ contains all the elements of $\mathfrak{P}$ which are centralized by I, we have $X_3 \in H$. Since $\langle X_3 \rangle = C_{\mathfrak{P}_0}(I)$, it follows that J normalizes $\langle X_3 \rangle$, so that $X_3^J = X_3^{-1}$, as J inverts $\mathfrak{H}/\mathfrak{Z}$. Let $X_2 = X_3^Q$. Since $[X_3, X_4] = 1$, so also $[X_2, X_4^Q] = 1$. But $X_4^Q = X_5^{Q^2} = X_5^{-1}$, so X_2 centralizes X_5, that is, $X_2 \in \mathfrak{P}_0$. Let $X_1 = X_2^{Q^*}$, and let $\mathfrak{X}_i = \langle \mathfrak{X}_i \rangle$, $1 \leqq i \leqq 6$. We obtain the following data:

Table 1

	J	I
X_1	X_1	X_1^{-1}
X_2	X_2^{-1}	X_2^{-1}
X_3	X_3^{-1}	X_3
X_4	X_4^{-1}	X_4
X_5	X_5^{-1}	X_5^{-1}
X_6	X_6	X_6^{-1}

Table 2

	Q	Q^*
X_1	—	X_2^{-1}
X_2	X_3^{-1}	X_1
X_3	X_2	X_3
X_4	X_5^{-1}	—
X_5	X_4	X_6^{-a}
X_6	X_6	X_5^a

Here $a^2 = 1$, and the last two entries in Table 2 are at our disposal since Q^* normalizes $\langle I, J \rangle$ and since $\mathfrak{X}_5$, $\mathfrak{X}_6$ are the only subgroups of $\mathfrak{D}$ of order 3 which admit $\langle I, J \rangle$. In addition we have the following commutation relations:

$$[X_i, X_6] = 1, 1 \leqq i \leqq 6, [X_4, X_5] = X_6 ,$$
$$[X_i, X_5] = 1, 1 \leqq i \leqq 3, [X_3, X_4] = [X_2, X_4] = 1 .$$

Furthermore, $[X_2, X_3] = X_6^b$, so by Table 2, we get $[X_1, X_3] = X_5^{ab}$. Here $b^2 = 1$, for if $b = 0$, we get $X_3 \in Z(\mathfrak{H})$, which is not the case. The as yet undetermined commutation relations are:

$$[X_1, X_4] = X_2^x X_3^y X_5^z X_6^t , \qquad [X_1, X_2] = X_3^c X_5^d X_6^e .$$

Use Table 1 and conjugate the second relation by J, obtaining $e = bc$. Then conjugation by I yields $d = abc$. Conjugation of the first relation by J yields $t = xyb + z$. Conjugation of the first relation by I yields $y = cx$.

Assume $c \neq 0$. Then

$$\mathfrak{P}_0' = \langle X_3, X_5, X_6 \rangle, [\mathfrak{P}_0', \mathfrak{P}_0] = \langle X_5, X_6 \rangle = \mathfrak{D} = Z(\mathfrak{P}_0) .$$

We see that $\mathfrak{P}_0'$ is elementary abelian. If $A \in \mathfrak{P}_0$, $B \in \mathfrak{P}_0'$, then $(AB)^3 = A^3 B^{A^2+A+1}$. But $cl(\mathfrak{P}_0) = 3$ and so $B^{A^2+A+1} = B^3[B, A]^3 = 1$. Hence, there is a map φ of $\mathfrak{P}_0/\mathfrak{P}_0'$ given by $\varphi(A\mathfrak{P}_0') = A^3$.

Clearly, $\varphi(X_1\mathfrak{P}_0') = 1$. But $\mathfrak{M}$ operates as $GL(2, 3)$ on $\mathfrak{D}$. Since $|\mathfrak{P}_0 : \mathfrak{P}_0'| = 3^2$, this forces $\mathfrak{M}$ to operate as $GL(2, 3)$ on $\mathfrak{P}_0/\mathfrak{P}_0'$. In particular, the four subgroups of $\mathfrak{P}_0/\mathfrak{P}_0'$ of order 3 are all conjugate under $\mathfrak{M}$. Hence, $\varphi(A\mathfrak{P}_0') = 1$ for all A in $\mathfrak{P}_0$ and $\mathfrak{P}_0$ is of exponent 3.

By [21, p. 324], the order of the Burnside group of exponent 3 on 2 generators is 27. Since $\mathfrak{P}_0$ must be a homomorphic image of this group, we get a contradiction, as $|\mathfrak{P}_0| = 3^5 > 27$. So $c = 0$.

Since $c = 0$, so also $c = d = e = y = 0$. Since $y = 0$, we also have $t = z$. Conjugation of the first relation by Q yields $[Q^{-1}X_1Q, X_5^{-1}] = X_3^{-x}X_4^zX_6^z$. Now $C(J) \cap O^{3'}(\mathfrak{N}) = \langle X_1, X_6, \mathfrak{Q}\rangle$, so $C(J) \cap O^{3'}(\mathfrak{N})$ is 2-closed, that is, X_1 normalizes $\mathfrak{Q}$. Hence, $(QX_1)^3 = J^u$, $u = 0$ or 1. Hence, $Q^{-1}X_1Q = JX_1^{-1}Q^{-1}X_1^{-1}J^u$. The previous commutation relation now yields $x = 0$.

Since $x = 0$, it follows that X_4 centralizes $\mathfrak{P}_0/\mathfrak{D}$. Hence, $\mathfrak{Q}^*$ is forced to centralize $\mathfrak{P}_0/\mathfrak{D}$. This is not the case, since $\mathfrak{D} \subseteq \mathfrak{P}_0'$. We conclude that $[X_4, X_5] = 1$.

Since I centralizes X_4, it follows that $\langle X_4, X_5, , X_6\rangle = \mathfrak{E} \lhd \mathfrak{M}$. Namely, $\mathfrak{D} \lhd \mathfrak{M}$, so we need to show that $\mathfrak{E}/\mathfrak{D} \lhd \mathfrak{M}/\mathfrak{D}$. Since X_4 centralizes $\mathfrak{D}$, we have $X_4 \in \mathfrak{P}_0$. Since $\mathfrak{P}_0/\mathfrak{D}$ is of order 27 and admits $\mathfrak{Q}^*$ as a group of automorphisms, it follows that $\mathfrak{E}/\mathfrak{D} = C_{\mathfrak{P}_0/\mathfrak{D}}(I) \lhd \mathfrak{M}/\mathfrak{D}$. Thus, $\langle \mathfrak{M}, Q\rangle \subseteq N(\mathfrak{E})$. Since $\langle \mathfrak{P}, Q\rangle = O^{3'}(\mathfrak{N})$, it follows that both $\mathfrak{M}$ and $O^{3'}(\mathfrak{N})$ are subgroups of $N(\mathfrak{E})$.

Let $\mathfrak{E}^* = O_3(N(\mathfrak{E}))$. Thus,

$$\mathfrak{E} \subseteq \mathfrak{E}^* \subseteq O_3(\mathfrak{M}) \cap O_3(\mathfrak{N}) = \mathfrak{P}_0 \cap \mathfrak{H} .$$

Suppose $\mathfrak{E}^* = \mathfrak{P}_0 \cap \mathfrak{H}$. Then $\mathfrak{Z} = \mathfrak{E}^{*\prime} \lhd N(\mathfrak{E})$, against $O^{3'}(\mathfrak{N}) \subset N(\mathfrak{E})$. Hence, $\mathfrak{E}^* \subset \mathfrak{P}_0 \cap \mathfrak{H}$. Since $|\mathfrak{E}| = 3^3$ and $|\mathfrak{P}_0 \cap \mathfrak{H}| = 3^4$, it follows that $\mathfrak{E}^* = \mathfrak{E}$. Thus, $N(\mathfrak{E})/\mathfrak{E}$ is isomorphic to a subgroup of Aut $(\mathfrak{E})$ which (a) is solvable, (b) contains a S_3-subgroup of Aut $(\mathfrak{E})$, (c) is 3-reduced. There are no such groups. The proof of the lemma is complete.

LEMMA 9.6. *Let $\mathfrak{B}$ be a subgroup of $\mathfrak{H}$ of type* (3, 3). *Then* $\mathfrak{B} \in \mathscr{D}$. (See Definition 7.3.)

Proof. We first show that if $B \in \mathfrak{B}$, then

(9.8) for some N in $\mathfrak{N}$, B centralizes an element of $\mathscr{U}(\mathfrak{P}^N)$.

Let

$$\mathfrak{H}_1 = \Omega_1(\mathfrak{H}),\ \mathfrak{V} = \mathfrak{H}_1/D(\mathfrak{H}_1),\ \mathfrak{V}_0 = C_{\mathfrak{V}}(\mathfrak{P}) .$$

Suppose $|\mathfrak{V}_0| > 3$. Then $\mathfrak{V}_0 = \mathfrak{W}/D(\mathfrak{H}_1)$ and every subgroup of $\mathfrak{W}$ which contains $D(\mathfrak{H}_1)$ is normal in $\mathfrak{P}$. Let $\mathfrak{W}_1 = \mathfrak{W} \cap C(B)$ so that $|\mathfrak{W} : \mathfrak{W}_1| \leqq 3$.

Since $|\mathfrak{B}_0| > 3$, so also $|\mathfrak{W}_1| \geqq 9$. Thus, B centralizes an element of $\mathscr{U}(\mathfrak{P})$ in this case. We may assume that $|\mathfrak{B}_0| = 3$.

Suppose $\mathfrak{N}/\mathfrak{H}$ has a normal subgroup $\mathfrak{K}/\mathfrak{H} = \mathfrak{X}$ of odd order $\neq 1$. Let k be a field of characteristic 3 which contains all $|\mathfrak{X}|^{\text{th}}$ roots of 1. Let $\widetilde{\mathfrak{B}} = k \otimes_{F_3} \mathfrak{B}$. Thus, $\widetilde{\mathfrak{B}}$ admits $\mathfrak{N}/\mathfrak{H}$ and $k \otimes \mathfrak{B}_0$ is the set of all fixed points of $\mathfrak{P}/\mathfrak{H}$ on $\widetilde{\mathfrak{B}}$. Let $\widetilde{\mathfrak{B}} = \bigoplus_\rho \widetilde{\mathfrak{B}}(\rho)$, where $\widetilde{\mathfrak{B}}(\rho)$ is the largest $\mathfrak{X}$-submodule of $\widetilde{\mathfrak{B}}$ on which $\mathfrak{X}$ acts as a multiple of the irreducible representation ρ. Since $\widetilde{\mathfrak{B}}$ inherits the non singular symplectic structure of $\mathfrak{B}$, it follows that ρ and ρ^* appear with the same multiplicity in $\widetilde{\mathfrak{B}}$, ρ^* denoting the contragredient representation of ρ. Since $|\mathfrak{P}|$ is odd, $\widetilde{\mathfrak{B}}(\rho)$ and $\widetilde{\mathfrak{B}}(\rho^*)$ are not conjugate under $\mathfrak{P}$. Hence, $\mathfrak{B}_0$ is not 1-dimensional in this case.

We may now assume that

$$(9.9) \qquad F(\mathfrak{N}/\mathfrak{H}) \text{ is a 2-group .}$$

If $\mathfrak{H} = \mathfrak{P}$, then (9.8) is obvious, so suppose $\mathfrak{H} \subset \mathfrak{P}$. Set $\mathfrak{N}^* = C(\mathfrak{Z})$, so that $|\mathfrak{N}: \mathfrak{N}^*| \leqq 2$. By Lemma 9.3 (i), together with (9.9), we conclude that $\mathfrak{N}$ is a 2, 3-group, and that a S_2 subgroup of $\mathfrak{N}^*$ is quaternion. Hence, $|\mathfrak{P}: \mathfrak{H}| = 3$. Since $|\mathfrak{B}_0| = 3$, we get that the width of $\mathfrak{H}$ is 1, against Hypothesis 9.2. Thus, (9.8) holds.

Suppose $\mathfrak{B} \notin \mathscr{D}$. Then $\mathsf{M}(\mathfrak{B}; 2)$ contains a four-group $\mathfrak{Q}$ which is not centralized by $\mathfrak{B}$. Hence, $[\mathfrak{B}, \mathfrak{Q}] = \mathfrak{Q}$, and $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{Q})$ is of order 3.

Let $\mathfrak{C} = C(\mathfrak{B}_0)$, $\mathfrak{H}_0 = C_{\mathfrak{H}}(\mathfrak{B}_0)$. By Lemma 7.2 applied to $\langle \mathfrak{B}_0, \mathfrak{Z} \rangle$, it follows that $\mathfrak{Z}$ centralizes $O_{3'}(\mathfrak{C})$. Hence, $[O_{3'}(\mathfrak{C}), \mathfrak{H}_0] \subseteqq \mathfrak{H} \cap O_{3'}(\mathfrak{C}) = 1$. This implies that $O_{3'}(\mathfrak{C}) = 1$ by Lemma 9.4.

Let $\mathfrak{P}_0$ be a S_3-subgroup of $\mathfrak{C}$ containing $\mathfrak{H}_0$ and let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_0$. Then $\mathfrak{P}^* = \mathfrak{P}^G$, so that with $\mathfrak{Z}^* = \mathfrak{Z}^G$, it follows that $\mathfrak{Z}^* \subseteqq Z(O_3(\mathfrak{C}))$. Let $\mathfrak{W} = \mathfrak{Z}^{*\mathfrak{C}}$, so that $\mathfrak{W}$ is 3-reducible in $\mathfrak{C}$. Set $\mathfrak{C}_1 = C_{\mathfrak{C}}(\mathfrak{W})$. We argue that $\mathfrak{C}_1 \cap \mathfrak{Q} = 1$. If not, then $\mathfrak{Q} \subseteqq C(\mathfrak{W})$, as $\mathfrak{Q}$ is an irreducible $\mathfrak{B}$-module. Hence, $\mathfrak{Q} \subseteqq C(\mathfrak{Z}^*)$, against Lemma 9.3 (i). Hence,

$$(9.10) \qquad \mathfrak{C}_1 \cap \mathfrak{Q} = 1 .$$

By (B), elements of $\mathfrak{B} - \mathfrak{B}_0$ have minimal polynomial $(x-1)^3$ on $\mathfrak{W}$.

We next argue that $\mathfrak{Z} \subseteqq \mathfrak{C}_1$. If not, then $\mathfrak{H}_0$ contains an extra special subgroup of width $w-1$ disjoint from $\mathfrak{C}_1$. We get that $m(\mathfrak{W}) \geqq 2 \cdot 3^{w-1}$. Since $m(\mathfrak{W} \cap \mathfrak{H}^G) \leqq w+1$, we have $m(\mathfrak{W}/\mathfrak{W} \cap \mathfrak{H}^G) \geqq 2 \cdot 3^{w-1} - w - 1$. By [32], it follows that $w^2 \geqq 2 \cdot 3^{w-1} - w - 1$. This is false for $w \geqq 3$, so $w = 2$. Thus, $C(\mathfrak{Z})/\mathfrak{H}$ is isomorphic to a subgroup of $GL(4, 3)$ which (a) is solvable, (b) is 3-reduced, (c) has an elementary subgroup of order 27. There are no such groups. We conclude that $\mathfrak{Z} \subseteqq \mathfrak{C}_1$.

Since $\mathfrak{Z} \subseteqq \mathfrak{C}_1$, we have $\mathfrak{W} \subseteqq \mathfrak{N}$, so that $[\mathfrak{W}, \mathfrak{B}, \mathfrak{B}] \subseteqq \mathfrak{Z}$. Hence by (B),

$$[\mathfrak{W}, \mathfrak{B}, \mathfrak{B}] = \mathfrak{Z} \,. \tag{9.11}$$

Since $\mathfrak{W} \subseteq \boldsymbol{Z}(\mathfrak{C}_1)$, we get

$$\mathfrak{Z} \subseteq \boldsymbol{Z}(\mathfrak{C}_1) \,, \tag{9.12}$$

which implies that

$$\mathfrak{C}_1 \subseteq \mathfrak{N} \,. \tag{9.13}$$

By (9.13), we get $[\mathfrak{C}_1, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{Z}$, and in particular,

$$[\boldsymbol{O}_3(\mathfrak{C}), \mathfrak{B}, \mathfrak{B}] = \mathfrak{Z} \,, \tag{9.14}$$

equality holding by (9.11) and the obvious containment $\mathfrak{W} \subseteq \boldsymbol{O}_3(\mathfrak{C})$.
Now (9.14) and (9.11) yield

$$\boldsymbol{O}_3(\mathfrak{C}) = \mathfrak{W}_1 \times \mathfrak{W}_2 \,, \tag{9.15}$$

where

$$\mathfrak{Z} \subset \mathfrak{W}_1 = [\mathfrak{Q}, \boldsymbol{O}_3(\mathfrak{C})], \text{ and } \mathfrak{W}_1 \text{ is elementary of order } 27 \,, \tag{9.16}$$

$$\mathfrak{W}_2 = \boldsymbol{C}(\mathfrak{Q}) \cap \boldsymbol{O}_3(\mathfrak{C}) \,. \tag{9.17}$$

Let $\mathfrak{C}_2 = \boldsymbol{O}_{3'}(\mathfrak{C} \bmod \mathfrak{C}_1)$, $\mathfrak{C}_3 = \mathfrak{C}_2\boldsymbol{Z}(\mathfrak{H})$. Thus, $\mathfrak{C}_1\boldsymbol{Z}(\mathfrak{H})$ contains a S_3-subgroup of $\mathfrak{C}_3$. Thus, $\boldsymbol{Z}(\mathfrak{H})$ is normal in a S_3-subgroup of $\mathfrak{C}_3$. By Lemma 5.22, we get $\boldsymbol{Z}(\mathfrak{H}) \subseteq \boldsymbol{O}_3(\mathfrak{C}_3)$. Hence, $\boldsymbol{Z}(\mathfrak{H}) \subseteq \boldsymbol{O}_3(\mathfrak{C})$. From (9.16), we conclude that $\boldsymbol{Z}(\mathfrak{H}) = \mathfrak{Z}$, that is,

$$\mathfrak{H} \text{ is extra special} \,. \tag{9.18}$$

We argue that $\boldsymbol{O}_{3,3'}(\mathfrak{C})$ does not centralize $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C}))$. If it does, then since $\mathfrak{Z} \subseteq \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C}))$, it follows that $\boldsymbol{O}_{3,3'}(\mathfrak{C}) \subseteq \mathfrak{N}$, so $[\boldsymbol{O}_{3,3'}(\mathfrak{C}), \mathfrak{H}_0] \subseteq \mathfrak{H}$, which implies that $\mathfrak{H}_0 \subseteq \boldsymbol{O}_3(\mathfrak{C})$, which in turn gives $\mathfrak{Q} = [\mathfrak{B}, \mathfrak{Q}] \subseteq \boldsymbol{O}_3(\mathfrak{C})$. Since $\mathfrak{B}_0 \subseteq \boldsymbol{Z}(\mathfrak{C})$, it follows that

$$[\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})), \boldsymbol{O}_{3,3'}(\mathfrak{C})] \quad \text{and} \quad \boldsymbol{C}(\boldsymbol{O}_{3,3'}(\mathfrak{C})) \cap \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C}))$$

are disjoint nontrivial normal abelian subgroups of $\mathfrak{C}$. In particular, if $\mathfrak{P}_0$ is a S_3-subgroup of $\mathfrak{C}$ containing $\mathfrak{H}_0$, then $\boldsymbol{Z}(\mathfrak{P}_0)$ is not cyclic. By Lemma 9.4, we get that $\Omega_1(\boldsymbol{Z}(\mathfrak{P}_0)) = \mathfrak{B}_0 \times \mathfrak{Z}$, and in particular, $\mathfrak{P}_0 \subseteq \mathfrak{N}$.

Since $\mathfrak{P}_0 \subseteq \mathfrak{N}$, we get that $[\mathfrak{P}_0, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{Z}$. Thus, if $B \in \mathfrak{B}$, the minimal polynomial of B on the Frattini quotient group of $\boldsymbol{O}_{3,3',3}(\mathfrak{C})/\boldsymbol{O}_{3,3'}(\mathfrak{C})$ divides $(x-1)^2$. By (B), it follows that $\mathfrak{Q}$ centralizes $\boldsymbol{O}_{3,3',3}(\mathfrak{C})/\boldsymbol{O}_{3,3'}(\mathfrak{C})$, and so $\mathfrak{Q} \subseteq \boldsymbol{O}_{3,3'}(\mathfrak{C})$.

Let $\mathfrak{R} = \langle \mathfrak{Q}, \mathfrak{H}_0 \rangle \subseteq \mathfrak{C}$, let $\mathfrak{R}_0 = \boldsymbol{C}_{\mathfrak{R}}(\boldsymbol{O}_3(\mathfrak{C}))$ and for any subset $\mathfrak{S}$ of $\mathfrak{R}$, let $\bar{\mathfrak{S}} = \mathfrak{S}\mathfrak{R}_0/\mathfrak{R}_0$.

We argue that $\bar{\mathfrak{Q}} \subseteq \boldsymbol{O}_{3'}(\bar{\mathfrak{R}})$. Namely, $\mathfrak{Q} \subseteq \boldsymbol{O}_{3,3'}(\mathfrak{C})$, and so $\mathfrak{Q} \subseteq \boldsymbol{O}_{3,3'}(\mathfrak{R})$, Thus, it suffices to show that $[\boldsymbol{O}_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \mathfrak{K}_0$. But

$$[\boldsymbol{O}_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \boldsymbol{O}_3(\mathfrak{K}) \cap \boldsymbol{O}_{3,3'}(\mathfrak{C}) \subseteq \boldsymbol{O}_3(\mathfrak{K}) \cap \boldsymbol{O}_3(\mathfrak{C}) \ ,$$

and so

$$\begin{aligned}[\boldsymbol{O}_3(\mathfrak{K}), \mathfrak{Q}] &= [\boldsymbol{O}_3(\mathfrak{K}), \mathfrak{Q}, \mathfrak{Q}] \subseteq [\boldsymbol{O}_3(\mathfrak{K}) \cap \boldsymbol{O}_3(\mathfrak{C}), \mathfrak{Q}] \subseteq [\boldsymbol{O}_3(\mathfrak{C}), \mathfrak{Q}] \\ &= \mathfrak{W}_1 \subseteq \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})) \ ,\end{aligned}$$

whence $[\boldsymbol{O}_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \mathfrak{K} \cap \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})) \subseteq \mathfrak{K}_0$.

Case 1. $\langle \overline{\mathfrak{Q}, \mathfrak{Q}^H} \rangle$ is abelian for all $H \in \mathfrak{H}_0$.

Since $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}, \mathfrak{B}}]$ admits the abelian group $\bar{\mathfrak{H}}_0$, and since $\mathfrak{Q} \subseteq [\mathfrak{Q}^{\mathfrak{H}_0}, \mathfrak{B}]$, it follows that $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}, \mathfrak{B}}] = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$. Since $\mathfrak{W}_2 = \boldsymbol{C}(\mathfrak{Q}) \cap \boldsymbol{O}_3(\mathfrak{C})$ admits the abelian group $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$, (B) implies that $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ centralizes $\mathfrak{W}_2$. Hence, $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ is isomorphic to an elementary 2-subgroup of Aut $(\mathfrak{W}_1)$. Since $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}, \mathfrak{B}}] = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$, we get that $\bar{\mathfrak{Q}} = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$, so that $\bar{\mathfrak{Q}}$ is a S_2-subgroup of $\bar{\mathfrak{K}}$.

Let $\mathfrak{K}_1 = \boldsymbol{O}_3(\mathfrak{K} \bmod \mathfrak{K}_0)$. Thus, $\mathfrak{K}_1 \cap \mathfrak{H}_0$ is of index 3 in $\mathfrak{H}_0$ and $\langle \mathfrak{K}_1 \cap \mathfrak{H}_0, \mathfrak{B} \rangle = \mathfrak{H}_0$. Since $|\mathfrak{K}_0|$ is odd, it follows that $\mathfrak{Q}$ is a S_2-subgroup of $\mathfrak{K}$. Let $\mathfrak{L} = \mathfrak{K}_1 \boldsymbol{O}_3(\mathfrak{C})$ and let $\mathfrak{L}_3$ be a S_3-subgroup of $\mathfrak{L}$ which contains $\mathfrak{K}_1 \cap \mathfrak{H}_0$ and is normalized by $\mathfrak{H}_0$. Since $|\mathfrak{L}|$ is odd, it follows that S_2-subgroups of $\boldsymbol{N}(\mathfrak{L}_3) \cap \mathfrak{L}\mathfrak{K}$ are four-groups. If $\mathfrak{Z} \subseteq \boldsymbol{D}(\mathfrak{L}_3)$, then by (B), S_2-subgroups of $\boldsymbol{N}(\mathfrak{L}_3) \cap \mathfrak{L}\mathfrak{K}$ centralize $\mathfrak{L}_3$. This is not the case, as $\mathfrak{Q}$ does not centralize $\mathfrak{W}_1$. Hence, $\mathfrak{Z} \not\subseteq \boldsymbol{D}(\mathfrak{L}_3)$. In particular, $\mathfrak{Z} \not\subseteq \boldsymbol{D}(\mathfrak{K}_1 \cap \mathfrak{H}_0)$. But $\mathfrak{K}_1 \cap \mathfrak{H}_0$ is of index 3 in $\mathfrak{H}_0$. Since $\mathfrak{H}$ is extra special, it follows that $w = 2$. Clearly, $\mathfrak{H} \subset \mathfrak{P}$, since $\boldsymbol{O}_3(\mathfrak{C})$ contains an elementary subgroup of order 3^4. On the other hand, Lemma 9.3 implies that $|\mathfrak{P} : \mathfrak{H}| \leqq 3$. Hence, $|\mathfrak{P} : \mathfrak{H}| = 3$, and $|\mathfrak{P}| = 3^6$. Since $\mathfrak{B}_0$ is obviously not conjugate to $\mathfrak{Z}$, it follows that $\boldsymbol{O}_3(\mathfrak{C})$ is elementary of order 3^4 and $\mathfrak{P}_0 = \boldsymbol{O}_3(\mathfrak{C})\mathfrak{H}_0$, $|\mathfrak{H}_0 \cap \boldsymbol{O}_3(\mathfrak{C})| = 27$. Clearly, $\boldsymbol{O}_3(\mathfrak{C})$ char $\mathfrak{P}_0$, since$\boldsymbol{O}_3(\mathfrak{C})$ is the only elementary subgroup of its order in $\mathfrak{P}_0$.

Let $\mathfrak{M} = \boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{C}))$ so that $\mathfrak{M}$ contains a S_3-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{G}$ with $\tilde{\mathfrak{P}} \supset \mathfrak{P}_0$. Since $\mathfrak{Z} = \boldsymbol{Z}(\mathfrak{P}_0) \cap \mathfrak{P}_0'$ char $\mathfrak{P}_0$, we have $\mathfrak{Z} \lhd \tilde{\mathfrak{P}}$. In particular, $\mathfrak{H} \subset \mathfrak{M}$. We therefore assume without loss of generality that $\mathfrak{P} = \tilde{\mathfrak{P}}$.

It is clear that $\boldsymbol{O}_3(\mathfrak{C}) = \boldsymbol{O}_3(\mathfrak{M})$ and that $\mathfrak{M}$ is a 2, 3-group. It is equally clear that $l_3(\mathfrak{M}) = 2$, so that $\mathfrak{Q} \subseteq \boldsymbol{O}_{3,2}(\mathfrak{M})$. Hence, $\mathfrak{B}$ is a S_3-subgroup of $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}$, so we can choose a subgroup $\mathfrak{B}_1$ of $\mathfrak{B}$ of order 3 such that $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$ and such that $\mathfrak{B}_1$ normalizes some S_2-subgroup $\mathfrak{T}_0$ of $\boldsymbol{O}_{3,2}(\mathfrak{M})$. Let $\mathfrak{A} = \boldsymbol{N}(\mathfrak{T}_0) \cap \mathfrak{P}$. Thus, $\mathfrak{A}$ is elementary of order 9, $\mathfrak{B}_1 \subset \mathfrak{A}$ and $\mathfrak{P} = \mathfrak{A}\boldsymbol{O}_3(\mathfrak{M})$, $\mathfrak{A} \cap \boldsymbol{O}_3(\mathfrak{M}) = 1$. Since $\boldsymbol{O}_3(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{B}_1)$ is of order 9, it follows that $\boldsymbol{C}(\mathfrak{B}_1) \cap \mathfrak{P}$ is of order 3^4. Hence, $\boldsymbol{C}(\mathfrak{B}_1) \cap \mathfrak{P} = \boldsymbol{C}(\mathfrak{B}_1) \cap \mathfrak{H}$; since $\mathfrak{A}$ is elementary we get $\mathfrak{A} \subset \mathfrak{H}$.

We now choose $\mathfrak{A}_1$ of order 3 in $\mathfrak{A}$ so that $\mathfrak{A}$ does not centralize $\boldsymbol{C}_{\mathfrak{T}_0}(\mathfrak{A}_1)$. Let $\mathfrak{T}_1 = [\boldsymbol{C}_{\mathfrak{T}_0}(\mathfrak{A}_1), \mathfrak{A}]$. Thus, $\mathfrak{T}_1$ is faithfully represented on $\boldsymbol{C}(\mathfrak{A}_1) \cap \boldsymbol{O}_3(\mathfrak{M}) = \mathfrak{N}$. It is straightforward to verify that $|\mathfrak{N}| = 9$ and that $\mathfrak{T}_1$ is a quaternion group. Hence, $\mathfrak{R} = [O_3(\mathfrak{M}), \mathfrak{A}_1]$, so $\mathfrak{R} \subseteq \mathfrak{H}$. Since $\mathfrak{Z} \subset \mathfrak{R}$, it follows that $\mathfrak{Z}$ is not weakly closed in $\mathfrak{H}$. As this violates Lemma 9.5, we conclude that Case 1 does not hold.

Case 2. There is an element H of $\mathfrak{H}_0$ such that $\langle \overline{\mathfrak{Q}, \mathfrak{Q}^H} \rangle$ is nonabelian.

Set $\widetilde{\mathfrak{W}} = \langle \mathfrak{W}_1, \mathfrak{W}_1^H \rangle$, so that $\langle \mathfrak{Q}, \mathfrak{Q}^H \rangle$ normalizes $\widetilde{\mathfrak{W}}$ and centralizes $O_3(\mathfrak{C})/\widetilde{\mathfrak{W}}$. Since $\mathfrak{W}_1 \cap \mathfrak{H} \supset \mathfrak{Z}$, it follows that $|\mathfrak{W}_1 \cap \mathfrak{W}_1^H| \geqq 9$. Clearly, $\mathfrak{W}_1 \neq \mathfrak{W}_1^H$, since $\langle \overline{\mathfrak{Q}, \mathfrak{Q}^H} \rangle$ is nonabelian. Hence, $\widetilde{\mathfrak{W}}$ is elementary of order 3^4. Since $\langle \overline{\mathfrak{Q}, \mathfrak{Q}^H} \rangle$ is injected into Aut $(\widetilde{\mathfrak{W}})$ under the restriction map, it follows readily that $\langle \overline{\mathfrak{Q}, \mathfrak{Q}^H} \rangle$ is the central product of two quaternion groups, each of which necessarily admits $\mathfrak{B}$. In particular, $\langle \overline{\mathfrak{Q}, \mathfrak{Q}^H} \rangle'$ is of order 2 and inverts $\mathfrak{W}_1 \cap \mathfrak{H}$. Since no involution of $\mathfrak{G}$ centralizes $\mathfrak{W}_1 \cap \mathfrak{H}$, it follows that $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ is extra special of order 32. Hence, $[\mathfrak{Q}^{\mathfrak{H}_0}, O_3(\mathfrak{C})]$ is elementary of order 3^4. This implies that $O_3(\mathfrak{C})$ contains $[\mathfrak{Q}^{\mathfrak{H}_0}, O_3(\mathfrak{C})] \times \mathfrak{B}_0$, an elementary subgroup of order 3^5. Hence, $w \geqq 3$.

Write $O_3(\mathfrak{C}) = \mathfrak{X} \times \mathfrak{Y}$, where

$$\mathfrak{X} = [O_3(\mathfrak{C}), \mathfrak{Q}^{\mathfrak{H}_0}] \quad \text{and} \quad \mathfrak{Y} = O_3(\mathfrak{C}) \cap C(\mathfrak{Q}^{\mathfrak{H}_0}) .$$

Thus, $\mathfrak{H}_0$ normalizes both $\mathfrak{X}$ and $\mathfrak{Y}$. Suppose $Y \in \mathfrak{Y} \cap \mathfrak{H}$. Then

$$[Y, \mathfrak{H}_0] \subseteq \mathfrak{Z} \cap \mathfrak{Y} = 1 , \quad \text{so} \quad Y \in Z(\mathfrak{H}_0) = \mathfrak{Z} \times \mathfrak{B}_0 .$$

Hence, $\mathfrak{Y} \cap \mathfrak{H} = \mathfrak{B}_0$. Since $[\mathfrak{Y}, \mathfrak{H}_0] \subseteq \mathfrak{H}$, it follows that $[\mathfrak{Y}, \mathfrak{H}_0] \subseteq \mathfrak{B}_0$. Since $\mathfrak{Q}^{\mathfrak{H}_0}$ is absolutely irreducible on $\mathfrak{X}$, it follows that

$$[C_{\mathfrak{H}_0}(\overline{\mathfrak{Q}^{\mathfrak{H}_0}}), O_3(\mathfrak{C})] \subseteq \mathfrak{B}_0 , \quad \text{so} \quad C_{\mathfrak{H}_0}(\overline{\mathfrak{Q}_{\mathfrak{H}}{}^0}) \subseteq O_3(\mathfrak{C}) ,$$

since $O_3(\mathfrak{C}) = O_3(\mathfrak{C} \bmod \mathfrak{B}_0)$.

Clearly, $|\mathfrak{H}_0 : C_{\mathfrak{H}_0}(\overline{\mathfrak{Q}^{\mathfrak{H}_0}})| = 3^a$, $a = 1$ or 2, since $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ is extra special of order 32. If $a = 1$, then $\mathfrak{H}_0 \cap O_3(\mathfrak{C})$ is of index 9 in $\mathfrak{H}$, so is nonabelian since $w \geqq 3$. This is impossible, since $\mathfrak{Z} \nsubseteq D(O_3(\mathfrak{C}))$.

Suppose $a = 2$. Set $\mathfrak{A} = \mathfrak{H}_0 \cap O_3(\mathfrak{C})$. Since $\mathfrak{A}$ is abelian, $w = 3$. Thus, $\mathfrak{A} \in \mathscr{S}\!\mathscr{C}\!\mathscr{N}(\mathfrak{H})$. Let $\mathfrak{A}_1 = \mathfrak{X} \cap \mathfrak{A}$, so that $27 \geqq |\mathfrak{A}_1| \geqq 9$. Suppose $|\mathfrak{A}_1| = 9$. Let $\mathfrak{A}_2$ be a complement to $\mathfrak{A}_1$ in $\mathfrak{X}$, so that $|\mathfrak{A}_2| = 9$, and $\mathfrak{A}_2 \cap \mathfrak{H} = 1$. Since $\mathfrak{A}_2$ centralizes $\mathfrak{A}$, we get $[\mathfrak{H}, \mathfrak{A}_2] \subseteq \mathfrak{H} \cap C(\mathfrak{A}) = \mathfrak{A}$, so that $[\mathfrak{H}, \mathfrak{A}_2, \mathfrak{A}_2] = 1$. Thus, $[Q_3^1(\mathfrak{N}), \mathfrak{A}_2]$ is a 2-group on which $\mathfrak{A}_2$ is faithfully represented. This violates Lemma 9.3. Hence, $\mathfrak{A}_1$ is of order 3^3, so that $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{B}_0$. Suppose $\mathfrak{B}_0 \subset \mathfrak{Y}$. Let $\mathfrak{Y}_1$ be a subgroup of $\mathfrak{Y}$ of order 9 which contains $\mathfrak{B}_0$. Then $\mathfrak{X}\mathfrak{Y}_1$ is abelian of order 3^6, and $[\mathfrak{H}, \mathfrak{X}\mathfrak{Y}_1] \subseteq \mathfrak{H} \cap C(\mathfrak{A}) = \mathfrak{A}$, so that $[\mathfrak{H}, \mathfrak{X}\mathfrak{Y}_1, \mathfrak{X}\mathfrak{Y}_1] = 1$. It follows that $[Q_3^1(\mathfrak{N}), \mathfrak{X}\mathfrak{Y}_1]$ is a 2-group on which $\mathfrak{X}\mathfrak{Y}_1/\mathfrak{A}$ is faithfully represented. This again violates Lemma 9.3, so $\mathfrak{Y} = \mathfrak{B}_0$.

Since $\mathfrak{Y} = \mathfrak{B}_0$ and $a = 2$, $\boldsymbol{O}_3(\mathfrak{C})$ is elementary of order 3^5 and $|\mathfrak{P}_0| = 3^7$. Lemma 5.2 implies that if U is any element of $\mathfrak{C}/\boldsymbol{O}_3(\mathfrak{C})$ of order 3, then $\boldsymbol{C}(U) \cap \boldsymbol{O}_3(\mathfrak{C})$ is of order at most 3^3.

Suppose by way of contradiction that $\mathfrak{U}$ is an elementary subgroup of $\mathfrak{P}_0$ of order 3^5 which is distinct from $\boldsymbol{O}_3(\mathfrak{C})$. By the previous paragraph, we conclude that $\mathfrak{U} \cap \boldsymbol{O}_3(C)$ is of order 3^3, and that if $U \in \mathfrak{U} - \boldsymbol{O}_3(\mathfrak{C})$, then $\boldsymbol{O}_3(\mathfrak{C}) \cap \boldsymbol{C}(U) = \boldsymbol{O}_3(\mathfrak{C}) \cap \mathfrak{U}$. Let $\mathfrak{U}_0$ be a complement to $\mathfrak{U} \cap \boldsymbol{O}_3(\mathfrak{C})$ in $\mathfrak{U}$. Thus, $\mathfrak{U}_0$ is faithfully represented on $\boldsymbol{Q}_3^1(\mathfrak{K})$, the central product of two quaternion groups. Let $\mathfrak{R}$ be a quaternion subgroup of $\boldsymbol{Q}_3^1(\mathfrak{K})$, and let $\mathfrak{U}_1 = \boldsymbol{C}(\mathfrak{R}) \cap \mathfrak{U}_0$. Thus, $\mathfrak{U}_1$ is of order 3. By Lemma 3.7 of [20], $\mathfrak{R}$ is faithfully represented on $\boldsymbol{O}_3(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{U}_1)$. This is absurd, since $\mathfrak{U}_0$ centralizes $\boldsymbol{O}_3(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{U}_1)$. We conclude that $\boldsymbol{O}_3(\mathfrak{C})$ is the only elementary subgroup of its order in $\mathfrak{P}_0$.

Since $|\mathfrak{P}_0| = |\mathfrak{H}| = 3^7$ and since $\mathfrak{P}_0$ is obviously not extra special it follows that $\mathfrak{P}_0$ is not a S_3-subgroup of $\mathfrak{G}$. Hence, $\mathfrak{P}_0$ is not a S_3-subgroup of $N(\boldsymbol{O}_3(\mathfrak{C}))$. Hence, $A_{\mathfrak{G}}(\boldsymbol{O}_3(\mathfrak{C}))$ is a solvable subgroup of $GL(5, 3)$ with S_3-subgroups of order at least 27 and with no nonidentity normal 3-subgroups. There are no such groups. The proof of Lemma 9.6 is complete.

LEMMA 9.7. *Every involution in $\mathfrak{N}$ centralizes $\mathfrak{Z}$.*

Proof. Suppose false. Let $\mathfrak{H}_1 = \Omega_1(\mathfrak{H})$, so that $\mathfrak{H}_1$ is extra special of exponent 3 and width $w \geqq 2$. Let $\mathfrak{H}_0 = \boldsymbol{C}_{\mathfrak{H}_1}(I)$ and let $\mathfrak{H}_2$ be the set of elements of $\mathfrak{H}_1$ inverted by I. Here I is an involution of $\mathfrak{N}$ which does not centralize $\mathfrak{Z}$. Since $\mathfrak{Z} \subseteq \mathfrak{H}_2$, it follows that $\mathfrak{H}_0$ is abelian. Since I centralizes $[H, H']$ for all H, H' in $\mathfrak{H}_2$, it follows that $\langle \mathfrak{H}_2 \rangle$ is abelian. Hence, $\mathfrak{H}_2 = \langle \mathfrak{H}_2 \rangle$. As is well known, $\mathfrak{H}_1 = \mathfrak{H}_0 \mathfrak{H}_2$ and $\mathfrak{H}_0 \cap \mathfrak{H}_2 = 1$. Hence, $\mathfrak{H}_2$ is elementary of order 3^{w+1} and $\mathfrak{H}_0$ is elementary of order 3^w.

By Lemmas 7.5 and 9.6, there is a subgroup $\mathfrak{M}$ in $\mathscr{M}\mathscr{S}(\mathfrak{G})$ with $\mathfrak{H}_0 \subseteq \mathfrak{M}$ such that $\mathfrak{M}$ satisfies Hypothesis 7.2 and $p = 2$. Let w_1 be the width of $\boldsymbol{O}_2(\mathfrak{M})$. Hence, $w \leqq w_1 \leqq 4$, the first inequality holding since $\mathfrak{H}_0$ is faithfully represented on $\boldsymbol{O}_2(\mathfrak{M})$, the second inequality holding by Lemma 7.5.

Suppose $w \geqq 3$. Hence, $w_1 \geqq 3$. If $H \in \mathfrak{H}_0^\#$ and $\boldsymbol{C}(H) \cap \boldsymbol{O}_2(\mathfrak{M})$ contains a four-subgroup $\mathfrak{B}$ containing $l_1(\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{M})))$, then by Lemma 7.2, both $\langle H, \mathfrak{Z} \rangle$ and $\mathfrak{B}$ satisfy the hypothesis of Lemma 7.4, so $\boldsymbol{C}(H)$ is nonsolvable. This is impossible, so H is not available. This implies that $w = 2$.

Suppose $\mathfrak{H} = \mathfrak{P}$. In this case, if H is any element of order 3 in $\mathfrak{H}$, then $\mathfrak{Z}$ char $\boldsymbol{C}_{\mathfrak{H}}(H)$. This implies immediately that $\mathfrak{Z}$ is weakly closed in $\mathfrak{P}$, which in turn implies that $\mathfrak{N}$ contains the centralizer of each of its nonidentity 3-elements. This implies that $\boldsymbol{O}_2(\mathfrak{M}) \subseteq \mathfrak{N}$, which is not the case. Hence $\mathfrak{H} \subset \mathfrak{P}$.

Since every 3, 5-subgroup of $S_4(3)$ is either a 3-group or a 5-group, it follows from the preceding paragraph that $\mathfrak{N}$ is a 2, 3-group, $S_4(3)$ being a 2, 3, 5-group. By Lemma 9.3, it then follows that $O^{3'}(\mathfrak{N})/\mathfrak{H} \cong SL(2, 3)$. Furthermore, if J is an involution of $O^{3'}(\mathfrak{N})$, then $C_{\mathfrak{H}}(J) \lhd \mathfrak{N}$. It follows that J inverts $\mathfrak{H}/Z(\mathfrak{H})$.

If I centralizes $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$, we conclude that I centralizes $O^{3'}(\mathfrak{N})/\mathfrak{H}$. But $C(I) \subseteq \mathfrak{M}$, so in particular, $C_{\mathfrak{N}}(I) \subseteq \mathfrak{M}$. Since I centralizes $O^{3'}(\mathfrak{N})/\mathfrak{H}$, it follows that I centralizes a S_2-subgroup $\mathfrak{Q}$ of $O^{3'}(\mathfrak{N})$. Hence, $\mathfrak{Q}$ normalizes $\mathfrak{H}_0$. Hence, $\mathfrak{Q}\mathfrak{H}_0$ is of index 3 in $C(I) \cap O^{3'}(\mathfrak{N})$. Let $\mathfrak{Q}' = \langle J \rangle$. By the preceding paragraph, $\mathfrak{Q}$ is faithfully represented on $\mathfrak{H}_0$. Thus, $\mathfrak{D} = C(I) \cap N(\mathfrak{Q}) \cap O^{3'}(\mathfrak{N}) \cong SL(2, 3)$ and $\mathfrak{D}$ is faithfully represented on $\mathfrak{H}_0$.

Since $\mathfrak{H}_0$ is faithfully represented on $O_2(\mathfrak{M})$, so is $\mathfrak{H}_0\mathfrak{D}$. Since $\mathfrak{H}_0 \cap \mathfrak{D} = 1$, S_3-subgroups of $\mathfrak{H}_0\mathfrak{D}$ are of exponent 3. Since the four subgroups of $\mathfrak{H}_0$ of order 3 are permuted transitively by $\mathfrak{Q}$, it follows that $w_1 \geqq 4$. Hence, $w_1 = 4$ and $O_2(\mathfrak{M})$ is extra special. Let $\mathfrak{P}_0$ be a S_3-subgroup of $\mathfrak{H}_0\mathfrak{D}$. We can choose P in $\mathfrak{P}_0 - \mathfrak{H}_0$ such that $C(P) \cap O_2(\mathfrak{M})$ contains a four-group. Since $C_{\mathfrak{N}}(P)$ clearly contains an element of $\mathscr{U}(3)$, Lemma 7.4 is violated. We conclude that I does not centralize $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$.

Since Aut $(Z(\mathfrak{H}))$ is abelian, the preceding paragraph implies that $Z(\mathfrak{H}) = Z(\mathfrak{P})$.

Since $O_2(\mathfrak{M}) \not\subseteq \mathfrak{N}$, we can choose H in $\mathfrak{H}_0^{\#}$ such that $C(H) \not\subseteq \mathfrak{N}$.

Let $|Z(\mathfrak{H})| = 3^a$, and suppose $a \geqq 2$. Let $\tilde{\mathfrak{P}}$ be a S_3-subgroup of $C_{\mathfrak{N}}(H)$. Thus, $Z(\mathfrak{H}) \subseteq Z(\tilde{\mathfrak{P}})$, and $\mathfrak{Z} = \mho^{a-1}(Z(\tilde{\mathfrak{P}}))$ char $\tilde{\mathfrak{P}}$, whence $\tilde{\mathfrak{P}}$ is a S_3-subgroup of $C(H)$. By Lemma 7.2 applied to $\langle H, \mathfrak{Z} \rangle$, it follows that $\mathfrak{Z}$ centralizes $O_{3'}(C(H))$, and so

$$[O_{3'}(C(H)), C_{\mathfrak{H}}(H)] \subseteq \mathfrak{H} \cap O_{3'}(C(H)) = 1 .$$

By Lemma 9.4, we have $O_{3'}(C(H)) = 1$. Let $\tilde{\mathfrak{P}}_1 = O_3(C(H)) \subseteq \tilde{\mathfrak{P}}$. Thus, $Z(\mathfrak{H}) \subseteq Z(\tilde{\mathfrak{P}}_1)$, and we get $\mathfrak{Z} = \mho^{a-1}(Z(\tilde{\mathfrak{P}}_1))$, whence $C(H) \subseteq \mathfrak{N}$. This contradiction forces $a = 1$, $Z(\mathfrak{H}) = \mathfrak{Z}$, $|\mathfrak{P}| = 3^6$.

Throughout the remainder of this lemma, the following notation is used: $\mathfrak{Q}$ is a S_2-subgroup of $O^{3'}(\mathfrak{N})$ normalized by I. Since $\mathfrak{Q}$ is a quaternion group, our preceding information implies the existence of an element Q in $\mathfrak{Q}$ of order 4 such that $IQI = Q^{-1}$. Let $J = Q^2$. Thus, J centralizes $\mathfrak{Z}$ and inverts $\mathfrak{H}/\mathfrak{Z}$.

We argue that $\mathfrak{G}$ is not 3-normal. Namely, for some H in $\mathfrak{H}_0^{\#}$, we have $\mathfrak{C} = C(H) \not\subseteq \mathfrak{N}$. If $|\mathfrak{C}|_3 = |\mathfrak{G}|_3$, then $\langle H \rangle$ is a conjugate of $\mathfrak{Z}$ contained in $\mathfrak{H}$, and we are done. Otherwise, it is clear that $\mathfrak{C} \cap \mathfrak{N}$ contains a S_3-subgroup of $\mathfrak{C}$ and since $\mathfrak{Z} \subseteq \mathfrak{C}$, $O_3(\mathfrak{C})$ contains at least two conjugates of $\mathfrak{Z}$. As $O_3(\mathfrak{C}) \subseteq \mathfrak{N}$, we again are done.

We next argue that $\mathfrak{Z}$ is not weakly closed in $\mathfrak{H}$. Choose G in

$\mathfrak{G}$ such that $\mathfrak{Z}_1 = \mathfrak{Z}^G \subseteq \mathfrak{P}$ and $\mathfrak{Z} \neq \mathfrak{Z}_1$. If $\mathfrak{Z}_1 \subseteq \mathfrak{H}$, we are done. Otherwise, let $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{Z}_1) = \mathfrak{Z}_1 \times C_{\mathfrak{H}}(\mathfrak{Z}_1)$. Since $\mathfrak{P}_0$ is not a S_3-subgroup of $\mathfrak{G}$ but $\mathfrak{P}_0$ is a S_3-subgroup of $C_{\mathfrak{N}}(\mathfrak{Z}_1)$, it follows that $\mathfrak{Z}$ ch/ar $\mathfrak{P}_0$. This implies that $\mathfrak{P}_0$ is elementary. Clearly, $27 \leqq |\mathfrak{P}_0| \leqq 81$, since $w = 2$. We assume without loss of generality that $\mathfrak{P} \cap \mathfrak{P}^G = \mathfrak{P}_0$. If $\mathfrak{Z} \subseteq \mathfrak{H}^G$, we are done, so we may assume that $\mathfrak{Z} \not\subseteq \mathfrak{H}^G$, which yields $\mathfrak{P}_0 = \mathfrak{Z} \times (\mathfrak{P}_0 \cap \mathfrak{H}^G)$. In particular, $\mathfrak{H} \cap \mathfrak{H}^G \neq 1$; let $\mathfrak{B} = \mathfrak{H} \cap \mathfrak{H}^G$, a group of order at least 3. Suppose $|\mathfrak{B}| = 3$. Let $\mathfrak{K} = C(B)$, so that $|\mathfrak{K} \cap \mathfrak{P}| = |\mathfrak{K} \cap \mathfrak{P}^G| = 3^5$. If $|\mathfrak{K}|_3 = 3^6$, we are done, so we may assume that $|\mathfrak{K}|_3 = 3^5$. In this case, we see that $|O_3(\mathfrak{K})| = 3^4$, which implies that $|O_3(\mathfrak{K}) \cap \mathfrak{H}| \geqq 27$, $|O_3(\mathfrak{K}) \cap \mathfrak{H}^G| \geqq 27$. Hence, $|\mathfrak{H} \cap \mathfrak{H}^G| \geqq 9$, contrary to assumption. Thus, we may assume that $|\mathfrak{B}| = 9$. We may also assume that $\mathfrak{B}$ contains no conjugate of $\mathfrak{Z}$. We have $\mathfrak{P}_0 = \mathfrak{Z} \times \mathfrak{Z}_1 \times \mathfrak{B}$. We argue that $\mathfrak{P}_0 \lhd \langle \mathfrak{P}, \mathfrak{P}^G \rangle$. Namely, let $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}$. If $\mathfrak{P}_0$ char $\mathfrak{P}_1$, then clearly $\mathfrak{P}$ normalizes $\mathfrak{P}_0$. Suppose $\mathfrak{P}_0$ ch/ar $\mathfrak{P}_1$. Then $\mathfrak{P}_1 = \mathfrak{P}_0\mathfrak{P}_0^*$, where $\mathfrak{P}_0^*$ is elementary of order 3^4. Hence, $Z(\mathfrak{P}_1) = \mathfrak{P}_0 \cap \mathfrak{P}_0^*$ is of order 27. This implies that $\mathfrak{P}_1'$ is of order 3. Since $\mathfrak{P}_1 \cap \mathfrak{H}$ is nonabelian, we have $\mathfrak{P}_1' = \mathfrak{Z}$. Thus, $\mathfrak{Z}_1$ centralizes $(\mathfrak{P}_1 \cap \mathfrak{H})/\mathfrak{Z}$. This is not the case, since involutions of $O^{3'}(\mathfrak{N})$ invert $\mathfrak{H}/\mathfrak{Z}$, so that the action of $\mathfrak{Z}_1$ on $\mathfrak{H}/\mathfrak{Z}$ is given either by $J_3 \oplus J_1$ or by $J_2 \oplus J_2$. By symmetry, we have $\mathfrak{P}_0 \lhd \langle \mathfrak{P}, \mathfrak{P}^G \rangle$. It is easy to verify that $O_3(N(\mathfrak{P}_0))$ is of order 3^5, which implies that $|\mathfrak{H} \cap \mathfrak{H}^G| \geqq 27$, the desired contradiction.

Since all parts of Hypothesis 9.3 are satisfied, Lemma 9.5 is violated. The proof of the lemma is complete.

LEMMA 9.8. *$\mathfrak{N}$ is the only element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{H}$.*

Proof. Suppose false. Choose $\mathfrak{K} \in \mathscr{SL}(\mathfrak{G})$ so that $\mathfrak{H} \subseteq \mathfrak{K} \not\subseteq \mathfrak{N}$, and with this restriction, minimize $|\mathfrak{K}|$. Since $\mathsf{M}(\mathfrak{H})$ contains only 1, Lemma 0.7.6 implies that $l_3(\mathfrak{K}) \leqq 2$. If $l_3(\mathfrak{K}) = 1$, then $\mathfrak{Z} \lhd \mathfrak{K}$, contrary to assumption. Hence, $l_3(\mathfrak{K}) = 2$; and $\mathfrak{K}$ is a 3, p-group for some prime p. Furthermore, $\mathfrak{H}$ acts irreducibly on $O_{3,p}(\mathfrak{K})/D(O_{3,p}(\mathfrak{K} \bmod O_3(\mathfrak{K})))$. Let $\mathfrak{H}_0 = \mathfrak{H} \cap O_3(\mathfrak{K})$, $\mathfrak{B} = \Omega_1(Z(O_3(\mathfrak{K})))$. By Lemma 9.4, $\mathfrak{B} \subseteq \mathfrak{H}$, so $|\mathfrak{B}| \leqq 9$. Thus, $O_{3,p}(\mathfrak{K})/O_3(\mathfrak{K})$ is a quaternion group whose involution inverts $\mathfrak{B}$. Since $\mathfrak{Z} \subset \mathfrak{B}$, Lemma 9.7 is violated. The proof is complete.

LEMMA 9.9. *Every involution I of $\mathfrak{N}$ inverts $\mathfrak{H}/Z(\mathfrak{H})$.*

Proof. By Lemma 9.7, I centralizes $\mathfrak{Z}$, so centralizes $Z(\mathfrak{H})$. If the lemma is false, then $C_{\mathfrak{H}}(I)$ contains a subgroup $\mathfrak{A}$ of type (3, 3) with $\mathfrak{A} \supset \mathfrak{Z}$. This violates Lemma 9.3 (iii).

LEMMA 9.10. *If $\mathfrak{A} \in \mathscr{S}_4(\mathfrak{P})$, then $\mathfrak{N}$ is the only element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{A}$.*

Proof. As in O, let $\mathscr{A}_1 = \{\mathfrak{A} \mid \mathfrak{A}$ is a 3-subgroup of $\mathfrak{N}$, $\mathfrak{A}$ contains an element of $\mathscr{SCN}_3(\mathfrak{P}^N)$ for some N in $\mathfrak{N}$, $\mathscr{A}_{n+1} = \{\mathfrak{A} \mid \mathfrak{A}$ is a 3-subgroup of $\mathfrak{N}$, $\mathfrak{A}$ contains a subgroup $\mathfrak{B}$ of type (3, 3), $C_{\mathfrak{N}}(B)$ contains an element of $\mathscr{A}_n$ for all B in $\mathfrak{B}\}$. Among all $\mathfrak{A} \in \mathscr{A}_4$ which violate the conclusion of the lemma, maximize $|\mathfrak{A} \cap \mathfrak{H}|$, and with this restriction, maximize $|\mathfrak{A}|$. By Lemma 9.8, $\mathfrak{H} \not\subseteq \mathfrak{A}$. Let $\mathfrak{M} \in \mathscr{MS}(\mathfrak{G})$ with $\mathfrak{A} \subseteq \mathfrak{M}$, $\mathfrak{M} \neq \mathfrak{N}$. By *maximality of* $|\mathfrak{A}|$, *it follows that* $\mathfrak{A}$ is a S_3-subgroup of $\mathfrak{M}$. We can therefore choose a prime q and a q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ permutable with $\mathfrak{A}$ such that $\mathfrak{L} = \mathfrak{A}\mathfrak{Q}$ is not contained in $\mathfrak{N}$. Let $\mathfrak{Q}$ be minimal with these properties. By Lemma 0.7.6, $l_3(\mathfrak{L}) \leqq 2$.

We first show that $\boldsymbol{O}_q(\mathfrak{L}) \subseteq \mathfrak{N}$. Suppose $\mathfrak{A} \cap \mathfrak{H}$ is noncyclic. Let $\mathfrak{B}$ be a subgroup of $\mathfrak{A} \cap \mathfrak{H}$ of type (3, 3). It suffices to show that $\boldsymbol{C}(B) \subseteq \mathfrak{N}$ for all $B \in \mathfrak{B}^\#$. Suppose false. Then maximality of $|\mathfrak{A} \cap \mathfrak{H}|$ yields $|\mathfrak{H}: \mathfrak{A} \cap \mathfrak{H}| \leqq 3$. In this case, $\boldsymbol{\mathsf{V}}(\mathfrak{A} \cap \mathfrak{H}) = 1$, so $\boldsymbol{O}_q(\mathfrak{L}) = 1$. Thus, we may assume that $\mathfrak{A} \cap \mathfrak{H}$ is cyclic. Since $w \geqq 2$, it follows that if P is any element of $\mathfrak{P}$ of order 3, then $\boldsymbol{C}_{\mathfrak{H}}(P)$ is noncyclic. Hence, every subgroup of $\mathfrak{N}$ of type (3, 3) is in $\mathscr{A}_4$. Since $\mathfrak{A}$ contains a subgroup of type (3, 3), maximality of $|\mathfrak{A} \cap \mathfrak{H}|$ implies that $\boldsymbol{C}(A) \subseteq \mathfrak{N}$ for all elements A of $\mathfrak{A}$ of order 3. Thus, in all cases, we have $\boldsymbol{O}_q(\mathfrak{L}) \subseteq \mathfrak{N}$.

By minimality of $\mathfrak{Q}$, $\boldsymbol{O}_{q,3}(\mathfrak{L}) = \boldsymbol{O}_q(\mathfrak{L}) \times \boldsymbol{O}_3(\mathfrak{L})$. Since $l_3(\mathfrak{L}) \leqq 2$, it follows that $l_3(\mathfrak{L}) = 2$, by maximality of $|\mathfrak{A}|$ and the structure of $\boldsymbol{O}_{q,3}(\mathfrak{L})$. Since $\boldsymbol{D}(\mathfrak{Q})$ is permutable with $\mathfrak{A}$, we get $\boldsymbol{D}(\mathfrak{Q}) \subseteq \mathfrak{N}$, by minimality of $\mathfrak{Q}$.

Clearly, $\mathfrak{A}$ is a S_3-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{L}))$. Hence, $\mathfrak{Z} \subseteq \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L}))$. Since $\mathfrak{Q}\boldsymbol{O}_3(\mathfrak{L}) \lhd \mathfrak{L}$, and since $\mathfrak{A}$ is a S_3-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{L}))$, and since $\mathfrak{H} \not\subseteq \mathfrak{A}$, it follows that $\mathfrak{H} \cap \mathfrak{A}$ acts nontrivially on $\boldsymbol{Q}_3^1(\mathfrak{L})$, but trivially on every proper $\mathfrak{A}$-invariant subgroup of $\boldsymbol{Q}_3^1(\mathfrak{L})$. Since $\boldsymbol{D}(\mathfrak{Q})$ centralizes $\mathfrak{Z}$, it follows that $\boldsymbol{D}(\mathfrak{Q})$ centralizes $\mathfrak{W} = \mathfrak{Z}^{\mathfrak{L}}$.

If $q \geqq 5$, then maximality of $\mathfrak{A}$ and Theorem 1 of [39] imply that $\mathfrak{A} = \mathfrak{P}$, against Lemma 9.8. Hence, $q = 2$. We may apply Theorem 1 of [39] once again and conclude that $\boldsymbol{D}(\mathfrak{Q}) \neq 1$. By Lemma 9.9, each element of $\boldsymbol{D}(\mathfrak{Q})^\#$ inverts $\mathfrak{H}/\mathfrak{Z}(\mathfrak{H})$. Since $\boldsymbol{Z}(\mathfrak{H})$ is a normal cyclic subgroup of $\mathfrak{P}$, it follows that $\mathfrak{A} \cap \boldsymbol{Z}(\mathfrak{H}) \subseteq \boldsymbol{O}_3(\mathfrak{L})$. Since $\mathfrak{A} \cap \mathfrak{H} \not\subseteq \boldsymbol{O}_3(\mathfrak{L})$, choose $H \in \mathfrak{A} \cap \mathfrak{H} - \boldsymbol{O}_3(\mathfrak{L})$. Let I be the element in $\boldsymbol{D}(\mathfrak{Q})^\#$. Then $H^I = H^{-1}H_0$ with H_0 in $\boldsymbol{Z}(\mathfrak{H})$. Since $H_0 \in \mathfrak{A}$, it follows that $[H, I]$ is contained in $\mathfrak{A} \cap \mathfrak{H} \cap O_{3,q}(\mathfrak{L}) \subseteq \mathfrak{A} \cap \boldsymbol{O}_3(\mathfrak{L})$. This violates the fact that $\mathfrak{A} \cap \mathfrak{H} \not\subseteq \boldsymbol{O}_3(\mathfrak{L})$. The proof is complete.

It is now easy to show that Hypothesis 9.2 is not satisfied. Otherwise, $\mathfrak{N}$ contains a four-subgroup $\mathfrak{T}$. But by Lemma 9.9, each element of $\mathfrak{T}^\#$ inverts $\mathfrak{H}/\boldsymbol{Z}(\mathfrak{H})$. This is not possible, since $\mathfrak{H} = \langle \boldsymbol{C}_{\mathfrak{H}}(J) \mid J \in \mathfrak{T}^\# \rangle$.

The remaining lemmas in this section are proved on the hypothesis that $\mathfrak{H}$ contains a noncyclic characteristic abelian subgroup.

Among all noncyclic normal elementary subgroups of $\mathfrak{N}$, let $\mathfrak{E}$ be minimal. Thus, $\mathfrak{E}/\mathfrak{Z}$ is a chief factor of $\mathfrak{N}$. Let $\mathscr{E}: \mathfrak{E} \supset \mathfrak{Z} \supset 1$. We will show that $\boldsymbol{A}_{\mathfrak{G}}(\mathscr{E}) = \boldsymbol{A}(\mathscr{E})$. First, suppose $\mathfrak{E}$ is not 3-reducible in $\mathfrak{N}$. Let $\mathfrak{L} = \boldsymbol{O}_3(\mathfrak{N} \bmod \boldsymbol{C}(\mathfrak{E}))$. Since $\mathfrak{E}/\mathfrak{Z}$ is a chief factor of N, we have $[\mathfrak{L}, \mathfrak{E}] = \mathfrak{Z}$, and $\mathfrak{Z} = \boldsymbol{C}_{\mathfrak{E}}(\mathfrak{L})$. These equalities imply immediately that $\mathfrak{L}$ maps onto $\boldsymbol{A}(\mathscr{E})$. Suppose $\mathfrak{E}$ is 3-reducible in $\mathfrak{N}$. Let $\mathfrak{L} = \boldsymbol{O}_{3'}(\mathfrak{N} \bmod \boldsymbol{C}(\mathfrak{E}))$. Then $\mathfrak{Z} = \boldsymbol{C}_{\mathfrak{E}}(\mathfrak{L})$ and $[\mathfrak{L}, \mathfrak{E}]$ admits $\boldsymbol{N}_{\mathfrak{N}}(\mathfrak{L}) = \mathfrak{N}$. Since $\boldsymbol{A}_{\mathfrak{L}}(\mathfrak{E})$ is a $3'$-group it follows that $[\mathfrak{L}, \mathfrak{E}]$ is a normal subgroup of $\mathfrak{N}$ disjoint from $\mathfrak{Z}$. Hence, $[\mathfrak{L}, \mathfrak{E}] = 1$, since $\mathfrak{Z}$ is the only minimal normal subgroup of $\mathfrak{P}$. Since $\mathfrak{E}$ is 3-reducible in $\mathfrak{N}$ and $\boldsymbol{O}_{3'}(\mathfrak{N} \bmod \boldsymbol{C}(\mathfrak{E}))$, it follows that $\mathfrak{E} \subseteqq \boldsymbol{Z}(\mathfrak{N})$. This is absurd since $\mathfrak{Z}$ is the only minimal normal subgroup of $\mathfrak{P}$. Thus, $\boldsymbol{A}_{\mathfrak{G}}(\mathscr{E}) = \boldsymbol{A}(\mathscr{E})$.

Throughout the remainder of this section, the following notation is used: $\mathfrak{P}$, $\mathfrak{Z}$, $\mathfrak{N}$ are as before, and $\mathfrak{E}$ is a noncyclic normal elementary subgroup of $\mathfrak{N}$ such that $\mathfrak{E}/\mathfrak{Z}$ is a chief factor of $\mathfrak{N}$. Also, $\mathscr{E}: \mathfrak{E} \supset \mathfrak{Z} \supset 1$.

LEMMA 9.11. (i) *If $\mathfrak{S}$ is a 2, 3-subgroup of $\mathfrak{G}$ and $\mathfrak{S}$ contains an element $\mathfrak{A}$ of $\mathscr{E}(3)$, then $\boldsymbol{O}_2(\mathfrak{S}) = 1$.*

(ii) *If $\mathfrak{A} \in \mathscr{E}(3)$, then $|\boldsymbol{C}(\mathfrak{A})|$ is odd.*

Proof. (i) Suppose I is an involution in $\boldsymbol{O}_2(\mathfrak{S})$. Since $\mathfrak{A}$ centralizes $\boldsymbol{O}_2(\mathfrak{S})$, Lemmas 7.4 and 5.38 imply that $\boldsymbol{C}(I)$ is nonsolvable.

(ii) Suppose I is an involution of $\boldsymbol{C}(\mathfrak{A})$. Then $\mathfrak{A} \times \langle I \rangle$ violates (i).

LEMMA 9.12. (i) *If I is an involution of $\boldsymbol{C}(\mathfrak{Z})$, then I inverts $\mathfrak{E}/\mathfrak{Z}$.*

(ii) *$\boldsymbol{C}(\mathfrak{Z})$ contains no four-group.*

(iii) *If $\mathfrak{T}$ is an abelian 2-subgroup of $\mathfrak{N}$, then $\boldsymbol{A}_{\mathfrak{N}}(\mathfrak{T})$ is a 2-group.*

Proof. (i) is a consequence of Lemmas 9.11 and 7.3, and (ii), (iii) are consequences of (i).

LEMMA 9.13. *$\mathfrak{N}$ does not contain a noncyclic abelian subgroup of order* 8.

Proof. Suppose false. Let $\mathfrak{Q}_3^*$ be a S_2-subgroup of $\mathfrak{N}$ permutable with $\mathfrak{P}$, and let $\mathfrak{N}_0 = \mathfrak{P}\mathfrak{Q}_3^*$. Let $\mathfrak{Q}_0 = \mathfrak{Q}_3^* \cap \boldsymbol{O}^{3'}(\mathfrak{N}_0)$. Thus, $\mathfrak{Q}_0$ is either a quaternion group or $\mathfrak{Q}_0 = 1$. Let $\mathfrak{Q}$ be a subgroup of $\mathfrak{Q}_3^*$ which contains $\mathfrak{Q}_0$, is permutable with $\mathfrak{P}$, contains a noncyclic abelian subgroup of order 8, and is minimal with these properties. Let $\mathfrak{N}_1 = \mathfrak{P}\mathfrak{Q}$. Thus, $\mathfrak{Q}$ is abelian of type (2, 4) if and only if every 2, 3-subgroup of $\mathfrak{N}$ is 3-closed. If $\mathfrak{Q}_0 \neq 1$, then $|\mathfrak{Q}| = 2^4$ and $\mathfrak{Q}$ is either the direct product of a group of order 2 and $\mathfrak{Q}_0$ or $\mathfrak{Q}$ is the central product of a cyclic group of order 4 and $\mathfrak{Q}_0$. Let $\mathfrak{F}/\mathfrak{Z}$ be a chief factor of $\mathfrak{N}_1$

with $\mathfrak{F} \subseteq \mathfrak{C}$. Let $\mathfrak{P}_0 = \boldsymbol{O}_3(\mathfrak{N}_1)$, $\mathfrak{P}_1 = \boldsymbol{C}_{\mathfrak{P}_0}(\mathfrak{F})$. Since $\boldsymbol{A}_{\mathfrak{N}}(\mathscr{C}) = \boldsymbol{A}(\mathscr{C})$, so also $\boldsymbol{A}_{\mathfrak{N}_1}(\mathscr{C}_0) = \boldsymbol{A}(\mathscr{C}_0)$ where $\mathscr{C}_0: \mathfrak{F} \supset \mathfrak{Z} \supset 1$. Hence, $\mathfrak{P}_0/\mathfrak{P}_1$ is also a chief factor of $\mathfrak{N}_1$ with the same order as $\mathfrak{F}/\mathfrak{Z}$. If $\mathfrak{Q}' = 1$, then

(9.19) $$\mathfrak{P} \lhd \mathfrak{N}_1, \mathfrak{Q} \text{ is of type } (4, 2), \text{ and } |\mathfrak{F}: \mathfrak{Z}| = 9 .$$

Suppose $\mathfrak{Q}' \neq 1$. If $\mathfrak{Q} = \mathfrak{Q}_0 \times \mathfrak{Q}_1$, where $|\mathfrak{Q}_1| = 2$, then

(9.20) $$\mathfrak{N}_1/\mathfrak{P}_0 \cong SL(2, 3) \times Z_2, \text{ and } |\mathfrak{F}: \mathfrak{Z}| = 9 .$$

Suppose $\mathfrak{Q}$ is the central product of $\mathfrak{Q}_0$ and a cyclic group of order 4. Then

(9.21) $$\mathfrak{N}_1/\mathfrak{P}_0 \text{ is the central product of } SL(2, 3) \text{ and } Z_4, \text{ and } |\mathfrak{F}: \mathfrak{Z}| = 3^4 .$$

By Lemmas 5.41 and 9.12, (9.19), (9.20), (9.21) exhaust all possibilities.

It is clear from Lemma 9.12 that

(9.22) if (9.19) holds, a $S_{2,3}$-subgroup of $\mathfrak{N}$ is 3-closed .

We next will show that

(9.23) every subgroup of $\mathfrak{F}$ of order 9 is in $\mathscr{D}$.

To see this, let $\mathfrak{F}_0$ be a subgroup of $\mathfrak{F}$ of order 9. If $\mathfrak{Z} \subset \mathfrak{F}_0$, then $\mathfrak{F}_0 \in \mathscr{E}(3) \subseteq \mathscr{D}$. Thus, we may assume that $\mathfrak{F}_0 \cap \mathfrak{Z} = 1$. Let $\mathfrak{T}$ be an abelian subgroup in $\mathsf{V}\!\!\mathsf{I}(\mathfrak{F}_0; 2)$ and assume by way of contradiction that $[\mathfrak{T}, \mathfrak{F}_0] \neq 1$. We may assume that $\mathfrak{T}$ is a four-group. Let $\mathfrak{F}_1 = \mathfrak{F}_0 \cap \boldsymbol{C}(\mathfrak{T})$, a group of order 3. Let $\mathfrak{C} = \boldsymbol{C}(\mathfrak{F}_1) \supseteq \langle \mathfrak{F}, \mathfrak{T} \rangle$. Since $m(\mathfrak{F}) \geqq 3$ and $\mathfrak{F} \lhd \mathfrak{P}$, there is $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$ with $\mathfrak{F} \subseteq \mathfrak{A}$. Hence, $\mathfrak{A} \subseteq \mathfrak{C} = \boldsymbol{C}(\mathfrak{F}_1)$ implies $\boldsymbol{O}_{3'}(\mathfrak{C}) = 1$ by hypothesis (ii) of Theorem 9.1.

By Lemma 5.5, $\mathfrak{Z} \subseteq \boldsymbol{O}_3(\mathfrak{C})$. Let $\mathfrak{W} = \Omega_1(\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})))$, and let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$. Let $\mathfrak{P}^G$ be a S_3-subgroup of $\mathfrak{G}$ which contains $\mathfrak{P}^*$. Then $\mathfrak{Z}^G \subset \mathfrak{P}^*$, so $\mathfrak{Z}^G \subseteq \mathfrak{W}$. By Lemma 9.12 (iii), $\mathfrak{T}$ is faithfully represented on $\mathfrak{W}$. Hence, if $F \in \mathfrak{F}_0 - \mathfrak{F} \cap \boldsymbol{C}(\mathfrak{T})$, then the minimal polynomial of F on $\mathfrak{W}$ is $(x-1)^3$. On the other hand, $\mathfrak{Z}$ centralizes $\mathfrak{W}$. Since $\mathfrak{C} \lhd \mathfrak{N}$, the minimal polynomial of F on $\mathfrak{W}$ is a divisor of $(x-1)^2$. This contradiction establishes (9.23).

Since $\mathfrak{Q}$ contains an abelian subgroup of type (2, 4), we can choose an involution I of $\mathfrak{Q}$ such that $\mathfrak{F}_0 = \boldsymbol{C}_{\mathfrak{F}}(I)$ is noncyclic. By Lemmas 7.4 and 5.38, $\boldsymbol{C}(I)$ contains no element of $\mathscr{E}(3)$. Hence, I inverts $\mathfrak{Z}$. Thus, in cases (9.19), (9.20) respectively, we have

(9.19)′–(9.20)′ $$\mathfrak{F} = \mathfrak{F}_0 \times \mathfrak{Z} .$$

In case (9.21), we have

(9.21)′ $$|\mathfrak{F}_0| = |\boldsymbol{C}_{\mathfrak{P}_0/\mathfrak{P}_1}(I)| = 9 .$$

Thus, in case (9.21), we have $|C_{\mathfrak{P}_0}(I)| \geqq 3^4$.

Let $\mathfrak{L}$ be a $S_{2,3}$-subgroup of $C(I)$ which contains $\mathfrak{F}_0$. Since $\mathfrak{F}_0 \in \mathscr{D}$ and since $C(I)$ contains an element of $\mathscr{U}(2)$, there is an element $\mathfrak{M}$ of $\mathfrak{M}\mathfrak{S}(G)$ which satisfies all the conclusions of Lemma 7.5, contains $\mathfrak{F}_0$ and contains a S_2-subgroup of $\mathfrak{L}$. By Lemma 7.5 (f), $I \in O_2(\mathfrak{M})$.

We will show that

$$C_{\mathfrak{N}_1}(I) \subseteqq \mathfrak{M} . \tag{9.24}$$

By Lemma 7.5 (f), it suffices to show that $\mathfrak{M}$ contains an S_2-subgroup of $C(I)$. By construction, $\mathfrak{M}$ contains an S_2-subgroup of $\mathfrak{L}$, which is an $S_{2,3}$-subgroup of $C(I)$. This proves (9.24).

Suppose (9.19) holds. In this case, we have (9.22). Also, (9.19) implies that every element of $\mathfrak{G}$ of order 3 centralizes an element of $\mathscr{E}(3)$. Let $\mathfrak{F}_1$ be a subgroup of $\mathfrak{F}_0$ of order 3 such that

$$[O_2(\mathfrak{M}) \cap C(\mathfrak{F}_1), \mathfrak{F}_0] = \mathfrak{Q}^* \neq 1 .$$

Thus, $\mathfrak{Q}^*$ is a quaternion group and a $S_{2,3}$-subgroup of $C(\mathfrak{F}_1)$ is not 3-closed. By (9.22), $\mathfrak{F}_1 \not\sim \mathfrak{Z}$. Since $|C_{\mathfrak{N}_1}(\mathfrak{F}_1)|_3 = |\mathfrak{P}|/3$, it follows that $C_{\mathfrak{N}_1}(\mathfrak{F}_1)$ contains a S_3-subgroup of $C(\mathfrak{F}_1)$. Let $\mathfrak{P}^*$ be a S_3-subgroup of $C_{\mathfrak{N}_1}(\mathfrak{F}_1)$. Since $C(\mathfrak{F}_1)$ contains an element of $\mathscr{S}_{\mathscr{CN}3}(\mathfrak{P})$, it follows that $O_{3'}(\mathfrak{C}) = 1$, where $\mathfrak{C} = C(\mathfrak{F}_1)$. Let $\mathfrak{R} = O_3(\mathfrak{C})/\mathfrak{F}_1$. Thus, $\mathfrak{Q}^*\langle F\rangle$ is faithfully represented on $Z(\mathfrak{R})$ for each F in $\mathfrak{F}_0 - \mathfrak{F}_1$. But $[\mathfrak{R}, F] \subseteqq \langle \mathfrak{Z}, \mathfrak{F}_1\rangle/\mathfrak{F}_1$, so $\mathfrak{Q}^*$ centralizes a subgroup of $O_3(\mathfrak{R})$ of index 9.

Suppose (9.19) holds and $O_3(\mathfrak{C}) \cap C(I)$ is noncyclic, where I is the involution of $\mathfrak{Q}^*$. In this case, since $\mathfrak{F}_0$ centralizes I and $O_3(\mathfrak{C}) \cap \mathfrak{F}_0 = \mathfrak{F}_1$, it follows that $\mathfrak{C} \cap \mathfrak{M}$ contains a subgroup of order 27 and exponent 3. Since every element of $\mathfrak{G}$ of order 3 centralizes an element of $\mathscr{E}(3)$, it follows that a S_3-subgroup of $\mathfrak{M}$ is nonabelian of order 27 and the width of $O_2(\mathfrak{M})$ is 3. Since $|O_3(\mathfrak{C}): O_3(\mathfrak{C}) \cap C(I)| = 9$, it follows that $|O_3(\mathfrak{C})| \leqq 3^4$. Since $O_3(\mathfrak{C}) \cap C(I)$ is assumed noncyclic, and since $m(Z(O_3(\mathfrak{C}))) \geqq 3$, it follows that $O_3(\mathfrak{C})$ is elementary of order 3^4. Since $\mathfrak{Q}^* \subseteqq \mathfrak{C}$, and since $\langle I\rangle = O_2(\mathfrak{M}) \cap C(\mathfrak{F}_0)$, it follows that $O_3(\mathfrak{C})$ is of index 3 in $\mathfrak{P}^*$. Hence, $|\mathfrak{P}| = 3^6$, since $\mathfrak{P}^*$ is of index 3 in some S_3-subgroup of $\mathfrak{G}$.

Since $|\mathfrak{P}^*| = 3^5$, we have $\mathfrak{P}^* = O_3(\mathfrak{C})\mathfrak{F}_0$.

We argue that $C_{\mathfrak{P}^*}(F)$ is of index 9 in $\mathfrak{P}^*$ for every F in $\mathfrak{F}_0 - \mathfrak{F}_1$. This assertion is equivalent to the assertion that $O_3(\mathfrak{C}) \cap C(F)$ is of order 9, since $\mathfrak{P}^* = O_3(\mathfrak{C})\langle F\rangle$. Now $O_3(\mathfrak{C}) = \mathfrak{U}_1 \times \mathfrak{U}_2$, where $\mathfrak{U}_1 = C(I) \cap O_3(\mathfrak{C})$, $\mathfrak{U}_2$ is inverted by I, and $|\mathfrak{U}_1| = |\mathfrak{U}_2| = 9$. Since $\mathfrak{U}_i \lhd \mathfrak{P}^*\mathfrak{Q}^*$, $i = 1, 2$, we must show that F does not centralize either $\mathfrak{U}_1$ or $\mathfrak{U}_2$. It is obvious that F does not centralize $\mathfrak{U}_2$. If F centralizes $\mathfrak{U}_1$, then $\langle \mathfrak{U}_1, F\rangle$ is elementary of order 27 and is contained in $\mathfrak{M}$, whereas we already know that S_3-subgroups of $\mathfrak{M}$ are nonabelian of order 27. So

$|\mathfrak{P}^*: C_{\mathfrak{P}^*}(F)| = 9$.

Since $C_{\mathfrak{P}^*}(F)$ is of index 9 in $\mathfrak{P}^*$ for every F in $\mathfrak{F}_0 - \mathfrak{F}_1$, it follows that $O_3(\mathfrak{C})$ char $\mathfrak{P}^*$. Thus, $N(O_3(\mathfrak{C}))$ contains a S_3-subgroup of $\mathfrak{G}$ and $S_{2,3}$-subgroup of $N(O_3(\mathfrak{C}))$ are not 3-closed. This implies that if $\tilde{\mathfrak{P}}$ is a S_3-subgroup of $N(O_3(\mathfrak{C}))$, then $O_3(\mathfrak{C})$ is not characteristic in $\tilde{\mathfrak{P}}$. More explicitly, $N(O_3(\mathfrak{C})) \cap N(\tilde{\mathfrak{P}})$ does not contain a noncyclic abelian subgroup of order 8, while $N(\tilde{\mathfrak{P}})$ does. Let $\mathfrak{A}$ be an elementary subgroup of $\mathfrak{P}$ of order 3^4 with $\mathfrak{A} \neq O_3(\mathfrak{C})$. If $\mathfrak{A} \cap O_3(\mathfrak{C})$ is of order 9, then $\tilde{\mathfrak{P}} = \mathfrak{A}O_3(\mathfrak{C})$ and $Z(\tilde{\mathfrak{P}})$ is not cyclic. Hence, $\mathfrak{A} \cap O_3(\mathfrak{C})$ is of order 27. Hence, $\tilde{\mathfrak{P}} = \mathfrak{P}^*\mathfrak{A}$, and it follows that $N(O_3(\mathfrak{C})) \cap C(I)$ contains S_3-subgroups of order 3^4. Furthermore, every subgroup of $\tilde{\mathfrak{P}}$ of order 3 centralizes an element of $\mathscr{E}(3)$. Since the width of $O_2(\mathfrak{M})$ is 3, it follows that a S_3-subgroup of $\mathfrak{M}$ is of the shape $Z_3 \wr Z_3$. But we have already shown that S_3-subgroups of $\mathfrak{M}$ are of order 27.

Suppose (9.19) holds and $O_3(\mathfrak{C}) \cap C(I)$ is cyclic. Since S_3-subgroups of $\mathfrak{M}$ are of exponent 3 or 9, it follows that $|O_3(\mathfrak{C}) \cap C(I)| = 3$ or 9. Hence, $|O_3(\mathfrak{C})| \leqq 3^4$, so $O_3(\mathfrak{C})$ is abelian. Hence, $\mathfrak{P}^* = O_3(\mathfrak{C})\mathfrak{F}_0$. Since elements of $\mathfrak{F}_0 - \mathfrak{F}_1$ have quadratic minimal polynomial of $O_3(\mathfrak{C})$, it follows that $\mho^1(\mathfrak{P}^*) = \mho^1(O_3(\mathfrak{C})) = \mho^1(O_3(\mathfrak{C}) \cap C(I))$. Hence, $\mho^1(\mathfrak{P}^*) = 1$, since otherwise $\mho^1(\mathfrak{P}^*)$ is conjugate to $\mathfrak{Z}$, against (9.22). Hence, $O_3(\mathfrak{C})$ is elementary of order 27.

Since $|O_3(\mathfrak{C})| = 27$, we get $|\mathfrak{P}^*| = 3^4$, $|\mathfrak{P}| = 3^5$. Since (9.19) holds, $\mathfrak{Q}$ is of type (4, 2) and $\mathfrak{Q}$ normalizes $\mathfrak{P}$. Let $\mathfrak{Q} = \tilde{\mathfrak{Q}} \cap C(\mathfrak{Z})$. Thus, $\tilde{\mathfrak{Q}}$ is cyclic of order 4, by Lemma 9.12 (ii). Also, the involution Q of $\tilde{\mathfrak{Q}}$ inverts $\mathfrak{F}/\mathfrak{Z}$, so inverts $\mathfrak{P}/\mathfrak{Z}$. Hence, $\mathfrak{P}/\mathfrak{Z}$ is elementary of order 3^4 and is the direct sum of $\mathfrak{C}/\mathfrak{Z}$ and another irreducible $\mathfrak{Q}$-module. This implies that $\mathfrak{P}$ is of exponent 3 and is extra special. Thus, for each P in $\mathfrak{P}$, $\mathfrak{Z}$ char $C_{\mathfrak{P}}(P)$. This implies that $\mathfrak{Z}$ is weakly closed in $\mathfrak{P}$. But turning back to $\mathfrak{C}$, it follows that $\mathfrak{Q}^*$ does not normalize $\mathfrak{Z}$, so $\mathfrak{Z}$ is not weakly closed in $\mathfrak{P}$. This contradiction shows that (9.19) does not hold.

Suppose (9.20) holds. By (9.24) it follows that $\mathfrak{Q}\mathfrak{F}_0 \subseteq \mathfrak{M}$. Hence, the width of $O_2(\mathfrak{M})$ is four. Hence, $C_{\mathfrak{P}_0}(I) = \mathfrak{F}_0$. Thus, $C_{\mathfrak{N}_1}(I)$ contains a S_3-subgroup $\tilde{\mathfrak{P}}$ which is a nonabelian group of order 27 and exponent 3. This is not the case, since $C(P) \cap O_2(\mathfrak{M})$ contains no four-subgroup for any element P of $\tilde{\mathfrak{P}}^{\#}$.

Suppose (9.21) holds. By (9.24) and (9.21)′, it follows that S_3-subgroups of $\mathfrak{M}$ are of order at least 3^4. Hence, the width of $O_2(\mathfrak{M})$ is four, and $C(I) \cap \mathfrak{P}_0$ contains no subgroup of order 27 and exponent 3, and of course $C(I) \cap \mathfrak{P}_0$ is of exponent 9. This is absurd, since S_3-subgroups of Aut $(O_2(\mathfrak{M}))$ contain subgroups of index and exponent 3. This completes the proof of this lemma.

LEMMA 9.14. *$N(J(\mathfrak{P}))$ does not contain a noncyclic abelian subgroup* *of order* 8.

Proof. First, suppose that $\boldsymbol{J}(\mathfrak{P})$ is not elementary. Then $\mathfrak{W} = \boldsymbol{Z}(\boldsymbol{J}(\mathfrak{P})) \cap \boldsymbol{D}(\boldsymbol{J}(\mathfrak{P})) \neq 1$. If $\mathfrak{W}$ is cyclic, then $\Omega_1(\mathfrak{W}) = \mathfrak{Z}$ char $\boldsymbol{N}(\boldsymbol{J}(\mathfrak{P}))$, so $\boldsymbol{N}(\boldsymbol{J}(\mathfrak{P})) \subseteq \mathfrak{N}$, and this lemma follows from Lemma 9.13. We may assume that $\mathfrak{W}$ is noncyclic. Let $\mathfrak{W}_1$ be a noncyclic elementary subgroup of order 9. We will show that $\mathfrak{W}_1 \in \mathscr{E}(3)$. Choose $\mathfrak{Q} \in \mathcal{M}(\mathfrak{W}_1; 3')$, minimal subject to $[\mathfrak{Q}, \mathfrak{W}_1] \neq 1$. Let $\mathfrak{W}_0 = \boldsymbol{C}_{\mathfrak{W}_1}(\mathfrak{Q})$ so that $|\mathfrak{W}_0| = 3$. Let $\mathfrak{C} = \boldsymbol{C}(\mathfrak{W}_0)$, and let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$ which contains $\boldsymbol{J}(\mathfrak{P})$. Hence, $\boldsymbol{J}(\mathfrak{P}) = \boldsymbol{J}(\mathfrak{P}^*)$. Let $\mathfrak{P} = \mathfrak{P}^G$ be a S_3-subgroup of $\mathfrak{G}$ which contains $\mathfrak{P}^*$.

Since $\mathfrak{C}$ contains an element of $\mathscr{SCN}_3(\mathfrak{P})$, it follows that $\boldsymbol{O}_{3'}(\mathfrak{C}) = 1$. Since $\mathfrak{W}_1 \subseteq \boldsymbol{Z}(\boldsymbol{J}(\mathfrak{P})) = \boldsymbol{Z}(\boldsymbol{J}(\mathfrak{P}^*))$, we have $[\boldsymbol{O}_3(\mathfrak{C}), \mathfrak{W}_1] \subseteq \boldsymbol{J}(\mathfrak{P})$ and $[\boldsymbol{O}_3(\mathfrak{C}), \mathfrak{W}_1, \mathfrak{W}_1] = 1$. It follows that $\mathfrak{Q}$ is a quaternion group. Let $\mathfrak{V} = (\mathfrak{Z}^G)^{\mathfrak{C}}$. Thus, $\mathfrak{V}$ is a normal elementary 3-subgroup of $\mathfrak{C}$, and by Lemma 5.10, $\mathfrak{V}$ is 3-reducible in $\mathfrak{C}$. It is a straightforward consequence of Lemma 5.2 that $\mathfrak{V} \subseteq \boldsymbol{J}(\mathfrak{P}^*)$. Thus, $\mathfrak{Q}$ centralizes $\mathfrak{V}$, as $\mathfrak{W}_1$ centralizes $\mathfrak{V}$. Thus, it follows that $\mathfrak{W}_1 \not\subseteq \boldsymbol{O}_3(\mathfrak{N}_1)$, where $\mathfrak{N}_1$ is a $S_{2,3}$-subgroup of $\mathfrak{N}^G$ which contains $\mathfrak{P}^G$. By Lemma 9.12, $|\mathfrak{P}^G : \boldsymbol{O}_3(\mathfrak{N}_1)| \leqq 3$. Thus, $\mathfrak{W}_1 \not\subseteq \boldsymbol{D}(\mathfrak{P}^G)$. This is absurd, since $\mathfrak{W}_1 \subseteq \boldsymbol{D}(\boldsymbol{J}(\mathfrak{P}))$, and $\boldsymbol{J}(\mathfrak{P}) = \boldsymbol{J}(\mathfrak{P}^*) = \boldsymbol{J}(\mathfrak{P}^G)$.

It is an immediate consequence of the preceding argument and Lemmas 7.4 and 5.38 that if $\boldsymbol{D}(\boldsymbol{J}(\mathfrak{P})) \neq 1$, then this lemma holds.

Assume now that $\boldsymbol{J}(\mathfrak{P})$ is elementary. To complete the proof of the lemma, it suffices to show that each subgroup of $\boldsymbol{J}(\mathfrak{P})$ of order 9 is in $\mathscr{E}(3)$. Suppose false, and $\mathfrak{W} \subseteq \boldsymbol{J}(\mathfrak{P})$, $|\mathfrak{W}| = 9$, $\mathfrak{W} \notin \mathscr{E}(3)$. Let $\mathfrak{T}$ be an element of $\mathcal{M}(\mathfrak{W}; 3')$ minimal subject to $[\mathfrak{W}, \mathfrak{T}] \neq 1$. Let $\mathfrak{W}_0 = \mathfrak{W} \cap \boldsymbol{C}(\mathfrak{T})$, so that $|\mathfrak{W}_0| = 3$.

Let $\mathfrak{C} = \boldsymbol{C}(\mathfrak{W}_0) \supseteq \langle \boldsymbol{J}(\mathfrak{P}), \mathfrak{T} \rangle$. Let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$ which contains $\boldsymbol{J}(\mathfrak{P})$, and let $\mathfrak{P}^G$ be a S_3-subgroup of $\mathfrak{G}$ which contains $\mathfrak{P}^*$. Hence, $\boldsymbol{J}(\mathfrak{P}) = \boldsymbol{J}(\mathfrak{P}^G)$. Since $\boldsymbol{J}(\mathfrak{P}^G) = \boldsymbol{J}(\mathfrak{P})^G$, we get that $G \in \boldsymbol{N}(\boldsymbol{J}(\mathfrak{P}))$. Replacing $\mathfrak{W}$ by $\mathfrak{W}^{G^{-1}}$ and $\mathfrak{T}$ by $\mathfrak{T}^{G^{-1}}$, we assume without loss of generality that $\mathfrak{P}^* \subseteq \mathfrak{P}$.

Since $\mathfrak{P}^*$ contains an element of $\mathscr{SCN}_3(\mathfrak{P})$, it follows that $\boldsymbol{O}_{3'}(\mathfrak{C}) = 1$. Hence, $\mathfrak{Z} \subseteq \boldsymbol{O}_3(\mathfrak{C})$. Let $\mathfrak{V} = \mathfrak{Z}^{\mathfrak{C}}$, so that $\mathfrak{V}$ is a normal elementary subgroup of $\mathfrak{C}$. Since $\mathfrak{V}$ is 3-reducible in $\mathfrak{C}$, it follows that $\mathfrak{V} \subseteq \boldsymbol{J}(\mathfrak{P})$. Hence, $\mathfrak{T}$ centralizes $\mathfrak{V}$. In particular, $\mathfrak{T}$ centralizes $\mathfrak{Z}$.

Let $\tilde{\mathfrak{J}} = \boldsymbol{J}(\mathfrak{P}) \cap \boldsymbol{O}_3(\mathfrak{C})$. Thus, $\mathfrak{W} \not\subseteq \tilde{\mathfrak{J}}$, and $\boldsymbol{J}(\mathfrak{P})/\tilde{\mathfrak{J}}$ acts faithfully on $\boldsymbol{O}_{3,3'}(\mathfrak{C})/\boldsymbol{O}_3(\mathfrak{C})$. Let $\mathfrak{R} = [\boldsymbol{O}_{3,3'}(\mathfrak{C}), \boldsymbol{J}(\mathfrak{P})]\boldsymbol{O}_3(\mathfrak{C})$. Since

$$[\boldsymbol{O}_3(\mathfrak{C}), \boldsymbol{J}(\mathfrak{P}), \boldsymbol{J}(\mathfrak{P})] = 1 ,$$

it follows that $\bar{\mathfrak{R}} = \mathfrak{R}/\boldsymbol{O}_3(\mathfrak{C})$ is a 2-group, and that $\boldsymbol{J}(\mathfrak{P})$ centralizes every characteristic abelian subgroup of $\bar{\mathfrak{R}}$. Since $\boldsymbol{J}(\mathfrak{P})$ centralizes $\mathfrak{V}$, so does $\mathfrak{R}$. By Lemma 9.12 (ii), $\mathfrak{R}$ contains no four-group. So $\mathfrak{R}$ is a quaternion group and $\boldsymbol{J}(\mathfrak{P})/\tilde{\mathfrak{J}}$ is of order 3, whence $\boldsymbol{J}(\mathfrak{P}) = \tilde{\mathfrak{J}}\mathfrak{W}$, and so $\mathfrak{R} =$

$[O_{3,3'}(\mathfrak{C}), \mathfrak{W}]O_3(\mathfrak{C})$. Since $J(\mathfrak{P})O_3(\mathfrak{C})/O_3(\mathfrak{C})$ is of order 3, it follows that $J(\mathfrak{P}) \subseteq O_{3,3',3}(\mathfrak{C})$, and so $\mathfrak{T} \subseteq O_{3,3'}(\mathfrak{C})$, whence $\mathfrak{T} \subseteq \mathfrak{R}$, and so $\mathfrak{R} = \mathfrak{T}O_3(\mathfrak{C})$, and $\mathfrak{T} \cong \bar{\mathfrak{R}}$ is a quaternion group. Note that $\mathfrak{T}$ is permutable with $\mathfrak{P}^*$, as $\mathfrak{P}^*$ normalizes $[O_{3,3'}(\mathfrak{C}), J(\mathfrak{P})]$.

Enlarge $\mathfrak{P}^*\mathfrak{T}$ to a $S_{2,3}$-subgroup $\mathfrak{N}_0$ of $\mathfrak{N}$, and let $\mathfrak{N}_1 = O^{3'}(\mathfrak{N}_0)$. Thus, $\mathfrak{N}_1 = \mathfrak{T}\tilde{\mathfrak{P}}$, where $\tilde{\mathfrak{P}} = \mathfrak{P}^N$ for some N in $\mathfrak{N}$. Since $J(\mathfrak{P}) = J(\mathfrak{P}^N) = J(\mathfrak{P})^N$, replacing $\mathfrak{W}$ by $\mathfrak{W}^{N^{-1}}$ and $\mathfrak{T}$ by $\mathfrak{T}^{N^{-1}}$, we assume without loss of generality that $\tilde{\mathfrak{P}} = \mathfrak{P}$ is permutable with $\mathfrak{T}$.

Let $\mathfrak{W}_1$ be a subgroup of $\mathfrak{W}$ of order 3 different from $\mathfrak{W}_0$. Let $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$. Thus, $\mathfrak{P} = \mathfrak{P}_0\mathfrak{W}_1$ and $\mathfrak{W}_1\mathfrak{T}$ is a complement to $\mathfrak{P}_0$ in $\mathfrak{N}_1$. Let I be the involution of $\mathfrak{T}$, let T be an element of $\mathfrak{T}$ of order 4, let $\mathfrak{K} = J(\mathfrak{P}) \cap \mathfrak{P}_0$, and let $\mathfrak{L} = \langle \mathfrak{K}, \mathfrak{K}^T \rangle$. Since $T^2 = I$ normalizes $\mathfrak{P}$, and $J(\mathfrak{P})$ char $\mathfrak{P}$, it follows that T^2 normalizes $\mathfrak{K}$. Hence, T normalizes $\mathfrak{L}$. Of course, $\mathfrak{W}_1$ also normalizes $\mathfrak{L}$, since $[\mathfrak{W}_1, \mathfrak{L}] \subseteq \mathfrak{K} \subseteq \mathfrak{L}$. Since $\mathfrak{K} \lhd \mathfrak{P}_0$, it follows that $\mathfrak{L} \lhd \mathfrak{P}_0$, so $\mathfrak{L} \lhd \mathfrak{N}_1$. Since

$$\mathfrak{L}' = [\mathfrak{K}, \mathfrak{K}^T] \subseteq \mathfrak{K} \cap \mathfrak{K}^T \subseteq \mathfrak{K} \subseteq J(\mathfrak{P}) ,$$

$J(\mathfrak{P})$ centralizes $\mathfrak{L}'$. Since $\mathfrak{W}_1 \subseteq J(\mathfrak{P})$, it follows that $\mathfrak{T}$ centralizes $\mathfrak{L}'$. Hence, $\mathfrak{L}' \subseteq \mathfrak{Z}$, as otherwise I centralizes an element of $\mathscr{U}(3)$.

Clearly, $\mathfrak{L}$ is of exponent 3, being of class at most 2 and being generated by its elementary subgroups. The definition of $J(\mathfrak{P})$ forces $\mathfrak{K} = C_{\mathfrak{L}}(\mathfrak{K})$. Hence, $Z(\mathfrak{L}) = \mathfrak{Z} = \mathfrak{L}'$, so that $\mathfrak{L}$ is extra special, while $\mathfrak{K} \in \mathscr{SCN}(\mathfrak{L})$. The width of $\mathfrak{L}$ is at least 2, since otherwise, Hypothesis 9.1 would be satisfied.

Now I centralizes $\mathfrak{Z}$ and normalizes $\mathfrak{K}$. We argue that I inverts $\mathfrak{K}/\mathfrak{Z}$. Suppose false and $\mathfrak{F}$ is a subgroup of $\mathfrak{K}$ of order 9 which contains $\mathfrak{Z}$ and is centralized by I. Since $A_{\mathfrak{N}}(\tilde{\mathscr{C}}) = A(\tilde{\mathscr{C}})$, where $\tilde{\mathscr{C}}: \mathfrak{K} \supset \mathfrak{Z} \supset 1$, it follows from Lemma 5.5 that $\mathfrak{F} \in \mathscr{E}(3)$. Thus, $C(I)$ is nonsolvable by Lemmas 7.4 and 5.38. This contradiction shows that I inverts $\mathfrak{K}/\mathfrak{Z}$. Hence I inverts $\mathfrak{L}/\mathfrak{Z}$.

We next show that $\mathfrak{T}$ centralizes $\mathfrak{P}_0/\mathfrak{L}$. This is clear, since $[\mathfrak{P}_0, \mathfrak{W}_1] \subseteq \mathfrak{K} \subseteq \mathfrak{L}$, so that $\mathfrak{W}_1$ centralizes $\mathfrak{P}_0/\mathfrak{L}$.

Since I inverts $\mathfrak{L}/\mathfrak{Z}$, it follows that if $\tilde{\mathfrak{W}}$ is any subgroup of $J(\mathfrak{P})$ of order 9, and $\tilde{\mathfrak{T}}$ is any element of $\mathscr{N}(\tilde{\mathfrak{W}}; 3')$ which is minimal subject to $[\tilde{\mathfrak{W}}, \tilde{\mathfrak{T}}] \neq 1$, then $\tilde{\mathfrak{T}}$ is a quaternion group and $\tilde{\mathfrak{W}} \cap C(\tilde{\mathfrak{T}}) \sim \mathfrak{Z}$. In particular, $\mathfrak{W} \in \mathscr{D}$.

Let $\mathfrak{M}$ be the subgroup given by Lemma 7.5 which contains a S_2-subgroup of $C(I)$ and contains $\mathfrak{W}$. Then Lemma 9.12 (ii) implies that $O_2(\mathfrak{M})$ is extra special, and that $\langle I \rangle = O_2(\mathfrak{M})'$. Hence, $C_{\mathfrak{N}_1}(I) \subseteq \mathfrak{M}$.

If the width of $O_2(\mathfrak{M})$ is 2, then $\mathfrak{W} = C_{\mathfrak{P}}(I)$ is a S_3-subgroup of $C_{\mathfrak{N}_1}(I)$ so $\mathfrak{L} = \mathfrak{P}_0$ is extra special. Since $\mathfrak{P}_0 = O_3(\mathfrak{N})$, it follows that Hypothesis 9.2 is satisfied, an excluded case. Hence, the width of $O_2(\mathfrak{M})$ is 3 or 4. On the other hand, every element of $(C(I) \cap O_3(\mathfrak{N}_1))\mathfrak{W}$ of order 3 centralizes an element of $\mathscr{E}(3)$, so $C_{\mathfrak{N}_1}(I)$ contains no subgroup of exponent 3 and order 27. Hence, $C_{\mathfrak{P}_0}(I)$ is cyclic of order 3 or 9. If $C_{\mathfrak{P}_0}(I) = \mathfrak{Z}$, Hypothesis 9.2 is satisfied, an excluded case. Hence, $C_{\mathfrak{P}_0}(I)$ is cyclic of order 9. This means that if $\mathfrak{W}_1$ is any subgroup of $\mathfrak{W}$ of order 3 such that $C(\mathfrak{W}_1) \cap O_2(\mathfrak{M}) \supset \langle I \rangle$, then $\mathfrak{W}_1 = \langle P^3 \rangle$ for some P in $\mathfrak{M}$. This is absurd, since we get that $\mathfrak{W} \subseteq \Omega^1(\tilde{\mathfrak{P}})$ for some S_3-subgroup $\tilde{\mathfrak{P}}$ of $\mathfrak{M}$, while $\tilde{\mathfrak{P}}$ is isomorphic to a subgroup of $Z_3 \times (Z_3 \wr Z_3)$. The proof is complete.

LEMMA 9.15. *If* $\mathfrak{A} \in \mathscr{E}(3)$ *and* $\mathfrak{B}$ *is a noncyclic abelian subgroup of order* 8, *then* $\langle \mathfrak{A}, \mathfrak{B} \rangle$ *is nonsolvable.*

Proof. Suppose false. Let $\mathscr{S}$ be the set of all 2, 3-subgroups $\mathfrak{S}$ of $\mathfrak{G}$ such that

(i) $\mathfrak{S}$ contains an element of $\mathscr{E}(3)$.

(ii) $\mathfrak{S}/O_3(\mathfrak{S})$ satisfies the hypothesis of Lemma 5.41.

Thus, $\mathscr{S} \neq \emptyset$.

If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are elements of $\mathscr{S}$, we say that $\mathfrak{S}_1 \ll \mathfrak{S}_2$ if and only if either $|\mathfrak{S}_1|_3 < |\mathfrak{S}_2|_3$ or $\mathfrak{S}_1 = \mathfrak{S}_2$.

Let $\mathfrak{S}$ be a maximal element of $\mathscr{S}$ under $\ll$. Let $\mathfrak{S}_p$ be a S_p-subgroup of $\mathfrak{S}$, $p = 2, 3$. Since $\mathfrak{S}$ contains an element of $\mathscr{E}(3)$, it follows from Lemma 9.11 (ii) that $O_2(\mathfrak{S}) = 1$.

Replacing $\mathfrak{S}$ by a conjugate if necessary, we assume that $\mathfrak{S}_3 \subseteq \mathfrak{P}$. By Lemma 5.41, $\mathfrak{S}$ has 2-length 1. If $\mathfrak{S}_2' = 1$, then

$$\mathfrak{S} = C_{\mathfrak{S}}(Z(\mathfrak{S}_3)) \cdot N_{\mathfrak{S}}(J(\mathfrak{S}_3))$$

by Theorem 1 of [43]. Since $Z(\mathfrak{P}) \subseteq Z(\mathfrak{S}_3)$, S_2-subgroups of $C_{\mathfrak{S}}(Z(\mathfrak{S}_3))$ are cyclic. Thus, $C_{\mathfrak{S}}(Z(\mathfrak{S}_3)) \subseteq N_{\mathfrak{S}}(\mathfrak{S}_3) \subseteq N_{\mathfrak{S}}(J(\mathfrak{S}_3))$, so $J(\mathfrak{S}_3) \lhd \mathfrak{S}$. Maximality of $\mathfrak{S}$ forces $\mathfrak{S}_3 = \mathfrak{P}$, against Lemma 9.14. Hence, $\mathfrak{S}_2' \neq 1$.

Suppose $\mathfrak{S}_2$ is extra special of width at least 2. By Lemma 5.52, it follows that $\mathfrak{S} = C_{\mathfrak{S}}(Z(O_3(\mathfrak{S})))N_{\mathfrak{S}}(J(\mathfrak{S}_3))$. Thus, maximality of $\mathfrak{S}$ together with Lemmas 9.13 and 9.14 imply that neither $C_{\mathfrak{S}}(Z(O_3(\mathfrak{S})))$ nor $N_{\mathfrak{S}}(J(\mathfrak{S}_3))$ contains a noncyclic abelian subgroup of order 8. Let $\mathfrak{T}_0 = C_{\mathfrak{S}}(Z(O_3(\mathfrak{S}))) \cap \mathfrak{S}_2$, $\mathfrak{T}_1 = N(J(\mathfrak{S}_3)) \cap \mathfrak{S}_2$.

Since $\mathfrak{S}_2 = \mathfrak{T}_0\mathfrak{T}_1$ and $\mathfrak{T}_i$ has no noncyclic abelian subgroup of order 8, the width of $\mathfrak{S}_2$ is 2, and $4 \leqq |\mathfrak{T}_i| \leqq 8$, $i = 0, 1$.

Suppose $\mathfrak{S}_3 \cdot C_{\mathfrak{S}}(Z(O_3(\mathfrak{S})))$ is 3-closed. Then $\mathfrak{T}_0$ normalizes $\mathfrak{S}_3$, so normalizes $J(\mathfrak{S}_3)$. This yields $\mathfrak{T}_0 \subseteq \mathfrak{T}_1$, which is not the case. Thus, $\mathfrak{S}_3 \cdot C_{\mathfrak{S}}(Z(O_3(\mathfrak{S})))$ is not 3-closed. Since $\mathfrak{T}_0 \lhd \mathfrak{S}_2$, it follows that $\mathfrak{T}_0$ is a quaternion group. Suppose $N_{\mathfrak{S}}(J(\mathfrak{S}_3))$ is 3-closed. Since $Z(\mathfrak{S}_3) \subseteq Z(O_3(\mathfrak{S}))$, it follows that $Z(\mathfrak{S}_3) \lhd \mathfrak{S}$. Maximality of $|\mathfrak{S}|_3$ forces $\mathfrak{S}_3 = \mathfrak{P}$. This violates Lemma 9.13. Thus, $N_{\mathfrak{S}}(J(\mathfrak{S}_3))$ is not 3-closed. Since $\mathfrak{S}_2' \subseteq N_{\mathfrak{S}}(\mathfrak{S}_3) \subseteq N_{\mathfrak{S}}(J(\mathfrak{S}_3))$, it follows that $\mathfrak{T}_1$ is also a quaternion group. Since $l_2(\mathfrak{S}) = 1$, $\mathfrak{S}_2$ is the central product of $\mathfrak{T}_0$ and $\mathfrak{T}_1$.

Let $\mathfrak{P} = \mathfrak{P}_1, \cdots, \mathfrak{P}_r$ be all the S_3-subgroups of $\mathfrak{G}$ which contain $\mathfrak{S}_3$, and let $\mathfrak{Z}_i = \Omega_1(\boldsymbol{Z}(\mathfrak{P}_i))$. Thus, $\mathfrak{Z}_i \subseteq \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S}))$ for all i, so that $\mathfrak{T}_0$ centralizes each $\mathfrak{Z}_i$. Let $\langle T\rangle = \mathfrak{T}_0 \cap \mathfrak{T}_1$ so that T is an involution which centralizes each $\mathfrak{Z}_i$. Also, $\mathfrak{T}_0 \subseteq \boldsymbol{C}(\mathfrak{Z}_i)$ for each i, so for each i, $\mathfrak{T}_1 \not\subseteq \boldsymbol{C}(\mathfrak{Z}_i)$.

Suppose $\mathfrak{S}_3 = \mathfrak{P}$. Since $\mathfrak{S}_2' = \mathfrak{T}_0'$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S}))$, it follows that $\mathfrak{Z} = \Omega_1(\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S}))$; otherwise, $\mathfrak{S}_2'$ centralizes an element of $\mathscr{U}(3)$. Hence, $\mathfrak{Z} \lhd \mathfrak{S}$, against Lemma 9.13. We conclude that $\mathfrak{S}_3 \subset \mathfrak{P}$.

Enlarge $\mathfrak{S}_3\mathfrak{T}_1$ to a $S_{2,3}$-subgroup of $\boldsymbol{N}(\boldsymbol{J}(\mathfrak{S}_3))$ and enlarge this subgroup to a maximal 2, 3-subgroup $\mathfrak{L}_0$ of $\mathfrak{G}$. Let $\mathfrak{L} = \boldsymbol{O}^{3'}(\mathfrak{L}_0)$. Since $|\mathfrak{L}|_3 > |\mathfrak{S}|_3$, it follows that $\mathfrak{L}$ contains no noncyclic abelian subgroup of order 8. Since $\mathfrak{L} \supseteq \boldsymbol{O}^{3'}(\mathfrak{S}_3\mathfrak{T}_1) = \mathfrak{S}_3\mathfrak{T}_1$, it follows that $\mathfrak{T}_1$ is a S_2-subgroup of $\mathfrak{L}$. Let $\mathfrak{L}_3$ be a S_3-subgroup of $\mathfrak{L}$ which contains $\mathfrak{S}_3$. Thus, $\mathfrak{L}_3 \subseteq \mathfrak{P}_i$ for some i.

Let $\mathfrak{W}$ be the normal closure of $\mathfrak{Z}_i$ in $\mathfrak{L}$. Thus, $\boldsymbol{C}_{\mathfrak{L}}(\mathfrak{W})$ contains T. Since $\mathfrak{T}_0 \subseteq \boldsymbol{C}(\mathfrak{Z}_i)$, it follows that $\boldsymbol{C}_{\mathfrak{L}}(\mathfrak{W}) \cap \mathfrak{T}_1 = \langle T\rangle$, so S_2-subgroups of $\boldsymbol{A}_{\mathfrak{L}}(\mathfrak{W})$ are four-groups. It follows that $\boldsymbol{J}(\mathfrak{L}_3) \lhd \mathfrak{L}$. Hence, $\mathfrak{L}_3 = \mathfrak{P}_i$ and so T centralizes an element of $\mathscr{U}(\mathfrak{P}_i)$. Thus, by Lemmas 7.1 (i) and 7.4, $\boldsymbol{C}(T)$ is nonsolvable. This contradiction shows that $\mathfrak{S}_2$ is not extra special of width $\geqq 2$.

Suppose $\mathfrak{S}_2$ is the central product of a quaternion group and a cyclic group of order 4. If $\boldsymbol{J}(\mathfrak{S}_3) \subseteq \boldsymbol{O}_3(\mathfrak{S})$, then again $\mathfrak{S}_3 = \mathfrak{P}$ and Lemma 9.14 is violated. Hence, $\boldsymbol{J}(\mathfrak{S}_3) \not\subseteq \boldsymbol{O}_3(\mathfrak{S})$. If $\mathfrak{S}_2'$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S}))$, then we get $\mathfrak{S} = \boldsymbol{C}_{\mathfrak{S}}(\boldsymbol{Z}(\mathfrak{S}_3))\boldsymbol{N}_{\mathfrak{S}}(\boldsymbol{J}(\mathfrak{S}_3))$, so that either $\boldsymbol{Z}(\mathfrak{S}_3)$ or $\boldsymbol{J}(\mathfrak{S}_3)$ is normal in $\mathfrak{S}$. Both these possibilities are excluded by Lemmas 9.13 and 9.14, so we may assume that $[\mathfrak{S}_2', \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S}))] = \mathfrak{W} \neq 1$. Let $\mathfrak{X}$ be a minimal normal subgroup of $\mathfrak{S}$ with $\mathfrak{X} \subseteq \mathfrak{W}$. Thus, $\mathfrak{S}_2$ is faithfully represented on $\mathfrak{X}$. Since $|\mathfrak{S}_3 : \boldsymbol{O}_3(\mathfrak{S})| = 3$ and $\boldsymbol{J}(\mathfrak{S}_3) \not\subseteq \boldsymbol{O}_3(\mathfrak{S})$, it follows that elements of $\mathfrak{S}_3 - \boldsymbol{O}_3(\mathfrak{S})$ centralize a hyperplane of $\mathfrak{X}$. This is not the case, since $|\mathfrak{X}| = 3^4$. Thus, $\mathfrak{S}_2$ is not the central product of a quaternion group and a cyclic group of order 4.

By Lemma 5.41 and maximality of $\mathfrak{S}$ under $\ll$, it follows that $\mathfrak{S}_2$ is either the direct product of a quaternion group and a group of order 2 or $\mathfrak{S}_2$ is special with $|\mathfrak{S}_2'| = 4$. Let $\mathfrak{V} = \boldsymbol{Z}(\mathfrak{S}_2)$, so that in both cases, $\mathfrak{V}$ is a four-group. We will exploit $\mathfrak{V}$ by showing that $\mathfrak{S}_3 = \mathfrak{P}$, that is, by showing that $\mathfrak{S}_3$ is a S_3-subgroup of $\mathfrak{G}$. Suppose by way of contradiction that $\mathfrak{S}_3 \subset \mathfrak{P}$.

We argue that $\mathfrak{V}$ normalizes $\boldsymbol{J}(\mathfrak{S}_3)$. For if this is not the case, then $\mathfrak{V}$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S}))$, against Lemma 9.12 (ii).

Since $\mathfrak{V}$ normalizes $\boldsymbol{J}(\mathfrak{S}_3)$, we may enlarge $\mathfrak{S}_3\mathfrak{V}$ to a $S_{2,3}$-subgroup $\mathfrak{L}$ of $\boldsymbol{N}(\boldsymbol{J}(\mathfrak{S}_3))$. Since $\mathfrak{S}_3$ is not a S_3-subgroup of $\mathfrak{L}$, $\mathfrak{L}$ does not contain a noncyclic abelian subgroup of order 8.

Let $\mathfrak{L}_p$ be a S_p-subgroup of $\mathfrak{L}$, $p = 2, 3$, with $\mathfrak{V} \subseteq \mathfrak{L}_2$, $\mathfrak{S}_3 \subset \mathfrak{L}_3$.

Case 1. $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{V}/\boldsymbol{O}_3(\mathfrak{S}) \subseteqq \boldsymbol{Z}(\mathfrak{S}/\boldsymbol{O}_3(\mathfrak{S}))$.

Let $\tilde{\mathfrak{S}}_3$ be a maximal element of $\mathsf{M}_{\mathfrak{L}}(\mathfrak{V}; 3)$ with $\mathfrak{S}_3 \subseteqq \tilde{\mathfrak{S}}_3$. Suppose $\mathfrak{S}_3 \subset \tilde{\mathfrak{S}}_3$. Choose $\mathfrak{S}_3^*$ in $\mathsf{M}_{\mathfrak{L}}(\mathfrak{V}; 3)$ so that $|\mathfrak{S}_3^* : \mathfrak{S}_3| = 3$, and let $\mathfrak{V}_0 = \boldsymbol{C}_{\mathfrak{V}}(\mathfrak{S}_3^*/\mathfrak{S}_3)$. Hence, $[\mathfrak{S}_3^*, \mathfrak{V}_0] = [\mathfrak{S}_3, \mathfrak{V}_0]$ is normal in $\mathfrak{S}$ and in $\mathfrak{S}_3^*$. Maximality of $\mathfrak{S}$ in $\mathscr{S}$ forces $[\mathfrak{S}_3, \mathfrak{V}_0] = 1$, against $\boldsymbol{O}_2(\mathfrak{S}) = 1$. Thus, $\mathfrak{S}_3$ is a maximal element of $\mathsf{M}_{\mathfrak{L}}(\mathfrak{V}; 3)$. In particular, $\mathfrak{L}$ is not 3-closed. Hence, $\boldsymbol{O}_3(\mathfrak{L})$ is of index 3 in $\mathfrak{L}_3$ and $\boldsymbol{O}_3(\mathfrak{L}) \subseteqq \mathfrak{S}_3$. Hence, $\boldsymbol{O}_3(\mathfrak{L}) = \mathfrak{S}_3$. If $\mathfrak{V} = \mathfrak{L}_2$, then $[\mathfrak{V}, \mathfrak{S}_3] \lhd \mathfrak{L}$ so that $\langle \mathfrak{S}, \mathfrak{L} \rangle \subseteqq \boldsymbol{N}([\mathfrak{V}, \mathfrak{S}_3])$, since $[\mathfrak{V}, \mathfrak{S}_3] = [\mathfrak{V}, \boldsymbol{O}_3(\mathfrak{S})] \neq 1$. This is impossible, so $\mathfrak{V} \subset \mathfrak{L}_2$.

Case 1a. $\mathfrak{S}_2$ is special.

Since $\mathfrak{V} \subseteqq \boldsymbol{Z}(\mathfrak{S}_2)$, it follows that $\mathfrak{S}_3\mathfrak{V}$ is a maximal subgroup of $\mathfrak{S}$. Thus, $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{S}_2/\boldsymbol{O}_3(\mathfrak{S})\mathfrak{V}$ is a chief factor of $\mathfrak{S}$. Let $\mathfrak{V}_0$ be a subgroup of $\mathfrak{V}$ of order 2 and let $\mathfrak{S}_2^0 = \boldsymbol{Z}(\mathfrak{S}_2 \bmod \mathfrak{V}_0)$. Since $\boldsymbol{O}_3(S)\mathfrak{V}_0 \lhd \mathfrak{S}$, $\mathfrak{S}_3\mathfrak{S}_2^0$ is a group. Hence, $\mathfrak{S}_2^0 = \mathfrak{V}$ or $\mathfrak{S}_2^0 = \mathfrak{S}_2$. If $\mathfrak{S}_2^0 = \mathfrak{S}_2$, then $\mathfrak{S}_2' \subseteqq \mathfrak{V}_0$, so that $\mathfrak{S}_2/\mathfrak{V}_0$ is abelian. This is not the case, since $\mathfrak{S}_2' = \mathfrak{V}$. Hence, $\mathfrak{S}_2^0 = \mathfrak{V}$, so that $\mathfrak{S}_2/\mathfrak{V}_0$ is extra special of width $\geqq 2$. It follows from the proof of Lemma 5.52 that $\boldsymbol{J}(\mathfrak{S}_3) \lhd \mathfrak{S}$. Maximality of $\mathfrak{S}$ in $\mathscr{S}$ guarantees that $\mathfrak{S}_3$ is a S_3-subgroup of $\mathfrak{G}$, against our assumption that $\mathfrak{S}_3 \subset \mathfrak{P}$.

Case 1b. $\mathfrak{S}_2$ is the direct product of a quaternion group and a group of order 2.

Since $\mathfrak{L}_2$ has no noncyclic abelian subgroup of order 8, it follows that $\mathfrak{V}$ is a self centralizing subgroup of $\mathfrak{L}_2$. Hence, $\mathfrak{L}_2$ is of maximal class. Let $\tilde{\mathfrak{L}}_2 = \mathfrak{L}_2 \cap \boldsymbol{O}_{3,2}(\mathfrak{L})$. Thus, $\tilde{\mathfrak{L}}_2$ has an automorphism of order 3. Being a subgroup of a group of maximal class, $\tilde{\mathfrak{L}}_2$ is either a quaternion group or a four-group.

Suppose $\tilde{\mathfrak{L}}_2$ is a quaternion group. In this case, $\mathfrak{L}/\boldsymbol{O}_3(\mathfrak{L}) \cong GL(2, 3)$ and $\mathfrak{V}$ normalizes a S_3-subgroup of $\mathfrak{L}$, against our previous argument. we conclude that $\tilde{\mathfrak{L}}_2$ is a four-group.

If $\tilde{\mathfrak{L}}_2 = \mathfrak{V}$, then $[\boldsymbol{O}_3(\mathfrak{L}), \mathfrak{V}] \lhd \mathfrak{L}$. But $\boldsymbol{O}_3(\mathfrak{L}) = \mathfrak{S}_3$ and $[\mathfrak{S}_3, \mathfrak{V}] = [\boldsymbol{O}_3(\mathfrak{S}), \mathfrak{V}] \lhd \mathfrak{S}$. Hence, $[\boldsymbol{O}_3(\mathfrak{L}), \mathfrak{V}] \lhd \langle \mathfrak{L}, \mathfrak{S} \rangle$ against the maximality of $\mathfrak{S}$ in $\mathscr{S}$. We conclude that $\tilde{\mathfrak{L}}_2 \neq \mathfrak{V}$, so that $\mathfrak{L}_2$ is a dihedral group of order 8 whose two four-subgroups are $\mathfrak{V}$ and $\tilde{\mathfrak{L}}_2$.

Since $\tilde{\mathfrak{L}}_2$ does not centralize $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L}))$, it follows that $\boldsymbol{J}(\mathfrak{L}_3) \lhd \mathfrak{L}_0$. Hence, $\boldsymbol{J}(\mathfrak{L}_3) = \boldsymbol{J}(\mathfrak{S}_3)$, so by construction of $\mathfrak{L}$, we conclude that $\mathfrak{L}_3$ is a S_3-subgroup of $\mathfrak{G}$. We may therefore assume without loss of generality that $\mathfrak{P} = \mathfrak{L}_3$. Hence, $|\mathfrak{P} : \mathfrak{S}_3| = 3$.

Let $\mathfrak{X} = \Omega_1(\boldsymbol{Z}(\mathfrak{S}_3))$. Recalling that $\mathfrak{S}_3 = \boldsymbol{O}_3(\mathfrak{L})$, we get $\mathfrak{X} \lhd \mathfrak{L}$. Since $\tilde{\mathfrak{L}}_2$ does not centralize $\mathfrak{Z}$, we get $\mathfrak{Z} \subset \mathfrak{X}$. Since $\boldsymbol{Z}(\mathfrak{P})$ is cyclic and $|\mathfrak{P} : \mathfrak{S}_3| = 3$, we get $|\mathfrak{X}| \leqq 27$. Hence, $|\mathfrak{X}| = 27$ and $\mathfrak{X}$ is the only minimal normal subgroup of $\mathfrak{L}$.

Since $\boldsymbol{J}(\mathfrak{P}) \lhd \mathfrak{L}$, we get $\mathfrak{X}_0 \lhd \mathfrak{L}$, where $\mathfrak{X}_0 = \Omega_1(\boldsymbol{Z}(\boldsymbol{J}(\mathfrak{P})) \cap \boldsymbol{D}(\boldsymbol{J}(\mathfrak{P})))$, if

$D(J(\mathfrak{P})) \neq 1$, and $\mathfrak{X}_0 = J(\mathfrak{P})$ if $D(J(\mathfrak{P})) = 1$. If $|\mathfrak{X}_0| > 27$, then some element of $\mathfrak{V}^{\#}$ centralizes a noncyclic subgroup of $\mathfrak{X}_0$. This was shown to be impossible in the proof of Lemma 9.14. Hence, $|\mathfrak{X}_0| \leqq 27$. This implies that $\mathfrak{X}_0 = \mathfrak{X}$.

Suppose $A, B \in \mathfrak{X}_0$ and $A = B^B$ for some G in $\mathfrak{G}$. Thus, $\langle J(\mathfrak{P}), J(\mathfrak{P}^{G^{-1}})\rangle \subseteqq C(B)$, and we can choose C in $C(B)$ such that $J(\mathfrak{P})^C = J(\mathfrak{P}^{G^{-1}})$. Hence, $CG = N \in N(J(\mathfrak{P}))$ and $A = B^G = B^{C^{-1}}N = B^N$. Thus, elements of $\mathfrak{X}$ are $\mathfrak{G}$-conjugate only if they are $N(\mathfrak{X})$-conjugate.

Let $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3$, where $|\mathfrak{X}_i| = 3$ and where $\mathfrak{X}_i$ admits $\mathfrak{V}$, $i = 1, 2, 3$. Let $\mathfrak{V}_i = C(\mathfrak{X}_i) \cap \mathfrak{V} = \langle V_i \rangle$. Thus, $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ are the only subgroups of $\mathfrak{X}$ of order 3 which admit $\mathfrak{V}$. Let Z be a generator for $\mathfrak{Z}$. Then $Z = X_1X_2X_3$ with $X_i \in \mathfrak{X}_i$.

We argue that $\mathfrak{Z} \not\sim \mathfrak{X}_i$ for $i = 1, 2, 3$. Namely, if $\mathfrak{Z} \sim \mathfrak{X}_i$, there is $N \in N(\mathfrak{X})$ such that $\mathfrak{X}_i = \mathfrak{Z}^N$. Let $\mathfrak{A} = A_{\mathfrak{G}}(\mathfrak{X})$. Thus, $|\mathfrak{A}|_3 = 3$, $\mathfrak{A}$ is solvable, and $\mathfrak{A} \supseteqq \mathfrak{B} = A_{\mathfrak{L}}(\mathfrak{X}) \cong \Sigma_4$. So $\mathfrak{A} = \mathfrak{B}$ or $\mathfrak{A} = \mathfrak{B} \times \mathfrak{A}_0$, where $\mathfrak{A}_0 = \langle A \rangle$ and A inverts $\mathfrak{X}$. In neither case are $\mathfrak{Z}$ and $\mathfrak{X}_i$ in the same orbit under $\mathfrak{A}$.

We now return to our study of $\mathfrak{S}$. Let $\mathfrak{S}_2 = \mathfrak{Q} \times \langle V \rangle$, where $\mathfrak{Q}$ is a quaternion group and $V \in \mathfrak{V}$. Let $\langle V_0 \rangle = \mathfrak{Q}'$. Since V_0 does not centralize $Z(O_3(\mathfrak{S}))$, there is a minimal normal subgroup $\mathfrak{Y}$ of $\mathfrak{S}$ such that $\mathfrak{Q}$ is represented faithfully on $\mathfrak{Y}$. Let $\mathfrak{V}^* = C_{\mathfrak{V}}(\mathfrak{Y})$ so that $|\mathfrak{V}^*| = 2$, $\mathfrak{S}_2 = \mathfrak{Q} \times \mathfrak{V}^*$. We see that $|\mathfrak{Y}| = 9$ and that $\mathfrak{Y}_0 = \mathfrak{Y} \cap Z(\mathfrak{S}_3)$ is of order 3 and admits $\mathfrak{V}$. Thus, $\mathfrak{Y}_0 \subset \mathfrak{X}$, so $\mathfrak{Y}_0 = \mathfrak{X}_i$ for some $i = 1, 2, 3$. Since $\mathfrak{X}_i \not\sim \mathfrak{Z}$, it follows that $\mathfrak{S}_3$ is a S_3-subgroup of $C(\mathfrak{Y}_0)$. Since $[\mathfrak{Y}, \mathfrak{S}_3] \subseteqq \mathfrak{Y}_0$, it follows that $\mathfrak{Y} \subseteqq O_3(C(\mathfrak{Y}_0))$. Since $\mathfrak{Q}$ permutes transitively the subgroups of $\mathfrak{Y}$ of order 3, it follows that $\mathfrak{Y} \subseteqq O_3(C(\mathfrak{Y}^*))$ for every subgroup $\mathfrak{Y}^*$ of $\mathfrak{Y}$ of order 3. This implies that $\mathfrak{Y} \in \mathscr{E}(3)$. Now $C(\mathfrak{V}^*)$ contains $\mathfrak{Y}$ and also contains an element of $\mathscr{U}(2)$, so $C(\mathfrak{V}^*)$ is nonsolvable. This contradiction shows that this case does not arise.

Case 2. $O_3(\mathfrak{S})\mathfrak{V}/O_3(\mathfrak{S}) \not\subseteqq Z(\mathfrak{S}/O_3(\mathfrak{S}))$.

We conclude that $\mathfrak{S}_2$ is special and that $\mathfrak{V} = \mathfrak{S}_2'$. Since $\mathfrak{V} = Z(\mathfrak{S}_2)$, we get that $\mathfrak{S}_3\mathfrak{V}$ is a maximal subgroup of $\mathfrak{S}$. That is, $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{V}$ is a chief factor of $\mathfrak{S}$.

Let $\mathfrak{P}_0$ be a maximal element of $\mathscr{N}_{\mathfrak{S}}(\mathfrak{V}; 3)$ with $\mathfrak{P}_0 \subset \mathfrak{S}_3$. Hence, $\mathfrak{P}_0$ is of index 3 in $\mathfrak{S}_3$, and all involutions of $\mathfrak{V}$ are fused in $\mathfrak{S}_3\mathfrak{V}$. Also, $[\mathfrak{V}_0, \mathfrak{P}_0] = [O_3(\mathfrak{S}), \mathfrak{V}_0]$ for every subgroup $\mathfrak{V}_0$ of $\mathfrak{V}$.

Suppose $\mathfrak{P}_0$ is not a maximal element of $\mathscr{N}_{\mathfrak{L}}(\mathfrak{V}; 3)$. Choose $\mathfrak{P}_1$ in $\mathscr{N}_{\mathfrak{L}}(\mathfrak{V}; 3)$ so that $|\mathfrak{P}_1 : \mathfrak{P}_0| = 3$, and let $\mathfrak{V}_0 = C_{\mathfrak{V}}(\mathfrak{P}_1/\mathfrak{P}_0)$. Then $\mathfrak{P}_1$ and $\mathfrak{S}_2$ both normalize $[\mathfrak{V}_0, \mathfrak{P}_1]$. Let $\mathfrak{L}^*$ be a $S_{2,3}$-subgroup of $N([\mathfrak{V}_0, \mathfrak{P}_1])$ which contains $\mathfrak{V}\mathfrak{P}_1$, and let $\mathfrak{L}_p^*$ be a S_p-subgroup of $\mathfrak{L}^*$ with $\mathfrak{P}_1 \subseteqq \mathfrak{L}_3^*$, $\mathfrak{V} \subseteqq \mathfrak{L}_2^*$. Note that $\mathfrak{L}^*$ contains a conjugate of $\mathfrak{S}_2$.

By maximality, $\mathfrak{S}_3 = N_{\mathfrak{P}}(\Omega_1(Z(O_3(\mathfrak{S}))))$. Hence, $\mathfrak{Z} \subset \mathfrak{S}_3$, and so $\mathfrak{Z} \subseteqq \Omega_1(Z(O_3(\mathfrak{S})))$, since $O_2(\mathfrak{S}) = 1$. If $\mathfrak{U} \in \mathscr{U}(\mathfrak{P})$, then $[\mathfrak{P}, \mathfrak{U}] \subseteqq \mathfrak{Z}$, and so

$\mathfrak{U} \subseteq \mathfrak{S}_3$. Since $\mathfrak{V}$ is a 4-group and does not centralize $\Omega_1(\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{S})))$, we conclude from $[\mathfrak{P}, \mathfrak{U}, \mathfrak{U}] = 1$ that $\mathfrak{U}$ centralizes $\mathfrak{V}\boldsymbol{O}_3(\mathfrak{S})/\boldsymbol{O}_3(\mathfrak{S})$. That is, $\mathfrak{U} \subseteq \mathfrak{P}_0 \subseteq \mathfrak{L}^*$. Hence, $\mathfrak{L}^*$ contains an element of $\mathscr{S}$ so maximality of $\mathfrak{S}$ guarantees that $\mathfrak{P}_1 = \mathfrak{L}_3^*$, since $\mathfrak{P}_1$ and $\mathfrak{S}_3$ are of the same order.

Let $\mathfrak{L}_2^{**} = \mathfrak{L}_2^* \cap \boldsymbol{O}_{3,2}(\mathfrak{L}^*)$. Thus, $\mathfrak{L}_2^{**}$ contains a noncyclic abelian subgroup of order 8, since $\mathfrak{L}^*$ contains a conjugate of $\mathfrak{S}_2$.

Suppose every subgroup of $\mathfrak{L}_2^{**}$ which is characteristic and abelian is also cyclic. Let $\mathfrak{L}_2^{***}$ be a subgroup of $\mathfrak{L}_2^{**}$ which is minimal subject to (a) containing a noncyclic abelian subgroup of order 8 and (b) being permutable with $\mathfrak{L}_3^*$. Then since $\boldsymbol{D}(\mathfrak{L}_2^{***}) \subseteq \boldsymbol{D}(\mathfrak{L}_2^{**})$, it follows that $\mathfrak{L}_2^{***}$ is not a special group with center of order 4. Since $\mathfrak{L}_3^*\mathfrak{L}_2^{***} \in \mathscr{S}$, our previous reduction excludes this possibility.

Let $\tilde{\mathfrak{L}}_2$ be a noncyclic characteristic abelian subgroup of $\mathfrak{L}_2^{**}$. If $|\tilde{\mathfrak{L}}_2| > 4$, then $\mathfrak{P}_1\tilde{\mathfrak{L}}_2$ contains an element of $\mathscr{S}$, against our previous reduction. We may assume that $\tilde{\mathfrak{L}}_2$ is a four-group. If $\tilde{\mathfrak{L}}_2 \cap \mathfrak{V} \neq 1$, then $\boldsymbol{O}_3(\mathfrak{L}^*)\tilde{\mathfrak{L}}_2/\boldsymbol{O}_3(\mathfrak{L}^*)$ is centralized by $\mathfrak{P}_1$, since $\mathfrak{V}$ normalizes $\mathfrak{P}_1$. But in this case, there are maximal elements of $\mathscr{S}$ which do not satisfy our previous reduction. If $\tilde{\mathfrak{L}}_2 \cap \mathfrak{V} = 1$, then $\mathfrak{P}_1\mathfrak{V}\tilde{\mathfrak{L}}_2$ contains an element of $\mathfrak{S}$ which also violates our previous reduction. Hence, $\mathfrak{P}_0$ is a maximal element of $\boldsymbol{\mathcal{U}}_{\mathfrak{L}}(\mathfrak{V}; 3)$.

Since $\boldsymbol{O}_3(\mathfrak{L}) \in \boldsymbol{N}_{\mathfrak{L}}(\mathfrak{V}; 3)$, we have $\boldsymbol{O}_3(\mathfrak{L}) \subseteq \mathfrak{P}_0$. Maximality of $\mathfrak{S}$ in $\mathscr{S}$ implies that $\mathfrak{L}$ contains no noncyclic abelian subgroup of order 8. Since the involutions of $\mathfrak{V}$ are fused in $\mathfrak{S}_3\mathfrak{V}$, it follows that $\boldsymbol{O}_{3,2}(\mathfrak{L}) = \boldsymbol{O}_3(\mathfrak{L})\mathfrak{V}$. Hence, $\mathfrak{P}_0 = \boldsymbol{O}_3(\mathfrak{L})$ is of index 3 in $\mathfrak{L}_3$. This violates $|\mathfrak{L}_3| > |\mathfrak{S}_3| = 3\,|\mathfrak{P}_0|$.

Thus, in all cases, we have shown that $\mathfrak{S}_3 = \mathfrak{P}$.

Suppose that $\mathfrak{V}$ normalizes $\mathfrak{S}_3$. Let $\mathfrak{W}$ be a minimal normal subgroup of $\mathfrak{S}$. Clearly, $\mathfrak{W} \supset \mathfrak{Z}$. Since $\mathfrak{V}\boldsymbol{O}_3(\mathfrak{S})/\boldsymbol{O}_3(\mathfrak{S})$ is a central factor of $\mathfrak{S}$, some involution of $\mathfrak{V}$ centralizes $\mathfrak{W}$. But $\mathfrak{W}$ contains an element of $\mathscr{U}(\mathfrak{P})$, so Lemmas 7.4 and 5.38 imply that $\boldsymbol{C}(V)$ is nonsolvable for some involution V of $\mathfrak{V}$. Thus, $\mathfrak{V}$ does not normalize $\mathfrak{P}$. In particular, $\mathfrak{S}_2$ is special.

Let $\mathfrak{P}_0$ be the largest subgroup of $\mathfrak{P}$ normalized by $\mathfrak{V}$. Thus, $|\mathfrak{P}: \mathfrak{P}_0| = 3$ and $\boldsymbol{N}_{\mathfrak{P}}(\mathfrak{S}_2)$ permutes transitively the involutions of $\mathfrak{V}$.

Let $\mathfrak{W}$ be a minimal normal subgroup of $\mathfrak{S}$. Clearly, $\mathfrak{W} \supset \mathfrak{Z}$, so $\mathfrak{V}$ is faithfully represented on $\mathfrak{W}$. Hence, $\mathfrak{S}_2$ is faithfully represented on $\mathfrak{W}$, so $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{W}) = \boldsymbol{O}_3(\mathfrak{W})$. Let $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2 \times \mathfrak{W}_3$, where $\mathfrak{W}_i = \boldsymbol{C}_{\mathfrak{W}}(\mathfrak{V}_i)$ and V_1, V_2, V_3 are the involutions of $\mathfrak{V}$. Thus, $\mathfrak{S}_2\mathfrak{P}_0$ normalizes each $\mathfrak{W}_i$, and $\mathfrak{S}$ permutes $\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3$ transitively. Obviously each $\mathfrak{W}_i$ is an irreducible $\mathfrak{P}_0\mathfrak{S}_2$-module.

Let $\mathfrak{R}_i = \boldsymbol{C}_{\mathfrak{P}_0\mathfrak{S}_2}(\mathfrak{W}_i)$, and $\mathfrak{S}_{2i} = \{S \in \mathfrak{S}_2 \mid [S, \mathfrak{S}_2] \subseteq \langle V_i \rangle\}$, for $i = 1, 2, 3$. Then $\boldsymbol{N}_{\mathfrak{S}}(\mathfrak{S}_2)$ permutes $\{\mathfrak{S}_{21}, \mathfrak{S}_{22}, \mathfrak{S}_{23}\}$ transitively, and so one of the following holds:

(a) $\mathfrak{S}_{2i} = \mathfrak{V}$, $i = 1, 2, 3$,

(b) $\mathfrak{S}_{2i} \supset \mathfrak{V}$, $i = 1, 2, 3$.

If (a) holds, then $\mathfrak{S}_2/\langle V_i\rangle$ is extra special, and since V_j inverts $\mathfrak{W}_i$ for $j \neq i$, it follows that $\mathfrak{K}_i \cap \mathfrak{S}_2 = \langle V_i\rangle$, whence $\mathfrak{K}_i = \boldsymbol{O}_3(\mathfrak{S})\langle V_i\rangle$. The proof of Lemma 5.52 now shows that $\boldsymbol{O}_3(\mathfrak{S}) \supseteq \boldsymbol{J}(\mathfrak{S}_3)$, the desired contradiction.

Suppose (b) holds. The group $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}\mathfrak{S}_{23}$ is clearly $\mathfrak{S}_3$-invariant, and since $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{S}_2/\boldsymbol{O}_3(\mathfrak{S})\mathfrak{V}$ is a chief factor of $\mathfrak{S}$, we have $\mathfrak{S}_2 = \mathfrak{S}_{21}\mathfrak{S}_{22}\mathfrak{S}_{23}$. Obviously, $[\mathfrak{S}_{2i}, \mathfrak{S}_{2j}] \subseteq \langle V_i\rangle \cap \langle V_j\rangle = 1$ for $i \neq j$. Because $\mathfrak{S}_2$ is special, we conclude that $\mathfrak{S}_{2i'} = \langle V_i\rangle$ and hence that $\mathfrak{S}_{2j}/\langle V_i\rangle$ is extra special for $i \neq j$. If the width of $\mathfrak{S}_{2j}/\langle V_i\rangle$ is greater than 1, then since $\boldsymbol{J}(\mathfrak{S}_3) \not\subseteq \boldsymbol{O}_3(\mathfrak{S})$, and since $\mathfrak{S}_{2j}/\langle V_i\rangle$ acts faithfully on $\mathfrak{W}_i$, it follows that $\boldsymbol{J}(\mathfrak{S}_3)$ centralizes $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{S}_{2j}/\boldsymbol{O}_3(\mathfrak{S})\langle V_i\rangle$. But $\boldsymbol{J}(\mathfrak{S}_3) \lhd \mathfrak{S}_3$, and so $\boldsymbol{J}(\mathfrak{S}_3)$ centralizes $\boldsymbol{O}_3(S)\mathfrak{S}_2/\boldsymbol{O}_3(\mathfrak{S})\mathfrak{V}$, that is, $\boldsymbol{J}(\mathfrak{S}_3) \subseteq \boldsymbol{O}_3(\mathfrak{S})$. We may assume that if i $\neq j$, then $\mathfrak{S}_{2i}/\langle V_i\rangle$ is of width 1. But then $\mathfrak{S}_{21}\mathfrak{S}_{22}/\langle V_3\rangle$ is the central product of $\mathfrak{S}_{21}/\langle V_3\rangle$ and $\mathfrak{S}_{22}/\langle V_3\rangle$, so is extra special of width 2, acts faithfully on $\mathfrak{W}_3$, and $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}$ admits $\boldsymbol{J}(\mathfrak{S}_3)$. By Lemma 5.52, $\boldsymbol{J}(\mathfrak{S}_3)$ centralizes $\boldsymbol{O}_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}/\boldsymbol{O}_3(\mathfrak{S})$, so again we get the contradiction $\boldsymbol{J}(\mathfrak{S}_3) \subseteq \boldsymbol{O}_3(\mathfrak{S})$. The proof is complete.

LEMMA 9.16. *If* $\mathfrak{A}$ *is a subgroup of* $\mathfrak{G}$ *of type* (3, 3) *and each element of* $\mathfrak{A}$ *centralizes an element of* $\mathscr{U}(3)$, *then*

(i) $\mathfrak{A} \in \mathscr{D}$.

(ii) $4 \nmid |\boldsymbol{C}(\mathfrak{A})|$.

Proof. (i) Suppose false, and $\mathfrak{T}$ is a four-group normalized but not centralized by $\mathfrak{A}$. Let $\mathfrak{A}_0 = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{T})$, so that $|\mathfrak{A}_0| = 3$. Let $\mathfrak{L}_0$ be a maximal 2, 3-subgroup of $\mathfrak{G}$ which contains a $S_{2,3}$-subgroup of $\boldsymbol{C}(\mathfrak{A}_0)$ containing $\mathfrak{A}\mathfrak{T}$. Let $\mathfrak{L} = \boldsymbol{O}^{3'}(\mathfrak{L}_0)$. Since $\mathfrak{L}$ contains an element of $\mathscr{E}(3)$, Lemma 9.15 implies that $\mathfrak{L}$ contains no noncyclic abelian subgroup of order 8. Hence, $\mathfrak{T}$ is a S_2-subgroup of $\mathfrak{L}$ and $\mathfrak{L}/\boldsymbol{O}_3(\mathfrak{L}) \cong A_4$. Let $\mathfrak{L}_3$ be a S_3-subgroup of $\mathfrak{L}$. Since $\mathfrak{T}$ does not centralize $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L}))$, it follows that $\boldsymbol{J}(\mathfrak{L}_3) \lhd \mathfrak{L}$. Hence, $\mathfrak{L}_3$ is a S_3-subgroup of $\mathfrak{G}$, and we may assume that $\mathfrak{L}_3 = \mathfrak{P}$.

Let $\mathfrak{X}$ be a minimal normal subgroup of $\mathfrak{L}$ with $\mathfrak{X} \subseteq \boldsymbol{Z}(\boldsymbol{J}(\mathfrak{P}))$. Thus, $\mathfrak{X}$ is elementary of order 27 and $\boldsymbol{C}_{\mathfrak{X}}(\mathfrak{T}) = 1$. Choose T in $\mathfrak{T}^\#$ so that $\mathfrak{X}_1 = \boldsymbol{C}_{\mathfrak{X}}(T)$ is of order 3 and is inverted by the generator of $\mathfrak{T}/\langle T\rangle$. Hence, $\langle\mathfrak{A}_0, \mathfrak{X}_1\rangle = \mathfrak{A}^*$ is elementary of order 9 and is normalized by $\mathfrak{T}$, and every element of $\mathfrak{A}^*$ centralizes an element of $\mathscr{U}(\mathfrak{P})$.

Let $\mathfrak{C}$ be a $S_{2,3}$-subgroup of $\boldsymbol{C}(T)$ which contains $\mathfrak{A}^*\mathfrak{T}$. By Lemma 5.38, $\mathfrak{C}$ contains an element of $\mathscr{U}(2)$, so $|\boldsymbol{O}_3(\mathfrak{C})| \leqq 3$, and $\boldsymbol{O}_3(\mathfrak{C}) \cap \mathfrak{A}^* = 1$. Hence, $\mathfrak{A}^*$ is faithfully represented on $\boldsymbol{O}_2(\mathfrak{C})$. Let $\mathfrak{C}_0$ be a characteristic abelian subgroup of $\boldsymbol{O}_2(\mathfrak{C})$. Suppose $\mathfrak{A}^*$ does not centralize $\mathfrak{C}_0$. Hence, there is an element A in $\mathfrak{A}^{*\#}$ such that $\boldsymbol{C}(A)$ contains an elementary subgroup of order 8. This is not the case, so $\mathfrak{A}^*$ centralizes $\mathfrak{C}_0$. If $|\mathfrak{C}_0| > 2$, then some element A of $\mathfrak{A}^{*\#}$ centralizes a noncyclic abelian subgroup of $\boldsymbol{O}_2(\mathfrak{C})$ of order 8. This is not the case, by Lemma 9.15. Hence, $\boldsymbol{O}_2(\mathfrak{C})$ is extra special of width at least 2 and $\langle T\rangle = \boldsymbol{O}_2(\mathfrak{C})'$. Hence, $\mathfrak{C}$ contains a S_2-subgroup of $\mathfrak{G}$.

By Lemma 9.15, no element of $\mathfrak{A}^{*\#}$ centralizes any noncyclic abelian subgroup of order 8. Hence, $\langle T\rangle = \boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{A}^*)$. For each A in $\mathfrak{A}^{*\#}$, $\boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(A)$ is either $\langle T\rangle$ or is extra special. Thus, $\boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(A)$ is either $\langle T\rangle$ or is a quaternion group, so no element of $\mathfrak{A}^{*\#}$ centralizes any four-subgroup of $\boldsymbol{O}_2(\mathfrak{C})$. Thus, the width of $\boldsymbol{O}_2(\mathfrak{C})$ is at most 4. Since $\boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{A}_0)$ is centralized by $\mathfrak{A}^*$, it follows that $\boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{A}_0) = \langle T\rangle$, and so the width of $\boldsymbol{O}_2(\mathfrak{C})$ is at most 3.

Consider $\boldsymbol{C}^*(\mathfrak{X}_1) = \{G \in \mathfrak{G},\ G$ either centralizes or inverts $\mathfrak{X}_1\}$. By construction, $|\boldsymbol{C}_{\mathfrak{L}}(\mathfrak{X}_1)|_3 = |\mathfrak{P}|/3$. Also, $\mathfrak{T} \subseteq \boldsymbol{C}^*(\mathfrak{X}_1)$, and $\boldsymbol{C}^*(\mathfrak{X}_1)$ contains no noncyclic abelian subgroup of order 8. Suppose $\mathfrak{A}_0 \not\subseteq \boldsymbol{O}_3(\tilde{\mathfrak{L}})$, where $\tilde{\mathfrak{L}}$ is a $S_{2,3}$-subgroup of $\boldsymbol{C}^*(\mathfrak{X}_1)$ which contains $\boldsymbol{C}^*_{\mathfrak{L}}(\mathfrak{X}_1)$. Then $\tilde{\mathfrak{L}}/\boldsymbol{O}_3(\tilde{\mathfrak{L}})$ contains a subgroup isomorphic to $\mathfrak{A}_0 \times \mathfrak{T}$. This is obviously impossible, since S_2-subgroups of $\tilde{\mathfrak{L}}/\boldsymbol{O}_3(\tilde{\mathfrak{L}})$ are of maximal class. Hence, $\mathfrak{A}_0 \subseteq \boldsymbol{O}_3(\tilde{\mathfrak{L}})$. This implies that $\mathfrak{A}_0$ centralizes $\boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{X}_1)$, so the width of $\boldsymbol{O}_2(\mathfrak{C})$ is 2. Hence, $\mathfrak{X}_1 \times \mathfrak{A}_0$ is a S_3-subgroup of $\boldsymbol{C}(T)$, so $\mathfrak{X}_1 \times \mathfrak{A}_0$ is a S_3-subgroup of $\boldsymbol{C}_{\mathfrak{L}}(T)$. By a formula of Wielandt [40],

$$|\boldsymbol{O}_3(\mathfrak{L})| = |\boldsymbol{O}_3(\mathfrak{L}) \cap \boldsymbol{C}(T)|^3/|\boldsymbol{O}_3(\mathfrak{L}) \cap \boldsymbol{C}(\mathfrak{T})|^2 .$$

Hence, $|\boldsymbol{O}_3(\mathfrak{L})| = 3^6/3^2 = 3^4$, so that $\boldsymbol{O}_3(L) = \mathfrak{A}_0 \times \mathfrak{X}$. This implies that $\boldsymbol{Z}(\mathfrak{P})$ is noncyclic, since $|\mathfrak{P}: \boldsymbol{O}_3(\mathfrak{L})| = 3$. The proof of (i) is complete.

As for (ii), suppose $\mathfrak{T}$ is a subgroup of $\boldsymbol{C}(\mathfrak{A})$ of order 4. Then $\boldsymbol{C}(T)$ contains an element of $\mathscr{U}(2)$, T being an involution of $\mathfrak{T}$. Thus, by Lemma 7.5, there is a subgroup $\mathfrak{M}$ in $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{A}\mathfrak{T}$ and satisfies $\boldsymbol{O}_{2'}(\mathfrak{M}) = 1$, while $\boldsymbol{O}_2(\mathfrak{M})$ is of symplectic type. Since $\mathfrak{A}$ acts faithfully on $\boldsymbol{O}_2(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{T})$, we can therefore choose A in $\mathfrak{A}^\#$ such that $\mathfrak{A}$ does not centralize $\boldsymbol{O}_2(\mathfrak{C}) \cap \boldsymbol{C}(\mathfrak{T}) \cap \boldsymbol{C}(\mathfrak{A})$. Thus, $\boldsymbol{C}(A)$ contains a noncyclic abelian subgroup of order 8, against Lemma 9.15. The proof of (ii) is complete.

LEMMA 9.17. *Suppose*

(a) $\mathfrak{R}$ *is a maximal* 2, 3-*subgroup of* $\mathfrak{G}$.

(b) $\mathfrak{R}$ *contains an element of* $\mathscr{D}$.

(c) $\mathfrak{R}$ *contains a noncyclic abelian subgroup of order* 8.

Then $\boldsymbol{O}_2(\mathfrak{R}) \neq 1$.

Proof. Let $\mathfrak{R}_p$ be a S_p-subgroup of $\mathfrak{R}$, $p = 2, 3$. We assume without loss of generality that $\mathfrak{R}_3 \subseteq \mathfrak{P}$. Suppose by way of contradiction that $\boldsymbol{O}_2(\mathfrak{R}) = 1$. Then $\boldsymbol{O}_3(\mathfrak{R}) \neq 1$, so by maximality of $\mathfrak{R}$, $\mathfrak{R}_3 = \boldsymbol{N}_{\mathfrak{P}}(\boldsymbol{O}_3(\mathfrak{R}))$. Hence, $\mathfrak{Z} \subseteq \mathfrak{R}_3$. Since $\boldsymbol{O}_2(\mathfrak{R}) = 1$, we get $\mathfrak{Z} \subseteq \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{R}))$. Hence, $\mathfrak{R}_3$ contains every element of $\mathscr{U}(\mathfrak{P})$. This contradicts Lemma 9.15 and completes the proof.

We now begin the construction of the final configuration.

By hypothesis, $2 \sim 3$. Let $\mathfrak{A}$ be a noncyclic abelian subgroup of $\mathfrak{G}$ of order 8 and let $\mathfrak{B}$ be an elementary subgroup of order 9 each of whose elements centralizes an element of $\mathscr{U}(3)$, chosen so that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is solvable. We assume without loss of generality that $\langle \mathfrak{A}, \mathfrak{B} \rangle$ is a 2, 3-group. Let $\mathfrak{L}$ be a maximal 2, 3-subgroup of $\mathfrak{G}$ which contains $\langle \mathfrak{A}, \mathfrak{B} \rangle$.

Let $\mathfrak{L}_p$ be a S_p-subgroup of $\mathfrak{L}$, $p = 2, 3$, with $\mathfrak{B} \subseteq \mathfrak{L}_3$. By Lemma 9.17, $\boldsymbol{O}_2(\mathfrak{L}) \neq 1$.

Let J be an involution in $\boldsymbol{Z}(\mathfrak{L}_2) \cap \boldsymbol{O}_2(\mathfrak{L})$. Since $\mathfrak{B} \in \mathscr{D}$, by Lemma 9.16, $\mathfrak{B}$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{L}))$. Hence, $\boldsymbol{C}(J)$ is a solvable subgroup containing $\mathfrak{B}$, $\mathfrak{L}_2$, and an element of $\mathscr{U}(2)$.

Since $\mathfrak{B} \in \mathscr{D}$, we may apply Lemma 7.5. Let $\mathfrak{M}$ be an element of $\mathscr{MS}(\mathfrak{G})$ which contains $\mathfrak{B}$ and $\mathfrak{L}_2$ and which satisfies all the conclusions of Lemma 7.5. Let $\mathfrak{K}$ be a $S_{2,3}$-subgroup of $\mathfrak{M}$ and let $\mathfrak{K}_0 = \boldsymbol{O}_2(\mathfrak{K})$. Since $\boldsymbol{O}_{2'}(\mathfrak{M}) = 1$, so also $\boldsymbol{O}_3(\mathfrak{K}) = 1$. Since no element of $\mathfrak{B}^\#$ centralizes any noncyclic abelian subgroup of order 8, it follows that $\mathfrak{K}_0$ is extra special of width 2, 3 or 4, and $\boldsymbol{C}_{\mathfrak{K}_0}(\mathfrak{B}) = \mathfrak{K}_0' = \langle I \rangle$, the last equality serving to define I. Hence, $\mathfrak{K}_0 = \boldsymbol{O}_2(\mathfrak{M})$. Let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$, $p = 2, 3$, with $\mathfrak{B} \subseteq \mathfrak{K}_3$. Let $\mathfrak{K}_3^* = \mathfrak{K}_3 \cap \boldsymbol{O}_{2,3}(\mathfrak{K})$, $\mathfrak{K}^* = \boldsymbol{N}_{\mathfrak{K}}(\mathfrak{K}_3^*)$. Thus, $\mathfrak{K} = \mathfrak{K}_0\mathfrak{K}^*$ and $\mathfrak{K}_0 \cap \mathfrak{K}^* = \boldsymbol{C}_{\mathfrak{K}_0}(\mathfrak{K}_3^*)$. Let $\mathfrak{K}_2^* = \mathfrak{K}^* \cap \mathfrak{K}_2$ so that $\mathfrak{K}^* = \mathfrak{K}_3\mathfrak{K}_2^*$. We assume without loss of generality that $\mathfrak{K}_3 \subseteq \mathfrak{P}$.

We argue that

$$(9.25) \qquad \mathfrak{K}_0 \cap \mathfrak{K}^* = \langle I \rangle .$$

Namely, choose $\mathfrak{U}$ in $\mathscr{U}(\mathfrak{P})$ and suppose $\boldsymbol{C}(\mathfrak{U}) \cap \mathfrak{K}_3^*$ is noncyclic. Then by Lemma 9.16 (ii), no noncyclic abelian subgroup of $\boldsymbol{C}(\mathfrak{U}) \cap \mathfrak{K}_3^*$ centralizes any subgroup of order 4, so (9.25) is clear. Suppose $\boldsymbol{C}(\mathfrak{U}) \cap \mathfrak{K}_3^*$ is cyclic. Hence, $\mathfrak{K}_3^*$ has a cyclic subgroup of index 3. Assume that (9.25) does not hold. Then $\mathfrak{B} \not\subseteq \mathfrak{K}_3^*$, so the 3-length of $\mathfrak{K}$ is at least 2. Hence, $\mathfrak{K}_3^*$ is elementary of order 9 and all elements of $\mathfrak{K}_3^{*\#}$ are fused in $\mathfrak{K}$. But then every element of $\mathfrak{K}_3^*$ centralizes an element of $\mathscr{U}(3)$, so again (9.25) holds. Thus, (9.25) holds.

LEMMA 9.18. *If $\mathfrak{N}$ is any 2, 3-subgroup of $\mathfrak{G}$ which contains $\mathfrak{B}$, I, and also contains a noncyclic abelian subgroup of order 8, then $\mathfrak{N} \subseteq \mathfrak{M}$.*

Proof. We may assume that $\mathfrak{N}$ is a maximal 2, 3-subgroup of $\mathfrak{G}$. By Lemma 9.17, we have $\boldsymbol{O}_2(\mathfrak{N}) \neq 1$. Since $\mathfrak{B} \in \mathscr{D}$, $\mathfrak{B}$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{N}))$. Since $\langle I \rangle$ is a S_2-subgroup of $\boldsymbol{C}(\mathfrak{B})$, by Lemma 9.16 (ii), it follows that $\langle I \rangle = \boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{N}))$, so $\mathfrak{N} \subseteq \boldsymbol{C}(I) = \mathfrak{M}$.

LEMMA 9.19. *$\mathfrak{K}_2^*$ contains no noncyclic abelian subgroup of order 8.*

Proof. Suppose false. In this case, $\mathfrak{K}^*$ is a $S_{2,3}$-subgroup of $N(\mathfrak{K}_3^*)$, by the preceding paragraph. Hence, the 3-length of $\mathfrak{K}^*$ is at least 2. But in this case, $\mathfrak{Z} \subset \mathfrak{K}_3^*$, so $\mathfrak{K}_3$ contains every element of $\mathscr{U}(\mathfrak{P})$, against Lemma 7.4. The proof is complete.

LEMMA 9.20. *If* $\tilde{\mathfrak{K}}_3$ *is a* S_3*-subgroup of* $N(\mathfrak{K}_3)$, *then*

$$\tilde{\mathfrak{K}}_3 = \mathfrak{K}_3 \cdot C_{\tilde{\mathfrak{K}}_3}(\mathfrak{K}_3) .$$

Proof. Let $\mathfrak{C}_1 = C(\mathfrak{K}_3)\mathfrak{K}_3$, $\mathfrak{N}_1 = N(\mathfrak{K}_3)$. Since $\mathfrak{B} \subseteqq \mathfrak{K}_3$, it follows that $\langle I \rangle$ is a S_2-subgroup of $\mathfrak{C}_1$, by Lemma 9.16 (ii). Hence, $\mathfrak{M}$ covers $\mathfrak{N}_1/\mathfrak{C}_1$, so $\mathfrak{C}_1$ contains $\mathfrak{K}_3$, which is equivalent to our assertion.

LEMMA 9.21.

(a) *If* $\tilde{\mathfrak{P}}$ *is any* 3-subgroup of $\mathfrak{M}$, *then no* S_3*-subgroup of* $N(\tilde{\mathfrak{P}})$ *is contained in any conjugate of* $\mathfrak{M}$.

(b) *If* P *is an element of* $\mathfrak{M}$ *of order* 3, *then* $C(P)$ *contains a subgroup* $\mathfrak{A}^*$ *of type* (3, 3) *such that* $C(A)$ *contains an element of* $\mathscr{U}(3)$ *for each* A *in* $\mathfrak{A}^*$.

(c) *If* $\tilde{\mathfrak{P}}$ *is any nonidentity* 3*-subgroup of* $\mathfrak{M}$, *then* $N(\tilde{\mathfrak{P}})$ *contains no noncyclic abelian group of order* 8.

(d) $\mathfrak{K}_3$ *contains no abelian subgroup of order* 27.

(e) $\mathfrak{K}_3$ *is isomorphic to one of the following groups*:

(i) *an elementary group of order* 9.

(ii) *a nonabelian group of order* 27.

Proof. Let $\mathfrak{P}^*$ be a S_3-subgroup of $N(\tilde{\mathfrak{P}})$. Suppose $\mathfrak{P}^* \subseteqq \mathfrak{M}^G$. Since $\tilde{\mathfrak{P}} \subseteqq \mathfrak{P}^*$, we get $\tilde{\mathfrak{P}} \subseteqq \mathfrak{M}^G$. Let $\mathfrak{P}_0 = \tilde{\mathfrak{P}}^{G^{-1}}$, $\mathfrak{P}_1 = \mathfrak{P}^{*G^{-1}}$. Then $\mathfrak{P}_0$ is a 3-subgroup of $\mathfrak{M}$ and $\mathfrak{P}_1$ is a S_3-subgroup of $N(\mathfrak{P}_0)$ which is contained in $\mathfrak{M}$. This violates Lemma 9.20, since $\mathfrak{K}_3 \subset \tilde{\mathfrak{K}}_3$. Hence, (a) holds.

Since $\mathfrak{B} \subseteqq \mathfrak{K}_3$, it follows from Lemma 9.16 (ii) that $\langle I \rangle$ is a S_2-subgroup of $\mathfrak{K}_3 C(\mathfrak{K}_3)$. Thus, $\mathfrak{K}_3 C(\mathfrak{K}_3)$ has a normal 2-complement. We assume without loss of generality that I normalizes $\tilde{\mathfrak{K}}_3$. Let $\hat{\mathfrak{K}}_3/\mathfrak{K}_3$ be a chief factor of $\tilde{\mathfrak{K}}_3\langle I \rangle$. Hence, $\hat{\mathfrak{K}}_3 = \mathfrak{K}_3 \times \bar{\mathfrak{K}}_3$, where $|\bar{\mathfrak{K}}_3| = 3$. This implies that $C_{\hat{\mathfrak{K}}_3}(P)$ contains an elementary subgroup of order 27. Let $\mathfrak{P}^G$ be a S_3-subgroup of $\mathfrak{G}$ containing $\tilde{\mathfrak{K}}_3$, and let $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$. Then $C(\mathfrak{U}) \cap C_{\tilde{\mathfrak{K}}_3}(P)$ is noncyclic, and any noncyclic subgroup of $C(\mathfrak{U}) \cap C_{\tilde{\mathfrak{K}}_3}(P)$ of order 9 may play the role of $\mathfrak{A}^*$ in (b).

Let $\mathfrak{R}$ be a maximal 2, 3-subgroup of $\mathfrak{G}$ which contains a $S_{2,3}$-subgroup of $N(\bar{\mathfrak{P}})$. By (b), $\mathfrak{R}$ contains an element of $\mathscr{D}$. Assume that $\mathfrak{R}$ contains a noncyclic abelian subgroup of order 8. Then by Lemma 9.17, $O_2(\mathfrak{R}) \neq 1$. By Lemma 9.16 (ii), we get $|Z(O_2(\mathfrak{R}))| = 2$. Clearly, $\mathfrak{R}$ is a $S_{2,3}$-subgroup of $C(Z(O_2(\mathfrak{R})))$, and so it contains an element $\mathfrak{D}$ of $\mathscr{D}$ and one of $\mathscr{U}(2)$. Applying Lemma 7.5, we get a conjugate $\mathfrak{M}^G$ of $\mathfrak{M}$ containing $\mathfrak{D}$ and a S_2-subgroup of $\mathfrak{R}$. By Lemma 7.5 (f), $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{R})) \subseteq \boldsymbol{C}(\mathfrak{D}) \cap \boldsymbol{O}_2(\mathfrak{M}^G)$. By Lemma 9.16 (ii), the last group is of order 2, and so equals $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{R}))$. Hence, $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{R})) = \boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{M}^G))$, and so $\mathfrak{R} \subseteq \mathfrak{M}^G$. This violates (a).

Suppose $\mathfrak{E}$ is an abelian subgroup of $\mathfrak{K}_3$ of order 27. Then there is an element E in $\mathfrak{E}^\#$ of order 3 such that $\boldsymbol{C}(E) \cap \boldsymbol{O}_2(\mathfrak{K})$ contains a noncyclic abelian subgroup of order 8. This violates (c) and establishes (d).

(e) is an immediate consequence of (d).

LEMMA 9.22. *$\mathfrak{K}_2 - \mathfrak{K}_0$ contains an involution.*

Proof. Suppose false. By a result of Glauberman [16], $\mathfrak{K}_2$ contains an involution J such that $J = I^G \neq I$. Since the lemma is false, $J \in \mathfrak{K}_0$. Let $\mathfrak{T} = \boldsymbol{C}_{\mathfrak{K}_0}(J)$. Then $\mathfrak{T}$ is generated by involutions, and $\mathfrak{T} \subseteq \mathfrak{M}^G = \boldsymbol{C}(J)$. Since the lemma is false, $\mathfrak{T} \subseteq \mathfrak{K}_0^G$. In particular, $I \in (\mathfrak{K}_0^G)'$. Hence, $(\mathfrak{K}_0^G)' = \langle J \rangle = \langle I \rangle$, a contradiction.

LEMMA 9.23. *The 3-length of $\mathfrak{M}$ is 1.*

Proof. Suppose false. By Lemma 9.21 (e), it follows that $\mathfrak{K}_3^*$ is elementary of order 9. Consider $\mathfrak{K}^*/\langle I \rangle$. Since $\mathfrak{K}^* \cap \mathfrak{K}_0 = \langle I \rangle$ by (9.25), it follows that $\mathfrak{K}_3^*\langle I \rangle/\langle I \rangle = \boldsymbol{F}(\mathfrak{K}^*/\langle I \rangle)$. This implies that $\mathfrak{K}^*/\langle I \rangle$ contains a quaternion subgroup $\mathfrak{Q}/\langle I \rangle$. Thus, $\mathfrak{Q}$ is not of maximal class, since no group of maximal class and order 16 has a quaternion factor group. Hence, $\mathfrak{Q}$ contains a noncyclic abelian subgroup of order 8. This violates Lemma 9.21 (c) with $\mathfrak{K}_3^*$ in the role of $\tilde{\mathfrak{P}}$.

LEMMA 9.24. *Each involution J of $\mathfrak{K}_2 - \mathfrak{K}_0$ normalizes a S_3-subgroup of $\mathfrak{K}$.*

Proof. Since $J \notin \mathfrak{K}_0$, Lemma 5.36 implies that J inverts an element P of $\mathfrak{K}$ of order 3. Let $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{K}}(\mathfrak{P})$. Suppose $\mathfrak{C} \cap \mathfrak{K}_0 = \langle I \rangle$. Then since $\mathfrak{K}_3\mathfrak{K}_0 \lhd \mathfrak{K}$, it follows that $\mathfrak{C}$ is 3-closed. Let $\mathfrak{C}_3$ be the S_3-subgroup of $\mathfrak{C}$. Thus, $\boldsymbol{C}_3$ is noncyclic. Since $\boldsymbol{N}(\mathfrak{C}_3) \cap \mathfrak{K}_0 = \langle I \rangle$, $\boldsymbol{N}_{\mathfrak{K}}(\mathfrak{C}_3)$ contains a S_3-subgroup of $\mathfrak{K}$ as a normal subgroup. Since $J \in \boldsymbol{N}(\boldsymbol{C}_3)$, we are done.

We may assume that $\mathfrak{C} \cap \mathfrak{K}_0 \supset \langle I \rangle$. Hence, $\mathfrak{C} \cap \mathfrak{K}_0 = \mathfrak{Q}$ is a quaternion group. Let $\tilde{\mathfrak{P}}$ be a S_3-subgroup of $\mathfrak{C}$. Thus, $\tilde{\mathfrak{P}}\mathfrak{Q} = \boldsymbol{O}^{3'}(\mathfrak{C})$ char $\mathfrak{C}$, so $\langle J \rangle\tilde{\mathfrak{P}}\mathfrak{Q}$ is a group. Let $\tilde{\mathfrak{P}}_0 = \boldsymbol{O}_3(\tilde{\mathfrak{P}}\mathfrak{Q})$. Thus, $\tilde{\mathfrak{P}}\mathfrak{Q}/\tilde{\mathfrak{P}}_0 \cong SL(2, 3)$, and J stabilizes $\tilde{\mathfrak{P}}\mathfrak{Q}/\tilde{\mathfrak{P}}_0$. If J does not centralize $\mathfrak{Q}\tilde{\mathfrak{P}}/\mathfrak{Q}\tilde{\mathfrak{P}}_0$, it follows from Lemma 5.36 that J normalizes a S_3-subgroup of $\mathfrak{Q}\tilde{\mathfrak{P}}$. Suppose J centralizes $\tilde{\mathfrak{P}}\mathfrak{Q}/\mathfrak{Q}\tilde{\mathfrak{P}}_0$. Then $\langle J \rangle\tilde{\mathfrak{P}}\mathfrak{Q}/\mathfrak{Q}\tilde{\mathfrak{P}}_0$ is a cyclic group of order 6, so J centralizes $\mathfrak{Q}/\mathfrak{Q}'$. This implies that $|\boldsymbol{C}(J) \cap \mathfrak{Q}| \geqq 4$, so that $\langle J, \mathfrak{Q} \rangle$ contains a noncyclic abelian subgroup of order 8. Since $\langle J, \mathfrak{Q} \rangle \subseteq \boldsymbol{N}(\langle P \rangle)$, Lemma 9.21 (c) is violated. We conclude that J

normalizes a S_3-subgroup of $\tilde{\mathfrak{P}}\mathfrak{Q}$. We assume without loss of generality that J normalizes $\tilde{\mathfrak{P}}$. Since $\boldsymbol{N}(\tilde{\mathfrak{P}}) \cap \mathfrak{K}_0 = \langle I \rangle$, it follows that $\boldsymbol{N}_{\mathfrak{K}}(\tilde{\mathfrak{P}})$ contains a S_3-subgroup of $\mathfrak{K}$ as a normal subgroup. The proof is complete.

LEMMA 9.25. (a) *If T is an involution of $\mathfrak{G}$, $\boldsymbol{C}(T)$ contains a noncyclic abelian subgroup of order* 8.

(b) *If the width of $\mathfrak{K}_0$ is* 2, *then for each involution T of $\mathfrak{G}$,* $|\boldsymbol{C}(T)|_3 \leqq 9$.

Proof. (a) By Lemma 5.38, $\boldsymbol{C}(T)$ contains an element $\mathfrak{U}$ of $\mathscr{U}(2)$. If $T \notin \mathfrak{U}$, then $\langle \mathfrak{U}, T \rangle$ is a noncyclic abelian subgroup of order 8 which is contained in $\boldsymbol{C}(T)$. Suppose $T \in \mathfrak{U}$. Since $\mathscr{SCN}_3(2) \neq \emptyset$, $\boldsymbol{C}(T)$ contains an element of $\mathscr{SCN}_3(2)$ by Lemma 0.8.9.

Suppose (b) is false, and T is an involution of $\mathfrak{G}$ with $|\boldsymbol{C}(T)|_3 \geqq 27$. Let $\mathfrak{S}$ be a maximal 2, 3-subgroup of $\mathfrak{G}$ which contains a $S_{2,3}$-subgroup of $\boldsymbol{C}(T)$. By Lemma 5.38, $\mathfrak{S}$ contains an element $\mathfrak{U}$ of $\mathscr{U}(2)$. Let $\mathfrak{S}_p$ be a S_p-subgroup of $\mathfrak{S}$, $p = 2, 3$. We assume without loss of generality that $\mathfrak{S}_2 \subseteqq \mathfrak{K}_2$.

Case 1. $\boldsymbol{O}_3(\mathfrak{S}) \neq 1$.

Since $\mathfrak{U} \in \mathscr{E}(2)$, $\mathfrak{U}$ centralizes $\boldsymbol{O}_3(\mathfrak{S})$. Since $\mathfrak{U}$ contains a conjugate of I, it follows that $|\boldsymbol{O}_3(\mathfrak{S})| \leqq 9$. Suppose $|\boldsymbol{O}_3(\mathfrak{S})| = 9$. Then $\boldsymbol{O}_3(\mathfrak{S})$ is conjugate to $\mathfrak{B}$, since $\mathfrak{B}$ is a S_3-subgroup of $\mathfrak{M}$. But then Lemma 9.16 (ii) is violated. Hence, $|\boldsymbol{O}_3(\mathfrak{S})| = 3$.

Since $\mathfrak{U}$ centralizes $\boldsymbol{O}_3(\mathfrak{S})$, $\boldsymbol{O}_3(\mathfrak{S})$ is conjugate to a subgroup of $\mathfrak{B}$. By Lemma 9.21 (b), $\boldsymbol{C}(\boldsymbol{O}_3(\mathfrak{S}))$ contains an elementary subgroup $\mathfrak{A}^*$ such that $\boldsymbol{C}(A)$ contains an element of $\mathscr{U}(3)$ for each A in $\mathfrak{A}^*$. Since $\mathfrak{S}$ is a $S_{2,3}$-subgroup of $\boldsymbol{N}(\boldsymbol{O}_3(\mathfrak{S}))$, we assume without loss of generality that $\mathfrak{A}^* \subseteqq \mathfrak{S}$. By Lemma 9.16 (i), $\mathfrak{A}^* \in \mathscr{D}$. Now Lemma 9.17 yields $\boldsymbol{O}_2(\mathfrak{S}) \neq 1$. Since $\mathfrak{A}^* \subseteqq \mathfrak{S}$, Lemma 9.16 (ii) forces $|\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{S}))| = 2$, and forces $\boldsymbol{Z}(\boldsymbol{O}_2(\mathfrak{S}))$ to be a maximal characteristic abelian subgroup of $\boldsymbol{O}_2(\mathfrak{S})$. Since $|\boldsymbol{O}_3(\mathfrak{S})| = 3$, it follows that $|\boldsymbol{O}_2(\mathfrak{S})| > 2$. Hence, $\boldsymbol{O}_2(\mathfrak{S})$ is extra special. Thus, $\boldsymbol{O}_2(\mathfrak{S})'$ is of order 2 and is normalized by every element of $\mathscr{U}(\mathfrak{K}_2)$. Hence, every element of $\mathscr{U}(\mathfrak{K}_2)$ is contained in $\mathfrak{S}_2$. Thus, I centralizes $\boldsymbol{O}_3(\mathfrak{S})$. Since $I \in \boldsymbol{Z}(\mathfrak{S}_2)$, we get $I \in \boldsymbol{O}_2(\mathfrak{S})$, so that $\langle I \rangle = \boldsymbol{O}_2(\mathfrak{S})'$. Hence, $\mathfrak{S} \subseteqq \mathfrak{M}$, against $|\mathfrak{S}|_3 \geqq 27$ and $|\mathfrak{M}|_3 = 9$.

Case 2. $\boldsymbol{O}_3(\mathfrak{S}) = 1$.

Since $|\mathfrak{S}|_3 \geqq 27$, it follows that $m(\boldsymbol{O}_2(\mathfrak{S})) \geqq 6$. Since the width of $\mathfrak{K}_0$ is 2, it follows that $\mathfrak{K}_2$ has no elementary subgroup of order 2^6. Thus, $\boldsymbol{O}_2(\mathfrak{S})$ is not elementary.

Now $\mathfrak{S}$ is clearly not contained in any conjugate of $\mathfrak{M}$, since $|\mathfrak{S}|_3 > |\mathfrak{M}|_3$. Since $\langle I \rangle = \boldsymbol{Z}(\mathfrak{K}_2)$, it follows that $\mathfrak{S}$ is not 2-closed.

Since $|\mathfrak{R}_2| \leqq 2^8$, we get $|O_2(\mathfrak{S})| = 2^7$. Hence, $|O_2(\mathfrak{S})| = 2^7$, so that $D(O_2(\mathfrak{S}))$ is a subgroup of order 2 and $\mathfrak{S}_2$ is of order 2^8. Hence, $\mathfrak{S}_2 = \mathfrak{R}_2$ and $D(O_2(\mathfrak{S})) = \langle I \rangle$. This shows that $\mathfrak{S} \subseteq \mathfrak{M}$. This contradiction completes the proof.

LEMMA 9.26. *If* $\tilde{\mathfrak{P}}$ *is any subgroup of* $\mathfrak{R}_3$ *of order* 3, *then*
(a) $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}})$ *is either* $\langle I \rangle$ *or a quaternion group;*
(b) *if* $\tilde{\mathfrak{P}} \not\subseteq Z(\mathfrak{R}_3)$, *then* $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}})$ *is quaternion.*

Proof. (a) Suppose $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}}) \supset \langle I \rangle$. Then $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}})$ is extra special and does not contain a noncyclic abelian subgroup of order 8. Thus, $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}})$ is either dihedral or quaternion. Now $C_{\mathfrak{R}_3}(\tilde{\mathfrak{P}})$ contains an elementary subgroup $\mathfrak{E}$ of order 9 with $\tilde{\mathfrak{P}} \subset \mathfrak{E}$. Hence, $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}})$ admits $\mathfrak{E}$. Since no element of $\mathfrak{E}^\#$ centralizes a noncyclic abelian subgroup of $\mathfrak{R}_0$ of order 8, $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{P}})$ is quaternion.

(b) Let $\mathfrak{E} = \langle \tilde{\mathfrak{P}}, Z(\mathfrak{R}_3) \rangle$, so that by Lemma 9.21 (e), $\mathfrak{E}$ is elementary of order 9 and $\mathfrak{E} \lhd \mathfrak{R}_3$. It follows that the three subgroups of $\mathfrak{E}$ of order 3 which are distinct from $Z(\mathfrak{R}_3)$ are conjugate in $\mathfrak{R}_3$. We can choose E in $\mathfrak{E}^\#$ such that $\mathfrak{R}_0 \cap C(E)$ is not centralized by $Z(\mathfrak{R}_3)$. By (a), $\mathfrak{R}_0 \cap C(E)$ is a quaternion group; so (b) holds.

LEMMA 9.27. $\mathfrak{R}_2^*$ *is a four-group.*

Proof. Suppose false. By Lemma 9.22, $\mathfrak{R}_2 - \mathfrak{R}_0$ contains an involution J. By Lemma 9.24, J normalizes a S_3-subgroup of $\mathfrak{R}$. Thus, we can choose M in $\mathfrak{M}$ such that $J^M = T_0$ normalizes $\mathfrak{R}_3$. Since $\mathfrak{R}_3$ permutes transitively by conjugation the S_2-subgroups of $\mathfrak{R}$, we may choose K in $\mathfrak{R}_3$ such that $T = T_0^K$ lies in $\mathfrak{R}_2$. Thus, $T \in N(\mathfrak{R}_3) \cap \mathfrak{R}_2$.

By Lemma 9.23, $\mathfrak{R}_3^* = \mathfrak{R}_3$. Thus, $T \in \mathfrak{R}_2^*$. If $\langle T, I \rangle = \mathfrak{R}_2^*$, we are done, so suppose $\langle T, I \rangle \subset \mathfrak{R}_2^*$. Let $\mathfrak{F}$ be a subgroup of $\mathfrak{R}_2^*$ of order 8 which contains $\langle T, I \rangle$. By Lemma 9.19, $\mathfrak{F}$ is dihedral of order 8. Let $\mathfrak{F}_0, \mathfrak{F}_1$ be the four-subgroups in $\mathfrak{F}$.

Suppose $\mathfrak{X}$ is a subgroup of $\mathfrak{R}_3$ of order 3 which admits $\mathfrak{F}$ and that $C(\mathfrak{X}) \cap \mathfrak{R}_0$ is a quaternion group. Hence, $C_{\mathfrak{M}}(\mathfrak{X})$ contains a normal quaternion subgroup and S_2-subgroups of $N_{\mathfrak{M}}(\mathfrak{X})$ are of order at least 2^5. Thus, $N_{\mathfrak{M}}(\mathfrak{X})$ contains a noncyclic abelian subgroup of order 8. This is impossible, by Lemma 9.21 (c). Hence, $C_{\mathfrak{R}_0}(\mathfrak{X}) = \langle I \rangle$, by Lemma 9.26 (a).

By Lemma 9.26 (b) and the preceding paragraph, it follows that $\mathfrak{F}$ normalizes no noncentral subgroup of $\mathfrak{R}_3$ of order 3.

Suppose $\mathfrak{R}_3$ is nonabelian. Then $\mathfrak{F}$ normalizes $\Omega_1(\mathfrak{R}_3)$, a group of exponent 3. Since $\langle I \rangle$ centralizes $\mathfrak{R}_3$, it follows that $\Omega_1(\mathfrak{R}_3)\mathfrak{F}$ is supersolvable. Thus, $\Omega_1(\mathfrak{R}_3)\mathfrak{F}$ contains a normal subgroup of order 9, so $\mathfrak{F}$ normalizes a noncentral subgroup of $\mathfrak{R}_3$ of order 3. This contradicts the preceding paragraph, so we conclude that $\mathfrak{R}_3$ is abelian, $\mathfrak{R}_3 = \mathfrak{B}$.

Let $\mathfrak{F}_i = \langle J_i, I\rangle$, $i = 0, 1$. If both J_0 and J_1 invert $\mathfrak{K}_3$, then J_0J_1 centralizes $\mathfrak{K}_3$, so $J_0J_1 \in \langle I\rangle$. This is not the case, since $\mathfrak{E}_0\mathfrak{E}_1$ is of order 8. Thus, we may assume notation is chosen so that J_0 centralizes $\mathfrak{X}_0$ and inverts $\mathfrak{X}_1$. Here, $|\mathfrak{X}_i| = 3$, and $\mathfrak{K}_3 = \mathfrak{X}_0 \times \mathfrak{X}_1$. Since $\mathfrak{K}_0 \cap C(\mathfrak{X}_i) = \langle I\rangle$, for $i = 0, 1$, the width of $\mathfrak{K}_0$ is 2.

Let $\mathfrak{E}$ be a $S_{2,3}$-subgroup of $N(\mathfrak{K}_3)$ which contains $\mathfrak{K}_2^*$. Since $\mathfrak{K}_3 \subset \mathfrak{P}$, we get $|\mathfrak{K}_3| = 9 < |\mathfrak{E}|_3$. By Lemma 9.16 (ii), $|C(\mathfrak{K}_3)|_2 = 2$. By Lemma 9.20, we get that $\mathfrak{E}$ is 3-closed. Let $\mathfrak{E}_3 = O_3(\mathfrak{E}) \supset \mathfrak{K}_3$.

Let F_0, F_1, F_2 be the three involutions of $\mathfrak{F}_0$, and set

$$3^{f_i} = |\mathfrak{E}_3 \cap C(F_i)|,\ i = 0, 1, 2 .$$

By Lemma 9.25 (b), we have $f_i \leqq 2$. Since $\mathfrak{E}_3 \cap C(\mathfrak{F}_0) = \mathfrak{X}_0$, a formula of Wielandt [44] yields

$$|\mathfrak{E}_3| = 3^{f_0+f_1+f_2-2} \leqq 3^4 .$$

Since the dihedral group $\mathfrak{F}$ is faithfully represented on $\mathfrak{E}_3/\mathfrak{K}_3$, it follows that $|\mathfrak{E}_3| = 3^4$.

Let $\mathfrak{D}$ be a $S_{2,3}$-subgroup of $N(\mathfrak{E}_3)$. Let $\mathfrak{D}_p$ be a S_p-subgroup of $\mathfrak{D}$, with $\mathfrak{K}_2^* \subseteq \mathfrak{D}_2$. By the formula of Wielandt [44] applied to $\mathfrak{F}_0$ acting on $O_3(\mathfrak{D})$, we get $O_3(\mathfrak{D}) = \mathfrak{E}_3$. If $\mathfrak{D}_3 = \mathfrak{E}_3$, then $\mathfrak{E}_3$ is a S_3-subgroup of $\mathfrak{G}$. But the center of $\mathfrak{E}_3$ contains $\mathfrak{K}_3 = \mathfrak{B}$ by Lemma 9.20, and hence is noncyclic. This contradicts hypothesis (iii) of Theorem 9.1. Therefore $\mathfrak{D}_3 \supset \mathfrak{E}_3$ and $\mathfrak{D}$ is not 3-closed.

By Lemma 9.25 (b), we get $O_2(\mathfrak{D}) = 1$. Since $\mathfrak{D}$ is quite obviously contained in no conjugate of $\mathfrak{M}$, Lemma 9.18 implies that $\mathfrak{D}$ contains no noncyclic abelian subgroups of order 8. Thus, $\mathfrak{D}_2$ is of maximal class. Hence, $\langle I\rangle = Z(\mathfrak{D}_2)$, so $\mathfrak{D}_2 = \mathfrak{K}_2^*$ is of order at most 16. Suppose $|\mathfrak{D}_2| = 16$. Since $O_2(\mathfrak{D}) = 1$, and $\mathfrak{D}$ is not 3-closed, it follows that $O_{3,2}(\mathfrak{D}) \cap \mathfrak{D}_2$ is a quaternion group. But then $\mathfrak{M}$ covers $\mathfrak{D}/O_3(\mathfrak{D})$. This is not the case, since $O_3(\mathfrak{D})$ contains a S_3-subgroup of $\mathfrak{M}$, and since $3 \mid |\mathfrak{D}: O_3(\mathfrak{D})|$. Hence, $\mathfrak{D}_2 = \mathfrak{F}$ is dihedral of order 8. Let $\tilde{\mathfrak{D}}_2 = O_{3,2}(\mathfrak{D}) \cap \mathfrak{D}_2$. Thus, $\tilde{\mathfrak{D}}_2$ is a four-group and $\mathfrak{D}/O_3(\mathfrak{D}) \cong \Sigma_4$.

Since $\mathfrak{D}/O_3(\mathfrak{D}) \cong \Sigma_4$, some chief factor of $\mathfrak{D}$ is of order 3^3. Thus, $O_3(\mathfrak{D})$ is necessarily elementary, and elements of $\mathfrak{D}_3 - O_3(\mathfrak{D})$ induce automorphisms of $O_3(\mathfrak{D})$ with minimal polynomial $(x-1)^3$. Hence, $O_3(\mathfrak{D}) = J(\mathfrak{D}_3)$ $O_3(\mathfrak{D}) = J(\mathfrak{D}_3)$ char $\mathfrak{D}_3$, so $\mathfrak{D}_3$ is a S_3-subgroup of $\mathfrak{G}$. This is not the case, since $Z(\mathfrak{D}_3)$ is noncyclic. The proof is complete.

LEMMA 9.28. *If the width of $\mathfrak{K}_0$ exceeds 2, then I is the only conjugate of I in $\mathfrak{K}_0$.*

Proof. Suppose $T = I^G \neq I$, $T \in \mathfrak{K}_0$. Then $C(T) \cap \mathfrak{K}_0 \subseteq C(T) = \mathfrak{M}^G$. By Lemma 9.27, $C(T) \cap \mathfrak{K}_0 \cap \mathfrak{K}_0^G$ is of index at most 2 in $C(T) \cap \mathfrak{K}_0$.

Since $C_{\mathfrak{K}_0}(T)$ is of index 2 in $\mathfrak{K}_0$, we get $|\mathfrak{K}_0: C(T) \cap \mathfrak{K}_0 \cap \mathfrak{K}_0^G| \leqq 4$. Since the width of $\mathfrak{K}_0$ is at least 3, it follows that $C(T) \cap \mathfrak{K}_0 \cap \mathfrak{K}_0^G$ is nonabelian. Hence, $\langle I \rangle = (C(T) \cap \mathfrak{K}_0 \cap \mathfrak{K}_0^G)' = \langle T \rangle$. This contradiction completes the proof.

LEMMA 9.29. *$\mathfrak{K}_3 = \mathfrak{B}$ is of order 9.*

Proof. Suppose false. By Lemma 9.21 (e), $\mathfrak{K}_3$ is nonabelian of order 27. Since $\mathfrak{K}_3$ is faithfully represented on $\mathfrak{K}_0$, the width of $\mathfrak{K}_0$ is at least 3. By a result of Glauberman [14], $\mathfrak{K}_2$ contains a conjugate T of I distinct from I, $T = I^G \neq I$. By Lemma 9.28, $T \in \mathfrak{K}_2 - \mathfrak{K}_0$, so by Lemma 9.24, we may assume that $T \in \mathfrak{K}_2^*$. Thus, by Lemma 9.27, $\mathfrak{K}_2^* = \langle I, T \rangle$.

Since $\mathfrak{K}_3$ is nonabelian, it follows that $\mathfrak{X}_1 = \mathfrak{K}_3 \cap C(T)$ is of order 3. By Lemma 1.3 of [17], $\mathfrak{K}_3$ has a subgroup $\mathfrak{X}_0$ of order 3 which centralizes $\mathfrak{X}_1$ and is inverted by T. Let $\mathfrak{X} = \mathfrak{X}_0 \times \mathfrak{X}_1$ and let $\mathfrak{X}_2, \mathfrak{X}_3$ be the remaining subgroups of $\mathfrak{X}$ of order 3.

Suppose $C(\mathfrak{X}_0) \cap \mathfrak{K}_0 \supset \langle I \rangle$. By Lemma 9.26 (a), $C(\mathfrak{X}_0) \cap \mathfrak{K}_0 = \mathfrak{Q}$ is a quaternion group. Since $C(\mathfrak{X}) \cap \mathfrak{K}_0 = \langle I \rangle$, it follows that $\mathfrak{X}_1$ is faithfully represented on $\mathfrak{Q}$. Since Aut $(\mathfrak{Q})$ has no element of order 6, T centralizes $\mathfrak{Q}$. But then $\langle Q, T \rangle = \mathfrak{Q} \times \langle T \rangle$ contains a noncyclic abelian subgroup of order 8. This violates Lemma 9.21 (c) with $\mathfrak{X}_0$ in the role of $\mathfrak{P}$. Hence, $C(\mathfrak{X}_0) \cap \mathfrak{K}_0 = \langle I \rangle$.

Since $C(\mathfrak{X}_0) \cap \mathfrak{K}_0 = \langle I \rangle$, the width of $\mathfrak{K}_0$ is at most 3. By Lemma 9.26 (b), we get $\mathfrak{X}_0 = Z(\mathfrak{K}_3)$. Thus, if we set $\mathfrak{Q}_i = \mathfrak{K}_0 \cap C(\mathfrak{X}_i)$, $i = 1, 2, 3$, then by Lemma 9.26, it follows that each $\mathfrak{Q}_i$ is quaternion. Hence, $\mathfrak{K}_0$ is the central product of $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$. Since T centralizes $\mathfrak{X}_1$ and interchanges $\mathfrak{X}_2$ and $\mathfrak{X}_3$, it follows that T normalizes $\mathfrak{Q}_1$ and interchanges $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$. Since $\mathfrak{X}_0$ is faithfully represented on $\mathfrak{Q}_1$, it follows that $\mathfrak{Q}_1\mathfrak{X}_0\langle T \rangle \cong GL(2, 3)$. Thus, we can choose generators A_i, B_i for $\mathfrak{Q}_i$ such that $A_1^T = B_1$, $A_2^T = A_3$, $B_2^T = B_3$. It follows that $C(T) \cap \mathfrak{K}_0 = \langle A_2A_2, B_2B_3, I \rangle$, an elementary group of order 8. Let $\mathfrak{F} = C(T) \cap \mathfrak{K}_2 = \langle T \rangle \times C(T) \cap \mathfrak{K}_0$. It now follows that $\tilde{\mathfrak{K}}_2 = N_{\mathfrak{K}_2}(\mathfrak{F}) = \langle \mathfrak{Q}_2, \mathfrak{Q}_3, A_1B_1, T \rangle$, a group of index 2 in $\mathfrak{K}_2$. Since $Z(\tilde{\mathfrak{K}}_2) = \langle I \rangle$, it follows that $\tilde{\mathfrak{K}}_2$ is a S_2-subgroup of $N(\mathfrak{F})$.

Now $T = I^G$, so $\mathfrak{F} \subseteqq \mathfrak{M}^G$. By symmetry, $N(\mathfrak{F}) \cap \mathfrak{M}^G$ contains a S_2-subgroup of $N(\mathfrak{F})$. This implies that $O_2(N(\mathfrak{F}))$ centralizes both T and I. Hence, $O_2(N(\mathfrak{F})) = \mathfrak{F}$.

Now $\tilde{\mathfrak{K}}_2$ permutes transitively the elements of $(\mathfrak{F} \cap \mathfrak{K}_0)T$, so $\mathfrak{S} = \{(\mathfrak{F} \cap \mathfrak{K}_0)T, I\}$ is the set of all the elements of $\mathfrak{F}$ which are conjugate to I in $\mathfrak{G}$. Since $N(\mathfrak{F}) \cap \mathfrak{M}^G$ normalizes $\mathfrak{S}$ but does not centralize I, it follows that $N(\mathfrak{F})$ permutes $\mathfrak{S}$ transitively.

Since $N(\mathfrak{F})$ is transitive on $\mathfrak{S}$, it follows that $N(\mathfrak{F}) = 9 \cdot |N(\mathfrak{F}) \cap \mathfrak{M}|$. Since $C(\mathfrak{F}) = C_{\mathfrak{M}}(\mathfrak{F})$, it follows that $\mathfrak{F} = C(\mathfrak{F})$. Since T centralizes $\mathfrak{X}_1$, it follows that $\mathfrak{X}_1$ normalizes $\mathfrak{K}_0 \cap C(T)$, so normalizes $\mathfrak{F} = \langle T \rangle \times \mathfrak{K}_0 \cap C(T)$. But it now follows that $27 \mid |N(\mathfrak{F})|$. Since $\mathfrak{F}$ is elementary of order 2^4, $\mathrm{Aut}(\mathfrak{F})$ has no subgroup of order 27. This violates the equality $\mathfrak{F} = C(\mathfrak{F})$, and the proof is complete.

LEMMA 9.30. *If T is any involution of $\mathfrak{G}$, then* $|C(T)|_3 \leqq 9$.

Proof. Since $|\mathfrak{K}_2 : \mathfrak{K}_0| = 2$, Lemma 5.38 implies that every involution of $\mathfrak{G}$ is conjugate to an involution of $\mathfrak{K}_0$. Thus, we may assume that $T \in \mathfrak{K}_0$. If $T \sim I$, we are done by Lemma 9.29, so from now on we suppose $T \not\sim I$.

Let $\mathfrak{U} \in \mathscr{U}(\mathfrak{K}_2)$, and let $\tilde{\mathfrak{K}}_0 = \mathfrak{K}_0 \cap C(\mathfrak{U})$. Thus, $\tilde{\mathfrak{K}}_0$ is of index 2 in $\mathfrak{K}_0$. Since $\mathfrak{K}_3$ has no fixed points on $\mathfrak{K}_0/\langle I \rangle$, Lemma 5.38 implies that for some X in $\mathfrak{K}_3$, $T^X \in \tilde{\mathfrak{K}}_0$. Thus, we assume without loss of generality that $T \in \tilde{\mathfrak{K}}_0$.

We argue that $C_{\mathfrak{M}}(T)$ contains a S_2-subgroup of $C(T)$. This is clear if $|\mathfrak{M} : C_{\mathfrak{M}}(T)|_2 = 2$, since $T \not\sim I$. So suppose $|\mathfrak{M} : C_{\mathfrak{M}}(T)|_2 = 4$. In this case, $C_{\mathfrak{K}_0}(T)$ is a S_2-subgroup of $C_{\mathfrak{M}}(T)$. Since $\langle I \rangle = C_{\mathfrak{K}_0}(T)'$ char $C_{\mathfrak{K}_0}(T)$, it follows that $C_{\mathfrak{K}_0}(T)$ is a S_2-subgroup of $C(T)$.

Let $\mathfrak{N}$ be a $S_{2,3}$-subgroup of $C(T)$ which contains $C_{\mathfrak{K}}(T)$. Suppose $O_3(\mathfrak{N}) \neq 1$. Since $\mathfrak{U} \subseteq \mathfrak{N}$, $\mathfrak{U}$ centralizes $O_3(\mathfrak{N})$, so $O_3(\mathfrak{N}) \subseteq \mathfrak{M}$. Since no element of $\mathfrak{K}_3^\#$ centralizes a four-subgroup of $\mathfrak{K}_0$ by Lemma 9.26 (a), we conclude that $O_3(\mathfrak{N}) = 1$.

Since $O_3(\mathfrak{N}) = 1$ and since $\mathfrak{N} \cap \mathfrak{M}$ contains a S_2-subgroup of $C(T)$, it follows that $I \in Z(O_2(\mathfrak{N}))$. Suppose X is a 3-element of $\mathfrak{N}$ and $\mathfrak{X}$ centralizes I. Then $X \in C(\langle T, I \rangle)$, so $X = 1$ by Lemma 9.26 (a). Thus, a S_3-subgroup $\mathfrak{N}_3$ of $\mathfrak{N}$ is faithfully represented on $Z(O_2(\mathfrak{N}))$.

Let $\Omega_1(Z(O_2(\mathfrak{N}))) = \mathfrak{Y}_1 \times \mathfrak{Y}_2$, where $\mathfrak{Y}_1 = \Omega_1(Z(O_2(\mathfrak{N}))) \cap C(\mathfrak{N}_3)$, and $\mathfrak{Y}_2 = [\Omega_1(Z(O_2(\mathfrak{N}))), \mathfrak{N}_3]$. Thus, $T \in \mathfrak{Y}_1$ and $\mathfrak{N}_3$ is faithfully represented on $\mathfrak{Y}_2$. Hence, $m(Z(O_2(\mathfrak{N}))) = m(\mathfrak{Y}_1) + m(\mathfrak{Y}_2) \geqq 7$. Thus, $\mathfrak{K}_0$ has an elementary subgroup of order 2^6, by Lemma 9.27. This is impossible, since the width of $\mathfrak{K}_0$ is at most 4. The proof is complete.

LEMMA 9.31. $|\mathfrak{G}|_3 > 3^4$.

Proof. Let $\mathfrak{X}$ be a subgroup of $\mathfrak{K}_3$ of order 3 such that $\mathfrak{K}_0 \cap C(\mathfrak{X}) = \mathfrak{Q}$ is quaternion. Let $\mathfrak{C}$ be a $S_{2,3}$-subgroup of $C(\mathfrak{X})$ which contains $\mathfrak{K}_3\mathfrak{Q}$. Since $\mathfrak{K}_3 = \mathfrak{B} \in \mathscr{B}$, it follows that $\mathfrak{K}_3$ centralizes $Z(O_2(\mathfrak{C}))$. Since $\langle I \rangle$ is a S_2-subgroup of $C(\mathfrak{K}_3)$, it follows that $O_2(\mathfrak{C}) = 1$, by Lemma 9.21(a).

Since $O_2(\mathfrak{C}) = 1$, $\mathfrak{Q}$ is faithfully represented on $O_3(\mathfrak{C})$, so is faithfully represented on $O_3(\mathfrak{C})/\mathfrak{X}$. Hence, $|O_3(\mathfrak{C}) : \mathfrak{X}| \geqq 9$. Since $[\mathfrak{K}_3 \cap O_3(\mathfrak{C}), \mathfrak{Q}] \subseteq O_3(\mathfrak{C}) \cap \mathfrak{Q} = 1$, it follows that $\mathfrak{K}_3 \cap O_3(\mathfrak{C}) = \mathfrak{X}$. Hence, $|\mathfrak{C}|_3 \geqq 3^4$. Suppose the lemma is false. Then $\mathfrak{C}$ contains a S_3-subgroup of $\mathfrak{G}$, and $O_3(\mathfrak{C})$ is of order 3^3, while $\mathfrak{X} \sim \mathfrak{Z}$. If $O_3(\mathfrak{C})$ is nonabelian, then Hypothesis 9.1 is satisfied. This is not the case, so $O_3(\mathfrak{C})$ is elementary. Hence, $O_3(\mathfrak{C}) = \mathfrak{X} \times [O_3(\mathfrak{C}), \mathfrak{Q}]$. Hence, the center of a S_3-subgroup of

$\mathfrak{G}$ is noncyclic. This is not the case. The proof is complete.

LEMMA 9.32. *Choose* J *in* $\mathfrak{R}_2^* - \langle I \rangle$. *If* J *inverts* $\mathfrak{R}_3$, *then* $A_{\mathfrak{G}}(\mathfrak{R}_2^*) = \mathrm{Aut}\,(\mathfrak{R}_2^*)$.

Proof. Let $\mathfrak{X}$ be any four-subgroup of $\mathfrak{M}$ which contains I. We will show that

$$(9.25) \qquad | A_{\mathfrak{M}}(\mathfrak{X}) | = 2 \, .$$

This is clear if $\mathfrak{X} \subseteq \mathfrak{R}_0$. If $\mathfrak{X} \not\subseteq \mathfrak{R}_0$, then by Lemmas 9.27 and 9.24, we see that $\mathfrak{X}$ is conjugate to $\mathfrak{R}_2^*$ in $\mathfrak{M}$. Let $\mathfrak{Y}$ be a subgroup of $\mathfrak{R}_3$ such that $\mathfrak{Q} = \mathfrak{R}_0 \cap C(\mathfrak{Y})$ is quaternion. Since J inverts $\mathfrak{R}_3$, $\mathfrak{Q}$ admits $\langle J \rangle \mathfrak{R}_3/\mathfrak{Y}$ as a group of automorphisms. Hence, J inverts an element Q of $\mathfrak{Q}$ of order 4. Then $JQJ = Q^{-1}$, that is, $Q^{-1}JQ = JI$, so $Q \in N_{\mathfrak{M}}(\mathfrak{R}_2^*)$. Thus, (9.25) holds.

Suppose that $\mathfrak{R}_2^* - \langle I \rangle$ contains a conjugate J of I. By (9.25), we can choose M in $\mathfrak{M} \cap N(\mathfrak{R}_2^*)$ such that $M^{-1}JM = JI$. By (9.25) again, this time applied to the group $C(J)$, we can choose M_0 in $C(J)$ with $M_0^{-1}IM_0 = IJ$. Thus, the lemma follows in this case.

We may now assume that

$$(9.26) \qquad I \text{ is the only conjugate of } I \text{ in } \mathfrak{R}_2^* \, .$$

By a result of Glauberman [16], $\mathfrak{R}_2$ contains a conjugate T of I with $T \neq I$. If the width of $\mathfrak{R}_0$ exceeds 2, then by Lemma 9.28, $T \notin \mathfrak{R}_0$, so by Lemma 9.24, (9.26) is violated. So suppose the width of $\mathfrak{R}_0$ is 2. In this case, $\mathfrak{R}_0$ has exactly 18 noncentral involutions and they are permuted transitively in $\mathfrak{M}$. Since T lies in no $\mathfrak{M}$-conjugate of $\mathfrak{R}_2^*$, Lemma 9.24 implies that $T \in \mathfrak{R}_0$. Thus, every involution of $\mathfrak{R}_0$ is conjugate to I in $\mathfrak{G}$. But by Lemma 5.38, every involution of $\mathfrak{G}$ is conjugate to an element of $\mathfrak{R}_0$. The proof is complete.

LEMMA 9.33. *There is a* S_3*-subgroup of* $\mathfrak{G}$ *which contains* $\mathfrak{R}_3$ *and is normalized by* $\mathfrak{R}_2^*$.

Proof. Let $\tilde{\mathfrak{P}}$ be a maximal element of $N(\mathfrak{R}_2^*; 3)$ which contains $\mathfrak{R}_3$. Suppose by way of contradiction that $|\tilde{\mathfrak{P}}| < |\mathfrak{G}|_3$. Let $\mathfrak{C}$ be a $S_{2,3}$-subgroup of $N(\tilde{\mathfrak{P}})$ which contains $\mathfrak{R}_2^*$. Let $\mathfrak{C}_p$ be a S_p-subgroup of $\mathfrak{C}$, $p = 2, 3$, with $\mathfrak{R}_2^* \subseteq \mathfrak{C}_2$.

Suppose $O_2(\mathfrak{C}) \neq 1$. Then since $\mathfrak{R}_3 \in \mathscr{D}$, we get $Z(O_2(\mathfrak{C})) \sim \langle I \rangle$, so $\mathfrak{C}$ is in a conjugate of $\mathfrak{M}$. This is not the case, by Lemma 9.21(a). Clearly, the maximality of $\tilde{\mathfrak{P}}$ forces $\tilde{\mathfrak{P}} = O_3(\mathfrak{C})$. Since $O_2(\mathfrak{C}) = 1$, the proof of Lemma 9.17 implies that $\mathfrak{C}$ has no noncyclic abelian subgroup of order 8. Thus, $\bar{\mathfrak{C}} = \mathfrak{C}/O_3(\mathfrak{C})$ is a 2, 3-group of order divisible by 3 such that

(a) $O_3(\bar{\mathfrak{C}}) = 1$.

(b) $\bar{\mathfrak{C}}$ contains a four-group.

(c) $\bar{\mathfrak{C}}$ contains no noncyclic abelian subgroup of order 8.

It is routine to verify that $\bar{\mathfrak{C}} \cong GL(2, 3)$ or $\bar{\mathfrak{C}} \cong \Sigma_4$ or $\bar{\mathfrak{C}} \cong A_4$. If $GL(2, 3) \cong \bar{\mathfrak{C}}$, then every four-subgroup of $\bar{\mathfrak{C}}$ normalizes a S_3-subgroup of $\bar{\mathfrak{C}}$, against the maximality of $\bar{\mathfrak{P}}$. Hence,

$$\mathfrak{C}/O_3(\mathfrak{C}) \cong \Sigma_4 \quad \text{or} \quad A_4 .$$

Let $\mathfrak{K}_2^* = \langle I, J \rangle$.

Case 1. J does not invert $\mathfrak{K}_3$. Let $\mathfrak{X} = C(J) \cap \mathfrak{K}_3$, so that $\mathfrak{X} = C(\mathfrak{K}_2^*) \cap \mathfrak{K}_3 = O_3(\mathfrak{C}) \cap C(\mathfrak{K}_2^*)$ is of order 3. By a formula of Wielandt [40], together with Lemma 9.30, we get $|O_3(\mathfrak{C})| \leqq 3^4$. Since $O_3(\mathfrak{C}) = F(\mathfrak{C})$, it follows from (B) that $m(O_3(\mathfrak{C})) \geqq 3$. Hence, $O_3(\mathfrak{C})$ is elementary of order 3^3 or 3^4. If $|O_3(\mathfrak{C})| = 3^3$, then $O_3(\mathfrak{C})$ char $\mathfrak{C}_3$, and so $\mathfrak{C}_3$ is a S_3-subgroup of $\mathfrak{G}$, against Lemma 9.31. Hence, $O_3(\mathfrak{C})$ is elementary of order 3^4. This implies that $O_3(\mathfrak{C})$ char $\mathfrak{C}_3$. Hence, $\mathfrak{C}_3$ is a S_3-subgroup of $\mathfrak{G}$. This is not the case, since $Z(\mathfrak{C}_3)$ is noncyclic.

Case 2. J inverts $\mathfrak{K}_3$ and $\bar{\mathfrak{C}} \cong \Sigma_4$.

Let $\mathfrak{V} = \mathfrak{C}_2 \cap O_{3,2}(\mathfrak{C})$. Thus, $\mathfrak{V}$ is a four-group. Suppose $\mathfrak{V} = \mathfrak{K}_2^*$. Let $\mathfrak{S} = \mathfrak{M} \cap \mathfrak{C}$. Then $|\mathfrak{S}| = 8.9$, and $\mathfrak{K}_3 \triangle \mathfrak{S}$. This is not the case, since $\mathfrak{K}_2^*$ is a four-group, by Lemma 9.27.

Since $\mathfrak{V} \neq \mathfrak{K}_2^*$, it follows that $\mathfrak{V}$ and $\mathfrak{K}_2^*$ are the four-subgroups of $\mathfrak{C}_2$ By Lemma 9.32, $A_{\mathfrak{G}}(\mathfrak{K}_2^*) = \mathrm{Aut}\,(\mathfrak{K}_2^*)$. Thus, $\mathfrak{V} \cap \mathfrak{K}_2^* = \langle V \rangle$ with $V \sim I$. Hence, all involutions of $\mathfrak{V}$ are conjugate to I in $\mathfrak{G}$.

Choose V in $\mathfrak{V}^\#$. Suppose $|C(V) \cap O_3(\mathfrak{C})| > 3$. Then $C(V) \cap O_3(\mathfrak{C})$ is a S_3-subgroup of $C(V)$, by Lemma 9.29, together with $V \sim I$. Hence, $|C_{\mathfrak{G}}(V)| = 8.9$, and $C_{\mathfrak{G}}(V)$ is 3-closed. This violates Lemma 9.27 applied to $C(V)$. Hence, $|O_3(\mathfrak{C}) \cap (V)| \leqq 3$.

Since $N(\mathfrak{V}) \cap \mathfrak{C}$ permutes transitively the involutions of $\mathfrak{V}$, we get $|C(V) \cap O_3(\mathfrak{C})| = 3$ for all V in $\mathfrak{V}^\#$. Hence, $|O_3(\mathfrak{V})| = 27$ and $O_3(\mathfrak{C}) = J(\mathfrak{C}_3)$ char $\mathfrak{C}_3$. But then $|\mathfrak{C}|_3 = |\mathfrak{G}|_3$, against Lemma 9.31.

Case 3. J inverts $\mathfrak{K}_3$ and $\bar{\mathfrak{C}} \cong A_4$.

Let $\mathfrak{C}_0 = O_3(\mathfrak{C})$. Then $\mathfrak{K}_3 = C_{\mathfrak{C}_0}(I)$ is elementary of order 3^2 and inverts by each element of $\mathfrak{K}_2^* - \langle I \rangle$. Since the involutions of $\mathfrak{K}_2^*$ are fused in $\mathfrak{C}$, we conclude

(a) for each $K \in (\mathfrak{K}_2^*)^\#$, the group $C_{\mathfrak{C}_0}(K)$ is elementary of order 3^2 and is inverted by each element of $\mathfrak{K}_2^* - \langle K \rangle$.

It follows that

(b) $\mathfrak{C}_0$ contains two chief factors of $\mathfrak{C}$, each of order 3^3.

Suppose that $\mathfrak{E}_0$ is abelian. By (a) and (b), it is elementary of order 3^6 and each element of $\mathfrak{E}_3 - \mathfrak{E}_0$ has minimal polynomial $(x-1)^3$ on $\mathfrak{E}_0$. Hence, $\mathfrak{E}_0$ char $\mathfrak{E}_3$. So $\mathfrak{E}_3$ is a S_3-subgroup of $\mathfrak{G}$. But $\boldsymbol{Z}(\mathfrak{E}_3) = [\mathfrak{E}_0, \mathfrak{E}_3, \mathfrak{E}_3]$ is not cyclic, against hypothesis (iii) of Theorem 9.1. Therefore, $\mathfrak{E}_0$ is not abelian. So (a) and (b) imply:

(c1) $\mathfrak{E}_0$ is special of order 3^6 and exponent 3,

(c2) $\boldsymbol{D}(\mathfrak{E}_0) = \boldsymbol{Z}(\mathfrak{E}_0)$ is a chief factor of $\mathfrak{E}$ of order 3^3,

(c3) $\mathfrak{E}_0/\boldsymbol{D}(\mathfrak{E}_0)$ is a chief factor of $\mathfrak{E}$ of order 3^3,

(c4) every element of $\mathfrak{E}_3 - \mathfrak{E}_0$ has minimal polynomial $(x-1)^3$ on both $\boldsymbol{D}(\mathfrak{E}_0)$ and $\mathfrak{E}_0/\boldsymbol{D}(\mathfrak{E}_0)$,

(c5) if $P \in \mathfrak{E}_3 - \mathfrak{E}_0$, then $|\boldsymbol{C}_{\mathfrak{E}_3}(P)| \leqq 3^3$.

This implies

(d) $\mathfrak{E}_0$ char $\mathfrak{E}_3$.

Indeed, if $\mathfrak{E}_1$ is any subgroup of index 3 in $\mathfrak{E}_3$ different from $\mathfrak{E}_0$, then $\mathfrak{E}_1 \cap \mathfrak{E}_0 \supseteqq \boldsymbol{D}(\mathfrak{E}_0)$. Hence, (c4) implies that the exponent of $\mathfrak{E}_1$ is 9. This proves (d), and gives

(e) $\mathfrak{E}_3$ is a S_3-subgroup of $\mathfrak{G}$.

Now let $\mathfrak{A}_0$ be a subgroup of $\mathfrak{K}_3$ of order 3 such that $\boldsymbol{C}_{\mathfrak{K}_0}(\mathfrak{A}_0) = \mathfrak{Q} \supset \langle I \rangle$. Let $\mathfrak{E}_3 = A_0 \times A_1$. Thus, $\mathfrak{Q}$ is a quaternion group and $\mathfrak{Q}\mathfrak{A}_1\langle J \rangle \cong GL(2, 3)$. Let $\mathfrak{L}$ be a $S_{2,3}$-subgroup of $\boldsymbol{C}^*_{\mathfrak{G}}(\mathfrak{A}_0)$ with $\mathfrak{Q}\mathfrak{K}_3\langle J \rangle \subseteqq L$. Let $\mathfrak{L}_0 = \boldsymbol{O}_3(\mathfrak{L})$. Since $\boldsymbol{D}(\mathfrak{E}_0)\mathfrak{K}_3$ is elementary of order 3^4 and contains an element of $\mathscr{U}(3)$, it follows that $\boldsymbol{O}_{3'}(\mathfrak{L}) = 1$. Since $\mathfrak{L}$ contains no noncyclic abelian subgroup of order 8, we get that $\mathfrak{Q}\mathfrak{A}_1\langle J \rangle$ is a complement to $\mathfrak{L}_0$ in $\mathfrak{L}$. Since I inverts $\mathfrak{L}_0/\mathfrak{A}_0$, it follows that $|\mathfrak{L}_0 : \mathfrak{A}_0| = 3^{2d}$ for some integer $d \geqq 1$. If $d = 1$, then $\boldsymbol{D}(\mathfrak{E}_0)\mathfrak{K}_3$ is a S_3-subgroup of $\boldsymbol{C}(\mathfrak{A}_0)$ and so S_3-subgroups of $\mathfrak{L}$ are abelian. This is absurd, so $d \geqq 2$. Since $|\mathfrak{G}|_3 = 3^7$ by (e), and since $|\mathfrak{L}|_3 = 3^{2d+2}$, we get $d = 2$.

Let $\mathfrak{L}_3$ be a S_3-subgroup of $\mathfrak{L}$ containing $\mathfrak{K}_3$. Since $\mathfrak{L}_3$ is not a S_3-subgroup of $\mathfrak{G}$, and since $\mathfrak{A}_0 \triangle \mathfrak{L}$, it follows that $\boldsymbol{Z}(\mathfrak{L}_3)$ is noncyclic. In particular, $\boldsymbol{Z}(\mathfrak{L}_0) \supseteqq \boldsymbol{Z}(\mathfrak{L}_3)$, so that $\boldsymbol{Z}(\mathfrak{L}_0)$ is not cyclic. Hence, $|\boldsymbol{Z}(\mathfrak{L}_0)| \geqq 3^3$. This implies that if $L \in \mathfrak{L}_0$, then $|\boldsymbol{C}_{\mathfrak{L}_0}(L)| \geqq 3^4$. Choose G in $\mathfrak{G}$ so that $\mathfrak{L}_3 \subseteqq \mathfrak{E}_3^G$, which is possible by (e). By (c5), we get $\mathfrak{L}_0 \subseteqq \mathfrak{E}_0^G$. Hence, $\mathfrak{L}_0 = \mathfrak{L}_{00} \times \mathfrak{L}_{01}$, where $\mathfrak{L}_{00}, \mathfrak{L}_{01}$ admit $\mathfrak{Q}$, $\mathfrak{L}_{00}$ is nonabelian of exponent 3 and order 3^3 and $\mathfrak{L}_{01}$ is elementary of order 3^2. Now $\mathfrak{A}_1 \subseteqq \mathfrak{L}_3$ and $|\boldsymbol{C}_{\mathfrak{L}_3}(\mathfrak{A}_1) \geqq 3^4$, so we get that $\mathfrak{A}_1 \subseteqq \mathfrak{E}_0^G$. Hence, $\mathfrak{L}_3 = \mathfrak{L}_0\mathfrak{A}_1 \subseteqq \mathfrak{E}_0^G$, and so $\mathfrak{L}_3 = \mathfrak{E}_0^G$. This is impossible, since $|\boldsymbol{Z}(\mathfrak{L}_3)| = 3^2$, $|\boldsymbol{Z}(\mathfrak{E}_0)| = 3^3$.

LEMMA 9.34. *Each involution of* $\mathfrak{K}_2^* - \langle I \rangle$ *inverts* $\mathfrak{K}_3$.

Proof. Let $\mathfrak{P}^*$ be a S_2-subgroup of $\mathfrak{G}$ which contains $\mathfrak{K}_3$ and is normalized by $\mathfrak{K}_2^*$, set $\mathfrak{X} = \mathfrak{K}_3 \cap \boldsymbol{C}(\mathfrak{K}_2^*)$. Suppose $\mathfrak{X} \neq 1$. Then $|\mathfrak{X}| = 3$, so by a formula of Wielandt [44], $|\mathfrak{P}^*| \leqq 3^4$. This contradicts Lemma 9.31. Hence, $\mathfrak{X} = 1$. As $\mathfrak{K}_2^* = \langle I, J \rangle$ for some involution J, the proof is complete.

LEMMA 9.35. (a) *If* $\mathfrak{X}$ *is a subgroup of* $\mathfrak{R}_3$ *of order* 3 *and* $\boldsymbol{C}(\mathfrak{X}) \cap \mathfrak{R}_0$ *is quaternion, then* $|\boldsymbol{C}(\mathfrak{X})|_3 = 3^4$.

(b) $|\boldsymbol{C}(\mathfrak{R})|_3 = 3^3$.

Proof. (a) Set $\mathfrak{Q} = \boldsymbol{C}(\mathfrak{X}) \cap \mathfrak{R}_0$, and let $\mathfrak{Y}$ be a subgroup of $\mathfrak{R}_3$ of order 3 distinct from $\mathfrak{X}$. Let J be an involution of $\mathfrak{R}_2^* - \langle I \rangle$. Thus, J inverts $\mathfrak{R}_3$ by Lemma 9.34. Also, $\langle J \rangle \mathfrak{Y}\mathfrak{Q} \cong GL(2, 3)$.

Let $\mathfrak{C}$ be a $S_{2,3}$-subgroup of $N(\mathfrak{X})$ which contains $\mathfrak{R}_3\mathfrak{Q}\mathfrak{R}_2^*$. Thus, $\mathfrak{Q}\langle J \rangle$ is a S_2-subgroup of $\mathfrak{C}$ and $\boldsymbol{O}_2(\mathfrak{C}) = 1$. Since

$$[\boldsymbol{O}_3(\mathfrak{C}) \cap \mathfrak{R}_3, \mathfrak{Q}] \subseteq \boldsymbol{O}_3(\mathfrak{C}) \cap \mathfrak{Q} = 1 ,$$

it follows that $\boldsymbol{O}_3(\mathfrak{C}) \cap \mathfrak{R}_3 = \mathfrak{X}$. Hence, I inverts $\boldsymbol{O}_3(\mathfrak{C})/\mathfrak{X}$. Hence, $\boldsymbol{O}_3(\mathfrak{C})/\mathfrak{X}$ is the direct sum of a certain number, say k, of modules each isomorphic to the faithful irreducible $F_3\mathfrak{Q}$-module, so that $|\boldsymbol{O}_3(\mathfrak{C}) : \mathfrak{X}| = 3^{2k}$. Hence, $|\mathfrak{C}|_3 = |\boldsymbol{C}(\mathfrak{R})|_3 = 3^{2(k+1)}$. Suppose $k \geqq 2$. Then by Lemma 9.33, we get $|\mathfrak{C}|_3 = |\mathfrak{G}|_3 = 3^6$.

We argue that $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})) = \mathfrak{X}$. Suppose false. We get $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})) = (\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})) \cap \boldsymbol{C}(I)) \times [\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})), I]$. Since $\mathfrak{R}_3 \cap \boldsymbol{O}_3(\mathfrak{C}) =$, we get

$$\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})) \cap \boldsymbol{C}(I) = \mathfrak{X} ;$$

also $[\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})), I]$ is normalized by $\mathfrak{Y}$, so if $[\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C})), I] \neq 1$, then a S_3-subgroup of $\mathfrak{G}$ has a noncyclic center. We conclude that $\mathfrak{X} = \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{C}))$. This implies that $\boldsymbol{O}_3(\mathfrak{C})$ is extra special of width 2. Since $\mathfrak{Q}\langle J \rangle$ is a S_2-subgroup of $N(\mathfrak{X})$, it follows that $\boldsymbol{O}_3(\mathfrak{C}) = \boldsymbol{O}_3(N(\mathfrak{X}))$. Thus, Hypothesis 9.2 is satisfied. Since this is not the case, we get $k = 1$. Thus (a) holds.

By Lemma 9.20, we have $|\boldsymbol{C}(\mathfrak{R}_3)|_3 \geqq 27$. Since $\mathfrak{R}_3$ is not central in a S_3-subgroup of $\mathfrak{C}$, (b) follows.

LEMMA 9.36. *Let* $\mathfrak{P}$ *be a* S_3*-subgroup of* $\mathfrak{G}$. *Then*

(a) $|\mathfrak{P}| = 3^5$.

(b) $\mathfrak{P}/\boldsymbol{Z}(\mathfrak{P})$ *is of maximal class and order* 3^4.

Proof. By Lemma 9.33, there is a conjugate $\mathfrak{B}$ of $\mathfrak{R}_2^*$ which normalizes $\mathfrak{P}$. By Lemma 9.32, all involutions of $\mathfrak{B}$ are conjugate to I. Let V_1, V_2, V_3 be the involutions of $\mathfrak{B}$. By Lemma 9.34, $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{B}) = 1$. By Lemma 9.29, $|\boldsymbol{C}_{\mathfrak{P}}(V_i)| \leqq 9$ for $i = 1, 2, 3$. Then by Wielandt [44], $|\mathfrak{P}| \leqq 3^6$.

Set $\mathfrak{Z} = \boldsymbol{Z}(\mathfrak{P})$. Since $\mathfrak{Z}$ is cyclic, $\boldsymbol{C}(\mathfrak{Z}) \cap \mathfrak{B} \neq 1$. We may assume notation is chosen so that V_1 is a generator for $\boldsymbol{C}(\mathfrak{Z}) \cap \mathfrak{B}$. Thus, $|\mathfrak{Z}| = 3$. Suppose V_1 inverts $\mathfrak{P}/\mathfrak{Z}$. Then, $|\mathfrak{P}| \leqq 3^5$, so by Lemma 9.31, $|\mathfrak{P}| = 3^5$. In this case, since $\mathfrak{P}$ is generated by elements of order 3, we get that $\mathfrak{Z} = \mathfrak{P}' = \boldsymbol{D}(\mathfrak{P})$. Since $\boldsymbol{O}_{3'}(N(\mathfrak{Z}) = 1$, so also $\boldsymbol{O}_{3'}(\mathfrak{Z})/\mathfrak{Z}) = 1$.

Hence, $\mathfrak{P} \lhd N(\mathfrak{Z})$. Thus, Hypothesis 9.2 is satisfied. Since this is not the case, we conclude that V_1 does not invert $\mathfrak{P}/\mathfrak{Z}$.

Let $\mathfrak{U}$ be a subgroup of $C_{\mathfrak{P}}(V_1)$ of order 3 distinct from $\mathfrak{Z}$. Thus, $C_{\mathfrak{P}}(V_1) = \mathfrak{Z}\mathfrak{U}$.

By Lemma 9.35(b), we get $|C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})| \leqq 27$. Since $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$, we have $|C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})| = 27$. Again, since $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$, if follows that $|N_{\mathfrak{P}/\mathfrak{Z}}(\mathfrak{Z}\mathfrak{U}/\mathfrak{Z})| = 9$. Thus, $\mathfrak{P}/\mathfrak{Z}$ is of maximal class, and $\mathfrak{U}\mathfrak{Z}/\mathfrak{Z} \not\subseteq (\mathfrak{P}/\mathfrak{Z})'$. Since $\mathfrak{U}\mathfrak{Z}/\mathfrak{Z}$ is the set of fixed points of V_1 on $\mathfrak{P}/\mathfrak{Z}$, it follows that $\mathfrak{P}/\mathfrak{Z}$ has a subgroup $\mathfrak{P}_0/\mathfrak{Z}$ of index 3 which is inverted by $\mathfrak{V}_1$. Since $\mathfrak{P}_0/\mathfrak{Z}$ is generated by elements of order 3, $\mathfrak{P}_0/\mathfrak{Z}$ is elementary. If $|\mathfrak{P}_0/\mathfrak{Z}| \geqq 3^4$, then $\mathfrak{P}/\mathfrak{Z}$ is not of maximal class. Hence, $|\mathfrak{P}_0/\mathfrak{Z}| \leqq 27$, so by Lemma 9.31, we have $|\mathfrak{P}_0 : \mathfrak{Z}| = 27$. This establishes both (a) and (b).

We may now complete the proof of Theorem 9.1. Let $\mathfrak{P}, \mathfrak{P}_0, \mathfrak{Z}, \mathfrak{U}, \mathfrak{V}, V_1$ be as above. Thus, $|\mathfrak{P}_0| = 3^4$, $\mathfrak{P}_0/\mathfrak{Z}$ is elementary of order 27 and is inverted by V_1. Being generated by elements of order 3, $\mathfrak{P}_0$ is of exponent 3. It follows that $Z(\mathfrak{P}_0)$ is not cyclic. Hence, we can choose a subgroup $\mathfrak{W}$ of $Z(\mathfrak{P}_0)$ of order 9 which is normal in $\mathfrak{P}\mathfrak{V}$. Set $\mathfrak{Y} = \mathfrak{W}\mathfrak{U}$. Thus, $\mathfrak{Y}$ is of order 27 and $\mathfrak{Y}$ admits V_1. Thus, $\mathfrak{Z}\mathfrak{U} \triangle \mathfrak{Y}$, so $\mathfrak{Y}$ is abelian, since $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$. This implies that $\mathfrak{W} \subset Z(\mathfrak{P})$, since $\mathfrak{P} = \mathfrak{P}_0\mathfrak{U}$. This contradiction completes the proof of Theorem 9.1.

Theorems 8.1 and 9.1 provide a proof of Theorem ES.

A substantial number of simplifications and corrections have been supplied by E. C. Dade.

Received October 23, 1964, and in revised form August 8, 1969. Research supported by a National Science Foundation Grant, GN-530, to the American Mathematical Society. The author also thanks the Sloan Foundation for its extended support.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

* Acting Managing Editor.

Pacific Journal of Mathematics
Vol. 33, No. 2 April, 1970