WILD POINTS OF CELLULAR ARCS IN 2-COMPLEXES IN $E^3$
AND CELLULAR HULLS

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Loveland has established that if $W$ is the set of wild points of a cellular arc that lies on a 2-sphere in $E^3$, then either $W$ is empty, $W$ is degenerate, or $W$ contains an arc. This note considers 2-complexes rather than 2-spheres. Making strong use of Loveland's results and others, it is proved that a cellular arc in a 2-complex in $E^3$ either contains an arc of wild points or has at most one wild point that has a neighborhood in the 2-complex homeomorphic to an open 2-cell. In the case of noncellular arcs in $E^3$, one can investigate "minimal cellular sets" containing the arc. A cellular hull of a subset $A$ of $E^3$ is a cellular set containing $A$ such that no proper cellular set also contains $A$. A characterization is given of those arcs in $E^3$ that have cellular hulls that lie in tame 2-complexes in $E^3$.

A 2-complex in $E^3$ is the homeomorphic image of a 2-dimensional finite Euclidean polyhedron. A subset $X$ of $E^3$ is said to be locally tame at a point $p$ of $X$ if there is a neighborhood $N$ of $p$ in $E^3$ and a homeomorphism $h$ of $\text{Cl}(N)$ ($\text{Cl} = \text{closure}$) onto a polyhedron in $E^3$ such that $h(\text{Cl}(N \cap X))$ is a finite Euclidean polyhedron. A point $p$ of a subset $X$ of $E^3$ is said to be a wild point of $X$ if $X$ is not locally tame at $p$. A subset $G$ of $E^3$ is said to be cellular (in $E^3$) if there exists a sequence $Q_i, Q_{i+1}, \cdots$ of 3-cells in $E^3$ such that for each positive integer $i$, $Q_{i+1} \subset \text{Interior} Q_i$ and $G = \bigcap_{i=1}^{\infty} Q_i$. If $A$ and $B$ are two arcs in $E^3$, then $A$ is said to be equivalent to $B$ if there is a homeomorphism $h$ mapping $E^3$ onto $E^3$ such that $h(A) = B$.

**Theorem 1.** Let $A$ be a cellular arc in a 2-complex in $E^3$. If the set of wild points of $A$ does not contain an arc, then $A$ has at most one wild point that has a neighborhood in the 2-complex homeomorphic to an open 2-cell.

**Proof.** Assume that $A$ has two wild points $p$ and $q$ that have neighborhoods in the 2-complex homeomorphic to an open 2-cell and contradict the hypothesis that $A$ is cellular. Then $p$ lies on a subarc of $A$ that is contained in the interior of a closed 2-cell. The argument of Theorem 5 of [3] then establishes that $p$ lies on a subarc $C$ of $A$ that is contained in a 2-sphere in $E^3$. Since $C$ is a cellular arc by [6], it follows from [5] that $p$ is the only wild point of $C$. Thus $p$ and $q$ are isolated wild points of $A$.  

551
If $p$ and $q$ are the endpoints of $A$, it follows from Theorem 10 of [8] that $A$ is not cellular, so this case cannot occur.

Next consider the case when $p$ is an interior point of $A$ and $q$ is an endpoint of $A$. As above, we obtain that $p$ lies interior to a subarc $C$ of $A$ whose only wild point is $p$ and that $C$ is contained in a 2-sphere $S$. By [4] and [2] we may assume that $S$ is locally polyhedral except at $p$. If $C_1$ and $C_2$ are subarcs of $C$ such that $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = p$, then Theorem 5 of [4] implies that $C_1$ and $C_2$ are equivalent. An application of Theorem 1 of [4] yields that if $C_1$ and $C_2$ are both locally tame at $p$ then $C$ is locally tame at $p$. Hence $p$ is a wild point of both $C_1$ and $C_2$. Let $B$ be a subarc of $A$ with endpoints $p$ and $q$. Then $B$ is a cellular arc whose endpoints are isolated wild points, by [8] this case cannot occur.

By arguments as in the above two cases, it follows that the last case, in which both $p$ and $q$ are interior points of $A$, can also not occur.

For the following theorem we need to define a particular 2-complex called a 3-book. A 3-book is defined to be a subset of $E^3$ which is the union of three closed 2-cells which meet precisely on a single arc on the boundary of each.

**Theorem 2.** An arc $A$ in $E^3$ has a cellular hull that lies in a tame 2-complex in $E^3$ if and only if $A$ is equivalent to an arc in a tame 3-book.

**Proof.** If $A$ has a cellular hull that lies in a tame 2-complex, then the set of wild points of $A$ is a closed totally disconnected set. It follows easily from [7] that such an arc is equivalent to an arc in a tame 3-book.

Conversely, suppose that $A$ lies in a tame 3-book $B$. Consider a maximal chain (ordered by inclusion) that has $B$ as a member and also has the property that each member of the chain is a cellular set that contains $A$. The intersection of the members of this maximal chain then yields a cellular hull of $A$ that lies in the tame 2-complex $B$.

The arc in [1] is an example of an arc that does not have a cellular hull that lies in any tame 2-complex.

**References**


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*Western Washington State College*
Charles A. Akemann, *Approximate units and maximal abelian C*-subalgebras* .................................................. 543

Gail Atneosen, *Wild points of cellular arcs in 2-complexes in E^3 and cellular hulls* .................................................. 551

John Logan Bryant and De Witt Sumners, *On embeddings of 1-dimensional compacta in a hyperplane in E^4* .................. 555

H. P. Dikshit, *On a class of Nörlund means and Fourier series* ................................. 559

Nancy Dykes, *Generalizations of realcompact spaces* .................................................. 571

Hector O. Fattorini, *Extension and behavior at infinity of solutions of certain linear operational differential equations* ........... 583

Neal David Glassman, *Cohomology of nonassociative algebras* ............................................. 617

Neal Hart, *Ulrm’s theorem for Abelian groups modulo bounded groups* ......................... 635

Don Barker Hinton, *Continuous spectra of second-order differential operators* .................. 641

Donald Gordon James, *On Witt’s theorem for unimodular quadratic forms. II* ................. 645

Melvin F. Janowitz, *Principal multiplicative lattices* .................................................. 653

James Edgar Keesling, *On the equivalence of normality and compactness in hyperspaces* ............ 657

Adalbert Kerber, *Zu einer Arbeit von J. L. Berggren über ambivalente Gruppen* ................. 669

Keizō Kikuchi, *Various m-representative domains in several complex variables* .................. 677

Jack W. Macki and James Stephen Muldowney, *The asymptotic behaviour of solutions to linear systems of ordinary differential equations* ......................... 693

Andy R. Magid, *Locally Galois algebras* .................................................. 707

T. S. Ravisankar, *On differentiably simple algebras* .................................................. 725

Joseph Gail Stampfli, *The norm of a derivation* .................................................. 737

Francis C.Y. Tang, *On uniqueness of central decompositions of groups* ......................... 749

Robert Charles Thompson, *Some matrix factorization theorems. I* ................................. 763

Robert Charles Thompson, *Some matrix factorization theorems. II* ................................. 811