Pacific Journal of Mathematics

CONTINUOUS SPECTRA OF SECOND-ORDER DIFFERENTIAL OPERATORS

Don Barker Hinton

Vol. 33, No. 3 May 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 3, 1970

CONTINUOUS SPECTRA OF SECOND-ORDER DIFFERENTIAL OPERATORS

DON HINTON

We consider the differential operator $l(y) = y'' + qy$, where q is a positive, continuously differentiable function defined on a ray $[a, \infty)$. The operator l determines, with appropriate restrictions, self-adjoint operators defined in the hilbert space $\mathscr{L}_2[a, \infty)$ of quadratically summable, complexvalued functions on $[a, \infty)$. In this note, we prove that if L is such a self-adjoint operator, then the conditions $q(t) \to \infty$ and $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$ are sufficient for the continuous spectrum $C(L)$ of L to cover the entire real axis.

Similar results are well-known; however, monotonicity conditions on q and q' are usually required. For example, in [1], p. 116, it is proved that if q tends monotonically to ∞ as $t \to \infty$, preserving the direction of convexity for large t, then the condition $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$ is sufficient to imply $C(L) = (-\infty, \infty)$ for every self-adjoint operator L determined by l.

THEOREM. *If $q(t) \to \infty$ as $t \to \infty$, $q'(t)q(t)^{-1/2} \to 0$ as $t \to \infty$, and L is a self-adjoint operator in $\mathscr{L}_2[a, \infty)$ determined by l, then $C(L) = (-\infty, \infty)$.*

Proof. To prove that the real number λ belongs to $C(L)$, it is sufficient to construct a bounded noncompact sequence $y_1, y_2, \cdots$ such that $||(L - \lambda)y_n|| \to 0$ as $n \to \infty$. The domain of L includes the set $\mathscr{M}$ of all y satisfying (i) y has compact support contained in the open interval (a, ∞), (ii) y' is absolutely continuous, and (iii) $y'' \in \mathscr{L}_2[a, \infty)$ (cf. [3], Chapter V). Hence, it follows that $\lambda \in C(L)$, if we prove that for each $\eta > 0$ and $N > a$, there is a nontrivial $y \in \mathscr{M}$ such that the support of y is contained in $[N, \infty)$ and $||(L - \lambda)y|| < \eta||y/|$. To establish this, we recall Lemma 2 of [2]:

Suppose f is a continuously differentiable positive function on $[b, \infty)$, and $f'(t)f(t)^{-1/2} \to 0$ as $t \to \infty$. If ε and K are positive numbers, then there is a number B such that if t and s are $\geqq B$ and $|t - s| \leqq Kf(s)^{1/2}$, then $|f(t)f(s)^{-1} - 1| < \varepsilon$.

We choose $0 < \varepsilon < \eta^2/25$, $K > 6400/\eta^2$ (assume $\eta < 1$), and apply the lemma to $f = q - \lambda$ on an interval $[b, \infty)$ such that $f(t) \geqq \Pi^2$ for $t \geqq b$. Let $s_0 \geqq \max\{N, B\}$ be such that $|f'(t)|f(t)^{-1/2} < \varepsilon$ for $t \geqq s_0$.

Define $s_1, s_2, \cdots$ by

$$s_{i+1} = s_i + \Pi f(s_i)^{-1/2} \qquad (i = 0, 1, \cdots),$$

and denote $f(s_i)^{1/2}$ by α_i. Since for $s_i \leqq s_0 + K\alpha_0$, we have $\alpha_i^2/\alpha_0^2 \leqq 1 + \varepsilon < 4$, it follows that for such s_i,

$$s_i - s_0 = \sum_{j=0}^{i-1} \Pi/\alpha_j \geqq \Pi i/2\alpha_0 ;$$

thus there is an integer p so that $s_p \leqq s_0 + K\alpha_0 < s_{p+1}$. We now construct a $y \in \mathscr{M}$ with support $[s_0, s_p]$.

Since $K > 9$ and $\alpha_i \geqq \Pi$ for each i, there exist $\tau_1, \tau_2 \in \{s_0, \cdots, s_p\}$ such that $s_0 < \tau_1 < \tau_2 < s_p$, $\alpha_0 \leqq \tau_1 - s_0 \leqq 2\alpha_0$, $\alpha_0 \leqq s_p - \tau_2 \leqq 2\alpha_0$, and $\tau_2 - \tau_1 \geqq K\alpha_0/2$. Define h and g on $[a, \infty)$ to be zero exterior to $[s_0, s_p]$ and otherwise by

$$g(t) = (-1)^i \alpha_i^{-1} \sin \alpha_i(t - s_i) \text{ for } s_i \leqq t \leqq s_{i+1}, \qquad (i = 0, \cdots, p-1)$$

and

$$h(t) = \begin{cases} (t - s_0)/(\tau_1 - s_0), & s_0 \leqq t \leqq \tau_1 , \\ 1, & \tau_1 \leqq t \leqq \tau_2 , \\ (s_p - t)/(s_p - \tau_2), & \tau_2 \leqq t \leqq s_p . \end{cases}$$

If $y = gh$, then a calculation yields that $y \in \mathscr{M}$.

Since $\varepsilon < 1/4$, from the lemma above we conclude that

$$(1) \qquad f(t)/f(s) = \{f(t)/f(s_0)\}/\{f(s)/f(s_0)\} < (5/4)/3/4) < 2$$

for all $t, s \in [s_0, s_p]$. Applying the mean value theorem, it follows that for $t \in [s_i, s_{i+1}]$,

$$(2) \qquad \begin{aligned} |f(t) - f(s_i)| &= |f'(t^*)(t - s_i)| \\ &\leqq \{|f'(t^*)|f(t^*)^{-1/2}\}\{\Pi f(t^*)^{1/2} f(s_i)^{-1/2}\} \\ &< \Pi(2)^{1/2}\varepsilon < 5\varepsilon . \end{aligned}$$

For $t \in [s_i, s_{i+1}] \subset [s_0, \tau_1]$, we have by application of (1), (2), and $\tau_1 - s_0 \geqq \alpha_0$ that

$$\begin{aligned} |y''(t) + f(t)y(t)| &= |2(\tau_1 - s_0)^{-1}(-1)^i \cos \alpha_i(t - s_i) \\ &\qquad + (t - s_0)(\tau_1 - s_0)^{-1}[f(t) - f(s_i)]g(t)| \\ &\leqq 2(\tau_1 - s_0)^{-1} + 5\varepsilon\alpha_i^{-1} \\ &< 2/\alpha_0 + 5(2)^{1/2}/\alpha_0 < 10/\alpha_0 . \end{aligned}$$

From this inequality and $\tau_1 - s_0 \leqq 2\alpha_0$, it follows that

$$(3) \qquad \int_{s_0}^{\tau_1} |y'' + fy|^2 dt \leqq (100/\alpha_0^2)(\tau_1 - s_0) \leqq 200/\alpha_0 .$$

Similarly, we have

$$\int_{\tau_2}^{s_p} |y'' + fy|^2 dt \leqq 200/\alpha_0 . \tag{4}$$

For $[s_i, s_{i+1}] \subset [\tau_1, \tau_2]$, the definition of y and (1) yield

$$\int_{s_i}^{s_{i+1}} y^2 dt = (s_{i+1} - s_i)/2\alpha_i^2 \geqq (s_{i+1} - s_i)/4\alpha_0^2 ,$$

hence, this inequality and $(\tau_2 - \tau_1) \geqq K\alpha_0/2$ imply that

$$\int_{\tau_1}^{\tau_2} y^2 dt \geqq (\tau_2 - \tau_1)/4\alpha_0^2 \geqq K/8\alpha_0 . \tag{5}$$

Since on $[s_i, s_{i+1}]$,

$$|y''(t) + f(t)y(t)| = |[f(t) - f(s_i)]y(t)| \leqq 5\epsilon |y(t)| ,$$

we have

$$\begin{gathered}\int_{s_i}^{s_{i+1}} |y'' + fy|^2 dt < 25\varepsilon^2 \int_{s_i}^{s_{i+1}} y^2 dt; \textit{ thus} \\ \left\{\int_{\tau_1}^{\tau_2} |y'' + fy|^2 dt\right\}\left\{\int_{\tau_1}^{\tau_2} y^2 dt\right\}^{-1} < 25\varepsilon^2 < \varepsilon .\end{gathered} \tag{6}$$

From the definition of ε and K, (3), (4), (5), and (6), we obtain

$$\left\{\int_{s_0}^{s_p} |y'' + fy|^2 dt\right\}\left\{\int_{s_0}^{s_p} y^2 dt\right\}^{-1} < \{3200/K\} + \varepsilon < \eta^2;$$

therefore the proof is complete.

In [3], p. 235, asymptotic methods are used to obtain criteria for $C(L) = (-\infty, \infty)$. In this development much of the argument depends on the divergent integral $\int_a^\infty q^{-1/2} dt = \infty$. The condition $q'(t)q(t)^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$ implies the divergence of this integral. We raise the following question for a class $C^{(1)}$ function q: Are the conditions $q(t) \rightarrow \infty$ as $t \rightarrow \infty$ (perhaps monotonically) and $\int_a^\infty q^{1/2} dt = \infty$ sufficient to imply $C(L) = (-\infty, \infty)$?

REFERENCES

1. I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Israel program for scientific translation, Jerusalem, 1965.
2. D. B. Hinton, *Some stability conditions for a nonlinear differential equation*, Trans. Amer. Math. Soc. **139** (1969), 349-358.
3. M. A. Naimark, *Linear differential operators*, part II, Ungar, New York, 1968.

Received October 3, 1969.

UNIVERSITY OF GEORGIA
ATHENS, GEORGIA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539–1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 33, No. 3 May, 1970