WILD POINTS OF CELLULAR ARCS IN 2-COMPLEXES IN $E^3$
AND CELLULAR HULLS

GAIL ATNEOSEN
Loveland has established that if $W$ is the set of wild points of a cellular arc that lies on a 2-sphere in $E^3$, then either $W$ is empty, $W$ is degenerate, or $W$ contains an arc. This note considers 2-complexes rather than 2-spheres. Making strong use of Loveland's results and others, it is proved that a cellular arc in a 2-complex in $E^3$ either contains an arc of wild points or has at most one wild point that has a neighborhood in the 2-complex homeomorphic to an open 2-cell. In the case of noncellular arcs in $E^3$, one can investigate "minimal cellular sets" containing the arc. A cellular hull of a subset $A$ of $E^3$ is a cellular set containing $A$ such that no proper cellular set also contains $A$. A characterization is given of those arcs in $E^3$ that have cellular hulls that lie in tame 2-complexes in $E^3$.

A 2-complex in $E^3$ is the homeomorphic image of a 2-dimensional finite Euclidean polyhedron. A subset $X$ of $E^3$ is said to be locally tame at a point $p$ of $X$ if there is a neighborhood $N$ of $p$ in $E^3$ and a homeomorphism $h$ of $\text{Cl}(N)$ (Cl = closure) onto a polyhedron in $E^3$ such that $h(\text{Cl}(N \cap X))$ is a finite Euclidean polyhedron. A point $p$ of a subset $X$ of $E^3$ is said to be a wild point of $X$ if $X$ is not locally tame at $p$. A subset $G$ of $E^3$ is said to be cellular (in $E^3$) if there exists a sequence $Q_1, Q_2, \cdots$ of 3-cells in $E^3$ such that for each positive integer $i$, $Q_{i+1} \subset \text{Interior} \ Q_i$ and $G = \bigcap_{i=1}^{\infty} Q_i$. If $A$ and $B$ are two arcs in $E^3$, then $A$ is said to be equivalent to $B$ if there is a homeomorphism $h$ mapping $E^3$ onto $E^3$ such that $h(A) = B$.

**Theorem 1.** Let $A$ be a cellular arc in a 2-complex in $E^3$. If the set of wild points of $A$ does not contain an arc, then $A$ has at most one wild point that has a neighborhood in the 2-complex homeomorphic to an open 2-cell.

**Proof.** Assume that $A$ has two wild points $p$ and $q$ that have neighborhoods in the 2-complex homeomorphic to an open 2-cell and contradict the hypothesis that $A$ is cellular. Then $p$ lies on a subarc of $A$ that is contained in the interior of a closed 2-cell. The argument of Theorem 5 of [3] then establishes that $p$ lies on a subarc $C$ of $A$ that is contained in a 2-sphere in $E^3$. Since $C$ is a cellular arc by [6], it follows from [5] that $p$ is the only wild point of $C$. Thus $p$ and $q$ are isolated wild points of $A$. 

551
If $p$ and $q$ are the endpoints of $A$, it follows from Theorem 10 of [8] that $A$ is not cellular, so this case cannot occur.

Next consider the case when $p$ is an interior point of $A$ and $q$ is an endpoint of $A$. As above, we obtain that $p$ lies interior to a subarc $C$ of $A$ whose only wild point is $p$ and that $C$ is contained in a 2-sphere $S$. By [4] and [2] we may assume that $S$ is locally polyhedral except at $p$. If $C_1$ and $C_2$ are subarcs of $C$ such that $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = p$, then Theorem 5 of [4] implies that $C_1$ and $C_2$ are equivalent. An application of Theorem 1 of [4] yields that if $C_1$ and $C_2$ are both locally tame at $p$ then $C$ is locally tame at $p$. Hence $p$ is a wild point of both $C_1$ and $C_2$. Let $B$ be a subarc of $A$ with endpoints $p$ and $q$. Then $B$ is a cellular arc whose endpoints are isolated wild points, by [8] this case cannot occur.

By arguments as in the above two cases, it follows that the last case, in which both $p$ and $q$ are interior points of $A$, can also not occur.

For the following theorem we need to define a particular 2-complex called a 3-book. A 3-book is defined to be a subset of $E^3$ which is the union of three closed 2-cells which meet precisely on a single arc on the boundary of each.

**THEOREM 2.** An arc $A$ in $E^3$ has a cellular hull that lies in a tame 2-complex in $E^3$ if and only if $A$ is equivalent to an arc in a tame 3-book.

**Proof.** If $A$ has a cellular hull that lies in a tame 2-complex, then the set of wild points of $A$ is a closed totally disconnected set. It follows easily from [7] that such an arc is equivalent to an arc in a tame 3-book.

Conversely, suppose that $A$ lies in a tame 3-book $B$. Consider a maximal chain (ordered by inclusion) that has $B$ as a member and also has the property that each member of the chain is a cellular set that contains $A$. The intersection of the members of this maximal chain then yields a cellular hull of $A$ that lies in the tame 2-complex $B$.

The arc in [1] is an example of an arc that does not have a cellular hull that lies in any tame 2-complex.

**REFERENCES**


Received October 22, 1969.

WESTERN WASHINGTON STATE COLLEGE
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. Sameelson
Stanford University
Stanford, California 94305

J. Dugundji
Department of Mathematics
University of Southern California
Los Angeles, California 90007

Richard Pierce
University of Washington
Seattle, Washington 98105

Richard Arens
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. Beckenbach B. H. Neumann F. Wole K. Yoshida

SUPPORTING INSTITUTIONS

University of British Columbia Stanford University
California Institute of Technology University of Tokyo
University of California University of Utah
Montana State University Washington State University
University of Nevada University of Washington
New Mexico State University
Oregon State University
University of Oregon
Osaka University
University of Southern California

* * *

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial “we” must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. 36, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
Pacific Journal of Mathematics  
Vol. 33, No. 3 May, 1970

<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charles A. Akemann, <em>Approximate units and maximal abelian C</em>-subalgebras*</td>
<td>543</td>
</tr>
<tr>
<td>Gail Atneosen, <em>Wild points of cellular arcs in 2-complexes in E^3 and cellular hulls</em></td>
<td>551</td>
</tr>
<tr>
<td>John Logan Bryant and De Witt Sumners, <em>On embeddings of 1-dimensional compacta in a hyperplane in E^4</em></td>
<td>555</td>
</tr>
<tr>
<td>H. P. Dikshit, <em>On a class of Nörlund means and Fourier series</em></td>
<td>559</td>
</tr>
<tr>
<td>Nancy Dykes, <em>Generalizations of realcompact spaces</em></td>
<td>571</td>
</tr>
<tr>
<td>Hector O. Fattorini, <em>Extension and behavior at infinity of solutions of certain linear operational differential equations</em></td>
<td>583</td>
</tr>
<tr>
<td>Neal David Glassman, <em>Cohomology of nonassociative algebras</em></td>
<td>617</td>
</tr>
<tr>
<td>Neal Hart, <em>Ulm’s theorem for Abelian groups modulo bounded groups</em></td>
<td>635</td>
</tr>
<tr>
<td>Don Barker Hinton, <em>Continuous spectra of second-order differential operators</em></td>
<td>641</td>
</tr>
<tr>
<td>Donald Gordon James, <em>On Witt’s theorem for unimodular quadratic forms. II</em></td>
<td>645</td>
</tr>
<tr>
<td>Melvin F. Janowitz, <em>Principal multiplicative lattices</em></td>
<td>653</td>
</tr>
<tr>
<td>James Edgar Keesling, <em>On the equivalence of normality and compactness in hyperspaces</em></td>
<td>657</td>
</tr>
<tr>
<td>Adalbert Kerber, <em>Zu einer Arbeit von J. L. Berggren über ambivalente Gruppen</em></td>
<td>669</td>
</tr>
<tr>
<td>Keizô Kikuchi, <em>Various m-representative domains in several complex variables</em></td>
<td>677</td>
</tr>
<tr>
<td>Jack W. Macki and James Stephen Muldowney, <em>The asymptotic behaviour of solutions to linear systems of ordinary differential equations</em></td>
<td>693</td>
</tr>
<tr>
<td>Andy R. Magid, <em>Locally Galois algebras</em></td>
<td>707</td>
</tr>
<tr>
<td>T. S. Ravisankar, <em>On differentiably simple algebras</em></td>
<td>725</td>
</tr>
<tr>
<td>Joseph Gail Stampfli, <em>The norm of a derivation</em></td>
<td>737</td>
</tr>
<tr>
<td>Francis C.Y. Tang, <em>On uniqueness of central decompositions of groups</em></td>
<td>749</td>
</tr>
<tr>
<td>Robert Charles Thompson, <em>Some matrix factorization theorems. I</em></td>
<td>763</td>
</tr>
<tr>
<td>Robert Charles Thompson, <em>Some matrix factorization theorems. II</em></td>
<td>811</td>
</tr>
</tbody>
</table>