

Pacific Journal of Mathematics

**CONTINUOUS SPECTRA OF SECOND-ORDER DIFFERENTIAL
OPERATORS**

DON BARKER HINTON

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We consider the differential operator $l(y) = y'' + qy$, where q is a positive, continuously differentiable function defined on a ray $[a, \infty)$. The operator l determines, with appropriate restrictions, self-adjoint operators defined in the hilbert space $\mathcal{L}_2[a, \infty)$ of quadratically summable, complexvalued functions on $[a, \infty)$. In this note, we prove that if L is such a self-adjoint operator, then the conditions $q(t) \rightarrow \infty$ and $q'(t)q(t)^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$ are sufficient for the continuous spectrum $C(L)$ of L to cover the entire real axis.

Similar results are well-known; however, monotonicity conditions on q and q' are usually required. For example, in [1], p. 116, it is proved that if q tends monotonically to ∞ as $t \rightarrow \infty$, preserving the direction of convexity for large t , then the condition $q'(t)q(t)^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$ is sufficient to imply $C(L) = (-\infty, \infty)$ for every self-adjoint operator L determined by l .

THEOREM. *If $q(t) \rightarrow \infty$ as $t \rightarrow \infty$, $q'(t)q(t)^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$, and L is a self-adjoint operator in $\mathcal{L}_2[a, \infty)$ determined by l , then $C(L) = (-\infty, \infty)$.*

Proof. To prove that the real number λ belongs to $C(L)$, it is sufficient to construct a bounded noncompact sequence y_1, y_2, \dots such that $\|(L - \lambda)y_n\| \rightarrow 0$ as $n \rightarrow \infty$. The domain of L includes the set \mathcal{M} of all y satisfying (i) y has compact support contained in the open interval (a, ∞) , (ii) y' is absolutely continuous, and (iii) $y'' \in \mathcal{L}_2[a, \infty)$ (cf. [3], Chapter V). Hence, it follows that $\lambda \in C(L)$, if we prove that for each $\eta > 0$ and $N > a$, there is a nontrivial $y \in \mathcal{M}$ such that the support of y is contained in $[N, \infty)$ and $\|(L - \lambda)y\| < \eta\|y\|$. To establish this, we recall Lemma 2 of [2]:

Suppose f is a continuously differentiable positive function on $[b, \infty)$, and $f'(t)f(t)^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$. If ε and K are positive numbers, then there is a number B such that if t and s are $\geq B$ and $|t - s| \leq Kf(s)^{1/2}$, then $|f(t)f(s)^{-1} - 1| < \varepsilon$.

We choose $0 < \varepsilon < \eta^2/25$, $K > 6400/\eta^2$ (assume $\eta < 1$), and apply the lemma to $f = q - \lambda$ on an interval $[b, \infty)$ such that $f(t) \geq \Pi^2$ for $t \geq b$. Let $s_0 \geq \max\{N, B\}$ be such that $|f'(t)f(t)^{-1/2} < \varepsilon$ for $t \geq s_0$.

Define s_1, s_2, \dots by

$$s_{i+1} = s_i + \Pi f(s_i)^{-1/2} \quad (i = 0, 1, \dots),$$

and denote $f(s_i)^{1/2}$ by α_i . Since for $s_i \leq s_0 + K\alpha_0$, we have $\alpha_i^2/\alpha_0^2 \leq 1 + \varepsilon < 4$, it follows that for such s_i ,

$$s_i - s_0 = \sum_{j=0}^{i-1} \Pi/\alpha_j \geq \Pi i/2\alpha_0;$$

thus there is an integer p so that $s_p \leq s_0 + K\alpha_0 < s_{p+1}$. We now construct a $y \in \mathcal{M}$ with support $[s_0, s_p]$.

Since $K > 9$ and $\alpha_i \geq \Pi$ for each i , there exist $\tau_1, \tau_2 \in \{s_0, \dots, s_p\}$ such that $s_0 < \tau_1 < \tau_2 < s_p$, $\alpha_0 \leq \tau_1 - s_0 \leq 2\alpha_0$, $\alpha_0 \leq s_p - \tau_2 \leq 2\alpha_0$, and $\tau_2 - \tau_1 \geq K\alpha_0/2$. Define h and g on $[a, \infty)$ to be zero exterior to $[s_0, s_p]$ and otherwise by

$$g(t) = (-1)^i \alpha_i^{-1} \sin \alpha_i(t - s_i) \text{ for } s_i \leq t \leq s_{i+1}, \quad (i = 0, \dots, p - 1)$$

and

$$h(t) = \begin{cases} (t - s_0)/(\tau_1 - s_0), & s_0 \leq t \leq \tau_1, \\ 1, & \tau_1 \leq t \leq \tau_2, \\ (s_p - t)/(s_p - \tau_2), & \tau_2 \leq t \leq s_p. \end{cases}$$

If $y = gh$, then a calculation yields that $y \in \mathcal{M}$.

Since $\varepsilon < 1/4$, from the lemma above we conclude that

$$(1) \quad f(t)/f(s) = \{f(t)/f(s_0)\}/\{f(s)/f(s_0)\} < (5/4)/3/4 < 2$$

for all $t, s \in [s_0, s_p]$. Applying the mean value theorem, it follows that for $t \in [s_i, s_{i+1}]$,

$$(2) \quad \begin{aligned} |f(t) - f(s_i)| &= |f'(t^*)(t - s_i)| \\ &\leq \{|f'(t^*)|f(t^*)^{-1/2}\}\{\Pi f(t^*)^{1/2}f(s_i)^{-1/2}\} \\ &< \Pi(2)^{1/2}\varepsilon < 5\varepsilon. \end{aligned}$$

For $t \in [s_i, s_{i+1}] \subset [s_0, \tau_1]$, we have by application of (1), (2), and $\tau_1 - s_0 \geq \alpha_0$ that

$$\begin{aligned} |y''(t) + f(t)y(t)| &= |2(\tau_1 - s_0)^{-1}(-1)^i \cos \alpha_i(t - s_i) \\ &\quad + (t - s_0)(\tau_1 - s_0)^{-1}[f(t) - f(s_i)]g(t)| \\ &\leq 2(\tau_1 - s_0)^{-1} + 5\varepsilon\alpha_i^{-1} \\ &< 2/\alpha_0 + 5(2)^{1/2}/\alpha_0 < 10/\alpha_0. \end{aligned}$$

From this inequality and $\tau_1 - s_0 \leq 2\alpha_0$, it follows that

$$(3) \quad \int_{s_0}^{\tau_1} |y'' + fy|^2 dt \leq (100/\alpha_0^2)(\tau_1 - s_0) \leq 200/\alpha_0.$$

Similarly, we have

$$(4) \quad \int_{\tau_2}^{s_p} |y'' + fy|^2 dt \leq 200/\alpha_0 .$$

For $[s_i, s_{i+1}] \subset [\tau_1, \tau_2]$, the definition of y and (1) yield

$$\int_{s_i}^{s_{i+1}} y^2 dt = (s_{i+1} - s_i)/2\alpha_i^2 \geq (s_{i+1} - s_i)/4\alpha_0^2 ,$$

hence, this inequality and $(\tau_2 - \tau_1) \geq K\alpha_0/2$ imply that

$$(5) \quad \int_{\tau_1}^{\tau_2} y^2 dt \geq (\tau_2 - \tau_1)/4\alpha_0^2 \geq K/8\alpha_0 .$$

Since on $[s_i, s_{i+1}]$,

$$|y''(t) + f(t)y(t)| = |[f(t) - f(s_i)]y(t)| \leq 5 \epsilon |y(t)| ,$$

we have

$$(6) \quad \begin{aligned} & \int_{s_i}^{s_{i+1}} |y'' + fy|^2 dt < 25\epsilon^2 \int_{s_i}^{s_{i+1}} y^2 dt; \text{ thus} \\ & \left\{ \int_{\tau_1}^{\tau_2} |y'' + fy|^2 dt \right\} \left\{ \int_{\tau_1}^{\tau_2} y^2 dt \right\}^{-1} < 25\epsilon^2 < \epsilon . \end{aligned}$$

From the definition of ϵ and K , (3), (4), (5), and (6), we obtain

$$\left\{ \int_{s_0}^{s_p} |y'' + fy|^2 dt \right\} \left\{ \int_{s_0}^{s_p} y^2 dt \right\}^{-1} < \{3200/K\} + \epsilon < \eta^2;$$

therefore the proof is complete.

In [3], p. 235, asymptotic methods are used to obtain criteria for $C(L) = (-\infty, \infty)$. In this development much of the argument depends on the divergent integral $\int_a^\infty q^{-1/2} dt = \infty$. The condition $q'(t)q(t)^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$ implies the divergence of this integral. We raise the following question for a class $C^{(1)}$ function q : Are the conditions $q(t) \rightarrow \infty$ as $t \rightarrow \infty$ (perhaps monotonically) and $\int_a^\infty q^{1/2} dt = \infty$ sufficient to imply $C(L) = (-\infty, \infty)$?

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