ON EMBEDDINGS OF 1-DIMENSIONAL COMPACTA IN A HYPERPLANE IN $E^4$

John Logan Bryant and De Witt Sumners
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J. L. BRYANT AND D. W. SUMNERS

In this note a proof of the following theorem is given.

**THEOREM 1.** Suppose that $X$ is a 1-dimensional compactum in a 3-dimensional hyperplane $E^3$ in euclidean 4-space $E^4$, that $\varepsilon > 0$, and that $f: X \to E^3$ is an embedding such that $d(x, f(x)) < \varepsilon$ for each $x \in X$. Then there exists an $\varepsilon$-push $h$ of $(E^4, X)$ such that $h|X = f$.

The proof of Theorem 1 is based on a technique exploited by the first author in [3]. This method requires that one be able to push $X$ off of the 2-skeleton of an arbitrary triangulation of $E^3$ using a small push of $E^4$. This could be done very easily if it were possible to push $X$ off of the 1-skeleton of a given triangulation of $E^3$ via a small push of $E^3$. Unfortunately, this cannot be accomplished unless $X$ has some additional property (such as local contractibility) as demonstrated by the examples of Bothe [2] and McMillan and Row [9]. However, we are able to overcome this difficulty by using a property of twisted spun knots obtained by Zeeman [10].

In the following theorem let $B^3$ denote the unit ball in $E^4$, $B^3$ the intersection of $B^3$ with the 3-plane $x_4 = 0$, and $D^2$ the intersection of $B^3$ with the 2-plane $x_3 = x_2 = 0$.

**THEOREM 2.** Let $X$ be a 1-dimensional compactum in $B^3$ such that $X \cap \text{Bd } D^2 = \emptyset$. Then there exists an isotopy $h_t: B^4 \to B^4$ ($t \in [0, 1]$) such that

1. $h_0 = \text{identity}$,
2. $h_t|\text{Bd } B^4 = \text{identity for each } t \in [0, 1]$, and
3. $h_t(X) \cap D^2 = \emptyset$.

**Proof.** Let $I = D^2 \cap B^3$. Since $X$ does not separate $B^3$, there exists a polygonal arc $J$ in $B^3 - X$ joining one endpoint of $I$ to the other. We may assume, by applying an appropriate isotopy of $B^4$, that $J_+$, the intersection of $J$ with the half-space $x_3 \geq 0$ is contained in $I$. Let $F$ be a 3-cell in $B^3$ such that $F \cap J = J_+$ and $F \cap X = \emptyset$, and let $J_-$ be the intersection of $J$ with the half-space $x_3 \leq 0$. Now spin the arc $J_-$ about the plane $x_3 = x_4 = 0$, twisting once, so that at time $t = \pi$, $J_-$ lies in $F$. (See Zeeman [10] for the details of this construction.) Observe that the boundary of the 2-cell $C$ traced out by $J_-$ is the same as $\text{Bd } D^2$. 

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It follows from [10, Corollary 2] that the pair \((B^4, C)\) is equivalent to the pair \((B^4, D^2)\) by an isotopy that keeps \(\text{Bd} B^4\) fixed. Such an isotopy, of course, will push \(X\) off of \(D^2\).

**Theorem 3.** Let \(X\) be a 1-dimensional compactum in a 3-plane \(E^3\) in \(E^4\). Then for each 2-complex \(K\) in \(E^4\) and each \(\varepsilon > 0\), there exists an \(\varepsilon\)-push \(h\) of \((E^4, X)\) such that \(h(X) \cap K = \emptyset\).

**Proof.** Given a 2-complex \(K\) and \(\varepsilon > 0\), we may assume first of all that none of the vertices of \(K\) lies in \(E^3\). Also, we may move the 1-simplexes of \(K\) slightly so that they do not meet \(X\).

Let \(\sigma\) be a 2-simplex of \(K\) such that \(\sigma \cap X \neq \emptyset\). By moving \(X\) an arbitrarily small amount, keeping it in \(E^3\), we can ensure that each component of \(\sigma \cap X\) not only lies in \(\text{Int} \sigma\), but has diameter less than \(\varepsilon\). Hence, we can get \(\sigma \cap X\) into a finite number of mutually exclusive line segments \(I_1, \cdots, I_n\) in \(\text{Int} \sigma \cap E^3\), each of which having diameter less than \(\varepsilon\). Let \(B_1, \cdots, B_n\) be a collection of mutually exclusive 4-cells in \(E^4\), each of diameter less than \(\varepsilon\), such that each triple \((B_j, B_j \cap E^3, B_j \cap \sigma)\) is equivalent to the triple \((B^4, B^3, D^2)\) (as defined above) and such that \(B_j \cap \sigma \cap E^3 = I_j\). Now apply Theorem 2 to each of the \(B_j (j = 1, \cdots, n)\).

**Lemma.** Suppose that \(X \subset E^3 \subset E^4\) and \(f: X \rightarrow E^3\) are as in the statement of Theorem 1 with \(d(x, f(x)) < \varepsilon\) for each \(x \in X\). Then for each \(\delta > 0\) there exists an \(\varepsilon\)-push \(h\) of \((E^4, X)\) such that \(d(h(x)), f(x)) < \delta\) for each \(x \in X\).

**Proof.** Apply the proof of Lemma 2 of [3] with \(p = 2\) and \(q = 1\).

The proof of Theorem 1 is now obtained by applying the technique employed in the proof of Theorem 4.4 of [7]. The only additional observation that should be made is that if \(X\) is a compactum in \(E^4\) satisfying the conclusion of Theorem 3 and if \(g\) is a homeomorphism of \(E^4\), then \(g(X)\) also satisfies the conclusion of Theorem 3 with respect to 2-complexes in the piecewise linear structure on \(E^4\) induced by \(g\).

**Corollary.** Let \(X\) be a 1-dimensional compactum in a 3-hyperplane in \(E^4\). Then for each \(\varepsilon > 0\) there exists a neighborhood of \(X\) in \(E^4\) that \(\varepsilon\)-collapses to a 1-dimensional polyhedron.

This follows from the fact that every 1-dimensional compactum can be embedded in \(E^3\) so as to have this property in \(E^3\).

Bothe [2] and McMillan and Row [9] have examples which show that not every embedding of the Menger universal curve in \(E^3\) has
small neighborhoods with 1-spines.

**Remark 1.** Notice that Theorem 1 is a consequence of a special case of a theorem of Bing and Kister [1] if $X$ is either a 1-dimensional polyhedron or a 0-dimensional compactum. If $X$ is a 2-dimensional polyhedron, then Theorem 1 is false in general as pointed out by Gillman [6]. It would be interesting to know for what 2-dimensional compacta Theorem 1 holds. For example, this theorem is true if $X$ is a compact 2-manifold [5].

**Remark 2.** One of the important properties of a compactum $X$ in a hyperplane in $E^n$ is that $E^n - X$ is 1-ALG (see [8]). If $n - \dim X \geq 3$, this is equivalent to saying that $E^n - X$ is 1-ULC. In [3] and [4] it is shown that any two such embeddings of $X$ into $E^n$ (regardless of whether they lie in a hyperplane) are equivalent, provided $n \geq 5$ and $2 \dim X + 2 \leq n$. Although there is no hope of improving this theorem by lowering the codimension of the embedding (at least for arbitrary compacta), Theorem 1 lends credence to the conjecture that this result holds when $n = 4$.

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