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Let X be a topological space and 2^X the space of all closed subsets of X with the finite topology. Assuming the continuum hypothesis it is shown that 2^X is normal if and only if X is compact. It is not known if the continuum hypothesis is a necessary assumption, but it is shown that for X a k-space, 2^X normal implies X compact. A theorem about the compactification of the n-th symmetric product of a space X is first proved which then plays an important part in the proof of the above results.

Throughout this paper we will assume that X is any completely regular T_1 space. By 2^X we will mean the space of all closed subsets of X with the finite topology [13, Definition 1.7, p. 153] except that we include the empty set as an isolated point as in [12]. The finite topology is also known as the exponential or Vietoris topology. Let $\mathscr{F}_n(X)$ be the subspace of 2^X consisting of all nonempty subsets of X with n points or less. This space is known as the n-th symmetric product of X.

In this paper the normality of 2^X is studied. If X is compact, it is known that 2^X is compact Hausdorff [13, Th. 4.2, p. 161] and thus normal. The main result of this paper is that if we assume the continuum hypothesis (CH), then 2^X is normal if and only if X is compact. The first result in this direction was obtained by Ivanova [9] who proved that if X is a well ordered space with the order topology, then 2^X normal implies X compact. In [10] it is shown that 2^{2^X} is normal if and only if X is compact. These results were obtained without the use of CH.

The paper is divided into three sections. In the first section our main result is that $\mathscr{F}_n(\beta X) = \beta\mathscr{F}_n(X)$ if and only if $\mathscr{F}_n(X)$ is pseudocompact. This result is related to the work of Glicksberg in [7] and the proof makes use of his work. In the second section of the paper we investigate the normality of 2^X without the aid of CH using the results of the first section. One significant result in this section is that if 2^X is normal with X noncompact, then X is normal and countably compact, but X^n is not pseudocompact for some n. As a corollary one obtains that if X is first countable or locally compact, then 2^X normal implies that X is compact. Also $2^X \times \beta N$ is normal only when X is compact.

In the last section of the paper it is shown that if CH is assumed, then if 2^X is normal, then X is compact. This result is related to a result of N. Noble [15] who has shown that if every power X^α of X is normal, then X is compact. Noble's result does not require CH, however.

PRELIMINARIES. As remarked in the introduction we assume that X is completely regular and T_1. We denote the Stone-Čech compactification of X by βX. One can imbed the space $\mathscr{F}_n(X)$ into the space $\mathscr{F}_n(\beta X)$ by the map $i(F) = F$ for all $F \in \mathscr{F}_n(X)$. This imbedding can be easily seen to be onto a dense subset of $\mathscr{F}_n(\beta X)$. Since $\mathscr{F}_n(\beta X)$ is compact, we thus have a compactification of $\mathscr{F}_n(X)$ by $\mathscr{F}_n(\beta X)$. By $\beta\mathscr{F}_n(X) = \mathscr{F}_n(\beta X)$ we mean that this compactification is equivalent to the Stone-Čech compactification of $\mathscr{F}_n(X)$.

General background in hyperspaces is conveniently given in [12] and [13]. Use is also made of techniques and results in [10]. Let us recall at this point two results to be used subsequently in the paper. If K is a closed subset of X, then 2^K as a topological space has the same topology as 2^K has as a subspace of 2^X. If $X = K_1 \cup K_2$ with K_1 and K_2 disjoint closed sets, then 2^X is equivalent to $2^{K_1} \times 2^{K_2}$ by [12, Corollary 5(a), p. 166].

We consider the cardinals as a subset of the ordinals in the natural way. Infinite cardinals will be denoted by ω_α where α is an ordinal and where ω_0 is the cardinality of the integers, ω_1 the first uncountable ordinal, etc. By CH is meant $2^{\omega_0} = \omega_1$. This assumption is made only in the last section of the paper.

1. On the compactification of $\mathscr{F}_n(X)$. In this section we establish the result $\beta\mathscr{F}_n(X) = \mathscr{F}_n(\beta X)$ if and only if $\mathscr{F}_n(X)$ is pseudocompact. We first show that $\mathscr{F}_n(X)$ is pseudocompact if and only if X^n is. Our proof of this result is not the easiest possible; however, by establishing an important proposition at this time, the proof of our main result in this section is made easier.

PROPOSITION 1.1. *If X^n is not pseudocompact, then there is a collection of nonempty open sets in X $\{U_k^i\colon k = 1, \cdots, n;\ i = 1, 2, \cdots\}$ such that $\bar{U}_k^i \cap \bar{U}_h^j = \phi$ for $(i, k) \neq (j, h)$ and such that if*

$$O_i = U_1^i \times \cdots \times U_n^i\,,$$

then $\{O_i\}_{i=1}^\infty$ forms a discrete collection in X^n.

One should compare Proposition 1.1 with that in Isbell [8, 38, p. 139] for motivation. We will prove the following lemma before proving 1.1.

LEMMA 1.2. *Suppose that X^n has a countable closed discrete* *subset* $B = \{x^i\}_{i=1}^{\infty}$ *such that* (1) $x^i \in U_i$ *with* U_i *open in* X^n; (2) *for each subsequence* $B' = \{x^{i_j}\}_{j=1}^{\infty}$ *of* B *and each projection* π_k, $\mathrm{Cl}_X \pi_k[B']$ *is not compact; and* (3) *for each* i, $x^i = (x_1^i, \cdots, x_n^i)$ *with* $x_j^i \neq x_k^i$ *for* $j \neq k$. *Then there is a subsequence* $B' = \{x^{i_j}\}_{j=1}^{\infty}$ *of* B *and a collection of open sets in* X, $\{V_k^j\colon k = 1, \cdots, n; j = 1, 2, \cdots\}$ *such that* (a) $\bar{V}_k^j \cap \bar{V}_h^i = \phi$ *for* $(j, k) \neq (i, h)$; (b) $x^{i_j} \in V_1^j \times \cdots \times V_n^j$; *and* (c) $V_1^j \times \cdots \times V_n^j \subset U_{i_j}$.

Proof. Let $B = \{x^i\}_{i=1}^{\infty}$ satisfy the hypotheses of the lemma. Let O_1^1 be an open set containing x_1^1 such that there is an infinite number of i's such that $\pi_k(x^i) \notin \bar{O}_1^1$ for $k = 1, \cdots, n$ and $\bar{O}_1^1$ does not contain x_j^1 for $j = 2, \cdots, n$. Such an O_1^1 exists by (2) and (3) of the hypotheses of the lemma. Let $\mathrm{B}_1 = \{x^{i_j}\}_{j=1}^{\infty}$ be the set of all x^i's such that $\pi_k(x^i) \notin \bar{O}_1^1$ for $k = 1, \cdots, n$ or $i = 1$. Then let O_2^1 be an open set containing x_2^1 such that for an infinite subset of B_1, $\pi_k(x^i) \notin \bar{O}_2^1$ for $k = 1, \cdots, n$; $\bar{O}_2^1$ does not contain x_j^1 for $j \neq 2$; and $O_2^1 \cap O_1^1 = \phi$. Such an O_2^1 exists by (2) and (3) of the lemma. Let $B_2 = \{x^i\colon \pi_k(x^i) \notin \bar{O}_j^1$ for $k = 1, \cdots, n$ and $j = 1$ and 2 or $i = 1\}$. Continuing this process n times we arrive an n infinite subsequences of B, $\{B_1, \cdots, B_n\}$ and open sets in X, $\{O_1^1, \cdots, O_n^1\}$, with (1) $O_i^1 \cap O_j^1 = \phi$ for $i \neq j$; (2) $x_j^1 \in O_j^1$ for $j = 1, \cdots, n$; and (3) $B_j = \{x^i\colon \pi_k(x^i) \notin \bar{O}_q^1$ for $k = 1, \cdots, n$ and $q = 1, \cdots, j\} \cup \{x^1\}$. Now let $\{V_1^1, \cdots, V_n^1\}$ be open subsets of X with the property that $x_j^1 \in V_j^1 \subset \bar{V}_j^1 \subset O_j^1$ and $V_1^1 \times \cdots \times V_n^1 \subset U_1$.

Now let $C_1 = B_n - \{x^1\}$ and $X_1 = X - \bigcup_{i=1}^n O_i^1$. Then $C_1 \subset (\mathrm{int}_X X_1)^n$ and C_1 together with X_1 satisfies the three hypotheses of the lemma. Let x^{i_2} be the first element of C_1. Then repeating the construction described above we can get open sets in X_1 which we can also suppose are open in X, $\{O_1^2, \cdots, O_n^2\}$ and $\{V_1^2, \cdots, V_n^2\}$, and an infinite subsequence C_2 of C_1 such that (1) $x_j^{i_2} \in V_j^2 \subset \bar{V}_j^2 \subset O_j^2$ for all j; (2) $V_1^2 \times \cdots \times V_n^2 \subset U_{i_2}$; and (3) $C_2 \subset (\mathrm{int}_X X_2)^n$ where $X_2 = X_1 - \bigcup_{i=1}^n O_i^2$. Let x^{i_3} be the first element of C_2. Continuing this process inductively we get a subsequence $B' = \{x^{i_j}\}_{j=1}^{\infty}$ and open sets $\{V_k^j\colon k = 1, \cdots, n; j = 1, 2, \cdots\}$ satisfying the conclusion of the lemma.

Proof of Proposition 1.1. By induction on n. If $n = 1$, the proposition is clearly true. Suppose $n > 1$ and consider the following cases.

Case (i). X^{n-1} is not pseudocompact.

In this case we apply our induction hypothesis to get sets $\{U_1^i \times \cdots \times U_{n-1}^i\}_{i=1}^{\infty}$ satisfying the conclusion of Proposition 1.1 for X^{n-1}. Then define $\{V_j^i\colon j = 1, \cdots, n;\ i = 1, 2, \cdots\}$ such that $V_j^i = U_j^{2i}$ for $j = 1, \cdots, n-1$ and $V_n^i = U_1^{2i+1}$. Then $\{V_j^i\}$ can be easily seen to satisfy the conclusion of Porposition 1.1.

Case (ii). X^{n-1} is pseudocompact.

In this case let $B = \{x^i\}_{i=1}^{\infty}$ be a countably infinite C-imbedded subset of X^n with $x^i \in U_i$ an open set in X^n with $\{U_i\}_{i=1}^{\infty}$ a discrete collection in X^n. We claim that B satisfies the conditions of Lemma 1.2. Suppose that for some i and some subsequence B' of B, $\mathrm{Cl}_X\pi_i[B']$ is compact. Then $\mathrm{Cl}_X\pi_i[B'] \times X^{n-1}$ is pseudocompact [2, E 3.9. E, p. 151]. But $B' \subset \mathrm{Cl}_X\pi_i[B'] \times X^{n-1}$ is C-imbedded in X^n, hence in $\mathrm{Cl}_X\pi_i[B'] \times X^{n-1}$, a contradiction. Thus conditions (1) and (2) of 1.2 are satisfied. If we let $X_{ij} = \{(x_1, \cdots, x_n) \in X^n\colon x_i = x_j\}$ and $A = \bigcup_{i\neq j} X_{ij}$, then noticing that there are only a finite number of the X_{ij}'s and that each X_{ij} is homeomorphic to X^{n-1} we get that $U_i \cap A = \phi$ except for a finite number of i's or X^{n-1} would not be pseudocompact. By eliminating that finite number of i's we may assume $B \subset X^n - A$ and thus that B satisfies condition (3) of 1.2. Now let $\{V_k^i\colon k = 1, \cdots, n; j = 1, 2, \cdots\}$ and $B' = \{x^{i_j}\}_{j=1}^{\infty}$ satisfy the conclusion of Lemma 1.2. Then $O_i = V_1^i \times \cdots \times V_n^i$ satisfies the conclusion of Proposition 1.1.

THEOREM 1.3. *For all n, $\mathscr{F}_n(X)$ is pseudocompact if and only if X^n is pseudocompact.*

Proof. Let $p\colon X^n \to \mathscr{F}_n(X)$ be defined by

$$p((x_1, \cdots, x_n)) = \{x_1, \cdots, x_n\}\ .$$

Then p is continuous and closed. Also $p|(X^n - A)$ is a local homeomorphism, hence open onto $\mathscr{F}_n(X) - \mathscr{F}_{n-1}(X)$, where A is defined as in the previous proof (see [4]). If X^n is pseudocompact, then $\mathscr{F}_n(X)$ is since pseudocompactness is preserved under continuous transformation. Now suppose that X^n is not pseudocompact. Let $\{U_k^i\colon k = 1, \cdots, n; i = 1, 2, \cdots\}$ satisfy the conclusions of Proposition 1.1. Let $O_i = U_1^i \times \cdots \times U_n^i$. Then $\{p(O_i)\}_{i=1}^{\infty}$ can be seen to be a discrete collection of nonempty open sets in $\mathscr{F}_n(X)$. Thus $\mathscr{F}_n(X)$ is not pseudocompact.

THEOREM 1.4. *Let $n \geqq 2$. Then $\beta\mathscr{F}_n(X) = \mathscr{F}_n(\beta X)$ if and only if $\mathscr{F}_n(X)$ is pseudocompact.*

Proof. Note that for $n = 1$, $\beta\mathscr{F}_1(X) = \mathscr{F}_1(\beta X)$ with no assumptions. Suppose that $\mathscr{F}_n(X)$ is pseudocompact. Then by Theorem 1.3, X^n is also pseudocompact. Thus by [7, Th. 1, p. 371], $\beta(X^n) = (\beta X)^n$. Now let $f\colon \mathscr{F}_n(X) \to [0, 1]$ be continuous and let $F\colon X^n \to [0, 1]$ be defined by $F = f \circ p$ where $p\colon X^n \to \mathscr{F}_n(X)$ is as defined in the proof of Theorem 1.3. Now F has an extension $F^*\colon (\beta X)^n \to [0, 1]$ since $\beta(X^n) = (\beta X)^n$. Consider the map $p^*\colon (\beta X)^n \to \mathscr{F}_n(\beta X)$ defined by

$$p^*((x_1, \cdots, x_n)) = \{x_1, \cdots, x_n\}\ .$$

Clearly p^* is an extension of p and a quotient map [4]. If we can show that F^* is constant on the point inverses of p^*, then by defining f^* by $F^* \circ p^{*^{-1}}$, f^* will be well defined. Also f^* will be continuous by [1, Th. 3.2, p. 123] and an extension of f to $\mathscr{F}_n(\beta X)$. Thus

$$\mathscr{F}_n(\beta X) = \beta \mathscr{F}_n(X)$$

by [6, Th. 6.5, p. 86]. Thus it will be sufficient to show that F^* is constant on the point inverses of p^*. To that end let $\{x_1, \cdots, x_k\} \in \mathscr{F}_n(\beta X)$ with $x_i \neq x_j$ for $i \neq j$. Let

$$p^*((z_1, \cdots, z_n)) = p^*((y_1, \cdots, y_n)) = \{x_1, \cdots, x_k\} .$$

One can construct a net $\{x_1^\alpha, \cdots, x_k^\alpha\}$ of elements x_i^α in X converging to $\{x_1, \cdots, x_k\}$ in $\mathscr{F}_n(\beta X)$ such that $x_i^\alpha \neq {}_j^\alpha$ for $i \neq j$ for each α and $x_i^\alpha \rightarrow x_i$ for all i. Now if $z_i = x_j$ let $z_i^\alpha = x_j^\alpha$, and if $y_i = x_j$ let $y_i^\alpha = x_j^\alpha$, for all α. Then $(y_1^\alpha, \cdots, y_n^\alpha) \rightarrow (y_1, \cdots, y_n)$ in X^n and $(z_1^\alpha, \cdots, z_n^\alpha) \rightarrow (z_1, \cdots, z_n)$ in X^n. Thus $F^*((y_1^\alpha, \cdots, y_n^\alpha)) \rightarrow F^*((y_1, \cdots, y_n))$ and

$$F^*((z_1^\alpha, \cdots, z_n^\alpha)) \rightarrow F^*((z_1, \cdots z_n)) .$$

Since $F^*((z_1^\alpha, \cdots z_n^\alpha)) = F^*((y_1^\alpha, \cdots, y_n^\alpha))$ for each α, this implies

$$F^*((y_1, \cdots, y_n)) = F^*((z_1, \cdots z_n)) .$$

Thus F^* is constant on the point inverses of p^* and the first half of the theorem is proved.

For the converse we will draw upon Proposition 1.1. Suppose that $\mathscr{F}_n(X)$ is not pseudocompact. Then X^n is not pseudocompact. Let $\{U_k^i : k = 1, \cdots n; i = 1, 2, \cdots\}$ be as in Proposition 1.1. Let

$$\mathscr{U}_i = \langle U_1^i, \cdots, U_n^i \rangle \cap \mathscr{F}_n(X) = p[U_1^i \times \cdots \times U_n^i]$$

in $\mathscr{F}_n(X)$ (see [13] for notation). Then one can show that $\{\mathscr{U}_i\}_{i=1}^\infty$ is a discrete collection of open sets in $\mathscr{F}_n(X)$. Let $B_i \in \mathscr{U}_i$ and $f\colon \mathscr{F}_n(X) \rightarrow [0, 1]$ be defined so that $f(B_i) = 1$ and $f(B) = 0$ for all $B \notin \bigcup_{i=1}^\infty \mathscr{U}_i$. Now if $\mathscr{F}_n(\beta X)$ were equivalent to $\beta \mathscr{F}_n(X)$, there would be a continuous extension of f to some $f^*\colon \mathscr{F}_n(\beta X) \rightarrow [0, 1]$. We will show that no extension of f to $\mathscr{F}_n(\beta X)$ is continuous. Let B_0 be a limit point of $\{B_i\}_{i=1}^\infty$ in $\mathscr{F}_n(\beta X)$. Let $\mathscr{U} = \langle U_1, \cdots, U_n \rangle \cap \mathscr{F}_n(\beta X)$ be a neighborhood of B_0 in $\mathscr{F}_n(\beta X)$. Let B_{i_1} and B_{i_2} be distinct with B_{i_1} and B_{i_2} in $\mathscr{U}$. Let B be defined by $B = \{p_1, \cdots, p_n\}$ where $p_j \in B_{i_1} \cap U_j$ for j odd and $p_j \in B_{i_2} \cap U_j$ for j even. Then $B \in \mathscr{U}$. But also $B \notin \bigcup_{i=1}^\infty \mathscr{U}_i$. Thus $f^*(B) = f(B) = 0$. Thus in every neighborhood $\mathscr{U}$ of B_0 in $\mathscr{F}_n(\beta X)$, f^* takes on the value 0 and the value 1, a contradiction. Thus $\mathscr{F}_n(\beta X) \neq \beta \mathscr{F}_n(X)$.

2. Results without the aid of CH.

In [9] it is shown that if

X is a well ordered space with the order topology, then 2^X normal implies X compact. In [10] it is shown that 2^{2^X} is normal if and only if X is compact. In this section we use Theorem 1.4 to show that for certain classes of spaces X, 2^X is not normal. The most positive result in this paper assumes CH and will be proved in the next section making use of the results of this section and [10].

LEMMA 2.1. *If $\{F_i\}_{i=1}^{\infty}$ is a countable collection of closed sets in a normal countably compact space X, then*

$$\mathrm{Cl}_{\beta X}[\bigcap_{i=1}^{\infty} F_i] = \bigcap_{i=1}^{\infty} \mathrm{Cl}_{\beta X} F_i \, .$$

Proof. Clearly $\mathrm{Cl}_{\beta X}[\bigcap_{i=1}^{\infty} F_i] \subset \bigcap_{i=1}^{\infty} \mathrm{Cl}_{\beta X} F_i$. Now suppose the contrary and let $x \in \mathrm{Cl}_{\beta X} F_i$ for all i with $x \notin \mathrm{Cl}_{\beta X}[\bigcap_{i=1}^{\infty} F_i]$. Then let V be an open set in βX containing x such that $\mathrm{Cl}_{\beta X} V \cap \mathrm{Cl}_{\beta X}[\bigcap_{i=1}^{\infty} F_i] = \phi$. Let $U = V \cap X$ and note that $\mathrm{Cl}_{\beta X} V = \mathrm{Cl}_{\beta X} U$. Clearly $(\mathrm{Cl}_X U) \cap [\bigcap_{i=1}^{\infty} F_i] = \phi$. By the countable compactness of X there is an n such that $(\mathrm{Cl}_X U) \cap (\bigcap_{i=1}^{n} F_i) = \phi$. By the second lemma in [10], this implies that $(\mathrm{Cl}_{\beta X} U) \cap [\bigcap_{i=1}^{n} \mathrm{Cl}_{\beta X} F_i] = \phi$. However, $x \in \mathrm{Cl}_{\beta X} U$ and $x \in \mathrm{Cl}_{\beta X} F_i$ for $i = 1, \cdots, n$, a contradiction. Thus

$$\mathrm{Cl}_{\beta X}[\bigcap_{i=1}^{\infty} F_i] = \bigcap_{i=1}^{\infty} \mathrm{Cl}_{\beta X} F_i$$

as asserted.

PROPOSITION. *If 2^X is normal, then X is normal and countably compact. If, in addition, X is not compact, then there is an n such that X^n is not pseudocompact.*

Proof. Suppose that 2^X is normal. Then X is normal and countably compact [10, corollary to Th. 1]. Suppose that X is not compact and that X^n is pseudocompact for all n. Let $x \in \beta X - X$ and let $\mathscr{F}_x = \{F : F$ is closed in X and $\mathrm{Cl}_{\beta X} F$ contains $x\}$. Let $\hat{X}$ be the set of singletons $\{\{x\} : x \in X\}$. Then $\mathscr{F}_x$ and $\hat{X}$ are closed subsets of 2^X and disjoint. We will show that $\mathscr{F}_x$ and $\hat{X}$ cannot be separated by a continuous real valued function. Suppose that $f: 2^X \to [0, 1]$ is continuous with $f | X \equiv 0$. Let f_n be the restriction of f to $\mathscr{F}_n(X)$ for each n. By Theorem 1.3 $\mathscr{F}_n(X)$ is pseudocompact. Thus by Theorem 1.4, f_n has an extension f_n^* to $\mathscr{F}_n(\beta X)$ for each n. Clearly $f_n^*(x) = 0$ for all n. For each n, let U_n be a neighborhood of x in βX such that for $A \in 2^{U_n} \cap \mathscr{F}_n(\beta X)$ we have $f_n^*(A) \leqq 2^{-n}$. Let

$$F_n = \mathrm{Cl}_X(U_n \cap X) \, .$$

Then $x \in \mathrm{Cl}_{\beta X} F_n$ for each n. Thus $x \in \mathrm{Cl}_{\beta X}[\bigcap_{i=1}^{\infty} F_i]$ by Lemma 2.1. Thus $\bigcap_{i=1}^{\infty} F_i = F_0$ is an element of $\mathscr{F}_x$. Let B be any finite subset of F_0. Then if card $B = k$, then $B \in 2^{U_n} \cap \mathscr{F}_n(\beta X)$ for all $n \geqq k$. Thus $f_n(B) = f(B) \leqq 2^{-n}$ for all $n \geqq k$. Thus $f(B) = 0$. This implies that $f(F_0) = 0$. Therefore $\hat{X}$ and $\mathscr{F}_x$ cannot be separated, a contradiction. Thus X^n must be nonpseudocompact for some n.

REMARK 2.3. It is not known if X normal and countably compact implies X^n pseudocompact for all n. All of the examples known to the author, for example Frolík's [3], of a completely regular space X which is countably compact and such that X^n is not pseudocompact are obtained by choosing an appropriate dense subset A of

$$N^* = \beta N - N$$

and letting $X = N \cup A$. Assuming CH, all such examples are non-normal by the result of Gillman and Fine [5] that proper dense subsets of N^* are not C^*-imbedded in N^*. If the normality and countable compactness of X implies X^n pseudocompact for all n, then 2^X normal implies X compact without assuming CH. However, this would be an interesting result even if CH were required in proving it.

PROPOSITION 2.4. *If X is a countably compact k-space and Y is countably compact, then $X \times Y$ is countably compact. Thus X^n is countably compact for all n.*

Proof. Proof of the first part of the proposition can be found in [14, Th. 1.1]. The second part follows by induction.

DEFINITION 2.5. A space is *strongly countably compact* if the closure of every countable set is compact [10]. A space is *sequentially compact* if each sequence has a convergent subsequence.

COROLLARY 2.6. *If X has any of the following properties, then 2^X normal implies X compact.*

(a) *X first countable,*
(b) *X locally compact,*
(c) *X a k-space,*
(d) *X strongly countably compact, and*
(e) *X sequentially compact.*

Proof. For the definition of a k-space see [1, Definition 9.2, p. 248]. By [1, 9.3, p. 248] (c) implies (a) and (b). But (c) follows from Proposition 2.2 and Proposition 2.4.

For (d) and (e), one can show that these properties are finitely productive. Thus in these cases X^n is pseudocompact for all n and

Proposition 2.2 can be applied.

We conclude this section with a minor result.

LEMMA 2.7. *If X is separable and countably compact, then $X \times \beta N$ normal implies X compact.*

Proof. Let $f: \beta N \to \beta X$ be continuous and surjective. If $X \times \beta N$ is normal, then so is $X \times \beta X$ since the map $g = i \times f: X \times \beta N \to X \times \beta X$ is closed. But $X \times \beta X$ is normal if and only if X is paracompact [16, Th. 2, p. 1046]. But paracompactness and countable compactness imply compactness [1, Corollary 3.4, p. 230]. Thus X is compact.

THEOREM 2.8. *If $2^X \times \beta N$ is normal, then X is compact.*

Proof. Let $\hat{X} = \{\{x\}: x \in X\}$. Then $\hat{X}$ is a homeomorphic copy of X [12, Corollary 3a, p. 166] and closed in 2^X [13, Proposition 2.4.2, p. 156]. Let K be the closure of any countable subset of X. Then K is countably compact. Now $\hat{K} \times \beta N$ is a closed subset of $2^X \times \beta N$, hence normal. Thus K is compact by Lemma 2.6. Thus X is strongly countably compact. But 2^X is normal since $2^X \times \beta N$ is, and thus X is compact by Corollary 2.6(d).

3. Results assuming CH. In [10, proof of Th. 4] it is shown that if X is not compact, then there is an initial ordinal ω_α such that $[0, \omega_\alpha)$ can be imbedded as a closed subset of 2^X. If we let the imbedding be $f(\beta) = F_\beta$, then the set $\{F_\beta: \beta < \omega_\alpha\}$ has the property that (1) for $\gamma > \beta$, $F_\gamma \subsetneqq F_\beta$; (2) if γ is a limit ordinal $F_\gamma = \bigcap\{F_\beta: \beta < \gamma\}$; and (3) $\bigcap\{F_\beta: \beta < \omega_\alpha\} = \phi$. This result will form an important part of what follows.

Recall that a regular open set V is one which has the property that $V = \operatorname{int} \bar{V}$. If B is a dense subset of X and V is a regular open set in X, then $U = V \cap B$ is a regular open set in B.

LEMMA 3.1. *If A is a discrete subset of X with X separable, then* card $A \leqq 2^{\omega_0}$.

Proof. For each $x \in A$ let V_x be a regular open set in X such that $V_x \cap A = \{x\}$. Let $U_x = V_x \cap B$ where B is a countable dense set in X. Then for $x \neq y$, $U_x \neq U_y$. Thus the map $g(x) = U_X$ is one to one into the power set of B. Thus card $A \leqq 2^{\omega_0}$.

PROPOSITION 3.2. *Assume* CH. *Suppose that X is separable and* *countably compact but not compact. Then* $[0, \omega_1)$ *can be imbedded in* 2^X *as a closed subset.*

Proof. We make use of the results in [10] described above to say that $[0, \omega_\alpha)$ can be imbedded in 2^X for some initial ordinal ω_α. Since X is separable, so is 2^X. Let A be the nonlimit points of $[0, \omega_\alpha)$. Then card $A = \omega_\alpha$ and A is discrete. Thus $\omega_\alpha \leqq 2^{\omega_0}$ by Lemma 3.1. Assuming CH $\omega_\alpha = \omega_0$ or $\omega_\alpha = \omega_1$. If $\omega_\alpha = \omega_0$, then by (3) above, X would not be countably compact. Thus $\omega_\alpha = \omega_1$ and $[0, \omega_1)$ is a closed subset of 2^X.

PROPOSITION 3.3. *Assume* CH. *Suppose that* X *is separable, countably compact, and not first countable. Then* $[0, \omega_1]$ *can be imbedded in* 2^X.

Proof. Let $\{V_\alpha\}$ be a neighborhood basis for x in X where X is not first countable at x. Since X is separable we may assume that $\{V_\alpha\}$ has cardinality $\omega_\alpha \leqq 2^{\omega_0}$. Since X is not first countable at x, $\omega_\alpha > \omega_0$. Thus card $\{V_\alpha\} = \omega_1$ and we may assume that the V_α's are indexed by the countable ordinals. We now define closed sets $\{F_\beta: \beta < \omega_1\}$ having the following properties: (1) $F_{\beta+1} \subset \bar{V}_\beta$ for all β; (2) $\gamma > \beta$ implies that $F_\gamma \subsetneqq F_\beta$; (3) if γ is a limit ordinal, then

$$F_\gamma = \cap \{F_\beta: \beta < \gamma\} ;$$

and (4) $\cap \{F_\beta: \beta < \omega_1\} = \{x\}$. The construction is as follows: let $\alpha_0 = 1$. Having defined a subsequence of the countable ordinals $\{\alpha_\beta: \beta < \gamma\}$ let $\alpha_\gamma = \sup \{\alpha_\beta: \beta < \gamma\}$ if γ is a limit ordinal. Otherwise let α_γ be the first α such that if $F = \cap \{\bar{V}_\lambda: \lambda < \alpha_\beta$ some $\beta < \gamma\}$, then $F - \bar{V}_\alpha \neq \phi$. Note that by the countable compactness of X and the fact that X is not first countable at x, $F \neq \{x\}$ and thus such an α_γ exists. Continue the process inductively and let $\{\alpha_\beta: \beta < \omega_1\}$ be the sequence so defined. Then let $F_\beta = \cap \{\bar{V}_\alpha: \alpha < \alpha_\beta\}$. Then $\{F_\beta: \beta < \omega_1\}$ satisfies (1), (2), (3), and (4) above. Let us define $F_{\omega_1} = \{x\}$. Then we claim that $\{F_\beta: \beta \leqq \omega_1\}$ is our desired set.

CLAIM. The map $f(\beta) = F_\beta$ is a homeomorphism of $[0, \omega_1]$ into 2^X.

Proof of claim: Clearly $f: [0, \omega_1] \rightarrow \{F_\beta\}$ is one to one and onto. Suppose that α is a countable limit ordinal. Then $F_\alpha = \cap \{F_\beta: \beta < \alpha\}$ by (3) above. Let $F_\alpha \in \langle U_1, \cdots, U_n \rangle$. We may suppose that $\langle U_1, \cdots, U_n \rangle \cap \{F_\beta\} = \{F_\beta: \beta \leqq \alpha$ and $F_\beta \subset \bigcup_{i=1}^n U_i\}$ by supposing some $U_i = X - F_{\alpha+1}$. Suppose that $\beta_i \rightarrow \alpha$ with $F_{\beta_i} \not\subset \bigcup_{i=1}^n U_i$. Then letting $G_i = F_{\beta_i} - \bigcup_{j=1}^n U_j$, $\{G_i\}_{i=1}^\infty$ has the finite intersection property and empty intersection, contradicting the countable compactness of X. Thus there is no such sequence β_i converging to α and f is continuous at α. Now consider ω_1. Let U be any open set in X containing $\{x\}$. Let α be such that $\bar{V}_\alpha \subset U$. Then for all $\beta > \alpha$, $F_\beta \in 2^U$. Thus f is continuous at ω_1. Thus f is homeomorphism onto $\{F_\beta: \beta \leqq \omega_1\}$.

THEOREM 3.4. *Assume* CH. *Then* 2^X *is normal if and only if* X *is compact.*

Proof. We need only show that if 2^X is normal, then X is compact. Assume that 2^X is normal. Let K be the closure of any countable subset of X. Then 2^K is also hormal since it is a closed subspace of 2^X. If we can show that for any separable space Z, 2^Z normal implies that Z is compact, then K will have to be compact. Thus X will be strongly countably compact and compact by Corollary 2.6(d).

CLAIM. If Z is separable and 2^Z is normal, then Z is compact.

Proof of claim. Suppose that Z is separable and not compact with 2^Z normal. By Corollary 2.6(a) Z is not first countable. Suppose that Z is not first countable at the point x. Let O be an open set containing x such that $Z - O$ is not compact. Such an O exists since X is not compact. Let P be an open set containing x such that $\bar{P} \subset O$. Let $U = Z - \mathrm{Cl}(Z - \bar{P})$. Then U has the property that $Z - U$ is separable and not compact. Now let V be an open set containing x with $\bar{V} \subset U$. Let $K_1 = \bar{V}$ and $K_2 = X - U$. Then let $K = K_1 \cup K_2$. Now K is a closed subset of Z and 2^K is a closed subspace of 2^Z as remarked in the preliminaries. Also 2^K is homeomorphic to $2^{K_1} \times 2^{K_2}$. But by Proposition 3.2 $[0, \omega_1)$ can be imbedded as a closed subset of 2^{K_2}. By Proposition 3.3 we can imbed $[0, \omega_1]$ as a closed subset of 2^{K_1}. Thus $[0, \omega_1] \times [0, \omega_1)$ is a closed subset of 2^K and thus of 2^Z. But $[0, \omega_1] \times [0, \omega_1)$ is not normal by [16, Th. 2, p. 1046] or [5, 8M(4), p. 129]. This implies that 2^Z is not normal, a contradiction. Thus Z must be compact.

This proves the claim and completes the proof of Theorem 3.4.

THEOREM 3.5. *Assume* CH. *The following are equivalent.*

(a) X *is compact*,

(b) 2^X *is compact*,

(c) 2^X *is normal*,

(d) 2^X *is meta-Lindelöf, and*

(e) 2^{2^X} *is regular.*

Proof. The equivalence of (a), (b), and (d) is shown in [10] without CH. The equivalence of (c) and (e) is given in [13, Th. 4.9, p. 163]. By Theorem 3.4 (a) and (c) are equivalent.

REMARK 3.6. It is trivial to see that the assumption that X is completely regular can be reduced to X being Hausdorff in Theorem 3.4, since 2^X normal will then imply that X is completely regular since it is a subspace of 2^X. It would have been a nuisance to keep stating different hypotheses for X for each new theorem, but many can be trivially reduced as in this case.

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