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LOCALLY COMPACT SPACES AND TWO CLASSES OF C*-ALGEBRAS

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Let X be a topological space which is second countable, locally compact, and T_0. Fell has defined a compact Hausdorff topology on the collection $\mathscr{C}(X)$ of closed subsets of X. X may be identified with a subset of $\mathscr{C}(X)$, and in the first part of this paper, the original topology on X is related to that induced from $\mathscr{C}(X)$. The main result is a necessary and sufficient condition for X to be almost strongly separated. In the second part, these results are applied to the primitive ideal space Prim (A) of a separable C^*-algebra A, giving in particular a necessary and sufficient condition for Prim (A) to be almost separated. Further information concerning ideals in A which are central as C^*-algebras is obtained.

Most of the theorems in the paper were suggested by the results for simplex spaces recently obtained by Effros [10], Effros and Gleit [11], Gleit [14], and Taylor [17]. The notion of a simplex space was introduced by Effros in [9]. If $\mathfrak{A}$ is a simplex space, then max $\mathfrak{A}$, $P_1(\mathfrak{A})$, and $EP_1(\mathfrak{A})$ denote the closed maximal ideals in $\mathfrak{A}$, the bounded positive linear functionals on $\mathfrak{A}$ of norm at most one, and its set of extreme points, resp., the first set provided with the hull-kernel topology and the latter two sets with the weak* topology. The sets max $\mathfrak{A}$ and $EP_1(\mathfrak{A})$-{0} are in a natural one-to-one correspondence, but the topologies do not agree in general. Information about the simplex space $\mathfrak{A}$ can be obtained by comparing these two topologies (see [11], [14], [17]).

In trying to develop an analogous theory for a C^*-algebra A, the first problem is to decide on replacements for max $\mathfrak{A}$, $P_1(\mathfrak{A})$, and $EP_1(\mathfrak{A})$. For simplicity, assume that A is separable and has a T_1 structure space. An obvious substitute for max $\mathfrak{A}$ is the structure space of A, Prim (A) (the primitive ideals in A, or in this case the maximal proper closed two-sided ideals in A, with the hull-kernel topology). To replace $P_1(\mathfrak{A})$ and $EP_1(\mathfrak{A})$ by the corresponding sets of linear functionals on A does not seem to lead to a fruitful theory. Instead, $P_1(\mathfrak{A})$ and $EP_1(\mathfrak{A})$-{0} are replaced by $N(A)$ and $EN(A)$-{0}, resp., where $N(A)$ is the compact Hausdorff space of C^*-semi-norms on A, and $EN(A)$ is the set of "extreme" points of $N(A)$ (see [4; § 1. 9. 13], [8], [12]). Then Prim (A) and $EN(A)$-{0} are in a natural one-to-one correspondence which is in general not a homeomorphism. By identifying these sets, the primitive ideals in A are endowed with two topologies. Regarding Prim (A) as a subset of $\mathscr{C}(\text{Prim}(A))$, the identification of Prim (A) and $EN(A)$-$\{0\}$ extends naturally to a homeomorphism of $\mathscr{C}(\text{Prim}(A))$ and $N(A)$. Thus the second topology on Prim (A) is just its relative topology in $\mathscr{C}(\text{Prim}(A))$. It is therefore natural to attempt to formulate those theorems about a simplex space $\mathfrak{A}$ which involve only the two topologies on max $\mathfrak{A}$ in terms of a locally compact space X and the associated space $\mathscr{C}(X)$.

The paper is organized as follows. § 2 contains theorems which relate the topology of X to that of $\mathscr{C}(X)$. The applications to C^*-algebras are in § 3. Two classes of C^*-algebras, called GM- and GC-algebras, are investigated; they correspond to the GM- and GC-simplex spaces of [11]. A C^*-algebra is a GM-algebra if its structure space is almost strongly separated, and a GC-algebra if it has a composition series (I_α) of closed two-sided ideals such that the $I_{\alpha+1}/I_\alpha$ are all central C^*-algebras. These algebras were studied by Delaroche [2], who in particular showed that the GC-algebras are just the GM-algebras with only modular primitive ideals. A new proof of this fact (Theorem 3.7) is included. Finally, § 4 points out how the GM- and GC-algebras are related to some of the classes of C^*-algebras in the literature.

2. Locally compact spaces. Throughout this section X is assumed to be a locally compact topological space satisfying the T_0 separation axiom. Recall that X is T_0 means that if $x, y \in X$ are such that $\{x\}^- = \{y\}^-$ (bar indicates closure), then $x = y$, and that X is locally compact means that if $x \in X$, then each neighborhood of x contains a compact neighborhood of x. It is important to remember that although a closed subset of a compact set must be compact, the converse need not be true in a non-Hausdorff space. Let X_1 denote the *closed* points in X, i.e., those x for which $\{x\}^- = \{x\}$. If $X = X_1$, then X is said to be T_1.

The following construction is due to J. M. G. Fell [13]. Let $\mathscr{C}(X)$ denote the collection of all closed subsets of X. The function $\lambda = \lambda_X : X \to \mathscr{C}(X): x \to \{x\}^-$ is one-to-one. If C is a compact subset of X and if $\mathscr{F}$ is a (possibly empty) finite collection of open subsets of X, then $\mathscr{U}(C; \mathscr{F})$ will denote the collection of all those $F \in \mathscr{C}(X)$ such that $F \cap C = \varnothing$ and $F \cap G \neq \varnothing$ for each $G \in \mathscr{F}$. The sets $\mathscr{U}(C; \mathscr{F})$ form a basis for a compact Hausdorff topology on $\mathscr{C}(X)$ [13]. It is readily verified that a net (F_α) in $\mathscr{C}(X)$ will converge to an element F in $\mathscr{C}(X)$ if and only if (1) for each x in F and neighborhood N of x, eventually $F_\alpha \cap N \neq \varnothing$, and (2) if P is the complement of a compact set with $F \subset P$, then eventually $F_\alpha \subset P$. This topology is metrizable whenever X is second countable [6; Lemma 2] (see Corollary 2.7 for a partial converse). A simple argument will prove

LEMMA 2.1. (1) λ *is open onto its image, and* (2) X *is Hausdorff if and only if* $\lambda: X \to \lambda(X)$ *is a homeomorphism.*

The first object is to find sets on which λ restricts to a homeomorphism. A set $\mathscr{T} \subset \mathscr{C}(X)$ will be called *dilated* if $x \in F$ for some $F \in \mathscr{T}$ implies that $\lambda(x) \in \mathscr{T}$. In particular, if $F \in \mathscr{C}(X)$, the set $F^{\perp} = \{E \in \mathscr{C}(X): E \subset F\}$ is compact and dilated.

LEMMA 2.2. *If* $\mathscr{T}$ *is a compact and dilated subset of* $\mathscr{C}(X)$, *then* $\lambda^{-1}(\mathscr{T})$ *is closed.*

Proof. Suppose that $x_0 \in X$ and $x_0 \notin \lambda^{-1}(\mathscr{T})$. Say $F \in \mathscr{T}$. As $\mathscr{T}$ is dilated, $x_0 \notin F$, and so there is a compact neighborhood $C(F)$ of x_0 which is disjoint from F. The sets $\mathscr{U}(C(F); \varnothing)$, $F \in \mathscr{T}$, form an open covering for $\mathscr{T}$; hence there are sets $F_1, \cdots, F_n \in \mathscr{T}$ such that

$$\mathscr{T} \subset \bigcup_{i=1}^{n} \mathscr{U}(C(F_i); \varnothing) .$$

Suppose $x \in C = \bigcap_{i=1}^{n} C(F_i)$ and $\lambda(x) \in \mathscr{T}$. Then $\lambda(x) \cap C(F_i) = \varnothing$ for some i, hence $x \notin C(F_i)$, a contradiction. This shows that C is a neighborhood of x_0 which is disjoint from $\lambda^{-1}(\mathscr{T})$.

If T is a subset of X_1, then $\lambda(T)$ is dilated; hence

COROLLARY 2.3. *If* T *is a subset of* X_1 *for which* $\lambda(T)$ *is compact, then* λ *restricts to a homeomorphism of* T *onto* $\lambda(T)$.

The following shows that convergence in X is closely related to that in $\mathscr{C}(X)$. The trick employed in the proof of (ii) was used by both Gleit [14] and Taylor [17].

THEOREM 2.4. (i) *Let* (x_α) *be a net in* X *such that* $\lambda(x_\alpha) \to F$ *for some* $F \in \mathscr{C}(X)$. *Then* $x_\alpha \to x$ *for any* $x \in F$.

(ii) *Let* (x_n) *be a sequence in* X_1 *such that* $\lambda(x_n) \to F$ *for some* $F \in \mathscr{C}(X)$. *Then the limit points of the set* $\{x_n: x \geqq 1\}$ *lie in* F.

Proof. (i) Say $x \in F$, and let G be an open set containing x. Then since $F \cap G \neq \varnothing$, eventually $\lambda(x_\alpha) \cap G \neq \varnothing$, hence $x_\alpha \in G$.

(ii) For each m the set $\{\lambda(x_n): n \geqq m\} \cup F^{\perp}$ is both closed and dilated, hence its inverse image $F_m = \{x_n: n \geqq m\} \cup F$ is closed. If x

is a limit point of $\{x_n : n \geqq 1\}$, it must lie in each of the sets F_m, and thus is an element of F.

COROLLARY 2.5. *Suppose that X is second countable. If $\varnothing \in \lambda(X_1)^-$, then neither X_1 nor X can be compact.*

Proof. $\mathscr{C}(X)$ is metrizable, hence there is a sequence (x_n) in X_1 with $\lambda(x_n) \to \varnothing$. It follows from Theorem 2.4 (ii) that no subsequence of (x_n) can converge to a point in X.

COROLLARY 2.6. *Suppose that $\lambda(X)^-$ is first countable (this is the case if X is second countable), and that T is a compact subset of X_1. If $F \in \mathscr{C}(X)$ and $T \cap F = \varnothing$, then $\lambda(T)^- \cap F^\perp = \varnothing$.*

Proof. If $E \in \lambda(T)^- \cap F^\perp$, there is a sequence (x_n) in T with $\lambda(x_n) \to E$. Since T is compact, the set $\{x_n : n \geqq 1\}$ has a limit point x in T. Then $x \in E$ from Theorem 2.4 (ii), and since $E \in F^\perp$, $x \in F$. But this is a contradiction.

COROLLARY 2.7. *Suppose that X is locally compact and T_1. If $\lambda(X)^-$ is second countable, then so is X.*

Proof. Let $\mathscr{T}_1, \mathscr{T}_2, \cdots$ be a basis of open sets for the topology of $\lambda(X)^-$; with no loss in generality, the sets $\mathscr{T}_n$ may be assumed to be closed under finite unions. Suppose that an $x \in X$ and an $F \in \mathscr{C}(X)$ with $x \notin F$ are given. It is sufficient to show that for some n, $\lambda^{-1}(\mathscr{T}_n)$ contains x in its interior and is disjoint from F. Using the local compactness of X, choose a compact neighborhood C of x disjoint from F. Corollary 2.6 and the fact that $F^\perp$ is closed give

$$\lambda(C)^- \subset \lambda(X)^- - F^\perp = \bigcup_k \mathscr{T}_{n_k}$$

for suitable integers n_k. As $\lambda(C)^-$ is compact and as the $\mathscr{T}_n$ are closed under finite unions, there is an n for which $\mathscr{T}_n \cap F^\perp = \varnothing$ and $\lambda(C) \subset \mathscr{T}_n$. This completes the proof.

The following will be useful in § 3.

COROLLARY 2.8. *Suppose that X is second countable and that $f: \mathscr{C}(X) \to [0, \infty)$ is continuous and monotone in the sense that E, $F \in \mathscr{C}(X)$ and $E \subset F$ imply $f(E) \leqq f(F)$. Suppose further that $f(\lambda(x)) > 0$ for all x in some compact subset T of X_1. Then there is an $\alpha > 0$ such that $f(\lambda(x)) \geqq \alpha$ for all $x \in T$.*

Proof. If there is no such α, choose a sequence (x_n) in T such that $f(\lambda(x_n)) \to 0$. Using first the compactness of $\mathscr{C}(X)$ and then that of T, it may be assumed that $\lambda(x_n) \to F$ for some $F \in \mathscr{C}(X)$ and that $x_n \to x$ for some $x \in T$. From Lemma 2.4 (ii), it follows that $x \in F$. Consequently, $0 < f(\lambda(x)) \leqq f(F)$ and $f(F) = 0$, a contradiction.

For simplex spaces, the following result is due to P. D. Taylor.

COROLLARY 2.9. *Suppose that X is second countable and that f is a continuous complex-valued function on $\lambda(X_1)^-$. For each $x \in X_1$, let $c(x)$ denote the set of all those $F \in \lambda(X_1)^-$ which contain x. Then $f \circ \lambda$ is continuous on X_1 if and only if f is constant on the sets $c(x)$, $x \in X_1$.*

Proof. Notice that $\lambda(x) \in c(x)$ for each $x \in X_1$. Suppose that $f \circ \lambda$ is continuous on X_1. Say $x \in X_1$ and $F \in c(x)$. Then there is a sequence (x_n) in X_1 such that $\lambda(x_n) \to F$. From Theorem 2.4 (i), $x_n \to x$, and

$$f(F) = \lim_{n\to\infty} f(\lambda(x_n)) = f(\lambda(x)) .$$

Conversely, suppose that f is constant on the $c(x)$, $x \in X_1$. Let (x_n) be a sequence in X_1 converging to an $x \in X_1$. To show that

$$f(\lambda(x_n)) \to f(\lambda(x)) ,$$

it is sufficient (since $f(\lambda(X_1))$ lies in the compact set $f(\lambda(X_1)^-)$) to show that every convergent subsequence of $f(\lambda(x_n))$ converges to $f(\lambda(x))$. Passing to a subsequence, suppose that $f(\lambda(x_n)) \to \alpha$ for some complex number α. Using the fact that $\mathscr{C}(X)$ is a compact metric space and passing to a further subsequence, it may even be assumed that $\lambda(x_n) \to F$ for some $F \in \lambda(X_1)^-$. Then from Theorem 2.4, (ii), $x \in F$, i.e., $F \in c(x)$, and therefore

$$f(\lambda(x)) = f(F) = \lim_{n\to\infty} f(\lambda(x_n)) = \alpha .$$

If G is a nonempty open subset of X, then G is locally compact and T_0 in its relative topology. Let ρ_G be the map $F \to F \cap G$ of $\mathscr{C}(X)$ onto $\mathscr{C}(G)$, and let σ_G be its restriction to $\lambda_X(G)$. Then $\sigma_G \circ \lambda_X = \lambda_G$ and σ_G is a bijection of $\lambda_X(G)$ onto $\lambda_G(G)$. Using the fact that G is open in X, it is easily checked that ρ_G is continuous; however, σ_G is in general not a homeomorphism.

LEMMA 2.10. *Let G be a nonempty open subset of X, and suppose that $\lambda(X)^- \subset \lambda(X) \cup (X - G)^\perp$. If $\mathscr{T}$ is a subset of $\lambda_X(G)$ and if $\sigma_G(\mathscr{T})$ is compact, then so is $\mathscr{T}$.*

Proof. As ρ_G is continuous,

$$\rho_G(\mathscr{T}^-) \subset [\rho_G(\mathscr{T})]^- = [\sigma_G(\mathscr{T})]^- = \sigma_G(\mathscr{T}) \subset \lambda_G(G) ,$$

and since $\varnothing \notin \lambda_G(G)$, $\mathscr{T}^- \cap (X - G)^\perp = \varnothing$. But

$$\mathscr{T}^- \subset \lambda(X)^- \subset \lambda(X) \cup (X - G)^\perp \subset \lambda_X(G) \cup (X - G)^\perp ,$$

so that $\mathscr{T}^-$ is contained in $\lambda_X(G)$, the domain of σ_G. Since

$$\sigma_G(\mathscr{T}^-) = \rho_G(\mathscr{T}^-) \subset \sigma_G(\mathscr{T})$$

and σ_G is one-to-one, $\mathscr{T}$ must be closed in $\mathscr{C}(X)$.

A point x in X will be said to be *strongly separated* in X if for each $y \neq x$, there are disjoint neighborhoods of x and y (i.e., x is closed, and separated in the sense of [3; § 1]). A nonempty subset Y of X will be called *strongly separated* in X provided each of its points is strongly separated in X. Finally, X will be called *almost strongly separated* if each nonempty closed subset F of X contains a nonempty relatively open subset G which is strongly separated in F (equivalently, every open subset U of X distinct from X is properly contained in an open subset V such that $V - U$ is strongly separated in $X - U$).

PROPOSITION 2.11. *A nonempty open subset G of X is strongly separated in X if and only if* $\lambda(X)^- \subset \lambda(X_1) \cup (X - G)^\perp$.

Proof. Assume first that G is strongly separated in X. Suppose that there is a net (x_α) in X and an $F \notin \lambda(X_1) \cup (X - G)^\perp$ such that $\lambda(x_\alpha)$ converges to F. Then F must contain two distinct points, at least one of which is in G, which is impossible by Theorem 2.4 (i). Conversely, suppose that $\lambda(X)^- \subset \lambda(X_1) \cup (X - G)^\perp$. From this inclusion it is immediate that $G \subset X_1$. As $\rho_G(\lambda(X)^-)$ is compact and contains $\lambda_G(G)$,

$$\lambda_G(G)^- \subset \rho_G(\lambda(X)^-) \subset \lambda_G(G) \cup \{\varnothing\} ,$$

and therefore $\lambda_G(G) \cup \{\varnothing\}$ is compact. For any relatively closed subset $\mathscr{T}$ of $\lambda_G(G)$, $\mathscr{T} \cup \{\varnothing\}$ is compact and dilated, hence $\lambda_G^{-1}(\mathscr{T})$ is a closed subset of G in the relative topology (Lemma 2.2). This shows that λ_G is continuous; since it is always open onto its image, λ_G is a homeomorphism and G is Hausdorff. To show that G is strongly separated, suppose $x \in G$ and $y \notin G$ are given. Let $U \subset G$ be a compact neighborhood of x; it will suffice to show that U is closed in X. As $\lambda_G(U)$ is compact and as $\lambda_G(U) = \sigma_G(\lambda_X(U))$, $\lambda_X(U)$ is compact (Lemma 2.10). $\lambda_X(U)$ is dilated since $U \subset X_1$, and so $U =$

$\lambda_X^{-1}(\lambda_X(U))$ is closed, by Lemma 2.2.

A topological space which is a countable union of compact sets will be called a K_σ.

LEMMA 2.12. *If X is second countable and if G is an open nonempty strongly separated subset of X, then $\lambda_X(G)$ is K_σ.*

Proof. Since G is Hausdorff, $\lambda_G(G)^- \subset \lambda_G(G) \cup \{\varnothing\}$ by Proposition 2.11, and $\lambda_G(G)$ is locally compact. Now $\mathscr{C}(G)$ is second countable, for as G is second countable, $\mathscr{C}(G)$ is a compact metric space [6; Lemma 2]. Therefore $\lambda_G(G)$ is K_σ. The equality $\lambda_G(G) = \sigma_G(\lambda_X(G))$, Lemma 2.10 and Proposition 2.11 now imply that $\lambda_X(G)$ is K_σ.

LEMMA 2.13. *Let E be a nonempty closed subset of X. Then the map $\theta\colon E^\perp \to \mathscr{C}(E)$ defined by $\theta(F) = F$ for all $F \in E^\perp$ is a homeomorphism onto, where $E^\perp$ has the relative topology from $\mathscr{C}(X)$.*

Proof. That θ is a bijection is clear. Since $E^\perp$ is compact Hausdorff, it is enough to show that θ is continuous. But this follows from the definition of the topologies and the fact that E is closed.

LEMMA 2.14. *If X is almost strongly separated, so is any nonempty subset of X which is either open or closed.*

Proof. See [11; § 3].

THEOREM 2.15. *Suppose that X is second countable, locally compact, and T_0. Then X is almost strongly separated if and only if*

(1) *X is T_1,*

(2) *$\lambda(X)$ is K_σ, and*

(3) *every nonempty closed subset of X is second category in itself.*

Proof. Say that (1)-(3) hold. Let F be a nonempty closed subset of X. Then F is T_1 and second category, and $\lambda_F(F)$ is K_σ by Lemma 2.13. Replacing F by X, it is therefore sufficient to show that if X satisfies (1) and (2) and is second category, then X contains a nonempty open strongly separated set. Write $\lambda(X) = \bigcup_{n=1}^\infty \mathscr{T}_n$, where each $\mathscr{T}_n$ is compact. Since the $\mathscr{T}_n$ are dilated, the $\lambda^{-1}(\mathscr{T}_n)$ are closed by Lemma 2.2. X is second category, hence for some n, $\lambda^{-1}(\mathscr{T}_n)$ contains a nonempty set G which is open in X. As $\lambda^{-1}(\mathscr{T}_n)$ is closed in X and is Hausdorff in the relative topology (Corollary

2.3), G is strongly separated in X.

Conversely, suppose that X is almost strongly separated. By a transfinite induction (see [11; Proposition 3.1]), there is an ordinal α_0 and a family (G_α) of open subsets of X, indexed by those ordinals α with $0 \leqq \alpha \leqq \alpha_0$, such that: (i) $G_0 = \emptyset$, $G_{\alpha_0} = X$; (ii) if $\alpha \leqq \alpha_0$ is a limit ordinal, then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$; and (iii) if $\alpha < \alpha_0$, then $G_\alpha \subset G_{\alpha+1}$ and $G_{\alpha+1} - G_\alpha$ is a nonempty strongly separated subset of $X - G_\alpha$. To see that (1) holds, say $x \in X$. Let β be the least ordinal such that $x \in G_\beta$. By (ii), β cannot be a limit ordinal; let $\alpha + 1 = \beta$. Then $x \in G_{\alpha+1} - G_\alpha$, so that $\{x\}$ is closed in $X - G_\alpha$, and therefore in X.

The natural map θ_α of $(X - G_\alpha)^\perp$ onto $\mathscr{C}(X - G_\alpha)$ is a homeomorphism, where $(X - G_\alpha)^\perp$ has the relative topology from $\mathscr{C}(X)$ (Lemma 2.13). Since θ_α carries $\lambda_X(G_{\alpha+1} - G_\alpha)$ onto $\lambda_{X-G_\alpha}(G_{\alpha+1} - G_\alpha)$ and since the latter is K_σ by (iii) and Lemma 2.12, $\lambda_X(G_{\alpha+1} - G_\alpha)$ must be K_σ. Now

$$X = \bigcup_{\alpha<\alpha_0}(G_{\alpha+1} - G_\alpha)$$

by the above and α_0 is countable (see [16; § 19, II]), so (2) holds. If $F_1, F_2, \cdots$ are closed and nowhere dense subsets of X, then $F_1 \cap G_1, F_2 \cap G_1, \cdots$ are closed and nowhere dense in the relative topology of G_1. Being locally compact and Hausdorff, G_1 is Baire, so the $F_n \cap G_1$ do not cover G_1. Thus X is second category. By Lemma 2.14, this is enough to show that (3) holds.

COROLLARY 2.16. *If X is second countable and almost strongly separated, then all nonempty closed and all nonempty open subsets of X are Baire.*

Proof. This follows from Lemma 2.14 and Theorem 2.15.

Suppose that X is second countable. If all nonempty closed subsets of X are Baire, then $\lambda(X)$ is G_δ [6; Th. 7]; in view of [16; § 30, VI], this fact may be useful in deciding whether X satisfies (2) of Theorem 2.15. As examples in § 4 will show, (1) and (2) are independent of one another even if all nonempty closed subsets of X are Baire. The set of integers with the Zariski (or cofinite) topology is second countable, locally compact, T_0, and satisfies conditions (1) and (2), but not (3), of Theorem 2.15.

3. C*-Algebras. Let A be a C^*-algebra. Throughout this section and the next, an ideal in A will always mean a closed two-sided ideal. Let $Z(A)$ be the center of A, and let $\mathrm{Id}(A)$ [resp., Prim (A), Max (A), and Mod (A)] donote the set of all ideals [primitive ideals, maximal ideals, and modular ideals] in A. For $a \in A$ and $I \in \mathrm{Id}(A)$, define $a(I)$ as the canonical image of a in A/I and $I^{\perp}$ as the set of all those ideals J in A which contain I. Prim (A) with the hull-kernel topology (sometimes called the structure, or Jacobson, topology) is the *structure space* of A. The following facts about the structure space (see [4]) will be used frequently without explicit mention: its closed points are the elements of Max (A); it is locally compact and T_0; it is second countable whenever A is separable; and $I \to \mathrm{Prim}(A) \cap I^{\perp}$ is a one-to-one correspondence between Id (A) and the closed subsets of Prim (A). The weakest topology on Id (A) making each of the maps $I \to \|a(I)\|$, $a \in A$, continuous will be called the weak* topology on Id (A). It is not hard to show that $I \to \mathrm{Prim}(A) \cap I^{\perp}$ is a homeomorphism of Id (A) onto $\mathscr{C}(\mathrm{Prim}(A))$ which restricts to λ on Prim (A) and carries $I^{\perp}$ onto $(\mathrm{Prim}(A) \cap I^{\perp})^{\perp}$ (where the second $\perp$ is taken in the sense of § 2) [12, Th. 2.2]. In what follows, Id (A) and $\mathscr{C}(\mathrm{Prim}(A))$ will be identified. Recall that if A is separable, Id (A) and Prim (A) with the weak* topology may be identified with the spaces $N(A)$ and $EN(A)$-$\{0\}$ of § 1.

In view of the above, the results of § 2 may be applied to C^*-algebras. Save for one, these will not be explicitly mentioned. For any $a \in A$, $I \to \|a(I)\|$ is a function of the type described in Corollary 2.8. This has the following amusing consequence: If A is separable and if T is a structurally compact subset of Max (A), then $\bigcup\{P: P \in T\}$ is a norm-closed subset of A.

A nonzero ideal I in A will be called an *M-ideal* in A if Prim $(A) - I^{\perp}$ is a strongly separated subset of the structure space of A, and A will be called an *M-algebra* [resp., a *GM-algebra*] if the structure space of A is Hausdorff [almost strongly separated]. Clearly A is an M-algebra if and only if A is an M-ideal in itself. Using [4; § 3.2], it is easily verified that A is a GM-algebra if and only if every nonzero quotient of A contains a nonzero M-ideal.

PROPOSITION 3.1. *The following are equivalent for a nonzero ideal I in a C^*-algebra A:*

(1) *I is an M-ideal*

(2) Prim $(A)^- \subset \mathrm{Max}(A) \cup I^{\perp}$, *where* Prim $(A)^-$ *is the weak* closure of* Prim (A) *in* Id (A)

(3) *for each $a \in I$, $P \to \|a(P)\|$ is continuous on* Prim (A) *in the structure topology.*

Proof. (1) $\Leftrightarrow$ (2): This is Proposition 2.11.

(1), (2) $\Rightarrow$ (3): Suppose that an $a \in I$ and an $\alpha > 0$ are given. The map $p \to \|a(P)\|$ is lower semi-continuous on Prim (A) with the structure topology, so it is enough to show that $T = \{P \in \mathrm{Prim}(A)\colon \|a(P)\| \geqq \alpha\}$ is structurally closed. Now T is a structurally compact subset of $\mathrm{Prim}(A) - I^{\perp}$, and as I is an M-ideal in A, $\mathrm{Prim}(A) - I^{\perp}$ is Hausdorff in the relative structure topology. The map σ which sends P into $P \cap I$ is a homeomorphism of $\mathrm{Prim}(A) - I^{\perp}$ onto $\mathrm{Prim}(I)$ for the structure topologies, hence the structure space of I is Hausdorff. From Lemma 2.1, this means that the structure and weak* topologies coincide on $\mathrm{Prim}(I)$. Then $\sigma(T)$ is a weak* compact subset of $\mathrm{Prim}(I)$, and T is a weak* compact subset of $\mathrm{Prim}(A)$ (Lemma 2.10). Since T is contained in $\mathrm{Max}(A)$, it is dilated and therefore structurally closed by Lemma 2.2.

(3) $\Rightarrow$ (1): Say $P \in \mathrm{Prim}(A) - I^{\perp}$ and $Q \in \mathrm{Prim}(A)$ are distinct. If $Q \in I^{\perp}$, choose an $a \in I$ with $\|a(P)\| = 2$. Then $\{R \in \mathrm{Prim}(A)\colon \|a(R)\| > 1\}$ and $\{R \in \mathrm{Prim}(A)\colon \|a(R)\| < 1\}$ are disjoint structurally open sets containing P and Q, resp. Now suppose that $Q \notin I^{\perp}$. For $R \in \mathrm{Prim}(A) - I^{\perp}$ and $a \in I$, $R \cap I \in \mathrm{Prim}(I)$ and

$$\|a(R \cap I)\| = \max\{\|a(R)\|, \|a(I)\|\} = \|a(R)\| .$$

This equality together with the homeomorphism σ of the previous paragraph implies that the structure and weak* topologies on $\mathrm{Prim}(I)$ coincide, and therefore that $\mathrm{Prim}(A) - I^{\perp}$ is Hausdorff in the relative structure topology. As $\mathrm{Prim}(A) - I^{\perp}$ is a structurally open subset of $\mathrm{Prim}(A)$, there are disjoint structure neighborhoods of P and Q.

THEOREM 3.2. *If A is a separable C^*-algebra, then* $\mathrm{Prim}(A)$ *is a G_δ in the weak* topology, and A is a GM-algebra if and only if*

(1) $\mathrm{Max}(A) = \mathrm{Prim}(A)$, *i.e., the structure space of A is T_1, and*

(2) $\mathrm{Prim}(A)$ *is K_σ in the weak* topology.*

Proof. This is an immediate consequence of Theorem 2.15, [6; Th. 7], and the fact that all nonempty closed subsets of the structure space are Baire [4; Corollaire 3.4.13].

Section 4 contains examples which show that neither (1) nor (2) is a consequence of the other, even for separable C^*-algebras. This completes the analogy between GM-simplex spaces and GM-C^*-algebras. In studying the second class of C^*-algebras, the following two lemmas will be useful.

LEMMA 3.3. *For any ideal I in a C^*-algebra A, $Z(I) = I \cap Z(A)$.*

Proof. See [1; Lemma 6].

LEMMA 3.4. *The following are equivalent for a C^*-algebra A:*

(i) *$Z(A) \not\subset P$ for each $P \in \mathrm{Prim}(A)$ and the structure space of A is Hausdorff, and*

(ii) *$P \to P \cap Z(A)$ is a one-to-one map from* $\mathrm{Prim}(A)$ *into* $\mathrm{Prim}(Z(A))$.

If these conditions are satisfied, then the map in (ii) *is a homeomorphism of* $\mathrm{Prim}(A)$ *onto* $\mathrm{Prim}(Z(A))$ *for the structure topologies.*

Proof. For the equivalence of (i) and (ii), see [1; Proposition 3] or [18; Corollary 3.1.2]. The last statement is contained in [15; Th. 9.1].

A C^*-algebra satisfying one of the equivalent conditions of the last lemma is called *central;* for other equivalent definitions, see [1; Proposition 3].

Several results from [7; §4] will now be recalled. Consider an $a \in Z(A)$ and a primitive ideal P in A. Choose an irreducible representation π of A with kernel P. As $\pi(a)$ is in the center of $\pi(A)$, it must be a multiple α of the identity operator on the space of π. Then $\pi(a)\pi(b) = \alpha\pi(b)$, i.e., $ab - \alpha b \in P$, for all $b \in A$. This last condition determines α uniquely, and shows that it depends only on P (and not on π). Set $f_a(P) = \alpha$. The function f_a is clearly bounded on $\mathrm{Prim}(A)$. It is easy to show that $\varphi(a) = f_a(P)$ for any $\varphi \in \theta^{-1}(P)$, where θ is the natural mapping of $P(A)$, the pure states on A, onto $\mathrm{Prim}(A)$. Because θ is an open map,

$$\begin{aligned} f_a^{-1}(U) &= \{P \in \mathrm{Prim}(A) : f_a(P) \in U\} \\ &= \theta(\{\varphi \in P(A) : f_a(\theta(\varphi)) \in U\}) \\ &= \theta(\{\varphi \in P(A) : \varphi(a) \in U\}) \end{aligned}$$

is structurally open for any open set U of complex numbers. This shows that f_a is structurally continuous. If A is central, then $P \in \mathrm{Prim}(A)$ implies $P \cap Z(A) \in \mathrm{Max}(Z(A)) = \mathrm{Prim}(Z(A))$, and regarding $a \in Z(A)$ as a function on $\mathrm{Max}(Z(A))$, $f_a(P) = a(P \cap Z(A))$. Since $Z(A) \cong C_0(\mathrm{Max}\, Z(A))$, we may identify the functions f_a with $C_0(\mathrm{Prim}(A))$.

A C^*-algebra A will be said to have *local identities* if given $P_0 \in \mathrm{Prim}(A)$, there is an $a \in A$ such that $a(P)$ is an identity in A/P for all P in some structure neighbourhood of P_0. A nonzero ideal I in A will be called a *C-ideal* in A if I is a central C^*-algebra. A will be called a *C-algebra* if it is a C-ideal in itself (i.e., is central), and a *GC-algebra* if every nonzero quotient of A contains a nonzero C-ideal.

PROPOSITION 3.5. *A nonzero ideal I in A is a C-ideal if and* *only if it is an M-ideal with local identities.*

Proof. Suppose that I is a C-ideal. Let P and Q be distinct primitive ideals in A with $P \notin I^{\perp}$. If $Q \notin I^{\perp}$, then since I is central, $P \cap Z(I)$ and $Q \cap Z(I)$ are distinct maximal ideals in $Z(I)$ hence there is an $a \in Z(I) \subset Z(A)$ with $f_a(P) \neq 0$ and $f_a(Q) = 0$. If $Q \in I^{\perp}$, let a be any element of $Z(I)$ with $a(P) \neq 0$. Then f_a will provide disjoint neighborhoods for P and Q, and A is an M-ideal.

Thus it suffices to show that a C^*-algebra A is a C-algebra if and only if it is an M-algebra with local identities. If A is a C-algebra, $Z(A)$ may be identified with $C_0(\mathrm{Prim}\,(A))$, hence it is trivial that A has local identities. Conversely, suppose that A is an M-algebra with local identities. Say $P_0 \in \mathrm{Prim}\,(A)$, and choose an $a \in A$ such that $a(P)$ is an identity in A/P for all P in some neighborhood T of P_0. Consider a continuous bounded complex-valued function f on $\mathrm{Prim}\,(A)$ with $f(P_0) = 1$ and whose support is contained in T. From the Dauns-Hofmann theorem (see [7; § 7]), there is a $b \in A$ such that $b(P) = f(P)a(P)$ for all $P \in \mathrm{Prim}\,(A)$. Then $(bc - cb)(P) = 0$ if $c \in A$ and $P \in \mathrm{Prim}\,(A)$, so that $b \in Z(A)$. Since $b \notin P_0$, A must be a C-algebra.

LEMMA 3.6. *For a nonzero C-ideal I in A,*

(1) $P \to \| a(P) \|$ *is structurally continuous on* $\mathrm{Prim}\,(A) - I^{\perp}$ *for each* $a \in A$, *and*

(2) $\mathrm{Prim}\,(A)^{-} \subset [\mathrm{Max}\,(A) \cap \mathrm{Mod}\,(A)] \cup I^{\perp}$.

Proof. To prove (1), fix $a \in A$, and suppose $P_0 \in \mathrm{Prim}\,(A) - I^{\perp}$ is given. It is sufficient to show that $P \to \| a(P) \|$ is structurally continuous on some structure neighborhood of P_0. From the structure homeomorphism of $\mathrm{Prim}\,(A) - I^{\perp}$ onto $\mathrm{Prim}\,(I)$ and the fact that I has local identities, there is a structure neighborhood T of P_0 contained in $\mathrm{Prim}\,(A) - I^{\perp}$ and a $b \in I$ such that $b(P \cap I)$ is an identity in $I/(P \cap I)$ for each $P \in T$. As I is an M-ideal in A, each $P \in T$ is a structurally closed point in $\mathrm{Prim}\,(A)$, and so is a maximal ideal. Therefore $P + I = A$ and there is a $*$-isomorphism of A/P onto $I/(I \cap P)$ which carries $c(P)$ into $c(I \cap P)$, $c \in I$ [4; Corollaire 1.8.4]. Hence $b(P)$ is an identity in A/P for each $P \in T$, and since $ab \in I$, Proposition 3.1 implies that $P \to \| (ab)(P) \| = \| a(P) \|$ is structurally continuous on T. Turning to (2), suppose $P \in \mathrm{Prim}\,(A)^{-}$, $P \notin I^{\perp}$. Since I is an M-ideal in A, Proposition 3.1 gives $P \in \mathrm{Max}\,(A)$. As I is central, there is an $a \in Z(I) \subset Z(A)$ with $a \notin P$. Since $a(P)$ is a nonzero central element of A/P, P must be modular.

In the case of simplex spaces, the analogues of (1) and (2) of the previous lemma are each equivalent to I being a C-ideal. This is not the case for C^*-algebras. In fact, there is an example of a noncentral C^*-algebra A which satisfies (1) and (2) with I replaced by A, viz, the algebra of all functions a from $\{1, 2, \cdots\}$ into the two-by-two matrices with complex entries such that $\lim_{n\to\infty} a_{ij}(n)$ exists and is equal to zero unless $i = j = 1$ (this example was also used by Delaroche in [2; § 6]).

The following result is due to Delaroche [2, Proposition, 14].

THEOREM 3.7. *A separable C^*-algebra A is a GC-algebra if and only if*

(1) *A is a GM-algebra, and*

(2) *every primitive ideal in A is modular.*

Proof. Suppose that A is a GC-algebra. Then by Proposition 3.5, A is a GM-algebra. If $P \in \operatorname{Prim}(A)$, then since P is a maximal ideal in A (Theorem 3.2), A/P must be central. But then A/P is primitive and has a nontrivial center, implying that P is modular.

Conversely, suppose that (1) and (2) hold, and let $I \neq A$ be an ideal in A. From Lemma 2.14, A/I is a GM-algebra. Since any primitive ideal in A/I is of the form P/I for some $P \in \operatorname{Prim}(A) \cap I^\perp$ [4; Proposition 2.11.5 (i)], and since $(A/I)/(P/I) \cong A/P$ for such P, every primitive ideal in A/I is modular. So to show that A is a GC-algebra, it is only necessary to show that A possesses a nonzero C-ideal. Let I be a nonzero M-ideal in A. The structure space of I, being homeomorphic to $\operatorname{Prim}(A) - I^\perp$ with the relative structure topology [4; Proposition 3.2.1], is Hausdorff. Since any $P \in \operatorname{Prim}(A) - I^\perp$ is a maximal ideal in A, $P + I = A$ and $I/(P \cap I) \cong (P + I)/P = A/P$ [4; Corollaire 1.8.4]. So any primitive ideal in I, being of the form $P \cap I$ for some $P \in \operatorname{Prim}(A) - I^\perp$, must be modular. This and [4; Proposition 1.8.5] show that it is sufficient to establish the following: If A is a separable C^*-algebra all of whose primitive ideals are modular and whose structure space is Hausdorff, then A has a nonzero C-ideal.

For such a C^*-algebra A, the structure and weak* topologies coincide on $\operatorname{Prim}(A)$ (Lemma 2.1). Let 1_P be the identity in A/P, $P \in \operatorname{Prim}(A)$. Let (u_n) be an approximate identity in A indexed on the positive integers, and set

$$T_n = \{P \in \operatorname{Prim}(A) \colon \| u_n(P) - 1_P \| \leqq 1/2\} ,$$

$n = 1, 2, \cdots$. Since $u_n(P) \to 1_P$ as $n \to \infty$ for each P, $\operatorname{Prim}(A) = \bigcup_{n=1}^\infty T_n$. Let A' be the C^*-algebra obtained by adjoining an identity 1 to A. Then $\operatorname{Prim}(A') \cong \operatorname{Prim}(A) \cup \{A\}$ and $A^\perp = \{A\}$. Fix a $P' \in \operatorname{Prim}(A') - A^\perp$, and set $P = P' \cap A$. Then $a(P) \to a(P')$, $a \in A$, is an isomorphism of A/P onto $(A + P')/P'$. Choose a $b \in A$ such that

$b(P) = 1_P$. Then $b(P')$ must be an identity in $(A + P')/P'$. The latter is an ideal in A'/P', and from Lemma 3.3, $b(P')$ is a central idempotent in A'/P'. Since A'/P' is primitive, $b(P') = 1(P')$. Consequently,

$$\| (u_n - 1)(P') \| = \| (u_n - b)(P') \| = \| (u_n - b)(P) \| \\ = \| u_n(P) - 1_P \| .$$

Therefore

$$T_n = \{P' \cap A \colon P' \in \operatorname{Prim}(A') \quad \text{and} \quad \| (u_n - 1)(P') \| \leqq 1/2\} ,$$

and T_n is a closed subset of $\operatorname{Prim}(A)$. Since the structure space of A is Baire [4; Corollaire 3.4.13], some T_n contains a nonempty open set T. Because $u_n \geqq 0$ and $\| u_n \| \leqq 1$, $\operatorname{Sp} u_n(P) \subset [1/2, 1]$ for each $P \in T$. Choosing a continuous real-valued function f on $[0, 1]$ with $f(0) = 0$ and $f = 1$ on $[1/2, 1]$ and setting $a = f(u_n)$, $a(P) = 1_P$ for each $P \in T$ [4; Proposition 1.5.3]. Let I be the ideal in A with $\operatorname{Prim}(A) - I^{\perp} = T$. Say $P \in T$. Since $\operatorname{Prim}(A)$ is locally compact and Hausdorff, there is a continuous bounded function g on $\operatorname{Prim}(A)$ such that $g(P) = 1$ and g vanishes off T. From the Dauns-Hofmann theorem (see [7; § 7]), there is a $b \in A$ with $b(Q) = g(Q)a(Q)$ for all $Q \in \operatorname{Prim}(A)$. Then $b(Q) = 0$ if $I \subset Q \in \operatorname{Prim}(A)$ and $(bc - cb)(Q) = 0$ if $c \in A$ and $Q \in \operatorname{Prim}(A)$, which imply (by [4; Th. 2.9.7 (ii)] that $b \in Z(I)$. Therefore I satisfies condition (i) of Lemma 3.4, and so is a C-ideal in A. This completes the proof of Theorem 3.7.

It is not known whether the conclusion of Theorem 3.7 is true for nonseparable C^*-algebras.

4. **Concluding remarks.** Let A be a C^*-algebra. Recall that A is a CCR-algebra ("liminaire") if the image of A by any irreducible representation is contained in the algebra of compact operators on the representing Hilbert space. A nonzero ideal I in A is a CCR-ideal in A if it is a CCR-algebra, and A is a GCR-algebra ("postliminaire") if every nonzero quotient of A contains a nonzero CCR-ideal.

The spectrum of A is the set $\hat{A}$ of all equivalence classes of irreducible representations of A provided with the inverse image topology by the natural map $\pi \to \operatorname{Ker} \pi$ of $\hat{A}$ onto the structure space of A. Dixmier [4; § 4.5] has shown that the closure $J(A)$ of the finite linear combinations of those $a \in A^+$ for which $\pi \to \operatorname{Tr} \pi(a)$ is finite and continuous on $\hat{A}$ is an ideal in A. A nonzero ideal I in A will be called a CTC-ideal in A if $I \subset J(A)$, and A will be called a CTC-algebra [resp., GTC-algebra] if A is a CTC-ideal in itself [every nonzero quotient of A contains a nonzero CTC-ideal]. These algebras have been studied in the literature, where they are sometimes called "C^*-algèbre à trace continue" ["C^*-algèbrea à trace continue géneralisée"]. Recall that a CTC-algebra has Hausdorff structure space and that a GTC-algebra is CCR ([4; § 4]).

A CCR-algebra A with a Hausdorff structure space will be said to satisfy the *Fell condition* if the canonical field of C^*-algebras defined by A satisfies the Fell condition of Dixmier [4; § 10.5]. This amounts to saying that given $P_0 \in \mathrm{Prim}(A)$, there is an $a \in A$ such that $a(P)$ is a one-dimensional projection in A/P for all P in some structure neighborhood of P_0. The following are some of the relations between the various classes of C^*-algebras:

(1) if A is separable, then it is both GM and GCR if and only if it is GTC ([5; Proposition 4.2]),

(2) if A is separable, then it is both GC and GCR if and only if it is GTC and all its irreducible representations are finite-dimensional ((1) and Theorem 3.7),

(3) A is GCR and M and satisfies the Fell condition if and only if it is CTC ([4; Propositions 4.5.3 and 10.5.8]; recall that A is CCR if it is GCR and M),

(4) A is a central GCR-algebra and satisfies the Fell condition if and only if it is a CTC-algebra with local identities ((3) and Proposition 3.7), and

(5) if A is separable, then it is GM if either it is a CCR-algebra with compact structure space or its irreducible representations are all finite-dimensional ([3; § 1]).

Let H be a separable infinite-dimensional Hilbert space. Let B denote the C^*-algebra obtained by adjoining an identity to $CC(H)$, the compact operators on H. The structure space of B (see [4; Exercise 4.7.14 (a)]) fails to be T_1, and therefore is not almost strongly separated. Yet $\mathrm{Prim}(B)$ is K_σ in the weak* topology.

In [3; § 2], Dixmier has constructed a separable CCR-algebra D whose structure space contains no nonempty strongly separated subset. In particular, D is not GM. Nevertheless, there is an open subset of the structure space of D which is homeomorphic to $[0, 1]$, and D contains an ideal C isomorphic to the C^*-algebra of continuous maps of $[0, 1]$ into $CC(H)$. So C is an M-algebra, yet no nonzero ideal in C is an M-ideal in D. Since D is a CCR-algebra, $\mathrm{Prim}(D)$ is T_1 in the structure topology, so that $\mathrm{Prim}(D)$ cannot be K_σ in the weak* topology (Theorem 3.2). These two examples are the ones promised after Theorems 2.15 and 3.2.

Finally, one further point of contact between C^*-algebras and simplex spaces will be mentioned. Fell has shown that a C^*-algebra

A can be described (to within isomorphism) as the set of all functions on Prim $(A)^-$ satisfying certain conditions, the value of such a function at an $I \in \text{Prim}(A)^-$ being an element of A/I [12]. Moreover, the Dauns-Hofmann theorem (see [7; § 7]) may be deduced from this representation theorem [Fell, unpublished]. There is an analogous representation theorem for simplex spaces, due to Effros [10; Corollary 2.5]. The analogue of the Dauns-Hofmann theorem for simplex spaces can be deduced from this representation theorem (however, this is not the manner in which it is proven in the literature; cf. [10; Th. 2.1]).

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ON A CLASS OF TOPOLOGICAL ALGEBRAS

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This note introduces a class of topological algebras, called A-convex, which generalize the notion of locally m-convex algebras. They include a number of function space algebras which are not locally m-convex. Certain of these algebras admit a modified Gel'fand type representation in a space of vector-valued functions without invoking commutativity requirements. One seemingly obtains a new way of representing locally m-convex algebras. A-convex algebras are locally m-convex under the assumption of completeness of certain factor algebras in a suitable topology.

The definition of an A-convex algebra is given in § 2 together with some basic results. We define a condition, P-complete, such that every P-complete, A-convex algebra is locally m-convex. A class of important functions algebras whose seminorms are defined by certain types of weight functions is defined in § 3, see W. H. Summers [9]. Many of these are not locally m-convex, but are A-convex algebras. The definition and basic properties of an algebra of vector-valued functions where the index set is a completely regular Hausdorff space and the functions take values in (various) Banach algebras are given in § 4. Finally, the result is obtained in § 5 that each A-convex algebra is an inverse limit of A-normed (normed linear space with separately continuous multiplication) algebras. It is also shown that certain A-convex algebras can be represented as a subalgebra of an algebra of vector-valued functions. A sufficient condition for the representation to be valid is that A be barrelled. It is shown by means of an example that barrelled is not necessary for this representation to be valid.

Some of our results are analogous to various others given by P. D. Morris and D. E. Wulbert [7], G. R. Allan [1, 2], and R. M. Brooks [4, 5].

2. Basic definitions and results on multiplication. This paragraph is concerned with the introduction of some basic definitions and results on multiplications in a locally convex topological vector space. Let A be a locally convex topological vector space over the complex numbers K with a topology T determined either by a family N of absolutely convex neighborhoods of the origin or by a family P of seminorms on the space. Assume that a multiplication is defined for A with respect to which it is an algebra.

We wish to extend some of the additive concepts of locally convex topological vector spaces in the direction of multiplication.

DEFINITION. 2.1 A subset U of A is said to be *left (multiplicatively) absorbing* if aU is absorbed by U for every a in A. It is said to be *right (multiplicatively) absorbing* if Ua is absorbed by U for every a in A. It is said to be *(multiplicatively) absorbing* (m-absorbing) if it is both left and right absorbing.

Let p and q denote elements of the family P of seminorms of the algebra A.

DEFINITION 2.2. The seminorm p is said to *absorb* the seminorm q if there exists a positive real number M such that $q(x) \leqq Mp(x)$ for every x in A. The seminorms p and q are said to be *conjugate* if they are mutually absorbing. Let $[p]$ denote the class of all q which are conjugate to p.

DEFINITION 2.3. The *left-translate* ${}_ap$ (*right-translate* p_a) of any element p of P by the element a of A is the mapping from the algebra A into the real numbers R defined by ${}_ap(x) = p(ax)[p_a(x) = p(xa)]$ for x in A.

DEFINITION 2.4. A seminorm p is *left absorbing* if it absorbs all of its left translates, *right absorbing* if it absorbs all of its right translates, and *absorbing* if it is both right and left absorbing.

It is clear that the class $[p]$ of p is left-absorbing, right absorbing, and absorbing if and only if p enjoys these respective properties.

We use the term *locally convex algebra A* to designate a locally convex topological vector space A with a multipication such that A is an abstract algebra over the complex field K.

DEFINITION 2.5. A locally convex algebra A is an *A-convex algebra* (absorbing convex algebra) if there exists a family P of absorbing seminorms defining the topology of A.

Clearly every locally m-convex algebra is an A-convex algebra. The property of being m-absorbing is preserved with respect to taking convex hulls, inverse images under a homomorphism and images under a surjective homomorphism. The proof of the following theorem is straight-forward.

THEOREM 2.6. *Any subalgebra of an A-convex algebra is A-convex. The product of A-convex algebras is again A-convex.*

Let A be an algebra without identity and let A^* be the algebra obtained from A by adjoining the identity. If p is an absorbing seminorm on A then the seminorm $p*$ defined by

$$p*((x, \lambda)) = p(x) + |\lambda|, \; x \in A, \; \lambda \in K ,$$

on A^* is absorbing. Thus, every A-convex algebra can be topologically embedded in an A-convex algebra with identity.

LEMMA 2.7. *Let p be a seminorm on the A-convex algebra A. Then there exist positive constants M and N, depending on x, such that*

$$\text{(i)} \quad p(xy) = {}_xp(y) \leqq Mp(y) ,$$
$$\text{(ii)} \quad p(yx) = p_x(y) \leqq Np(y) ,$$

for any y in A. The greatest lower bound of the M for which (i) *holds is denoted by ${}_p\|x\|$ while the greatest lower bound of the N for which* (ii) *holds is denoted by $\|x\|_p$.*

The lemma follows directly from the definitions and implies that multiplication is continuous in the right (left) factor for a fixed left (right) factor.

The kernel $R(p)$ of a seminorm p on the A-convex algebra A is the set $\{x \in A: p(x) = 0\}$. It follows immediately from Lemma (2.7) that $R(p)$ is a closed, two-sided ideal of A. Thus one can define the factor algebra $A\backslash R(p)$ on which p induces a norm. Let P be a family of seminorms defining the topology on an A-convex algebra A such that $A\backslash R(\mathrm{p})$ is complete in the norm induced by p for each $p \in R$. Then A is said to be *P-complete*.

THEOREM 2.8. *Let A be an A-convex algebra which is P-complete in some family P of absorbing seminorms defining the topology of A. Then there exists a representation $O(p)$ of A by a Banach algebra B for each seminorm p of A.*

Proof. Denote by R, rather than $R(p)$, the kernel of the seminorm p of A. It follows from the definition that the factor algebra $A\backslash R$ is complete in the norm induced by p. The coset $x + R$ of $A\backslash R$

is denoted by x' for simplicity. The seminorm p induces a norm p' on $A\backslash R$ which is defined for x' in $A\backslash R$ by

$$p'(x') = \inf \{p(y): \quad y \in x'\} .$$

Since p is subadditive, $p'(x') = p(y)$ for any $y \in x'$. We note that

$$\begin{aligned} p'(x'y') &= p'((xy)') = p(xy) \\ &\leqq {}_p\|x\| \, p(y) = {}_p\|x\| \, p'(y') . \end{aligned}$$

Thus multiplication is continuous in the right factor with respect to a fixed left factor in the norm p'. Similarly it is continuous in the left factor with respect to a fixed right factor. It follows from a well-known theorem of Gel'fand that there exists a norm $|\ |$ on $A\backslash R$ equivalent to p' with respect to which $A\backslash R$ is a Banach algebra. The natural map $O(p)$ of A into $A\backslash R$ is the required representation.

COROLLARY 2.9. *Let A be a P-complete A-convex algebra. Then A is locally m-convex.*

Examples 3. The first example gives a complete A-convex algebra which is not locally m-convex. Let $C_b(\mathbf{R})$ denote the algebra of bounded continuous complex-valued functions on the real numbers $\mathbf{R}$ (pointwise operations). Denote the set of strictly positive real-valued continuous functions on $\mathbf{R}$ which vanish at infinity by $C_0^+(\mathbf{R})$. The family of seminorms $\{p_\phi: \phi \in C_0^+(\mathbf{R})\}$ determine a locally convex linear topology β on $C_b(\mathbf{R})$ where

$$P_\phi(f) = \sup \{|f(x)\phi(x)|: x \in \mathbf{R}\}, \ f \in C_b(\mathbf{R}) .$$

The space $(C_b(\mathbf{R}), \beta)$ is A-convex since

$$P_\phi(fg) \leqq M(f)\, P_\phi(g), \ g \in C_b(\mathbf{R}) ,$$

where $M(f)$ is the maximum of $|f|$. Completeness follows from Theorem 3.6 of [9]. It is easy to verify that each p_ϕ fails to be submultiplicative.

Suppose that $(C_b(\mathbf{R}), \beta)$ is locally m-convex and let Q be a set of submultiplicative seminorms which define β. We may assume that $\max (q_1, \cdots, q_n) \in Q$ and $\lambda q_1 \in Q$ whenever $q_1, \cdots q_n \in Q$ and $\lambda \geqq 1$. Thus, for $\phi \in C_0^+(\mathbf{R})$, there exists $q \in Q$ and $\psi \in C_0^+(\mathbf{R})$ such that

$$(*) \qquad V(\psi) \subseteq V(q) \subseteq V(\phi) ,$$

where $V(\psi) = \{f: p_\psi(f) \leqq 1\}$. Since $V(\psi) \subseteqq V(\phi)$, $\phi \leqq \psi$ (pointwise). Let $\theta \in \mathbf{R}$ with $0 < \theta < \min (1, M(\psi))$. Then for some $x \in \mathbf{R}$ it follows that $\psi(x) = \theta \geqq \phi(x)$, and $\theta^n < \phi(x)$ where n is a positive integer.

Consider the function f defined by

$$f(y) = \begin{cases} \dfrac{y-x+1}{\psi(y)}, & x-1 \leqq y \leqq x; \\ \dfrac{-y+x+1}{\psi(y)}, & x < y \leqq x+1; \\ 0 & \text{, otherwise.} \end{cases}$$

Then f is well-defined since $\psi \in C_0^+(\mathbf{R})$ and $f \in C_b(\mathbf{R})$. But $p_\psi(f) = 1$ and $p_\phi(f^n) \geqq f^n(x)\phi(x) = \theta^{-n}\phi(x) > 1$ which contradicts $(*)$ since q is submultiplicative. Hence $(C_b(\mathbf{R}), \beta)$ is not locally m-convex.

This example is a special case of a weighted space (see W. H. Summers [9, 10]). By slight modifications of the arguments given here, other examples can be constructed using algebras of weighted functions. The representation of the last section is valid for this example.

A second example is obtained from the algebra $C\,[0, 1]$ of all continuous complex valued functions on the closed interval $[0, 1]$. A norm p is defined on this algebra by

$$p(f) = \sup\{\,|\,f(x)\phi(x)\,| : x \in [0, 1]\}\ ,$$

where

$$\phi\,(x) = \begin{cases} x & 0 \leqq x \leqq 1\backslash 2; \\ 1-x & 1\backslash 2 \leqq x \leqq 1\ . \end{cases}$$

Then $(C\,[0, 1], p)$ is a normed linear space which is A-convex but not locally m-convex. This space is not complete and the topological completion is not an algebra.

Noncommutative examples may be obtained in the same manner as the first example where the range of the function space is a noncommutative space such as all bounded operators on a separable Hilbert space in the operator norm.

We now give a representation of an A-convex algebra in an algebra of vector-valued functions. The next section is concerned with the definition and basic properties of such an algebra.

4. Algebras of vector-valued functions. Suppose that T is a completely regular Hausdorff space such that to each point t of T there corresponds a complex Banach algebra $B(t)$. Let F denote the set of all vector-valued functions f from T to $\bigcup\,\{B(t): t \in T\}$ such that $f(t) \in B(t)$ for each $t \in T$ and such that the function $/f/$ defined by $/f/(t) = \|f(t)\|$ is continuous on T ($\|f(t)\|$ denoting the norm in $B(t)$). We consider any subset H of F which forms an algebra over K with the usual pointwise definition of sum, product and scalar multiplication. Consider as a subbasis of neighborhoods of the origin in H the sets $N(C, O)$, where C is a compact subset of T and O an open subset of the real numbers R, defined by

$$N(C, O) = \{f: f \in H,\ /f/(k) \in O \text{ for all } k \in C\} .$$

This system of neighborhoods determines a topology which by definition makes addition continuous and continuity of scalar multiplication is clear. This gives a topology on H such that H is a locally convex topological algebra. We consider H with this topology in the remaining section.

The following two theorems and their corollaries generalize results which are well known if T is compact and Hausdorff (and hence the elements of F are bounded functions). They also extend a result of Morris and Wulbert [7] given in a similar setting where only commutative algebras are considered.

THEOREM 4.1. *Let H be a locally convex algebra as defined above. Suppose that*

(i) $H_t = \{f(t): f \in H\} = B(t)$ *for all* $t \in T$, *and*

(ii) *The product hf belongs to the closed ideal generated by f for every choice of $f \in H$ and continuous real-valued function h on T.*

Then every closed (right, left, two-sided) ideal of F is given by a set of the form $\{f \in H: f(t) \in I_t$ for all $t \in T\}$ where I_t is a closed (right, left, two-sided) ideal in $B(t)$ for each $t \in T$. Conversely, every collection $\{I_t: t \in U\}$ (right, left, two-sided) ideals where I_t is an ideal in $B(t)$ determines a closed (right, left, two-sided) ideal in H.

The proof is a direct generalization of one given by Naimark [8, § 26, Subsection 2] of a similar theorem.

The following are now immediate:

COROLLARY 4.2. *Every maximal closed (left, right, two-sided) ideal in H consists of $\{f \in H: f(t) \in I_t\}$ where I_t is a maximal closed ideal of $B(t)$.*

COROLLARY 4.3. *If $B(t)$ is simple for each $t \in T$, then every closed two-sided ideal in H consists of all $f \in H$ which vanish on some closed subset of T.*

COROLLARY 4.4. *If $B(t)$ is simple for each $t \in T$ then there is a one-to-one correspondence between the closed subsets of U and closed ideals of H via $G \to I_G = \{f \in H: f(G) = 0\}$.*

COROLLARY 4.5. *If $B(t)$ is simple for each $t \in T$ there is a one-to-one correspondence between maximal closed ideals in H and points of T.*

Let $R(T)$ denote the algebra of all bounded real-valued functions which are continuous on T.

THEOREM 4.6. *Let T be a completely regular Hausdorff space such that*

(i) *$H_t = B(t)$ for all $t \in T$;*

(ii) *For each $w \in B(t)$, $w \in \overline{wB(t)}$;*

(iii) *H is closed with respect to multiplication by elements in $R(T)$.*

Then for any $g \in H$ and $a \in R(T)$, ag is an element of the closed ideal generated by g.

The proof is omitted.

5. **Representations of A-convex algebras.** An *A-normed algebra* is an A-convex algebra having a single absorbing norm defining the topology. The following theorem gives a generalization of a theorem of Michael [Proposition 2.7, 6].

THEOREM 5.1. *A locally convex algebra is A-convex if and only if it is isomorphic to a subalgebra of a product of A-normed algebras.*

The proof of this theorem is standard in view of the properties of A-convex algebras given in § 2. The spaces $A\backslash R(p)$ used in Theorem (2.8) are used to make up the product space into which A is embedded. Thus, an A-convex algebra is an inverse limit of A-normed algebras.

This theorem gives an alternate proof of Theorem (2.8). For if each factor algebra is complete in norm then each factor algebra is a Banach algebra and hence locally m-convex. Then A is a subalgebra of a locally m-convex algebra. It follows that A must be locally m-convex.

The second example of § 3 shows that every A-normed algebra cannot be completed as an algebra. Note that if each factor algebra $A\backslash R(p)$ can be completed then A is locally m-convex. This result follows from Theorem (5.1).

Suppose that A is an A-convex algebra with P a defining collection of seminorms. Let σ_p denote the canonical quotient map of A to $A\backslash R$ (p) $(=A_p)$ given in § 2. If each A_p has a completion we may consider A_p as a subalgebra of its completion. For any finite subset $\{x_1, \cdots, x_n\}$ of A and $\varepsilon > 0$ the sets

$$V_p\{x_1, \cdots, x_n; \varepsilon\} = \{q \in P\colon |\, \|\sigma_p(x_i)\| - \|\sigma_q(x_i)\| \,| < \varepsilon, 1 \leqq i \leqq n\}$$

give a base for a neighborhood system at $p \in P$. Then P with this topology, $\mathscr{T}(P)$, is a completely regular Hausdorff space. In fact, the sets

$$W(x_1, \cdots, x_n; \varepsilon) = \{(p,q);\ |\, \|\sigma_p(x_i)\| - \|\sigma_q(x_i)\| \,| < \varepsilon,\ 1 < i < n\}$$

give a base of a uniformity inducing $\mathscr{T}(P)$. For each $p \in P$ we correspond the A-convex algebra A_p and consider an algebra H of vector-valued functions from P to the A_p as described in § 4. We show that if A is barrelled then A may be represented as a subalgebra of H. By proposition 4.3 of [6], a barrelled A-convex algebra is locally m-convex. However, the space $(C_b(\mathbf{R}), \beta)$, which is not barrelled, can be represented by our procedure. Also, this representation, which replaces a directed index set with a topological space, is seemingly different from the usual projective limit type representation for locally m-convex spaces [6].

Consider the Gel′fand map $G\colon A \rightarrow H$ where $G(x)(\mathrm{p}) = \sigma_p(x)$. It is easy to verify that $|G(x)|$ is continuous on P for each x in A. Since P is Hausdorff, G is one-to-one and it is clearly linear.

THEOREM 5.2. *If A is barrelled then G is continuous.*

Proof. It suffices to show that G is continuous at 0. A basic neighborhood of $G(0) = 0$ is of the form $N(C, S_\varepsilon)$ where C is compact in P and $S_\varepsilon = \{r\colon |r| < \varepsilon\}$. The set, $M = \bigcap\, [p^{-1}[-\varepsilon/2, \varepsilon/2]\colon p \in C]$ is a barrel in A [use the compactness of C to see that M is absorbing] and hence a neighborhood of 0. But $G(M) \subseteqq N(C, S_\varepsilon)$ so that G is continuous.

THEOREM 5.3. *The map, G^{-1}, is continuous.*

Proof. Let $x \in A$, $G^{-1}(y) = x$. Then a basic neighborhood of x is of the form
$V = x + \bigcap\, [p_i^{-1}\{(-\varepsilon, \varepsilon)\}\colon \varepsilon > 0, p_i \in P, 1 \leqq i \leqq n]$. But the set $C = \{p_1, \cdots, p_n\}$ is compact and $D = x + N(C, S_\varepsilon)$ is a neighborhood of $G(x)$ such that $G^{-1}(D) = V$. Hence G^{-1} is continuous.

The last two theorems give the following modified Gel′fand type theorem.

THEOREM 5.4. *A barrelled A-convex algebra A can be represented as a subalgebra of an algebra of vector-valued functions as defined in* § 4.

For the space $(C_b(\mathbf{R}), \beta)$, the map G of Theorem (5.2) in continuous. Since each p is actually a norm, each quotient space is (setwise) $C_b(\mathbf{R})$. In the topology defined on P, if a set M is compact then there exists $\phi \in C_0^+(\mathbf{R})$ such that $\psi \leqq \phi$ for each $p_\psi \in M$. Then for $V = p_\phi^{-1}(-\varepsilon, \varepsilon)$ it follows that $G(V) \subseteqq N(M, S_\varepsilon)$ and the result follows. For function spaces of this nature, if each compact set is dominated by a single seminorm then G is continuous and the representation is valid for the space.

If an A-convex algebra has an involution, $*$, one can define the concept of a subset being $*$-absorbing in a similar fashion to our previous definitions. The *adjoint*, p^*, of a seminorm p, is defined by $p^*(x) = p(x^*)$ and an A^*-convex algebra can be defined to be an A-convex algebra which is defined by a family of absorbing seminorms, each of which absorbs its adjoint. Then Theorem (2.8) can be proven with the representation being in a symmetric algebra. Similarly, the representation of this section carries over to A^*-algebras with no difficulty, with the obvious restatements.

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INTEGRAL DOMAINS THAT ARE NOT EMBEDDABLE IN DIVISION RINGS

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A class of totally ordered rings V is constructed having the property $1 < \alpha \in V \Rightarrow 1/\alpha \in V$, but such that V cannot be embedded in any division ring.

1. Inverses in semigroup power series rings.

This note has only one objective—to construct the above class of counterexamples (see [6]).

NOTATION 1.1. Throughout Γ will be a totally ordered cancellative semigroup with identity e; R will denote any totally ordered division ring. If $\alpha: \Gamma \Rightarrow R$ is any function, then the *support* of α is the set $\operatorname{supp} \alpha = \{s \in \Gamma \mid \alpha(s) \neq 0\}$. The set $V = V(\Gamma, R)$ of all functions α such that $\operatorname{supp} \alpha$ satisfies the a.c.c. (ascending chain condition) form a totally ordered abelian group. If Γ is cancellative, then under the usual power series multiplication (see [3]), V is a totally ordered ring.

1.2. Any $1 < \alpha \in V$ with $\alpha(s) = 0$ for $s > e$ may be written as $\alpha = \alpha(e)(1 - \lambda)$, where $1 \leqq \alpha(e)$ and $\lambda = \Sigma\{\lambda(a)\, a \mid a < e\}$. It will be shown that

$$(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots = 1 + \textstyle\sum_s \sum' \lambda(a(1)\lambda(a(2)) \cdots \lambda(a(n)) ,$$

where the finite sum Σ' is over all integers and over all distinct n-tuples of Γ^n satisfying $s = a(1)a(2) \cdots a(n)$ with each $a(i) < e$; the sum Σ is over all $s < e$. To prove that $1/\alpha \in V$ it suffices to establish conditions (a) and (b) below.

(a) For each $s \in \Gamma$, there are only a finite number of n with $\lambda^n(s) \neq 0$;

(b) $\operatorname{supp}(1 - \lambda)^{-1}$ satisfies the a.c.c.

Assuming (a) and (b), the main theorem follows at once. By adjoining an identity as in [8; p. 158] to the semigroup in [2] a semigroup that actually satisfies the hypothesis in (ii) below can be constructed.

MAIN THEOREM 1.3. *If Γ is a totally ordered cancellative semigroup with identity e and R any totally ordered division ring, then the power series ring $V = V(\Gamma, R)$ has the following properties*:

(i) $1 < \alpha \in V$ and $\alpha(s) = 0$ for $s > e \Longrightarrow 1/\alpha \in V$.

(ii) *If in addition Γ cannot be embedded in a group, then V*

cannot be embedded in a division ring.

An already known result ([8; p. 135]) follows immediately from 1.3 (i).

COROLLARY 1.4. *If in addition Γ is a group, then $V(\Gamma, R)$ is a division ring.*

2. Proof of the main theorem. Assume 1.2 (a) or (b) fails. Then a lengthy but elementary argument shows there exists a doubly indexed matrix $\{a(i, j) \in \operatorname{supp}\lambda \mid 1 \leqq i < \infty; 1 \leqq j \leqq n(i)\}$ such that the products $u(i) = a(i, 1)a(i, 2) \cdots a(i, n(i))$ of the rows form an infinite properly ascending chain. Eventually a contradiction will be derived from this. Without loss of generality assume $\Gamma \leqq e$.

DEFINITION 2.1. For any totally ordered semigroup Γ with identity e and any element $a \in \Gamma$ with $a \leqq e$, define a semigroup by

$$\Gamma(a) = \{q \in \Gamma \mid \exists \text{ an integer } m > 0,\ q^m \leqq a\} .$$

LEMMA 2.2. *With Γ as above, for any $a(1), \cdots, a(m) \in \Gamma$ with each $a(j) \leqq e$, set $u = a(1)a(2) \cdots a(m)$ and define*

$$a^* = \min \{a(1), \cdots, a(m)\} .$$

Then $\Gamma(u) = \Gamma(a^*)$.

2.3. Consider a fixed subset $L \subseteqq \Gamma$ all of whose elements satisfy $L \leqq e$ and where L satisfies the a.c.c., e.g., $L = \operatorname{supp}\lambda < e$. Consider an array of elements $A = ||a(i, j)||$ with $\{a(i, j) \mid 1 \leqq i < \infty, 1 \leqq j \leqq n(i)\} \subseteqq L$, where repetitions in the $a(i, j)$ are allowed. Assume all $n(i) \geqq 2$. Define $u(i) = u(i, A)$ by

$$u(i) = u(i, A) = a(i, 1)a(i, 2) \cdots a(i, n(i)) .$$

Let $\mathscr{K}$ be the set of all such $A = ||a(i, j)||$ for which $u(1) < u(2) < \cdots < u(i) < \cdots$ is strictly ascending at each i. With each member $A = ||a(i, j)|| \in \mathscr{K}$, we next associate three objects

$$\{a(i)^* \mid 1 \leqq i < \infty\},\ m = m(A), \text{ and } G = G(A) .$$

Define $a(i)^* \equiv \min \{a(i, j) \mid 1 \leqq j \leqq n(i)\}$. Note that $u(1) < u(2) < \cdots$ implies that $\Gamma(a(1)^*) \subseteqq \Gamma(a(2)^*) \subseteqq \Gamma(a(i)^*) \subseteqq \cdots$. Thus since L satisfies the a.c.c., there is a unique smallest integer $m \equiv m(A)$ such that the semigroups $G \equiv \Gamma(a(m)^*) = \Gamma(a(m + 1)^*) = \cdots$ are all equal. The following schematic diagram of all these quantities may be helpful.

$$\begin{array}{ll}
\Gamma(a(1)^*) = \Gamma(u(1)) & u(1) = a(1,1)a(1,2)\cdots a(1)^*\cdots a(1,n(1)) \\
\quad\quad \cap\!\shortparallel & \\
\Gamma(a(2)^*) = \Gamma(u(2)) & u(2) = a(2,1)a(2,2)\cdots a(2)^*\cdots a(2,n(2)) \\
\quad\quad \cap\!\shortparallel & \\
\Gamma(a(m)^*) = \Gamma(u(m)) & u(m) = a(m,1)a(m,2)\cdots a(m)^*\cdots a(m,n(m)) \\
\quad\quad \| & \\
\quad G = \Gamma(u(m+1)) \,. &
\end{array}$$

2.4. Among the elements of $\mathscr{K}$, let $\mathscr{N} \subset \mathscr{K}$ be all those $A = ||a(i, j)||$ such that this associated $G = G(A)$ is as big as possible and call this particular $G \equiv M$. If $\mathscr{K} \neq \emptyset$, also $\mathscr{N} \neq \emptyset$. Define $\bar{a} = \max\{a(m)^* \mid A \in \mathscr{K}, m = m(A)\}$. Pick and element $B = ||b(i, j)|| \in \mathscr{N}$. Then by our choice of M, $\Gamma(\bar{a}) = M$. Thus $M = G(B) = \Gamma(b(i)^*) = \Gamma(b(i, j)) = \Gamma(u(i)) = \Gamma(\bar{a})$ for $i \geqq m(B) \equiv m$. Finally, with each element B of $\mathscr{N}$, we associate an integer $r = r(B)$. Since $\bar{a} \in \Gamma(u(m))$, there is a unique smallest integer $r \equiv r(B) \geqq 1$ such that $\bar{a}^r \leqq u(m) < \bar{a}^{r-1}$.

2.5. By omitting some of the rows of B and renumbering the remaining ones, it may be assumed as a consequence of the a.c.c. without loss of generality that $m = 1$, and also that $b(1)^* \geqq b(2)^* \geqq \cdots$ is not ascending. Each $u(i)$ is of one of the following three forms:

$$u(i) = q(i)b(i)^* \,, \tag{1}$$

$$u(i) = b(i)^*w(i) \,, \tag{2}$$

$$u(i) = q(i)b(i)^*w(i) \,, \tag{3}$$

where the $q(i)$, $w(i)$ are certain products of the $b(i, j)$. If there are an infinite number of $u(i)$ of the forms (1) or (2), then since

$$\begin{aligned} u(i+1) = q(i+1)b(i+1)^* > u(i) = q(i)b(i)^*,\ b(i+1)^* \leqq b(i)^* \\ \Longrightarrow q(i+1) > q(i) \,; \end{aligned}$$

it follows (after omitting some rows and renumbering) that there is a properly infinite ascending chain:

Case 1. $q(1) < q(2) < \cdots$;

Case 2. $w(1) < w(2) < \cdots$.

If neither Case 1 nor Case 2 applies, then

$$\begin{gathered} u(i+1) = q(i+1)b(i+1)^*w(i+1) > q(i)b(i)^*w(i) \\ \text{and } b(i+1)^* \leqq b(i)^* \end{gathered}$$

implies that one of the inequalities $q(i+1) > q(i)$ or $w(i+1) > w(i)$

must necessarily hold. It is asserted that there is a subsequence $\{i(k) \mid k = 1, 2, \cdots\}$ such that

$$\textit{Case } 3. \quad \text{either (a):} \quad q(i(1)) < q(i(2)) < \cdots$$
$$\text{or (b):} \quad w(i(1)) < w(i(2)) < \cdots.$$

For if not, then the a.c.c. must hold in both the sets $\{q(i)\}$ and $\{w(i)\}$. Then by omitting some rows and renumbering the remaining ones it may be assumed that we have an element B in $\mathscr{N}$ with $q(1) \geqq q(2) \geqq \cdots$ and $w(1) \geqq w(2) \geqq \cdots$. However, then

$$q(1)b(1)^*w(1) \geqq q(2)b(2)^*w(2) \geqq \cdots$$

gives a contradiction.

2.6. We may assume $q(1) < q(2) < \cdots$ or $w(1) < w(2) \cdots$ are properly ascending, depending on which of the Cases 1, 2, 3(a) or 3(b) is applicable. Set $t = r(B)$, so that $\bar{a}^t \leqq u(m) = u(1) \leqq u(i)$.

2.7. It is next shown that either $q(i) \geqq \bar{a}^{t-1}$ or $w(i) \geqq \bar{a}^{t-1}$ holds for all i. Suppose that the following holds.

$$\begin{array}{lllll} \textit{Case } 1. & q(1)b(1)^* & < q(2)b(2)^* & < \cdots; \\ & q(1) & < q(2) & < \cdots; \\ & b(1)^* \geqq & b(2)^* & \geqq \cdots. \end{array}$$

Then $\bar{a}^t \leqq u(1) \leqq u(i) = q(i)b(i)^*$, and $\bar{a} \geqq b(i)^*$ implies that

$$\bar{a}^{t-1} \leqq q(1) \leqq q(i) .$$

(For if $\bar{a}^{t-1} > q(i)$, then $\bar{a} \geqq b(i)^*$ implies that $\bar{a}^t > q(i)b(i)^*$.) (If $t = 1$, then $\bar{a}^0 = e$.) Similarly, in Case 2 also $\bar{a}^{t-1} \leqq w(1) \leqq w(i)$.

Only Case 3(b) will be proved, since 3(a) is entirely parallel.

$$\begin{array}{lllll} \textit{Case } 3(b). & q(1)b(1)^*w(1) & < q(2)b(2)^*w(2) & < \cdots; \\ & w(1) < & w(2) & < \cdots; \\ & b(1)^* \geqq & b(2)^* & \geqq \cdots. \end{array}$$

Then again $\bar{a}^t \leqq u(1) \leqq u(i) = q(i)b(i)^*w(i)$ and $\bar{a} \geqq b(i)^* \geqq q(i)b(i)^*$ imply that $\bar{a}^{t-1} \leqq w(1) \leqq w(i)$. (Otherwise, if $\bar{a}^{t-1} > w(i)$, then $\bar{a}^t > q(i)b(i)^*w(i)$.)

The basic idea motivating the proof is that for $B \in \mathscr{N}$, a new $C \in \mathscr{N}$ can be constructed with $r(C) \leqq r(B) - 1$.

2.8. Thus either $q(1) < q(2) < \cdots$ and all $q(i) \geqq \bar{a}^{t-1}$; or $w(1) < w(2) < \cdots$ and all $w(i) \geqq \bar{a}^{t-1}$. Assume the latter. Let

$$C = \|c(i, j)\| \in \mathscr{K}$$

be defined by taking as its i-th row all the $b(i, j)$ appearing in $w(i)$. (In view of $w(1) < w(2) < \cdots$, there does not exist an infinite number of rows of C containing only one element. By omitting a finite number of rows it may be assumed that all rows of C contain two or more elements of L.) Define $c(i)^* \equiv \inf\{c(i, j) \mid j \geqq 1\}$. Since $b(i)^* \leqq c(i)^* \leqq \bar{a}$, it follows that

$$\boldsymbol{M} = \Gamma(b(i)^*) \subseteqq \Gamma(c(i)^* \subseteqq \Gamma(\bar{a}) = \boldsymbol{M} .$$

Consequently, $\boldsymbol{G}(C) = \boldsymbol{M}$ and $C \in \mathscr{N}$. Since $w(1) \geqq \bar{a}^{t-1}$, $r(C) \leqq t - 1$. By repetition of this process, we may reduce the r to one so that finally $\bar{a}^r = \bar{a} \leqq w(1) < w(2) \cdots$. Since all $c(i, j) \in L$ satisfy $c(i, j) \leqq e$ and since $w(i)$ is a product of these, it follows that $\bar{a} \geqq c(i)^* \geqq w(i)$. Thus $\bar{a} = w(1) = w(2) = \cdots$ gives a contradiction. Thus $\mathscr{K} = \varnothing$ and the main theorem has been proved.

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ON THE NUMBER OF NONPIERCING POINTS IN CERTAIN CRUMPLED CUBES

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Let K denote the closure of the interior of a 2-sphere S topologically embedded in Euclidean 3-space E^3. If $K - S$ is an open 3-cell, McMillan has proved that K has at most one nonpiercing point. In this paper we use a more general condition restricting the complications of $K - S$ to describe the number of nonpiercing points. The condition is this: for some fixed integer n $K - S$ is the monotone union of cubes with n holes. Under this hypothesis we find that K has at most n nonpiercing points (Theorem 5). In addition, the complications of $K - S$ are induced just by these nonpiercing points. Generally, at least two such points are required, for otherwise $n = 0$ (Theorem 3).

A space K as described above is called a *crumpled cube*. The *boundary of* K, denoted $\mathrm{Bd}\, K$, is defined by $\mathrm{Bd}\, K = S$, and the *interior of* K, denoted $\mathrm{Int}\, K$, is defined by $\mathrm{Int}\, K = K - \mathrm{Bd}\, K$. We also use the symbol Bd in another sense: if M is a manifold with boundary, then $\mathrm{Bd}\, M$ denotes the boundary of M. This should not produce any confusion.

Let K be a crumpled cube and p a point in $\mathrm{Bd}\, K$. Then p is a *piercing point of* K if there exists an embedding f of K in the 3-sphere S^3 such that $f(\mathrm{Bd}\, K)$ can be pierced with a tame arc at $f(p)$.

Let U be an open subset of S^3. The *limiting genus of* U, denoted $LG(U)$, is the least nonnegative integer n such that there exists a sequence $H_1, H_2, \cdots$ of compact 3-manifolds with boundary satisfying (1) $U = \cup H_i$, (2) $H_i \subset \mathrm{Int}\, H_{i+1}$, and (3) genus $\mathrm{Bd}\, H_i = n$ ($i = 1, 2, \cdots$). If no such integer exists, $\mathrm{LG}(U)$ is said to be infinite. Throughout this paper the manifolds H_i described above can be obtained with connected boundary, in which case H_i is called a *cube with n holes*.

Applications of the finite limiting genus condition are investigated in [6] and [14]. For any crumpled cube K such that $\mathrm{LG}(\mathrm{Int}\, K)$ is finite and $\mathrm{Bd}\, K$ is locally peripherally collared from $\mathrm{Int}\, K$, it is shown that $\mathrm{Bd}\, K$ is locally tame (from $\mathrm{Int}\, K$) except at a finite set of points. Under the hypothesis of this paper, $\mathrm{Bd}\, K$ may be wild at every point; nevertheless, with a collapsing (in the sense of Whitehead [15]) argument comparable to [13, Th. 1], the problem of counting the nonpiercing points of K is reduced to one in which the results of [6] and [14] apply.

A subset X of the boundary of a crumpled cube K is said to be *semi-cellular in* K if for each open set U containing X there exists an open set V such that $X \subset V \subset U$ and loops in $V - X$ are null homotopic in $U - X$. In the last section of this paper semi-cellular sets are discussed in order to characterize those sewings of two crumpled cubes which yield S^3, in case the limiting genus of one of the crumpled cubes is finite.

A simple closed curve J is *essential in an annulus* A if J lies in A and bounds no disk in A.

If X is a set in a topological space, then $\mathrm{Cl}\, X$ denotes the closure of X.

2. A cellularity criterion.

LEMMA 1. *Let H be a sphere with n handles. Then there exists an integer $k(n)$ such that if $J_1, \cdots, J_{k(n)}$ are mutually exclusive simple closed curves in H, no one of which bounds a disk in H, then some pair $\{J_r, J_s\}$ bounds an annulus in H.*

Proof. The number $k(n) = 2$ is known to work if $n = 1$. Otherwise, the proof proceeds by induction, using $k(n) = 3n - 2$ whenever $n \geqq 2$.

THEOREM 2. *Let C be a crumpled cube such that $\mathrm{LG}(\mathrm{Int}\, C) = n < \infty$. Then there exists a finite set Q of points in $\mathrm{Bd}\, C$ such that for each open set $U \supset \mathrm{Bd}\, C$, each point of $\mathrm{Bd}\, C - Q$ has a neighborhood V such that any loop in $V - \mathrm{Bd}\, C$ is null-homotopic in $U - \mathrm{Bd}\, C$.*

Proof. Assume $n > 0$. Using Lemma 1 we associate with a sphere with n handles an integer $k(n)$. Let $k = \max\{3, k(n)\}$. Suppose $p_1, p_2, \cdots, p_{2k}$ are points in $\mathrm{Bd}\, C$ and U is an open set containing $\mathrm{Bd}\, C$. It suffices to show that one of these points has a neighborhood V such that each loop in $V - \mathrm{Bd}\, C$ is nullhomotopic in $U - \mathrm{Bd}\, C$.

Step 1. *Preliminary constructions.* There exists a collection of mutually exclusive disks $D_1, \cdots, D_{2k}$ on $\mathrm{Bd}\, C$ with $p_i \in \mathrm{Int}\, D_i$ $(i = 1, \cdots, 2k)$. Furthermore, $\mathrm{Bd}\, C$ contains another collection of mutually exclusive disks $E_1, \cdots, E_k$ such that for $i = 1, \cdots, k$

$$D_{2i-1} \cup D_{2i} \subset \mathrm{Int}\, E_i\,.$$

We consider C to be embedded in S^3 so that the closure of $S^3 - C$ is a 3-cell [8, 10]. We select a point b of $\mathrm{Int}\, C$ and construct arcs $B_1, \cdots, B_{2k}$ such that (1) distinct arcs B_i and B_j intersect only at the point b, (2) the endpoints of B_i are b and p_i, and (3) B_i is locally tame mod p_i (i $= 1, \cdots, 2k$).

By Theorem 1 of [3] there exist pairwise disjoint annuli

$$D_1^*, D_2^*, \cdots, D_{2k}^*, E_1^*, E_2^*, \cdots, E_k^*$$

in S^3 such that

(4) $\operatorname{Bd} D_i^* \supset \operatorname{Bd} D_i$ and $\operatorname{Bd} E_j^* \supset \operatorname{Bd} E_j$,

(5) $D_i^* \cap \operatorname{Bd} C \subset D_i$,

(5′) $E_j^* \cap \operatorname{Bd} C \subset E_j - (D_{2j-1} \cup D_{2j})$,

(6) $(\cup(\operatorname{Bd} D_i^* - \operatorname{Bd} D_i)) \cup (\cup(\operatorname{Bd} E_j^* - \operatorname{Bd} E_j)) \subset \operatorname{Int} C$,

(7) $D_i^*(E_j^*)$ is locally polyhedral mod $\operatorname{Bd} D_i$ $(\operatorname{Bd} E_j)$, and

(8) $((\cup D_i^*) \cup (\cup E_j^*)) \cap (\cup B_i) = \varnothing$.

If a surface approximating $\operatorname{Bd} C$ is to intersect the D_i^*'s and E_j^*'s properly, we must force it to lie very close to $\operatorname{Bd} C$. To do this, first we thicken certain subsets of $\operatorname{Bd} C$, thereby obtaining mutually exclusive open sets $W_0, W_1, \cdots, W_{3k}$ such that

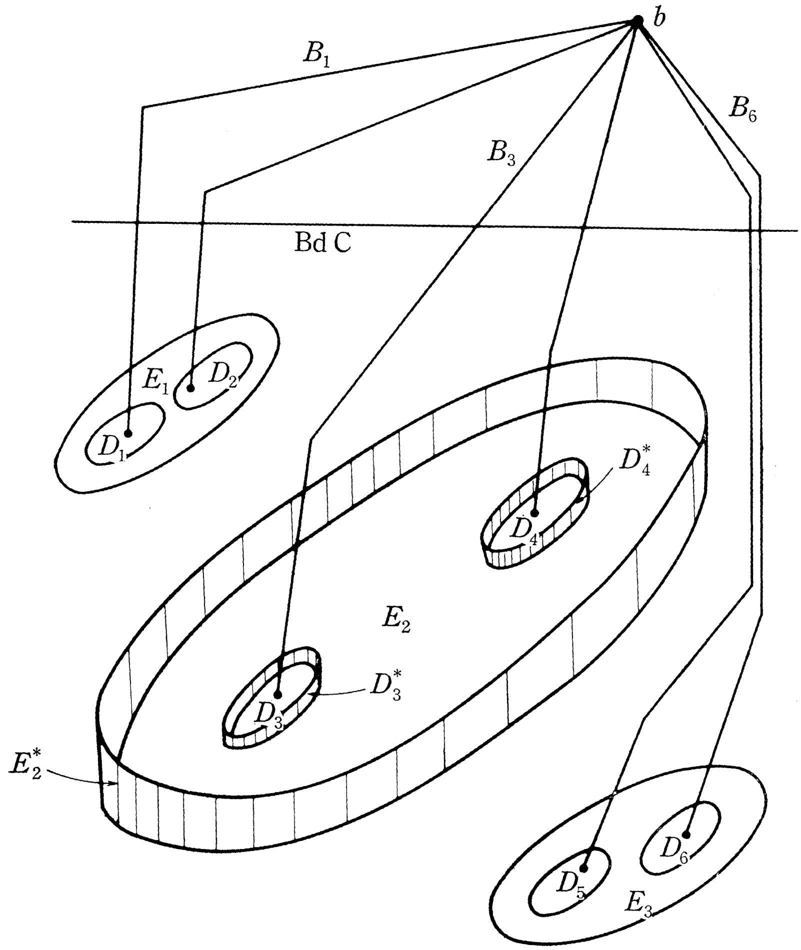

FIGURE 1

(9) $W_i \cap C \subset U - ((\cup \operatorname{Bd} D_i^*) \cup (\cup \operatorname{Bd} E_j^*))$,
(10) $W_0 \supset \operatorname{Bd} C - ((\cup D_i) \cup (\cup E_j))$,
(11) $W_i \supset \operatorname{Int} D_i$ $(i = 1, \cdots, 2k)$,
(12) $W_{2k+i} \supset \operatorname{Int} E_i - (D_{2i-1} \cup D_{2i})$ $(i = 1, \cdots, k)$,
(13) $(\cup W_j) \cap B_i = W_i \cap B_i$ $(i = 1, \cdots, 2k)$.

In addition, we require that $\operatorname{Bd} D_i \cap \operatorname{Cl} W_s \neq \varnothing$ only if $s = 2k + i$ or $s = i$ and $\operatorname{Bd} E_j \cap \operatorname{Cl} W_s \neq \varnothing$ only if $s = 0$ or $s = 2k + j$. Then we construct a neighborhood Y of $\operatorname{Bd} C - \cup W_i$ such that $Y \cap C \subset U$ and any arc in $\operatorname{Int} C \cap (Y \cup (\cup W_i))$ from a point of W_i to a point of W_j intersects all the annuli in between. For example, if A is an arc from W_0 to W_1, then A intersects both E_1^* and D_1^*.

By hypothesis $\operatorname{Int} C$ contains a cube with n holes M such that $C - (Y \cup (\cup W_i)) \subset \operatorname{Int} M$. Without loss of generality, we assume that $\operatorname{Bd} M$ is polyhedral and in general position with respect to

$$(\cup \operatorname{Int} E_j^*) \cup (\cup \operatorname{Int} D_i^*) .$$

Step 2. *A special disk in* $\operatorname{Bd} M$. Let G denote the collection of those components of $\operatorname{Bd} M \cap (\cup E_j^*) \cup (\cup D_i^*))$ which are essential simple closed curves in any annulus E_j^* or D_i^*. Each annulus $E_j^*(D_i^*)$ contains a curve in the collection G, because $\operatorname{Bd} M$ separates the components of $\operatorname{Bd} E_j^*(\operatorname{Bd} D_i^*)$.

In the next paragraphs we show that at least one of the curves in G bounds a disk in $\operatorname{Bd} M$. Suppose the contrary. From Lemma 1 we find that $\operatorname{Bd} M$ contains an annulus A such that $\operatorname{Bd} A = J_r \cup J_s$, where J_r and J_s are essential curves on E_r^* and E_s^*, respectively, and $r \neq s$. This reduces to the case in which each component of $\operatorname{Int} A \cap (\cup E_j^*)$ bounds a disk in $\cup E_j^*$. Assume $r \neq 1 \neq s$.

Case A. *No component of* $A \cap (\cup E_j^*)$ *separates the components of* $\operatorname{Bd} A$. Let L be a simple closed curve in $S^3 - (E_1^* \cup E_r^*)$ such that $L \cap C = B_2 \cup B_{2r}$. It follows from the constructions of Step 1 that each point of $L \cap A$ is separated (in A) from J_s by a component of $\operatorname{Int} A \cap (E_1^* \cup E_r^*)$; thus, by trading certain disks in $\operatorname{Int} A$ for disks in $E_1^* \cup E_r^*$, we see that J_r and J_s are homotopic in $S^3 - L$. But this is impossible, since J_r links L and J_s does not.

Case B. *Some component of* $A \cap (\cup E_j^*)$ *separates the components of* $\operatorname{Bd} A$. By considering all components of $A \cap ((\cup E_j^*) \cup (\cup D_i^*))$, we find that A contains an annulus A' such that no curve in

$$\operatorname{Int} A' \cap ((\cup E_j^*) \cup (D_i^*))$$

is essential in A' and $J_r \subset \operatorname{Bd} A'$. Let J' denote the other component of $\operatorname{Bd} A'$, and without loss of generality assume that $J' \cap D_{2r}^* = \varnothing$.

Let L' be a simple closed curve in $S^3 - ((\cup E_j^*) \cup (\cup D_i^*))$ such that $L' \cap C = B_2 \cup B_{2r}$. Each point of $L' \cap A'$ is separated in A' from either J_r or J' by $\text{Int}\, A'((\cup E_j^*) \cup (\cup D_i^*))$, and each curve of this intersection bounds disks in both A' and $(\cup E_j^*) \cup (\cup D_i^*)$. Hence, by the usual disk trading, we see that J_r is homotopic to J' in $S^3 - L'$. Again this leads to a contradiction, for J_r links L'; on the other hand, J' either is contained in D_{2r-1}^* or is an inessential curve in some E_j^*, which implies that J' does not link L'.

Neither of the two cases can occur. Consequently, some simple closed curve J in the collection G bounds a disk in $\text{Bd}\, M$.

Step 3. *A neighborhood V of one of the points p_i.* Corresponding to one of the points, say p_1, there exists a disk $D \subset \text{Bd}\, M$ such that $\text{Bd}\, D$ is an essential curve in D_1^*, but each component of Int $D \cap (\cup D_i^*)$ bounds a disk in $\cup D_i^*$. Repeating this process, it follows that for one of the p_i's, say p_1 again, and for each open set U' containing $\text{Bd}\, C$, there exists a polyhedral disk E in $U' \cap \text{Int}\, C$ such that $\text{Bd}\, E$ is an essential simple closed curve on D_1^* but each component of $(\text{Int}\, E \cap (\cup D_i^*))$ bounds a disk in $\cup D_i^*$.

To find the desired open set in C, let V' be a spherical neighborhood of p_1 such that $V' \cap C \subset W_1$, and define $V = V' \cap C$. For any loop L in $V - \text{Bd}\, C$, another linking argument shows that L is separated from $\text{Bd}\, C$ (in V) by some disk $E \subset U$ as described above. Since L is contractible in V', it follows from [5, Lemma 1] that L is contractible in $U - \text{Bd}\, C$. This completes the proof.

THEOREM 3. *Suppose C is a crumpled cube such that* $\text{LG}(\text{Int}\, C) < \infty$ *and C contains at most one nonpiercing point. Then* $\text{Int}\, C$ *is an open 3-cell.*

Proof. Assume C is embedded in S^3 so that the closure of $S^3 - C$ is a 3-cell K [8, 10]. Equivalently, we show that K is a cellular subset of S^3.

Let Q denote the finite set of points of $\text{Bd}\, C$ given by Theorem 2, p the nonpiercing point of C (the argument when C has no nonpiercing point is essentially the same), and U an open set containing K. There exists an open set V containing K such that loops in $V - K$ are null-homotopic in $U - (\text{Int}\, K \cup p)$. Let f be a map of a disk Δ into $U - (\text{Int}\, K \cup p)$ such that $f(\text{Bd}\, \Delta) \subset V - K$. It follows from [12, Th. 2] and techniques of [2, Th. 4.2] that f can be adjusted slightly at points of $\text{Int}\, \Delta$ so that $f(\Delta) \cap \text{Bd}\, C$ is 0-dimensional and $f(\Delta) \cap Q = \varnothing$. Finally, there exists a finite number of mutually exclusive simple closed curves $S_1, \cdots S_k$ in Δ whose union separates $\text{Bd}\, \Delta$ from $f^{-1}(f(\Delta)) \cap \text{Bd}\, C)$ and such that $f|S_i$ is null homotopic in

$U - K$ $(i = 1, \cdots, k)$. This implies that $f|\mathrm{Bd}\,\Delta$ extends to a map of Δ into $U - K$. According to McMillan's Cellularity Criterion [11, Th. 1'], K is a cellular subset of S^3.

3. Topological collapsing. The following result generalizes Theorem 1 of [13]. The argument below necessarily differs from McMillan's, since we have no mapping criterion to determine the finite limiting genus condition.

THEOREM 4. *Suppose K is a finite connected simplicial complex, L a subcomplex of K such that K collapses to L, and h a homeomorphism of K into S^3 such that* $\mathrm{LG}(S^3 - h(K)) = n$. *Then*

$$\mathrm{LG}(S^3 - h(L)) \leqq n .$$

Proof. It is sufficient to show that the result holds if L is obtained from K by a single elementary collapse. Suppose that σ is a principal simplex of K, τ is a proper face of σ such that τ is a proper face of no other simplex in K, and

$$L = K - \mathrm{Int}\,\sigma - \mathrm{Int}\,\tau .$$

We consider the case when σ is a 3-simplex, because the applications of Theorem 4 in this paper can be viewed as involving collapses of this type only; for the remaining cases a similar argument applies.

Let U be an open subset of S^3 containing $h(L)$. There exists a neighborhood U^* of $h(L)$ in U such that some component Z of $h(\sigma) - U^*$ contains $h(\sigma) - U$. Using [4, Th. 4] we find a tame disk D in $U^* - h(L)$ such that $\mathrm{Bd}\,D \cap h(K) = \varnothing$ and exactly one of the components of $D \cap h(\sigma)$ separates Z from $h(L \cap \sigma)$ in $h(\sigma)$.

There exists a neighborhood W of $h(K)$ such that $W \cap \mathrm{Bd}\,D = \varnothing$ and W can be deformed to $h(K)$ in S^3-$\mathrm{Bd}\,D$ by a homotopy keeping $h(K)$ pointwise fixed. For each point x in $U \cap h(K)$ define an open set N_x as

$$N_x = \{y \in S^3 \,|\, \rho(x, y) < \rho(x, \mathrm{Bd}\,U \cup \mathrm{Bd}\,W)\}$$

and for each point x in $h(\sigma) - U$ define N_x as

$$N_x = \{y \in S^3 \,|\, \rho(x, y) < \rho(x, D \cup \mathrm{Bd}\,W)\} .$$

Then let $V = \bigcup_{x \in h(K)} N_x$.

Claim. $D \cap V$ separates Z from $h(L)$ in V, and U contains the component Y of $V - D$ that contains $h(L)$.

Suppose there exists an arc α in $V - D$ from a point of Z to a point of $h(L)$. Then α is homotopic in $S^3 - \operatorname{Bd} D$ (with endpoints fixed) to a path α' in $h(K)$, and α' is homotopic in $h(K)$ (with endpoints fixed) to a path α^* such that $\alpha^* \cap D$ consists of a finite set of points at which α^* pierces D. But then the number of such points must be even, contradicting the separation properties of D in $h(K)$.

To establish the other part of the claim, suppose there exists a point y in $Y - U$. Then $y \in N_x$ for some x in $h(\sigma) - U$. Let A be the straight line segment from y to x in N_x, and let B denote an arc from y to $h(L)$ in Y. Since $A \cup B$ does not intersect D, deforming $A \cup B$ to a path in $h(K)$ leads to a contradiction as before. This completes the proof of the claim.

By hypothesis $S^3 - h(K)$ contains a polyhedral cube with n holes H such that $\operatorname{Int} H \supset S^3 - V$. We adjust H slightly so that $\operatorname{Bd} H \cap D$ consists of a finite number of simple closed curves. Note that $D \cup (\operatorname{Bd} H \cap U)$ separates $h(L)$ from $h(\sigma) - U$ (in S^3). Thus, the unicoherence of $S^3 - D$ implies that some component F of $\operatorname{Bd} H - D$, where $F \subset U$, separates $h(L)$ from $h(\sigma) - U$ in $S^3 - D$.

We observe that $\operatorname{Cl} F$ is a disk with k ($k \leqq n$) handles and (possibly) some holes. By attaching disks to $\operatorname{Bd} F$ near D, we see that F is contained in a sphere with k handles S_k in $\operatorname{Cl}(S^3 - h(L))$ and that S_k bounds a cube with k holes M satisfying

$$S^3 - U \subset M \subset S^3 - h(L) .$$

This implies that $\operatorname{LG}(S^3 - h(L)) \leqq n$.

4. The number of nonpiercing points.

THEOREM 5. *If C is a crumpled cube such that* $\operatorname{LG}(\operatorname{Int} C) = n$ $(1 \leqq n < \infty)$, *then C has at most n nonpiercing points.*

Proof. Suppose to the contrary that C contains at least $n+1$ nonpiercing points $p_1, \cdots, p_{n+1}$. As before we assume C is embedded in S^3 so that the closure of S^3 of $S^3 - C$ is a 3-cell H [8, 10]. Let h denote a homeomorphism of a 3-simplex Δ^3 onto H.

Some triangulation K of Δ^3 collapses to a subcomplex L such that $h(L)$ is a 3-cell locally tame except at $p_1, \cdots, p_{k+1}$; thus, each point p_i is a nonpiercing point of $\operatorname{Cl}(S^3 - h(L))$. Theorem 4 gives that $\operatorname{LG}(S^3 - h(L)) \leqq n$. This leads to a contradiction, however, for either [6, Th. 2] or [14, Th. 1] implies that $\operatorname{Cl}(S^3 - h(L))$ has at most n nonpiercing points.

COROLLARY. *If C is a crumpled cube such that* $\operatorname{LG}(\operatorname{Int} C) \leqq 1$, *then* $\operatorname{Int} C$ *is an open 3-cell.*

The techniques used to prove Theorem 5 can be reapplied to obtain the following result.

THEOREM 6. *If H is a cube with k handles in S^3 and*

$$\mathrm{LG}(S^3 - H) = n \, (1 \leqq n < \infty) ,$$

then Bd H *is pierced by a tame arc at all but (at most) $n - k$ of its points.*

To describe the number of nonpiercing points precisely requires some additional definitions. Let A be an arc in S^3 locally tame modulo an endpoint p. The *local enveloping genus of A at p,* denoted LEG (A, p), is the smallest nonnegative integer r (if there is no such integer r, $\mathrm{LEG}(A, p) = \infty$) such that there exist arbitrarily small neighborhoods of p, each of which is bounded by a surface of genus r (a sphere with r handles) that intersects A at exactly one point. Chapter 4 of [14] gives illustrations of arcs A_n, each locally tame mod an endpoint p_n, such that $\mathrm{LEG}(A_n, p_n) = n$ $(n = 1, 2, \cdots, \infty)$.

Let $B = \{(x, y, z) \in E^3 | x^2 + y^2 + z^2 \leqq 1\}$. Let f be a homeomorphism of B onto a 3-cell C in S^3, and p a point of Bd C. The *local enveloping genus of C at p,* denoted $\mathrm{LEG}(C, p)$, is defined by

$$\mathrm{LEG}(C, p) = \mathrm{LEG}(f(\alpha), p) ,$$

where α is the line segment in B from the origin to $f^{-1}(p)$.

THEOREM 7. *If C is a 3-cell in S^3 such that $\mathrm{LG}(S^3 - C) = n$ $(2 \leqq n < \infty)$ and $p_1, \cdots, p_k$ are the nonpiercing points of $S^3 - \mathrm{Int}\, C$, then*

$$n = \sum_{i=1}^{k} \mathrm{LEG}(C, p_i) .$$

Proof. As in the proof of Theorem 5, let h be a homeomorphism of a 3-simplex Δ^3 onto C. Some triangulation of Δ^3 collapses to a subcomplex L such that $h(L)$ is a 3-cell locally tame modulo $\cup p_i$. It follows from the definition of local enveloping genus that the subcomplex L can be chosen to satisfy

$$\mathrm{LEG}(C, p_i) = \mathrm{LEG}(h(L), p_i) \qquad (i = 1, \cdots, k) .$$

Since $\mathrm{LG}(S^3 - h(L)) \leqq n$, Theorem 6 of [14] implies

$$n \geqq \Sigma\, \mathrm{LEG}(h(L), p_i) = \Sigma\, \mathrm{LEG}(C, p_i) .$$

Let U be an open set containing C. To establish the inequality in the other direction, we shall find pairwise disjoint disks with handles

$G_1, \cdots, G_k$ in $U - \cup p_i$ subject to the following conditions: the number of handles on G_i is bounded by LEG(C, p_i), $\mathrm{Bd}\, G_i$ bounds an annulus A_i in G_i such that $G'_i = \mathrm{Cl}\,(G_i - A_i)$ is contained in $U - C$, $\mathrm{Int}\, A_i \cap \mathrm{Bd}\, C$ is contained both in a null sequence of pairwise disjoint disks in $\mathrm{Bd}\, C - \cup p_i$ and in a null sequence of such disks in $\mathrm{Int}\, A_i$, and $\cup \mathrm{Bd}\, G_i$ bounds a disk with $(k-1)$ holes in $\mathrm{Bd}\, C - \cup p_i$. Furthermore, G_i can be obtained arbitrarily close to p_i. Thus, in the next two paragraphs we describe how to find one such surface G_1 near p_1.

In $\mathrm{Bd}\, C$ there exists a Sierpinski curve X locally tame mod p_1 and containing p_1 in its inaccessible part. By removing a null sequence of nice 3-cells from C we obtain a 3-cell C^* such that $C^* \cap \mathrm{Bd}\, C = X$ and C^* is locally tame mod p_1. It follows from the definition of local enveloping genus that arbitrarily close to p_1 is a surface H such that $H \cap C^*$ is a disk D, with $D \cap \mathrm{Bd}\, C^* = \mathrm{Bd}\, D$, and p_1 lies interior to the small disk on $\mathrm{Bd}\, C^*$ bounded by $\mathrm{Bd}\, D$. Adjust H near $\mathrm{Bd}\, C^*$ so that $\mathrm{Bd}\, D$ lies in the inaccessible part of X. Without moving any point of D adjust H further so that the nondegenerate components of $(H - D) \cap \mathrm{Bd}\, C$ comprise a null sequence of simple closed curves and that $(H - D) \cap C^* = \varnothing$ [4, Th. 4]. Hence,

$$(H - D) \cap X = \varnothing \,.$$

Now consider the component K of $H - C$ whose closure contains $\mathrm{Bd}\, D$. Associate with each simple closed curve S_j of $(\mathrm{Bd}\, K - \mathrm{Bd}\, D)$ a disk F_j in $C - C^*$ such that

(1) $F_j \cap \mathrm{Bd}\, C = \mathrm{Bd}\, F_j = S_j$,

(2) $F_j \cap F_k = \varnothing$ if $S_j \cap S_k = \varnothing$,

(3) $\lim_{j\to\infty} \mathrm{diam}\, F_j = 0$.

Define $G_1 = (\cup F_j) \cup \mathrm{Cl}\, K$. Then G_1 is a disk with handles, and the number of handles is bounded by LEG(C, p_1). Note that $\mathrm{Bd}\, G_1 = \mathrm{Bd}\, D$. Since components of $(G_1 - \mathrm{Bd}\, G_1) \cup C$ are either arcs or points, we can readily obtain an annulus A_1 in G_1 such that $\mathrm{Bd}\, A_1$ contains $\mathrm{Bd}\, G_1$ and $\mathrm{Int}\, A_1$ contains $(G_1 - \mathrm{Bd}\, G_1) \cap C$, and now the remaining requirements on G_1 must be satisfied.

Applying Theorem 2 and techniques from the proof of Theorem 3, we find a map f of a disk with $(k-1)$ holes E into $U - C$ such that

$$f(E) \cap G'_i = f(\mathrm{Bd}\, E) \cap G'_i = \mathrm{Bd}\, G'_i \qquad (i = 1, \cdots, k)$$

and f has no singularities near $\mathrm{Bd}\, E$. According to [9, Lemma 1] there exists a homeomorphism f' of E into $U - C$ such that

$$f'(E) \cap G'_i = f'(\mathrm{Bd}\, E) \cap G'_i = \mathrm{Bd}\, G'_i \quad (i = 1, \cdots, k) \,.$$

Thus, if S denotes $f'(E) \cup (\cup G'_i)$, S is a sphere with handles, and the number of handles is bounded by $\Sigma\,\mathrm{LEG}(C, p_i)$. Moreover, S can be obtained so as to separate $S^3 - U$ from C. Finally, since U is an arbitrary open set, we have that

$$n \leqq \sum \mathrm{LEG}(C, p_i) .$$

5. Semi-cellular subsets.

THEOREM 8. *Suppose C is a crumpled cube such that*

$$2 \leqq \mathrm{LG}(\mathrm{Int}\, C) < \infty ,$$

and X is a nonseparating subcontinuum of Bd C *containing only piercing points of C. Then X is semi-cellular in C.*

Proof. Let $p_1, \cdots, p_k$ denote the nonpiercing points of C, and D a disk in $\mathrm{Bd}\, C - \cup p_i$ whose interior contains X. If C is embedded in S^3 so that $\mathrm{Cl}(S^3 - C)$ is a 3-cell K, then K collapses to a 3-cell K' which is locally tame mod $(D \cup p_1)$, with p_1 a nonpiercing point of $S^3 - \mathrm{Int}\, K' = C'$. According to Theorem 4, $\mathrm{LG}(\mathrm{Int}\, C') < \infty$. Since each point of D is a piercing point of C', it follows from Theorem 3 that $\mathrm{Int}\, C'$ is an open 3-cell. Then X is semi-cellular in C' [7, Lemma 2.7]; clearly X must also be semi-cellular in C.

Theorem 8 can be applied to characterize those sewings of two crumpled cubes which yield S^3, when one of the crumpled cubes has finite limiting genus. With minor changes, such as in the references to the number of nonpiercing points, we can use the proof of [7, Th. 5.7] to prove Theorem 9.

THEOREM 9. *Suppose C_1 and C_2 are crumpled cubes, h is a homeomorphism of* Bd C_1 *to* Bd C_2, *and* $\mathrm{LG}(\mathrm{Int}\, C_2) < \infty$. *Then $C_1 \bigcup_h C_2 = S^3$ if and only if each nonpiercing point of C_1 is identified by h with a piercing point of C_2.*

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CHARACTERIZATION OF SEPARABLE IDEALS

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A k-algebra A is called separable if the exact sequence of left $A^e = A_{\otimes_k} A^0$-modules: $0 \to J \to A^e \xrightarrow{\phi} A \to 0$ splits, where $\phi(a \otimes b^0) = a \cdot b$; a two-sided ideal $\mathfrak{A}$ of A is separable in case the k-algebra $A/\mathfrak{A}$ is separable.

In this note, we present two characterizations of separable ideals. In particular, one finds that a monic polynomial $f \in k[x]$ generates a separable ideal if, and only if, $f = g_1 \cdots g_s$, where the g_i are monic polynomials which generate pairwise comaximal indecomposable ideals in $k[x]$, and $f'(a)$ is a unit in $k[a] = k[x]/f \cdot k[x]$ $(a = x + f \cdot k[x])$.

Throughout this paper, we assume that all rings have units and all ring morphisms preserve units, further, all modules will be assumed unitary. We will denote the center of the ring A by $Z(A)$. Each k-algebra A induces an exact sequence of left $A^e = A \otimes_k A^0$-modules:

$$0 \longrightarrow J \longrightarrow A^e \xrightarrow{\ \phi\ } A \longrightarrow 0 \tag{1}$$

where $\phi(a \otimes b^0) = a \cdot b$.

DEFINITION 2 [1]. A will be called a *separable* k-algebra if the sequence (1) splits. More generally, a two-sided ideal $\mathfrak{A}$ in the k-algebra A will be called a *separable ideal* if the quotient algebra $A/\mathfrak{A}$ $(k \to A \to A/\mathfrak{A})$ is separable. Denote by $\mathrm{Sep}_k(A)$ the set of all such ideals in A; of particular interest is the subset $\mathrm{Sep}_k^*(A)$ of all separable ideals $\mathfrak{A}$ for which $A/\mathfrak{A}$ is a projective k-module.

PROPOSITION 3 [6]. *Let A be a k-algebra.*

(a) $\mathfrak{A} \in \mathrm{Sep}_k(A) \wedge \mathfrak{A} \leqq \mathfrak{A}' \Rightarrow \mathfrak{A}' \in \mathrm{Sep}_k(A)$ ($\mathfrak{A}'$ *is any two-sided ideal of A*).

(b) *If* $(\mathfrak{A}_i)_{i=1}^n \subset \mathrm{Sep}_k(A)$ *is a family of pairwise comaximal ideals, then* $\bigcap_{i=1}^n \mathfrak{A}_i \in \mathrm{Sep}_k(A)$.

The following result found in [1] provides a criterion for answering the question, is $\mathrm{Sep}_k(A) = \emptyset$ or $\mathrm{Sep}_k(A) \neq \emptyset$.

PROPOSITION 4. *Let A be a k-algebra, and let K be a commutative k-algebra. If $\phi(0: J) \otimes_k K$ generates $Z(A) \otimes_k K$ as an ideal, then $A \otimes_k K$ is a separable K-algebra.*

COROLLARY 5. (a) *If $\alpha < k$ is an ideal such that*

$$\alpha \cdot Z(A) + \phi(0: J) = Z(A) ,$$

then $\alpha A \in \mathrm{Sep}_k(A)$.

(b) *If* $Z(A) = k$, *and either* $\phi(0: J)$ *is not nil or* $\phi(0: J) \nleqq \mathrm{Rad}(k)$, *then* $\mathrm{Sep}_k(A) \neq \emptyset$, *where* $\mathrm{Rad}(k)$ *is the Jacobson radical of* k.

1. Representation of separable ideals.

THEOREM 1.1. *Let* A *be a* k*-algebra and* $\mathfrak{A} \in \mathrm{Sep}_k(A)$. *If the* k*-module* $A/\mathfrak{A}$ *if of finite type, then for each maximal ideals* $m < k$, *there is a family* $(M_i)_{i=1}^s \subset \mathrm{Sep}_k(A)$ *of maximal two-sided ideals such that*

$$\mathfrak{A} + (m \cdot A) = M_1 \cap \cdots \cap M_s . \tag{1.2}$$

Proof. For each maximal ideal $m < k$, the k/m-algebra $k/m \otimes A/\mathfrak{A}$ is separable and of finite type as a k/m-module, it follows from [2] Proposition 3.2 that $k/m \otimes A/\mathfrak{A} \cong (A/\mathfrak{A})/m(A/\mathfrak{A}) \cong A/(m \cdot A + \mathfrak{A}) \cong B_1 \oplus \cdots \oplus B_s$, where each B_i is a simple k/m-algebra with $Z(B_i)$ being a separable field extension of k/m; in particular, each B_i is a separable k-algebra. Denoting by M_i the kernel of the mapping $A \rightarrow A/(mA + \mathfrak{A}) \rightarrow B_i$, we find that the family $(M_i)_{i=1}^s$ has the desired properties.

REMARK 1.3. If, in (1.1), we assume $\mathfrak{A} \in \mathrm{Sep}_k^*(A)$, it follows from (1.1) of [9], that we can drop the assumption that $A/\mathfrak{A}$ is a k-module of finite type.

We obtain immediately from the local criteria for separability ([2], p. 100) the following theorem.

THEOREM 1.4. *Let* A *be a* k*-algebra with two-sided ideal* $\mathfrak{A}$ *such that the* k*-module* $A/\mathfrak{A}$ *is of finite type. Suppose either that* k *is Noetherian or that* $A/\mathfrak{A}$ *is a projective* k*-module.*

If, for each maximal ideal $m < k$, $\mathfrak{A} + m \cdot A$ *has a representation* (1.2) *with separable maximal ideals, then* $\mathfrak{A} \in \mathrm{Sep}_k(A)$.

COROLLARY 1.5. *Let* k *be a field.*

(a) $\mathfrak{A}$ *is a separable maximal ideal of* A *if, and only if,* $A/\mathfrak{A}$ *is a simple* k*-algebra whose center is a separable field extension of* k.

(b) $\mathfrak{A} \in \mathrm{Sep}_k(A)$ *if, and only if,* $\mathfrak{A}$ *is the intersection of a finite family of separable maximal ideals of* A.

REMARKS 1.6. (1.5) generalizes a result of [6] where a different definition of $\mathfrak{A} \in \mathrm{Sep}_k(A)$ is given. (1.5) also leads to the following fact. For a field k, $f \in k[x]$ generates a separable ideal if, and only if, $f'(a)$ is a unit in $k[a] = k[x]/f \cdot k[x]$, $a = x + fk[x]$, and f is the product of distinct polynomials of $k[x]$.

DEFINITION 1.7. [5]. A monic polynomial $f \in k[x]$ is separable if the ideal $fk[x]$ is separable.

PROPOSITION 1.8. *If $f \in k[x]$ is separable, then $f'(a)$ is a unit in $k[a]: = k[x]/f \cdot k[x]$.*

Proof. Assume, first, that k is local with maximal ideal m. Denote by $\bar{f}$ the reduction of f modulo m, then

$$k[x]/(m, f) \simeq k/m \otimes_k k[a] \simeq k/m[x]/\bar{f}k/m[x] = k/m[\bar{a}]$$

is a separable k/m-algebra, hence $\bar{f}$ is a separable polynomial. Whence, by (1.6), $\bar{f} = \bar{g}_1 \cdots \bar{g}_s$ in $k/m[x]$, where each $\bar{g}_i$ is irreducible and $\bar{f}'(\bar{a})$ is a unit in $k/m[\bar{a}]$.

Now suppose $f'(a)$ is a nonunit in $k[a]$; by [7], p. 29, Lemma 4, each maximal ideal of $k[a]$ has the form $(g_i(a), m)$, where $g_i \in k[x]$ has reduction $\bar{g}_i$ modulo m. Thus, $f'(a) \in (g_i(a), m)$ for some $i \in [1, s]$, and this implies $f'(x) \in (g_i(x), m)$. But then

$$f'(x) \in \ker (k[x] \to k[x]/fk[x] \to k[x]/(m, fk[x]) \to k[x]/(g_ik[x], m)) ,$$

so that $\bar{f}'(a)$ could not be a unit in $k/m[\bar{a}]$. This contradiction establishes our claim that $f'(a)$ is a unit when k is local.

In general, observe that $f'(a)$ is a unit in $k[a]$ if, and only if $f'_m(a_m)$ is a unit in $k_m[a_m] = k_m \otimes_k k[a]$ for each maximal ideal $m < k$, and then apply the foregoing result.

PROPOSITION 1.9. *Let $f \in k[x]$ be a monic polynomial satisfying the conditions.*

(i) *$f'(a)$ is a unit in $k[a] = k[x]/fk[x]$;*

(ii) *$f(x) = f_1 \cdots f_s$ in $k[x]$, where the monic polynomials f_j generate indecomposable ideals which are pairwise comaximal.*

Then f is separable.

Proof. Let $m < k$ be a maximal ideal of k, and denote by $\bar{f}$ the reduction of f modulo m, then $\bar{f}'(\bar{a})$ is a unit in $k/m[\bar{a}] = k/m \otimes_k k[a]$. Since $\bar{f} = \bar{f}_1 \cdots \bar{f}_s$ in $k/m[x]$, we see that $\bar{f}'_i = 0$ in $k/m[x]$ entails $\bar{f}' = \bar{f}'_i g + \bar{f}_i g' \in \bar{f}_i k/m[x]$. But this implies $\bar{f}'(\bar{a})$ in a nonunit in $k/m[\bar{a}]: = k/m[x]/\bar{f}k/m[x]$, since $\bar{f}_i k/m[x] < k/m[x]$. Thus, each of the $\bar{f}_i$ separable polynomials in $k/m[x]$ which generate pairwise comaximal by (ii). An application of [5] (2.3) shows that f is a separable polynomial in $k[x]$.

COROLLARY 1.10. *Suppose k has no proper idempotents and $f \in k[x]$ is monic. A necessary and sufficient condition that f be separable is that conditions* (i) *and* (ii) *of* (1.9) *holds.*

Proof. We need only verify that when f is separable, $f = f_1 \cdots f_s$ where each of the ideals $f_i k[x]$ is indecomposable and they are pairwise comaximal. But $k[x]/f \cdot k[x]$ has only a finite number of idempotents, since it is a free k-module of rank equal deg (f); hence $k[x]/f \cdot k[x] = B_1 \pi \cdots \pi B_s$, where each B_i is connected and separable as well as projective as a k-module. Then, by [5] (2.9), $B_i = k[x]/f_i k[x]$ and we see that $f = f_1 \cdots f_s$ as usual.

2. Another representation of separable ideals.

DEFINITION 2.1. Let A be a k-algebra. The two-sided ideal $\mathfrak{A} < A$ will be called *decomposable* if $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are proper two-sided comaximal ideals of A; otherwise $\mathfrak{A}$ will be called *indecomposable*. A will be called decomposable or indecomposable according to whether or not 0 is.

THEOREM 2.2. *Let k be a commutative ring without proper idempotents. Assume $\mathfrak{A} \in \mathrm{Sep}_k^*(A)$. Then there is a unique family $(M_i)_{i=1}^s$ of pairwise comaximal indecomposable separable ideals of A such that*

$$\mathfrak{A} = M_1 \cap \cdots \cap M_s . \tag{2.3}$$

Proof. Since the projective k-module $A/\mathfrak{A}$ has finite rank, we can write $A/\mathfrak{A} \simeq B_1 \pi \cdots \pi B_s$, where the B_i are indecomposable separable k-algebras. Putting $M_i = \ker [A \to A/\mathfrak{A} \to B_i]$ we obtain the desired family.

If $\mathfrak{A} = N_1 \cap \cdots \cap N_t$, where the N_j are as the M_i, then $A/\mathfrak{A} = A/M_1 \pi \cdots \pi A/M_s = A/N_1 \pi \cdots \pi A/N_t$ implies that

$$1 = e_1 + \cdots + e_s = f_1 + \cdots + f_t,\ e_i,\ f_j$$

being orthogonal central idempotents. Since all the factors are indecomposable, for each i there is a unique j such that $f_i = f_i e_j$; hence $t \leqq S$, and by symmetry, $s \leqq t$, so $s = t$. The indecomposability also implies (after reordering) that $e_i = f_i$, so that

$$M_i = \ker [A \to (A/\mathfrak{A})e_i] = \ker [A \to (A/\mathfrak{A})f_i] = N_i ,$$

completing the proof.

REMARK 2.4. (2.2) generalizes a result obtained in [5], see p. 471, (2.10).

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A COMPARISON OF TWO NATURALLY ARISING UNIFORMITIES ON A CLASS OF PSEUDO-PM SPACES

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In this paper, we shall consider an important class of probabilistic pseudometric spaces, the so-called pseudometrically generated spaces, i.e., spaces with a collection of pseudometrics on which a probability measure has been defined. Specifically, we shall examine the relationship between the uniformity introduced on the space probabilistically by means of the so-called ε, λ uniform neighborhoods and the uniformity obtained by considering all the uniform neighborhoods generated by each of the pseudometrics as a subbase.

A *probabilistic metric* (*PM*) *space* is a pair $(S, \mathscr{F})$ where S is a set, $\mathscr{F}$ is a mapping from $S \times S$ into Δ, the set of all one-dimensional left continuous distribution functions, whose value $\mathscr{F}(p, q)$ at any $(p, q) \in S \times S$ is usually denoted by F_{pq}, satisfying

(I) $F_{pp} = H$

(II) $F_{pq} = H$ implies $p = q$

(III) $F_{pq}(0) = 0$

(IV) $F_{pq} = F_{qp}$

(V) $F_{pq}(x) = F_{qr}(y) = 1$ implies $F_{pr}(x + y) = 1$,

where H is the distribution function defined by

$$H(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0 . \end{cases}$$

A *Menger space* is a triple $(S, \mathscr{F}, T)$ where $(S, \mathscr{F})$ is a *PM* space, T is a mapping (called a t-norm) from the unit square $[0, 1] \times [0, 1]$ into $[0, 1]$ which is nondecreasing in each place, symmetric, associative, satisfies boundary condition

$$T(a, 1) = a ,$$

and with the additional property

$$(\mathrm{Vm}) F_{pr}(x + y) \geqq T(F_{pq}(x), F_{qr}(y)) .$$

A *probabilistic pseudometric* (*pseudo-PM*) *space* is a pair $(S, \mathscr{F})$ satisfying (I), (III), (IV), and (V). Similarly, a *pseudo-Menger space* is a triple $(S, \mathscr{F}, T)$ satisfying (I), (III), (IV), and (Vm).

For further information on the basic properties of *PM* spaces, the reader is referred to Schweizer and Sklar [3].

DEFINITION 1. A *metrically generated* (*MG*) *space* is a *PM* space $(S, \mathscr{F})$ together with a probability space $(\mathscr{D}, \mathscr{B}, \mu)$ such that $\mathscr{D}$ is a set of metrics on S and such that for any $(p, q) \in S \times S$ and any $x > 0$

$$\{d \in \mathscr{D}: d(p, q) < x\} \in \mathscr{B} \tag{1}$$

and

$$F_{pq}(x) = \mu\{d \in \mathscr{D}: d(p, q) < x\} . \tag{2}$$

A *pseudometrically generated* (*pseudo-MG*) *space* is a pseudo-*PM* space $(S, \mathscr{F})$ together with a probability space $(\mathscr{D}, \mathscr{B}, \mu)$ of pseudometrics on S such that conditions (1) and (2) hold.

In the sequel, we will use the notation $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ to denote *MG* and pseudo-*MG* spaces.

In his paper [5], R. Stevens showed that any *MG* space is a Menger space under the t-norm T_m where

$$T_m(a, b) = \max\{a + b - 1, 0\} .$$

His proof may be easily generalized to show that any pseudo-*MG* space is a pseudo-Menger space under T_m.

DEFINITION 2. Let S be a set and let $\mathscr{D}$ be a collection of pseudometrics on S. Then the *gage uniformity* of $\mathscr{D}$ on S (denoted by $\mathscr{U}_{\mathscr{D}}$) is the uniformity generated by the following subbase

$$[\{(p, q) \in S \times S : d(p, q) < x\}]_{d \in \mathscr{D}, x > 0} .$$

It is shown in Kelley [1] that any uniformity on a set may be regarded as the gage uniformity of some collection of pseudometrics on that set.

THEOREM 1. *Let $(S, \mathscr{F}, T)$ be a pseudo-Menger space with the property that* $\sup_{x<1} T(x, x) = 1$. *Then the sets*

$$U(\varepsilon, \lambda) = \{(p, q) \in S \times S : F_{pq}(\varepsilon) > 1 - \lambda\}$$

form a base for a pseudometrizable uniformity on S.

The above theorem was proven by Schweizer, Sklar, and Thorp [4]. Since pseudo-*MG* spaces are pseudo-Menger spaces under T_m, a continuous t-norm, it follows that the sets

$$\begin{aligned} U(\varepsilon, \lambda) &= \{(p, q) \in S \times S : F_{pq}(\varepsilon) > 1 - \lambda\} \\ &= \{(p, q) : \mu\{d \in \mathscr{D}: d(p, q) < \varepsilon\} > 1 - \lambda\} \end{aligned}$$

form a base for a uniformity on the pseudo-*MG* space $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$. This uniformity will be referred to as the $\mathscr{F}$ uniformity and will be denoted by $\mathscr{U}_{\mathscr{F}}$.

Given a pseudo-*MG* space $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$, it follows from the above that we can put two uniformities on S, namely the gage uniformity $\mathscr{U}_{\mathscr{D}}$ and the $\mathscr{F}$ uniformity $\mathscr{U}_{\mathscr{F}}$. A natural question that arises is whether there is any relationship between the two uniformities. We shall first examine this question for pseudo-*MG* spaces generated by a countable family of pseudometrics.

THEOREM 2. *If $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ is a pseudo-MG space and $\mathscr{D}$ is countable, then $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.*

Proof. We shall first show that $\mathscr{B} = 2^{\mathscr{D}}$.

First of all, since for any $(p, q) \in S \times S$ and any $\varepsilon > 0$

$$\{d \in \mathscr{D}: d(p, q) < \varepsilon\} \in \mathscr{B} ,$$

it follows that its complement $\{d: d(p, q) \geqq \varepsilon\}$ is also μ-measurable. Similarly

$$\{d: d(p, q) \leqq \varepsilon\} = \bigcap_{n=1}^{\infty} \left\{d: d(p, q) < \varepsilon + \frac{1}{n}\right\} \in \mathscr{B} .$$

Hence, we have for any $(p, q) \in S \times S$ and any $\varepsilon > 0$

$$\{d: d(p, q) = \varepsilon\} = \{d: d(p, q) \leqq \varepsilon\} \cap \{d: d(p, q) \geqq \varepsilon\} \in \mathscr{B} .$$

Now pick any $d_0 \in \mathscr{D}$ and well order $\mathscr{D} - \{d_0\}$ as

$$\{d'_1, d'_2, \cdots\} .$$

Now since $d_0 \neq d'_k$, there is a pair $(p_k, q_k) \in S \times S$ for which

$$d_0(p_k, q_k) \neq d'_k(p_k, q_k) .$$

Hence it follows that

$$\{d_0\} = \bigcap_{k=1}^{\infty} \{d: d(p_k, q_k) = d_0(p_k, q_k)\} \in \mathscr{B} .$$

Since any subset of $\mathscr{D}$ is a countable union of unit sets $\{d\}$, it follows that $\mathscr{B} = 2^{\mathscr{D}}$.

To show that $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$, it suffices to show that any base element $U(\varepsilon, \lambda)$ of $\mathscr{U}_{\mathscr{F}}$, contains a base element

$$V = \bigcap_{d \in A} \{(p, q): d(p, q) < \varepsilon_d\} ,$$

of $\mathscr{U}_{\mathscr{D}}$, where A is a finite subset of $\mathscr{D}$ and each $\varepsilon_d > 0$. Well order $\mathscr{D}$ as

$$\{d_1, d_2, \cdots\} .$$

Clearly

$$\mu(\mathscr{D}) = \sum_{k=1}^{\infty} \mu\{d_k\} = 1 .$$

Pick n large enough so that

$$\mu\Big(\bigcup_{k=1}^{n} \{d_k\}\Big) = \sum_{k=1}^{n} \mu\{d_k\} > 1 - \lambda .$$

Let V be defined by

$$V = \bigcap_{k=1}^{n} \{(p, q): d_k(p, q) < \varepsilon\} .$$

Clearly, if $(p_0, q_0) \in V$, then

$$\bigcup_{k=1}^{n} \{d_k\} \subseteqq \{d: d(p_0, q_0) < \varepsilon\}$$

and

$$1 - \lambda < \mu\Big(\bigcup_{k=1}^{n} \{d_k\}\Big) \leqq \mu\{d: d(p_0, q_0) < \varepsilon\} = F_{p_0 q_0}(\varepsilon) ,$$

so that $(p_0, q_0) \in U(\varepsilon, \lambda)$. In other words,

$$V \subseteqq U(\varepsilon, \lambda) ,$$

which is what we wished to prove.

THEOREM 3. *Let $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ be a pseudo-MG space with the property that $\mathscr{D}$ is countable, and μ is nonzero on all nonempty measurable subsets of $\mathscr{D}$. Then $\mathscr{U}_{\mathscr{D}} = \mathscr{U}_{\mathscr{F}}$.*

Proof. In view of the preceding theorem, it is sufficient to show $\mathscr{U}_{\mathscr{D}} \subseteqq \mathscr{U}_{\mathscr{F}}$. In the proof of the preceding theorem we have already shown that all subsets of $\mathscr{D}$ are μ-measurable. It follows that $\mu(\{d_0\}) > 0$ for any $d_0 \in \mathscr{D}$. Now, for any $\varepsilon > 0$,

$$\mu\{d: d(p, q) < \varepsilon\} > 1 - \mu\{d_0\} \text{ implies } d_0(p, q) < \varepsilon .$$

It follows that

$$U(\varepsilon, \mu\{d_0\}) \subseteqq \{(p, q) : d_0(p, q) < \varepsilon\} .$$

Taking finite intersections, we have that any base element of $\mathscr{U}_{\mathscr{D}}$ contains a base element of $\mathscr{U}_{\mathscr{F}}$ and the desired result is an immediate consequence of this.

THEOREM 4. *Let $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ be a pseudo-MG space such that $\mathscr{D}$ is countable. Let $\mathscr{D}' \subseteqq \mathscr{D}$ be defined by*

$$\mathscr{D}' = \{d \in \mathscr{D}: \mu\{d\} > 0\} .$$

Then $\mathscr{U}_{\mathscr{F}} = \mathscr{U}_{\mathscr{D}'}$.

Proof. Let $(\mathscr{D}', \mathscr{B}', \mu')$ be the probability space naturally induced by $(\mathscr{D}, \mathscr{B}, \mu)$. By the previous theorem, the $\mathscr{F}'$ uniformity of $(S, \mathscr{F}'; \mathscr{D}', \mathscr{B}', \mu')$, $\mathscr{U}_{\mathscr{F}'}$, is equivalent to $\mathscr{U}_{\mathscr{D}'}$. Since $\mathscr{D} - \mathscr{D}'$ is a countable union of sets of μ-measure 0,

$$F'_{pq}(x) = \mu'\{d \in \mathscr{D}': d(p, q) < x\} = \mu\{d \in \mathscr{D}: d(p, q) < x\} = F_{pq}(x) ,$$

so that $\mathscr{U}_{\mathscr{F}} = \mathscr{U}_{\mathscr{F}'} = \mathscr{U}_{\mathscr{D}'}$.

Thus, we have essentially solved our problem for spaces generated by a countable family of pseudometrics. It is reasonable to ask whether any of these results can be extended to arbitrary pseudo-*MG* spaces. The following example shows that this is not the case.

EXAMPLE 1. Let S be the set of all real-valued measurable functions on the unit interval $[0, 1]$. For any $t \in [0, 1]$ define a pseudometric d_t on S by

$$d_t(f, f^*) = |f(t) - f^*(t)|$$

for any f, f^* in S. Let $\mathscr{D} = \{d_t: t \in [0, 1]\}$, and let μ be the probability measure on $\mathscr{D}$ induced by the Lebesgue measure on $[0, 1]$. Let $\mathscr{F}: S \times S \to \Delta$ be defined by

$$\mathscr{F}(f, f^*)(x) = \mu\{d_t: d_t(f, f^*) < x\} .$$

Hence $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ is a pseudo-*MG* space. (The pseudometrics d_t may be interpreted as giving the distance between two particles at time t)

It is easy to show that $\mathscr{U}_{\mathscr{F}}$ and $\mathscr{U}_{\mathscr{D}}$ are not even comparable. For two particles may be close to one another at any finite number of instants but still be far away from each other the rest of the time. Conversely, given our finite number of instants we can find two particles which are far apart at these instants but arbitrarily close to each other at all other times.

However, the question still remains, whether any of our results on countably generated pseudo-MG spaces can be generalized to the uncountable case when sufficiently strong restrictions are placed upon the generating family of pseudometrics. A natural restriction that comes to mind is the requirement that all the pseudometrics be comparable.

DEFINITION 3. Two pseudometrics d_1 and d_2 on a set S are said to be *comparable* if one of the following relations holds

(i) $d_1(p, q) \geqq d_2(p, q)$ for all $(p, q) \in S \times S$; or

(ii) $d_2(p, q) \geqq d_1(p, q)$ for all $p, q \in S \times S$.

DEFINITION 4. A linearly ordered set $(S, \leqq)$ is said to be *countably bounded* if there exists a countable subset $A \subseteqq S$ such that for every element $s \in S$, there exists an element $\alpha \in A$ such that $s \leqq \alpha$.

The real numbers with the usual ordering are countably bounded where as the collection of ordinals less than the first uncountable is not countably bounded.

THEOREM 5. *Let* $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ *be a pseudo-MG space, such that any two pseudometrics of* $\mathscr{D}$ *are comparable. If* $\mathscr{D}$ *is countably bounded under the induced linear ordering, then* $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.

Proof. If $\mathscr{D}$ has an upper bound, this result may be proven very easily. If $\mathscr{D}$ does not have an upper bound, then neither does A, the countable bounding set, and we can construct from A a strictly increasing sequence $\{d_k\}_{k=1}^{\infty}$ such that for every $d \in \mathscr{D}$, there exists a k such that $d < d_k$.

Let (p_k, q_k) be a point of $S \times S$ such that $d_k(p_k, q_k) < d_{k+1}(p_k, q_k)$. Let A_k be defined by

$$A_k = \{d \in \mathscr{D}: d(p_k, q_k) < d_{k+1}(p_k, q_k)\} .$$

It is obvious that $\{A_k\}_{k=1}^{\infty}$ forms an increasing sequence of μ-measurable sets. It is also obvious that $\lim_{k\to\infty} A_k = \mathscr{D}$, whence $\lim_{k\to\infty} \mu(A_k) = \mu(\mathscr{D}) = 1$. Thus for any $\lambda > 0$ there exists a N such that $\mu(A_N) > 1 - \lambda$. Hence for any $d \in A_N$

$$d_{N+1}(p, q) < \varepsilon \text{ implies } d(p, q) < \varepsilon$$

and

$$\mu\{d: d(p, q) < \varepsilon\} \geqq \mu(A_N) > 1 - \lambda ,$$

so that

$$\{(p, q): d_{N+1}(p, q) < \varepsilon\} \subseteqq U(\varepsilon, \lambda)$$

which proves that $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.

Theorem 5 might seem to indicate that perhaps $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$ holds for all pseudo-MG spaces with comparable pseudometrics. But even this is false as the following example shows.

EXAMPLE 2. Let Ω denote the set of all ordinal numbers less than the first ordinal having the power of the continuum. Let φ denote a one-to-one correspondence from the closed unit interval $I = [0, 1]$ onto Ω. Now define a function $f_y \colon I \to I$ for every $y \in I$ as follows:

$$f_y(x) = \begin{cases} 1, \text{ if } \varphi(x) \leqq \varphi(y) \\ x/4, \text{ if } \varphi(x) > \varphi(y) \,. \end{cases}$$

Also define for every $y \in I$ a function $d_y \colon I \times I \to R$ by

$$d_y(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = x_2 \\ f_y(x_1) + f_y(x_2), & \text{if } x_1 \neq x_2 \,. \end{cases}$$

Define a measure μ on the Boolean σ-algebra $\mathscr{B}$ of $2^{\mathscr{D}}$ (where $\mathscr{D} = \{d_y \colon y \in I\}$) consisting of all subsets of $\mathscr{D}$ which have a cardinal numbers less than that of the continuum and of the complements of these sets by

$$\mu(A) = \begin{cases} 0, \text{ if card } (A) < \mathfrak{C} \\ 1, \text{ if card } (\mathscr{D} - A) < \mathfrak{C} \,. \end{cases}$$

One may easily verify that μ satisfies all the conditions for a probability measure.

It may also be easily verified that d_y is a metric on I for every $y \in I$ and that $\varphi(y_1) < \varphi(y_2)$ implies that $d_{y_1}(x_1, x_2) \leqq d_{y_2}(x_1, x_2)$ for every $(x_1, x_2) \in I \times I$.

To show I, $\mathscr{D}$, and μ determine an MG-space, it suffices to show that for $(x_0, z_0) \in I \times I$ and any $\varepsilon_0 > 0$, the set

$$\{d_y \in \mathscr{D} \colon d_y(x_0, z_0) < \varepsilon_0\}$$

is μ-measurable. If $x_0 = z_0$ or $\varepsilon_0 > 2$, this is obviously true. If $\varepsilon_0 \leqq 2$, let $x' = \varphi^{-1}\{\max\{\varphi(x_0), \varphi(z_0)\}\}$. Then

$$A = \{d_y \in \mathscr{D} \colon d_y(x_0, z_0) < \varepsilon_0 \leqq 2\} \subseteqq \{d_y \in \mathscr{D} \colon \varphi(y) < \varphi(x')\} = B \,,$$

since if $\varphi(x') \leqq \varphi(y)$ held, then $\varphi(x_0) \leqq \varphi(y)$ and $\varphi(z_0) \leqq \varphi(y)$ and

$$d_y(x_0, z_0) = f_y(x_0) + f_y(z_0) = 1 + 1 = 2 \,.$$

We have card $(B) < \mathfrak{C}$ and so card $(A) < \mathfrak{C}$. Hence A is μ-measurable and $\mu(A) = 0$.

We shall now show that the proper inclusion $\mathscr{U}_{\mathscr{D}} \subsetneqq \mathscr{U}_{\mathscr{F}}$ holds (instead of $\mathscr{U}_{\mathscr{F}} \subsetneqq \mathscr{U}_{\mathscr{D}}$). We have

$$U(1, \tfrac{1}{2}) = \{x_1, x_2) \in I \times I\colon \mu\{d_y\colon d_y(x_1, x_2) < 1\} > \tfrac{1}{2}\} = D_I$$

where D_I is the diagonal set on $I \times I$, since, as shown in the preceding paragraph, $x_1 \neq x_2$ implies

$$\mu\{d_y\colon d_y(x_1, x_2) < 1\} = 0 .$$

To show that proper inclusion holds, assume the contrary. Then we would have to have

$$\bigcap_{i=1}^{n} \{(x_1, x_2)\colon d_{y_i}(x_1, x_2) < \varepsilon_i\} = D_I$$

for some $\{d_{y_i}\}_{i=1}^{n}$ and $\{\varepsilon_i\}_{i=1}^{n}$, $\varepsilon_i > 0$. Let

$$\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\} \text{ and } d_{y_0} = \max\{d_{y_1}, \cdots, d_{y_n}\} .$$

Then

$$B = \{(x_1, x_2)\colon d_{y_0}(x_1, x_2) < \varepsilon_0\} = D_I .$$

Since card $\{x\colon \varphi(x) \leqq \varphi(y_0)\} < \mathfrak{C}$, there exist two points $x_0 \neq z_0$ in the open interval $(0, \varepsilon_0)$ such that $\varphi(x_0) > \varphi(y_0)$ and $\varphi(z_0) > \varphi(y_0)$. Thus

$$d_{y_0}(x_0, z_0) = f_y(x_0) + f_y(z_0) = (x_0/4) + (z_0/4) \leqq (\varepsilon_0/4) + (\varepsilon_0/4) < \varepsilon_0 ,$$

so that $(x_0, z_0) \in B$, but $(x_0, z_0) \notin D_I$, which contradicts our assumption.

Hence Theorem 5 cannot be extended to arbitrary pseudo-*MG* spaces with comparable pseudometrics. However, Theorem 5 does admit generalization in another direction. For it may be easily seen that a pseudo-*MG* space $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ with comparable pseudometrics such that $\mathscr{D}$ is countably bounded, also has the property that the gage uniformity of $\mathscr{D}$ is also generated by some countable subfamily $\mathscr{A} \subseteqq \mathscr{D}$, for instance the countable bounding set; i.e., $\mathscr{U}_{\mathscr{D}} = \mathscr{U}_{\mathscr{A}}$. We shall now show that in any pseudo-*MG* space with this property, $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$ holds. We shall derive this result by first proving an even more general result.

THEOREM 6. *Let $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ be a pseudo-MG space. Let $\mathscr{R}$ be an arbitrary countable collection of pseudometrics upon S with the property that $\mathscr{U}_{\mathscr{D}} \subseteqq \mathscr{U}_{\mathscr{R}}$. Then $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{R}}$.*

Proof. Consider the countable collection of uniform neighborhoods

$$\mathscr{C} = \left\{\left\{(p, q): d(p, q) < \frac{1}{n}\right\}: d \in \mathscr{R}, n = 1, 2, \cdots\right\}.$$

Well-order $\mathscr{C}$ as

$$\{V_1, V_2, V_3, \cdots\}.$$

We shall now show that for every uniform neighborhood $U(\varepsilon, \lambda) \in \mathscr{U}_{\mathscr{F}}$, there exists an M such that

$$\bigcap_{i=1}^{M} V_i \subseteqq U(\varepsilon, \lambda).$$

For, consider the sets $A_m \subseteqq \mathscr{D}$ defined as follows

$$A_m = \left\{d \in \mathscr{D}: \bigcap_{i=1}^{m} V_i \subseteqq \{(p, q): d(p, q) < \varepsilon\}\right\}.$$

Obviously $\{A_m\}_{m=1}^{\infty}$ is an increasing sequence of sets. It is also very easy to show that $\lim_{m\to\infty} A_m = \mathscr{D}$. Now extend μ to an outer measure μ_0^* on $\mathscr{D}$ by defining

$$\mu_0^*(A) = \inf\{\mu(B): A \subseteqq B \text{ and } B \in \mathscr{B}\}.$$

It may be shown (see, for instance, Munroe [2], p. 99) that μ_0^* is a regular outer measure on $\mathscr{D}$. We then have

$$\lim_{m\to\infty} \mu_0^*(A_m) = \mu_0^*\left(\lim_{m\to\infty} A_m\right) = \mu_0^*(\mathscr{D}) = 1.$$

Therefore, there exists an M such that

$$\mu_0^*(A_M) > 1 - \lambda.$$

Now if $(p_0, q_0) \in \bigcap_{i=1}^{M} V_i$,

$$A_M \subseteqq \{d \in \mathscr{D}: d(p_0, q_0) < \varepsilon\}.$$

Hence

$$\mu\{d \in \mathscr{D}: d(p_0, q_0) < \varepsilon\} \geqq \mu_0^*(A_M) > 1 - \lambda,$$

so that $(p_0, q_0) \in U(\varepsilon, \lambda)$ and $\bigcap_{i=1}^{M} V_i \subseteqq U(\varepsilon, \lambda)$. This completes the proof that $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{R}}$.

COROLLARY. *Let $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ be a pseudo-MG space. If there exists a countable collection $\mathscr{R}$ of pseudometrics on S such that $\mathscr{U}_{\mathscr{R}} = \mathscr{U}_{\mathscr{D}}$, then $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.*

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APPROXIMATION OF WIENER INTEGRALS OF FUNCTIONALS CONTINUOUS IN THE UNIFORM TOPOLOGY

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The result obtained in this paper is a technique for the approximation and estimation of error of Wiener integrals of suitable functionals continuous in the uniform topology. For a certain class of functionals called third degree polynomials exact results occur at the first as well as each subsequent stage of approximation.

Similar results for functionals continuous in the Hilbert topology are given in [1], [4], [5], [6] and [7]. In each of these papers the functions $x(s)$ of Wiener space are approximated by linear combinations of the first n indefinite integrals $\{\beta_i(t)\}$ of a certain complete set of orthonormal functions $\{\alpha_i(t)\}$. The approximation for $x(t)$ turns out to be $\sum_{i=1}^n c_i(x)\beta_i(t)$ where the $c_i(x)$'s are Stieltjes integrals of $x(s)$ with respect to the α's. When $x(t)$ is replaced by this approximation in $F[x(\cdot)]$ a standard Wiener integration formula can be applied. If F is required to be continuous in the Hilbert topology, [4] and [5] show there is (as might be expected) considerable choice in the C.O.N. set. However the uniform topology seems more natural to use in Wiener space and when continuity in this topology is required it may be there is not so large a choice. The Haar functions seem a reasonable choice to try and it is these the author has used.

Let C be the space of real functions continuous on $[0, 1]$ and which vanish at zero. Let $\{h_n(s)\}$ be the Haar functions normalized to be right continuous and to vanish at $s = 1$. The approximation of this paper applies to $F[x]$ if

$$F[x_0 + x] = F[x_0] + \sum_{i=1}^{3} \int_0^1 (i) \int_0^1 x(s_1) \cdots x(s_i) d_{(i)} K_i(x_0 | s_1, \cdots, s_i) + Q[x_0, x]$$

where, with $||x|| = \max_t |x(t)|$, $|Q[x_0, x]| \leqq A||x||^D \exp B(||x_0||^2 + ||x||^2)$ with $B < 1/12$ and $D > 0$.

Notation. Let $\{h_n(s)\}$ be the Haar functions on $[0, 1]$ normalized to be right continuous and so $h_n(1) = 0$.

Let, for $n = 1, 2, 3, \cdots$,

$$c_n(x) = -\int_0^1 x(s) dh_n(s) ,$$

$$\beta_n(t) = \int_0^t h_n(s)ds \ ,$$

$$x^n(t) = \sum_{i=1}^n c_i(x)\beta_i(t) \ ,$$

$$\psi_n(\xi, t) = \sum_{i=1}^n \xi_i\beta_i(t) \ ,$$

$$e_n(\xi) = (2\pi)^{-n/2} \exp [(-\xi_1^2 - \cdots - \xi_n^2)/2] \ .$$

(This is the kernel commonly used now whereas that used in [1], [4] and [5] was $\pi^{-n/2} \exp(-\xi_1^2 - \cdots - \xi_n^2)$.)

Finally let

$$||x|| = \sup_{t \in [0,1]} |x(t)|$$

for $x \in C$ and let

$$\rho(s, t) = (2^{3/2}/\pi)\sum_{k=1}^{\infty} \sin (k - \tfrac{1}{2})\pi t h_k(s)/(2k - 1)$$

(that this last series converges for $(s, t) \in [0, 1] \times [0, 1]$ and is, for fixed s, continuous in t will be seen in Theorem 1. Also $\rho(s, 0) = 0$ and so, for fixed s, $\rho(s, t)$ is in C and $\rho^n(s, t)$ can be computed).

In connection with Radon integrals the symbol $\int_0^1$ will be used rather than $\int_0^1(n)\int_0^1$ and d subscripted with n subscripted s's will be replaced by $d_{(n)}$. Another abbreviation is given by the following equation:

$$\int_{-\infty}^{\infty} G(f(\xi), n)d\mu_m = \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty} e_n(\xi)G(f(\xi), n)d\xi_i \cdots d\xi_n \ .$$

If $F[x]$ is defined on C we define $I_n(F)$ and $J_n(F)$ by the following equations provided the right hand sides have meaning.

$$I_n(F) = \int_{-\infty}^{\infty} F[\psi_n(\xi, \cdot)]d\mu_n \ ,$$

$$J_n(F) = \tfrac{1}{2}\int_{-\infty}^{\infty}\int_0^1\{F[\psi_n(\xi, \cdot) + \rho(s, \cdot) - \rho^n(s, \cdot)] + F[\psi_n(\xi, \cdot) - \rho(s, \cdot) + \rho^n(s, \cdot)]\}dsd\mu_n \ .$$

2. The principal theorem. The following theorem and corollary are the main results of the paper.

THEOREM 6. *Let $F[x]$ be integrable on C and such that $J_n(F)$: $n = 1, 2, 3, \cdots$ exists as a finite quantity. For each $x_0 \in C$ let:*

$K_i(x_0 | s_1, \cdots, s_i)$, $i = 1, 2, 3,$ be right continuous and of bounded variation in any $j (j \leqq i)$ of the variables for the other $i - j$ variables fixed. For each pair $[x_0, x] \in C \times C$

$$let\ P[x_0, x] = F[x_0] + \sum_{i=1}^{3}\int_0^1 x(s_1)\cdots x(s_i)d_{(i)}K(x_0|s_1, \cdots, s_i)\ .$$

Let $F[x_0 + x] = P[x_0, x] + Q[x_0, x]$ *define* $Q[x_0, x]$. *Then if* $|Q[x_0, x]| \leqq A||x||^D \exp B[||x_0||^2 + ||x||^2]$, *where* $B < 1/12$, *and* $D > 0$ *and if* $\alpha \in (0, 1/2)$ *there follows*

$$\left|J_n(F) - \int_C F[x]dx\right| = o(n^{-\alpha D})\ as\ n \to \infty\ .$$

Furthermore, if $Q = 0$ *then*

$$J_n(F) = \int_C F[x]dx\ for\ each\ n\ .$$

COROLLARY. *Under the conditions of the above theorem a specific estimate of error is given by*

$$\begin{aligned}&\left|J_n(F) - \int_C F[x]dx\right|\\ &\leqq A\{M^{2/3}[2^{\alpha+1}/n^\alpha]^D[2/\sqrt{1-12B}]^{1/3}\\ &+ [2/\sqrt{1-4B}][31/\sqrt{n}]^D \exp[31^2 B/n]\end{aligned}$$

where M *is the constant given in Lemma* 2 *with* P *replaced by* $3D/2$.

The following theorems (except Theorem 2) and two lemmas are the main results used in the proof of Theorem 6. These theorems are analogous to correspondingly numbered theorems in [4]. In fact Theorems 3 and 5 are identical to those of [4] and so proofs for them will not be given.

THEOREM 1. (i) *The* $\rho(s, t)$ *series converges, the convergence being uniform in* $(s, t) \in [0, 1] \times [0, 1]$.

(ii) $\rho(s, t)$ *is continuous in* t *for each fixed* s.

(iii) $||\rho(s, \cdot) - \rho^n(s, \cdot)||$ *is measurable in* s.

(iv) $||\rho(s, \cdot) - \rho^n(s, \cdot)|| \leqq 31/n^{1/2}$.

(v) *If, for* $x \in C$, $F[x] = K_0 + \sum_{i=1}^{3}\int_0^1 x(s_1)\cdots x(s_i)d_{(i)}K_i(s_1, \cdots, s_i)$ *in which the* K_i*'s are right continuous and of bounded variation in any* $j(j \leqq i)$ *of the variables for the other* $i - j$ *variables fixed then*

$$\int_C F[x]dx = \frac{1}{2}\int_0^1\{F[\rho(s, \cdot)] + F[-\rho(s, \cdot)]\}ds\ . \tag{2.1}$$

(The reason $\sqrt{2}$ does not appear under the ρ's as in Theorem 1 of [4] is the change in kernel which results in $\int_C x(s)x(t)dx$ being $\frac{1}{2}\min(s, t)$ rather than $\min(s, t)$.)

LEMMA 1. (*Ciesielski* [2]). *For each $x \in C$, the graph of $x^n(t)$ is an inscribed polygon of the graph of $x(t)$. The graph of $x^{n+1}(t)$ has at least the same vertices as that of $x^n(t)$ and $\{x^n(t)\}$ converges uniformly to $x(t)$.*

Some notation, now to be given, is used in Lemma 2 below. For fixed $x \in C$ and $\alpha \in (0, \frac{1}{2})$ let $\varphi_\alpha[x]$ be the infimum of $h > 0$ such that $|x(t') - x(t'')| \leqq h|t' - t''|^\alpha$ for t' and t'' in $[0, 1]$. (that such h exists for almost all $x \in C$ has been shown by N. Wiener [10]).

LEMMA 2. (*Yeh* [8]). *For every $\alpha \in (0, 1/2)$ and $P > 0$, the functional $\{\varphi_\alpha[x]\}^P$ is Wiener integrable i.e., $\int_C \{\varphi_\alpha[x]\}^P dx < \infty$.*

In fact for any $N > \frac{1}{2} \max\{(1 + 2\alpha)/(1 - 2\alpha), P\}$,

$$\int_C \{\varphi_\alpha[x]\}^P dx \leqq M$$

where

$$M = (2N)^N e^{-N}\{1 - 2^{1/2+\alpha-N(1-2\alpha)}\}^{-1} \sum_{m=1}^{\infty} (m + 1)^P/(2N + 1) < \infty .$$

THEOREM 2. *If $F[x]$ is continuous in the uniform topology on C and if either*

(i) *$F[x]$ is bounded*

or

(ii) *there exist nondecreasing $G_1(u)$ and $G_2(u)$ defined on $[0, \infty)$ such that $G_1[\max_{t \in [0,1]} x(t)]$ and $G_2[\max_{t \in [0,1]}\{-x(t)\}]$ are Wiener integrable and such that*

$$|F[x] \leqq G_1[\max_{t \in [0,1]} x(t)] + G_2[\max_{t \in [0,1]} \{-x(t)\}]$$

then

$$\lim_{n \to \infty} I_n(F) = \int_C F[x] dx . \tag{2.2}$$

Particular suitable choices for G_1 and G_2 are

(2.3) $G_1(u) = G_2(u) = M \exp\{hu^p\}$ *for $p \in [0, 2)$ and arbitrary real M and h.*

(2.4) $G_1(u) = G_2(u) = M \exp\{hu^2\}$ *for $h < \frac{1}{2}$ and arbitrary real M.*

THEOREM 3. *If $F[x] \in L_1(C)$ then*

$$\int_C F[x] dx = \int_{-\infty}^{\infty} \int_C F[x(\cdot) - x^n(\cdot) + \psi_n(\xi, \cdot)] dx d\mu_n .$$

THEOREM 4. *For $\alpha \in (0, 1/2)$ and $P \geqq 0$,*

$$\int_C ||x - x^n||^P dx \leqq M[2^{\alpha+1}/n^\alpha]^P$$

where M is as in Lemma 2.

THEOREM 5. *For fixed $i \in \{1, 2, 3\}$, let $H(t_1, \cdots, t_i)$ be right continuous and of bounded variation in any $j (j \leqq i)$ of its variables for the other $i - j$ variables fixed. Then there exists $N(s_1, \cdots, s_i)$ of bounded variation and right continuous such that for all $x \in C$,*

$$\int_0^1 [x(t_1) - x^n(t_1)] \cdots [x(t_i) - x^n(t_i)] d_{(i)} H(t_1, \cdots, t_i)$$

is of the form

$$\int_0^1 x(s_1) \cdots x(s_i) d_{(i)} N(s_1, \cdots, s_i) .$$

The proof of Theorem 6 and its corollary follows. Let

$$\begin{aligned}
\varepsilon_n &= \int_C F[x]dx - J_n(F) \\
&= \int_C F[x]dx \\
&\quad - \int_{-\infty}^{\infty} \tfrac{1}{2}\int_0^1 \{F[\psi_n(\xi, \cdot) + \rho(s, \cdot) - \rho^n(s, \cdot)] \\
&\quad + F[\psi_n(\xi, \cdot) - \rho(s, \cdot) + \rho^n(s, \cdot)]\} ds d\mu_n .
\end{aligned}$$

If now F is replaced by $P + Q$ the integrals can be combined and, because of Theorems 1, 3 and 5, the part involving P disappears. (The detailed argument is exactly the same as that in [4, pp. 64-65] where all symbols and theorems used there are to be replaced by the corresponding ones of this paper. See also the note after (2.1)). What is left is

$$\begin{aligned}
\varepsilon_n = \int_{-\infty}^{\infty} \Big\{ &\int_C Q[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)] dx \\
&- \tfrac{1}{2}\int_0^1 (Q[\psi_n(\xi, \cdot), \rho(s, \cdot) - \rho^n(s, \cdot)] \\
&+ Q[\psi_n(\xi, \cdot), -\rho(s, \cdot) + \rho^n(s, \cdot)] ds \Big\} d\mu_n .
\end{aligned}$$

If

$$|Q[x_0, x]| \leqq A||x||^D \exp B[||x_0||^2 + ||x||^2]$$

then

$$|\varepsilon_n| \leqq A\int_{-\infty}^{\infty}\left\{\int_c \|x - x^n\|^D \exp B[\|\psi_n(\xi, \cdot)\|^2 + \|x - x^n\|^2]dx \right.$$
$$+ \int_0^1 \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^D \exp B[\|\psi_n(\xi, \cdot)\|^2$$
$$\left. + \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^2]ds\right\}d\mu_n .$$

Now steps almost identical to those of [4, pp. 66–67] with $\|x\|^2$ replacing $\int_0^1 [x(s)]^2 ds$ and $a^2 + b^2 = [(a+b)^2 + (a-b)^2]/2$ replaced by

$\|x\|^2 + \|y\|^2 \leqq \|x + y\|^2 + \|x - y\|^2$ yield

$$\int_{-\infty}^{\infty}\left\{\int_c \|x - x^n\|^D \exp B[\|\psi_n(\xi, \cdot)\|^2 + \|x - x^n\|^2]dx\right\}d\mu_n \tag{2.5}$$
$$\leqq \left[\int_c \|x - x^n\|^{3D/2}dx\right]^{2/3}\left[\int_c \exp 6B\|x\|^2 dx\right]^{1/3}$$

and

$$\int_{-\infty}^{\infty} \exp B\|\psi_n(\xi, \cdot)\|^2 d\mu_n \int_0^1 \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^D$$
$$\exp B\|\rho(s, \cdot) - \rho^n(s, \cdot)\|^2 ds \leqq \int_c \exp 2B\|x\|^2 dx \, . \tag{2.6}$$
$$\int_0^1 \|\rho(s, \cdot) - \rho^n(s, \cdot)\|^D \exp [B\|\rho(s, \cdot) - \rho^n(s, \cdot)\|^2]ds \, .$$

Finally one notes that

$$\|x\| = \max \{\max_{t \in [0,1]} x(t), \max_{t \in [0,1]} [-x(t)]\}$$

so that for $K \in (0, 1)$

$$\exp (K\|x\|^2) \leqq \exp (K\{\max x(t)\}^2) + \exp (K\{\max [-x(t)]\}^2)$$

and

$$\int_c \exp (K\|x\|^2)dx \leqq 2\int_c \exp (K\{\max x(t)\}^2)dx \tag{2.7}$$
$$= 4\int_0^{\infty} \exp [-(1 - 2K)u^2/2]du/\sqrt{(2\pi)} = 2/\sqrt{1 - 2K} \, .$$

(for the distribution of $\max x(t)$ see [3]).

The estimate (2.7) used first with Theorem 4 and then with Theorem 1 (iv) provides the estimates of the right sides of (2.5) and (2.6). The estimate given in the corollary follows at once as does also the order estimate of the theorem.

3. Proof of Theorems 1, 2 and 4. As noted in §2 after the statement of Theorem 6, only Theorems 1, 2, and 4 remain to be proved. Yeh's lemma [8] and Ciesielski's lemma [2] provide a proof for Theorem 4. The lemma due to Ciesielski will be used in the proofs of Theorems 1 and 2. An outline of the proof of this lemma will follow. First there will be noted that there is a natural double indexing of the Haar functions:

$$\begin{aligned}&\alpha_0^{(0)}(s)\,,\\&\alpha_n^{(k)}(s)\colon n = 0, 1, 2, \cdots, k = 1, 2, \cdots, 2^n\,.\end{aligned}$$

A corresponding double indexing applies to the β's. It will be convenient to speak of "the nth cycle of α's (β's): $n \geqq 0$" by which will be meant $\{\alpha_n^{(k)}\colon k = 1, 2, \cdots, 2^n\}$ (or similar for β's). Note that $\alpha_0^{(0)}$ is not in a cycle. Now it is fairly easy to prove by induction than any partial sum of the $c\beta$-series to at least the end of the $(N-1)^{\text{th}}$ cycle gives the value of $x(t)$ for all t of the form $l/2^N\colon l = 1, 2, \cdots, 2^N$ and that the graph of this partial sum is polygonal with vertices precisely those points where the graph of the partial sum agrees with the graph of $x(t)$. The conclusions of the lemma are thus obtained.

The proof of Theorem 2 follows:

First there is noted that a functional continuous in the uniform topology is Wiener measurable. Lemma 1 together with Lebesgue's bounded or dominated convergence theorem completes the proof of (i) or (ii) respectively. That (2.3) or (2.4) provide suitable choices for the G's follows from the formula for the integral of a functional of $\max x(t)$ which yields

$$\begin{aligned}\int_C G_1[\max x(t)]dx &= \int_C G_1[\max -\{x(t)\}]dx\\&= 2\int_0^\infty e^{-\xi^2/2}G_1(\xi)d\xi/\sqrt{(2\pi)}\\&= 2A\int_0^\infty e^{-\xi^2/2+h\xi^p}d\xi/\sqrt{(2\pi)}\end{aligned}$$

and the last integral clearly converges for the conditions given on p and h in (2.3) and (2.4).

Next is given the proof of Theorem 1.

(i) For any fixed s there is a most one $h_k(s)$ in "the n^{th} cycle of Haar Functions" (for this notion c.f. beginning of outline of proof of the lemma) which is not zero and $|h_k(s)| \leqq \sqrt{2^n}$. But the k for that $h_k(s)$ satisfies $k \geqq 1 + 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n$. A comparison of the series, after terms of value zero have been deleted, of

$$\sum_{k=1}^\infty |\sin(k-\tfrac{1}{2})\pi t h_k(s)/(2k-1)|$$

with the series

$$\sum_{n=1}^{\infty} (\sqrt{2^n}/2^n) ,$$

which converges, yields the conclusion of (i).

(ii) That $\rho(s, t)$ is continuous in t for each fixed s follows at once from uniform convergence of a series of continuous functions.

(iii) To show that $||\rho(s, \cdot) - \rho^n(s, \cdot)||$ is measurable in s, $\rho^n(s, t)$ will first be calculated.

$$\rho^n(s, t) = -\sum_{i=1}^{n} \int_0^1 \rho(s, u) dh_i(u) \beta_i(t)$$

(the Stieltjes integrals exist since $\rho(s, u)$ is continuous in u)

$$= (-2^{3/2}/\pi) \sum_{i=1}^{n} \int_0^1 \left[\sum_{k=1}^{\infty} \sin (k - \tfrac{1}{2}) \pi u h_k(s)/(2k - 1) \right] dh_i(u) \beta_i(t)$$

$$= (-2^{3/2}/\pi) \sum_{i=1}^{n} \sum_{k=1}^{\infty} \int_0^1 \sin (k - \tfrac{1}{2}) \pi u dh_i(u) h_k(s) \beta_i(t)/(2k - 1)$$

(because of uniform convergence of a series of continuous functions). Thus $\rho^n(s, t)$ is measurable in s for each fixed t and so of course is $\rho(s, t)$. Since $\rho^n(s, t) - \rho(s, t)$ is continuous in t, $||\rho^n(s, \cdot) - \rho(s, \cdot)||$ is determined by a countable number of t values and so is measurable in s.

(iv) That

$$||\rho(s, \cdot) - \rho^n(s, \cdot)|| \leqq 31/n^{1/2}$$

uniformly in s is seen as follows. Let k be such that

$$1 + 1 + 2 + 4 \cdots + 2^{n-1} (= 2^n) \leqq k$$
$$\leqq 1 + 1 + 2 + 4 + \cdots + 2^n (= 2^{n+1}) .$$

Note that $n \leqq \log_2 k \leqq n + 1$.

Now

$$|\rho(s, t) - \rho^k(s, t)| = (2^{3/2}/\pi) \Bigg| \sum_{i=1}^{\infty} \Bigg[\sin (i - \tfrac{1}{2}) \pi t + \sum_{j=1}^{k} \int_0^1 \sin (i - \tfrac{1}{2}) \pi u dh_j(u) \beta_j(t) \Bigg] h_i(s)/(2i - 1) \Bigg| .$$

Since $\sin (i - \frac{1}{2}) \pi t \in C$, there follows from Lemma 1 that

$$\left\| \sum_{j=1}^{k} \int_0^1 \sin (i - \tfrac{1}{2}) \pi u dh_j(u) \beta_j(\cdot) \right\| \leqq || \sin (i - \tfrac{1}{2}) \pi \cdot || = 1 \tag{3.1}$$

for all i. Now let the series (in i) for $\rho(s, t) - \rho^k(s, t)$ be split into two parts, viz. a finite sum from $i = 1$ to $i = k$ and the remainder of the series from $i = k + 1$ onward. The second of these two parts is estimated as follows:

$$(2^{3/2}/\pi)\left|\sum_{i=k+1}^{\infty}\cdots\right| \leqq 2^{5/2}\sum_{i=n}^{\infty}(1/\sqrt{2})^i/\pi$$

(because of (3.1), the comparison series mentioned in the proof of Theorem 1(i), and the relation between k and n)

$$=2^{5/2}(1/\sqrt{2})^n/[\pi(1-1/\sqrt{2})] \leqq 32/(\pi k^{1/2})\ .$$

To estimate the first part it will be noted that, for any i, the maximum difference between the graph of $\sin(i-\frac{1}{2})\pi t$ and the k^{th} polygonal approximation, viz. $(\sin(i-\frac{1}{2})\pi t)^k$, is no greater than the maximum slope of this sine curve multiplied by $1/2^n$. Thus

$$\left\|\sin(i-\tfrac{1}{2})\pi\cdot + \sum_{j=1}^{k}\int_0^1 \sin(i-\tfrac{1}{2})\pi u dh_j(u)\beta_j(\cdot)\right\| \leqq (i-\tfrac{1}{2})\pi/2^n \leqq i\pi/2^n$$

and therefore

$$(2^{3/2}/\pi)\left|\sum_{i=1}^{k}\cdots\right| \leqq (2^{3/2}/\pi)\left[\pi/2^n + \sum_{j=0}^{n}(2^{j+1}\pi/2^n)(1/\sqrt{2})^j\right]$$

(because, for given s, the one function in the j^{th} cycle of Haar functions which is not zero has index no greater than 2^{j+1}: the $\pi/2^n$ before the summation is due to $\alpha_0^{(0)}$ which is not in a cycle)

$$=(2^{3/2}/2^n)\left[1+2\sum_{j=0}^{n}\sqrt{2^j}\right] \leqq 20/k^{1/2}\ ,$$

and addition of the estimates completes the proof.

To prove (v) there is noted that the Fubini theorem for mixed Stieltjes and Wiener integrals will yield the required result if (2.1) can be shown to hold for $F[x]$ any one of the forms K_0, $x(s_1)$, $x(s_1)x(s_2)$ and $x(s_1)x(s_2)x(s_3)$. But (2.1) clearly does hold for K_0 (yielding K_0) and for $x(s_1)$ and $x(s_1)x(s_2)x(s_3)$ (yielding 0). That (2.1) holds for $x(s_1)x(s_2)$ is seen from the computation

$$\int_0^1 \rho(u, s_1)\rho(u, s_2)du$$
$$= (2^3/\pi^2)\sum_{k=1}^{\infty}\sin(k-\tfrac{1}{2})\pi s_1 \sin(k-\tfrac{1}{2})\pi s_2/(2k-1)^2 = \min(s_1, s_2)$$

(by Mercer's theorem for the integral equation

$$\phi_n(s) = \lambda_n\int_0^1 \min(s, t)\phi_n(t)dt\ .$$

[9, p. 136] or [7, p. 464]) and the proof is complete.

Finally there follows the proof of Theorem 4. Let k be such that

$$1+1+2+4+\cdots+2^{n-1}(=2^n)\leqq k$$
$$\leqq 1+1+2+4+\cdots+2^n(=2^{n+1})$$

and let $t\in[0,1]$ be such that

$$r/2^n\leqq t\leqq(r+1)2^n\colon 0\leqq r\leqq 2^n-1\,.$$

Now (see notation in Yeh's lemma) for almost all $x\in C$

$$|x(t)-x(r/2^n)|\leqq\phi_\alpha[x]/2^{\alpha n}.\quad\text{Also}\tag{3.2}$$

$$|x^k(r/2^n)-x^k(t)|\leqq|x^k(r/2^n)-x^k([r+1]/2^n)|\tag{3.3}$$

(because the graph of x^k is a chord of the graph of x on $[r/2^n,(r+1)/2^n]$ according to the Ciesielski lemma).

$$=|x(r/2^n)-x([r+1]/2^n)|$$

(since, from the Ciesielski lemma, x and x^k agree at $r/2^n$ and $[r+1]/2^n$)

$$\leqq\phi_\alpha[x]/2^{\alpha n}$$

for almost all x.

Thus

$$\begin{aligned}&|x(t)-x^k(t)|\\&\quad=|x(t)-x(r/2^n)+x(r/2^n)-x^k(t)|\\&\quad=|x(t)-x(r/2^n)+x^k(r/2^n)-x^k(t)|\\&\quad\leqq 2\phi_\alpha[x]/2^{\alpha n}\end{aligned}$$

(because of the Schwarz inequality and inequalities (3.2) and (3.3)). From the fact that $n\geqq\log_2 k-1$ there then follows for almost all x

$$||x-x^k||\leqq 2^{\alpha+1}\phi_\alpha[x]/k^\alpha$$

and an application of Yeh's lemma completes the proof.

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LOCALIZATION OF THE CORONA PROBLEM

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The corona problem for planar open sets D and the fibers of the maximal ideal space of $H^\infty(D)$ are discussed and shown to depend only on the local behavior of D.

Let $\boldsymbol{D}$ be an open subset of the Riemann sphere $\boldsymbol{C}^*$, and let $\boldsymbol{H}^\infty(\boldsymbol{D})$ be the uniform algebra of bounded analytic functions on D. We will assume always that $H^\infty(D)$ contains a nonconstant function, that is, that $\boldsymbol{C}^*\backslash D$ has positive analytic capacity. Our object is to study the maximal ideal space $\mathscr{M}(D)$ of $H^\infty(D)$, and the "fibers" $\mathscr{M}_\lambda(D)$ of $\mathscr{M}(D)$ over points $\lambda \in \partial D$. The basis for our investigation is the observation that the fiber $\mathscr{M}_\lambda(D)$ depends only on the behavior of D near λ. This localization principle is used to obtain information related to the corona problem.

The corona of D is the part of $\mathscr{M}(D)$ which does not lie in the closure of D. Our main positive results are that D has no corona under either of the following assumptions:

(1) that the diameters of the components of $\boldsymbol{C}^*\backslash D$ (in the spherical metric, if D is unbounded) be bounded away from zero; or

(2) that for some fixed $m \geqq 0$, the complement of each component of D has $\leqq m$ components.

The proofs rest on the localization principle, and on Carleson's solution of the corona problem for the open unit disc [2]. Each of the above conditions includes the extension of Carleson's theorem to finitely connected planar domains due to Stout [9].

In the negative direction, we present an example, due to E. Bishop, of a connected one-dimensional analytic variety W which is not dense in the maximal space of $H^\infty(W)$. The construction is similar to that of Rosay [8].

1. Two basic lemmas.

The localization process depends on the following two lemmas.

LEMMA 1.1. *Let $\lambda \in \partial D$, and let U be an open neighborhood of λ. If $f \in H^\infty(D \cap U)$, there is $F \in H^\infty(D)$ such that $F - f$ extends to be analytic at λ, and $(F - f)(\lambda) = 0$. Moreover, F can be chosen so that $\|F\|_D \leqq 33\|f\|_{D\cap U}$.*

Indication of proof. Suppose $U = \Delta(\lambda; \delta)$ is the disc of radius δ, centered at λ. Let g be a smooth function supported on U, such that $g = 1$ on $\Delta(\lambda; \delta/2)$, and $|\partial g/\partial \bar{z}| \leqq 4/\delta$. Define $f = 0$ off D, and set

$$(T_g f)(\zeta) = \frac{1}{\pi}\iint \frac{f(\zeta) - f(z)}{\zeta - z}\frac{\partial g}{\partial \bar{z}} dxdy\ .$$

For a description of the properties of $T_g f$, see II.1 or VIII.10 of [3]. The desired function is obtained by adjusting $T_g f$ by a constant:

$$F = T_g f - \frac{1}{\pi}\iint \frac{f(z)}{\lambda - z}\frac{\partial g}{\partial \bar{z}} dxdy\ .$$

LEMMA 1.2. *Let $\lambda \in \partial D$, and let $f \in H^\infty(D)$. Then there is a bounded sequence $f_n \in H^\infty(D)$ such that f_n extends to be analytic at λ, and $f_n(z) \to f(z)$ uniformly on any subset of D at a positive distance from λ. Moreover, if f extends continuously to $D \cup \{\lambda\}$, then the f_n converge uniformly to f on D.*

Proof. This is VIII.10.8 of [3]. The proof is the same as that of 1.1, except that one uses a sequence of g_n whose supports shrink to $\{\lambda\}$.

2. The fibers. In order to define the fibers, we prove the following lemma.

LEMMA 2.1. *If $\varphi \in \mathscr{M}(D)$, then there is a unique point $\lambda \in \bar{D}$ such that $\varphi(f) = f(\lambda)$ for all functions $f \in H^\infty(D)$ which are analytic at λ.*

Proof. If D is bounded, then the coordinate function z belongs to $H^\infty(D)$, and the point $\lambda = \varphi(z)$ is easily seen to have the desired properties. Since D may be unbounded, we must be more circumspect.

For convenience, we rotate the sphere so that $\infty \in D$, and so that φ is not "evaluation at ∞". Choose $h \in H^\infty(D)$ such that $h(\infty) = 0$ while $\varphi(h) = 1$. Then $zh \in H^\infty(D)$. We will show that $\lambda = \varphi(zh)$ has the desired properties. Note that $\varphi((z - \lambda)h) = 0$.

Suppose $f \in H^\infty(D)$ extends to be analytic in a neighborhood of λ. Then $(f - f(\lambda))/(z - \lambda) \in H^\infty(D)$, so that $\varphi(f - f(\lambda)) = \varphi(h(f - f(\lambda))) = \varphi((z - \lambda)h)\varphi((f - f(\lambda))/(z - \lambda)) = 0$, and $\varphi(f) = f(\lambda)$.

For the uniqueness, suppose that $\lambda' \neq \lambda$ belongs to $\bar{D}$. We must find $F \in H^\infty(D)$ which is analytic at λ and at λ', and which satisfies $F(\lambda) \neq F(\lambda')$. Using 1.2, we see that there is $f \in H^\infty(D)$ such that f is analytic at λ and at λ', $f(\infty) = 0$, and f is not identically zero on D. If $z_0 \in D$ is such that $f(z_0) \neq 0$, then one of the three functions f, zf, $(f - f(z_0))/(z - z_0) \in H^\infty(D)$ will separate λ and λ'. That does it.

The *fiber* $\mathscr{M}_\lambda(D)$ of $\mathscr{M}(D)$ over $\lambda \in \bar{D}$ consists of all $\varphi \in \mathscr{M}(D)$ such that $\varphi(f) = f(\lambda)$ for all $f \in H^\infty(D)$ which extend to be analytic in a neighborhood of λ. From the definition of $\mathscr{M}_\lambda(D)$, and 2.1, we conclude that the $\mathscr{M}_\lambda(D)$ form a partition of $\mathscr{M}(D)$ into disjoint closed subsets. If $\lambda \in D$, then $\mathscr{M}_\lambda(D)$ consists of the single homomorphism "evaluation at λ." If φ_α is a net in $\mathscr{M}(D)$ converging to $\varphi \in \mathscr{M}_\lambda(D)$, and if φ_α lies in the fiber over λ_α, then the λ_α converge to λ.

By 1.2, the functions in $H^\infty(D)$ which extend analytically across $\lambda \in \partial D$ are dense in the functions in $H^\infty(D)$ which extend continuously to λ. We conclude the following.

LEMMA 2.2. *If $f \in H^\infty(D)$ extends continuously to $D \cup \{\lambda\}$, then $\varphi(f) = f(\lambda)$ for all $\varphi \in \mathscr{M}_\lambda$.*

The next theorem shows that the fibers and fiber algebras depend only on the behavior of D near λ.

THEOREM 2.3. *Let $\lambda \in \partial D$, and let U be an open neighborhood of λ. The fibers $\mathscr{M}_\lambda(D)$ and $\mathscr{M}_\lambda(D \cap U)$ are homeomorphic. The restriction of $H^\infty(D)$ to $\mathscr{M}_\lambda(D)$ coincides (modulo this identification) with the restriction of $H^\infty(D \cap U)$ to $\mathscr{M}_\lambda(D \cap U)$.*

Proof. Since $H^\infty(D) \subset H^\infty(D \cap U)$, every homomorphism in $\mathscr{M}(D \cap U)$ determines a homomorphism in $\mathscr{M}(D)$ by restricting it to $H^\infty(D)$. The restrictions of the homomorphisms in $\mathscr{M}_\lambda(D \cap U)$ belong to the fiber $\mathscr{M}_\lambda(D)$. This determines a continuous map of $\mathscr{M}_\lambda(D \cap U)$ into $\mathscr{M}_\lambda(D)$, which we must show is one-to-one and onto.

For this, let $\varphi \in \mathscr{M}_\lambda(D)$, and $f \in H^\infty(D \cap U)$. Choose F as in 1.1, and define $\tilde{\varphi}(f) = \varphi(F)$. By the definition of the fiber, $\tilde{\varphi}(f)$ is independent of the function F, subject to the conditions of 1.1. Using 2.2 one sees that $\tilde{\varphi}$ is multiplicative on $H^\infty(D \cap U)$. Moreover, if φ is already the restriction of some $\psi \in \mathscr{M}_\lambda(D \cap U)$ to $H^\infty(D)$, then the definition of $\tilde{\varphi}$ shows that $\tilde{\varphi}$ coincides with ψ. It follows that the correspondence $\varphi \leftrightarrow \tilde{\varphi}$ is a homeomorphism, as was required. On account of 1.1, again, the fiber algebras are isomorphic.

COROLLARY 2.4. *With the above identification of $\mathscr{M}_\lambda(D)$ and $\mathscr{M}_\lambda(D \cap U)$, the adherence of D in $\mathscr{M}_\lambda(D)$ coincides with the adherence of $D \cap U$ in $\mathscr{M}_\lambda(D \cap U)$.*

Proof. A net in $D \cap U$ will converge to $\varphi \in \mathscr{M}_\lambda(D)$ in $\mathscr{M}(D)$ if and only if it converges to $\tilde{\varphi} \in \mathscr{M}_\lambda(D \cap U)$ in $\mathscr{M}(D \cap U)$.

As another consequence of 2.3, we have the following extension of a result in [10].

THEOREM 2.5. *The cluster set of $f \in H^\infty(D)$ at $\lambda \in \partial D$ coincides with the range (of the Gelfand transform) of f on $\mathscr{M}_\lambda(D)$.*

Proof. Every point in the cluster set of f at λ is assumed by f on $\mathscr{M}_\lambda(D)$. On the other hand, suppose that w does not belong to the cluster set of f at λ. Then there is an open neighborhood U of λ such that $|f - w| \geqq \varepsilon > 0$ on $D \cap U$. Consequently $f - w$ is invertible in $H^\infty(D \cap U)$, and f cannot assume the value w on $\mathscr{M}_\lambda(D \cap U) = \mathscr{M}_\lambda(D)$.

COROLLARY 2.6. *If $\lambda \in \partial D$ and $f \in H^\infty(D)$, then*

$$\sup_{\varphi \in \mathscr{M}_\lambda} |\varphi(f)| = \limsup_{D \ni z \to \lambda} |f(z)| .$$

THEOREM 2.7. *The restriction A_λ of $H^\infty(D)$ to $\mathscr{M}_\lambda(D)$ is a closed subalgebra of $C(\mathscr{M}_\lambda(D))$ whose maximal ideal space is $\mathscr{M}_\lambda(D)$.*

Proof. This follows readily from the following assertion: If $h \in A_\lambda$, then there is $F \in H^\infty(D)$ such that $F = h$ on $\mathscr{M}_\lambda$, and $||F|| \leqq 66||h||$. In order to establish this assertion, choose $f \in H^\infty(D)$ such that $f = h$ on $\mathscr{M}_\lambda$. By 2.6, there is an open neighborhood U of λ such that $|f| \leqq 2||h||$ on $D \cap U$. The desired function F is now the extension of $f|_{D \cap U}$ given by 1.1.

3. The corona problem. The open set D is dense in $\mathscr{M}(D)$ if and only if whenever $f_1, \cdots, f_n \in H^\infty(D)$ satisfy $|f_1| + \cdots + |f_n| \geqq \delta > 0$ on D, then there exist $g_1, \cdots, g_n \in H^\infty(D)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$. We wish to consider open sets D with the following property, which is (at least formally) stronger than the assertion that D be dense in $\mathscr{M}(D)$.

Property (*). For each integer $n \geqq 1$ and each $\delta > 0$, there are constants $C(n, \delta)$ such that whenever $f_1, \cdots, f_n \in H^\infty(D)$ satisfy $|f_j| \leqq 1$, $1 \leqq j \leqq n$, and $\sum |f_j| \geqq \delta$ on D, then there exist $g_1, \cdots, g_n \in H^\infty(D)$ such that $\sum f_j g_j = 1$ and $|g_j| \leqq C(n, \delta)$, $1 \leqq j \leqq n$.

LEMMA 3.1. *An open set D has the property (*) if and only if wherever E is a union of disjoint open sets, each one of which is conformally equivalent to D, then E is dense in $\mathscr{M}(E)$.*

Proof. Suppose D has property (*), and suppose that $f_1, \cdots, f_n \in H^\infty(E)$ satisfy $\sum |f_j| \geqq \delta > 0$. We can assume that $|f_j| \leqq 1$, $1 \leqq j \leqq n$. Using property (*), we can solve the relation $\sum f_j g_j = 1$ on each subset of E conformally equivalent to D. The uniform estimate on the g_j's guarantees that the resulting solutions belong to $H^\infty(E)$. So E is dense in $\mathcal{M}(E)$. On the other hand, if D does not have property (*), one easily constructs $f_1, \cdots, f_n \in H^\infty(E)$ such that $\sum |f_j| \geqq \delta > 0$, while $\sum f_j g_j = 1$ has no analytic solutions $g_1, \cdots, g_n$ which are bounded on all of E.

Now Carleson [2] has shown that the open unit disc has the property (*). From this, and localization, we can use a simple topological argument, as in [5], to deduce the following.

THEOREM 3.2. *If the diameters (in the spherical metric) of the components of the complement of D are bounded away from zero, then D is dense in $\mathcal{M}(D)$.*

Proof. By rotating the sphere, we can assume that $\infty \in D$. Suppose the diameters of the components ∂D are bounded below by $\varepsilon > 0$. If $\lambda \in \partial D$, then $D \cap \Delta(\lambda; \varepsilon/2)$ is simply connected, that is, each component of $D \cap \Delta(\lambda; \varepsilon/2)$ is conformally equivalent to a disc. By 3.1 and Carleson's theorem, $D \cap \Delta(\lambda; \varepsilon/2)$ is dense in $\mathcal{M}(D \cap \Delta(\lambda; \varepsilon/2))$. By 2.4, $\mathcal{M}_\lambda(D)$ belongs to the closure of D in $\mathcal{M}(D)$. Since this is true for all $\lambda \in \partial D$, D is dense in $\mathcal{M}(D)$.

The work of Behrens [1] shows that, under the hypotheses of 3.2, each fiber algebra A_λ is a logmodular algebra (on its Shilov boundary). In particular, the Gleason parts of A_λ are one point parts and analytic discs. Using a Melnikov criterion (cf. [4]), it can be seen that each $\mathcal{M}_\lambda$ is a peak set of $H^\infty(D)$, so that $\mathcal{M}_\lambda$ contains every part which it meets. Hence the Gleason parts of $H^\infty(D)$, under the assumptions of 3.2, are the distinct components of D, together with one-point parts and analytic discs.

Concerning the existence of the constants $C(n, \delta)$ for multiply connected domains, one can say the following.

THEOREM 3.3. *For each choice of integers $m, n \geqq 1$, and each $\delta > 0$, there exist constants $C_m(n, \delta)$ such that property (*) is valid, with the constants $C_m(n, \delta)$, for all domains D which have $\leqq m$ boundary components.*

Proof. Proceeding by induction, we can assume that the theorem is true, with m replaced by $m - 1$, so that the required constants $C_{m-1}(n, \delta)$ exist. We also assume that for some n and δ, the constant $C_m(n, \delta)$ fails to exist. From this we will obtain a contradiction.

By hypothesis, there are domains D_k, $1 \leqq k < \infty$, which have m boundary components, such that property (*) fails for D_k, with constant $C(n, \delta) = k$. We can assume that D_k is a circle domain, obtained from the open unit disc Δ by excising $m-1$ disjoint closed subdiscs, one of which is centered at 0. Let r_k be the smallest number such that the annulus $\{r_k < |z| < 1\}$ is contained in D_k. There will be two cases to consider: $\limsup r_k < 1$ and $\limsup r_k = 1$.

First, suppose that $\limsup r_k = 1$. By passing to a subsequence, we can assume that r_k converges to 1 sufficiently rapidly, so that D_k is conformally equivalent to a domain E_k obtained from the rectangle $\{2^{-k-1} < \operatorname{Im}(z) < 2^{-k}, -1 < \operatorname{Re}(z) < 1\}$ by excising $m-1$ holes, so that at least one of the excised holes meets $\{\operatorname{Re}(z) < -1/2\}$, and at least one meets $\{\operatorname{Re}(z) > 1/2\}$. If $E = \cup E_k$, then the proof of 3.1 shows that E cannot be dense in $\mathscr{M}(E)$. Now the open sets $E_+ = \{z \in E: \operatorname{Re}(z) > -1/2\}$ and $E_- = \{z \in E: \operatorname{Re}(z) < 1/2\}$ are unions of domains, each of which has a complement with $\leqq m-1$ components. In view of the induction assumption, E_+ and E_- are dense in $\mathscr{M}(E_+)$ and $\mathscr{M}(E_-)$ respectively. If $\lambda \in \bar{E}$ satisfies $\operatorname{Re}(\lambda) > -1/2$, then $\mathscr{M}_\lambda(E) = \mathscr{M}_\lambda(E_+)$ while if $\lambda \in \bar{E}$ satisfies $\operatorname{Re}(\lambda) < 1/2$, then $\mathscr{M}_\lambda(E) = \mathscr{M}_\lambda(E_-)$. In any event, every $\mathscr{M}_\lambda(E)$ is adherent to E, so that E is dense in $\mathscr{M}(E)$. This contradiction allows us to reject the case $\limsup r_k = 1$.

Hence we can assume that there is an $r < 1$ such that each D_k contains the annulus $\{r < |\lambda| < 1\}$. Let D be the disjoint union of the sets D_k, and let $H^\infty(D)$ be the algebra of bounded functions on D which are analytic on each D_k. Again the proof of 3.1 shows that D cannot be dense in the maximal ideal space $\mathscr{M}(D)$ of $H^\infty(D)$.

Let N be the set of positive integers, and let Δ be the open unit disc. It will be convenient to regard D as a subset of $\Delta \times N$, so that $H^\infty(\Delta \times N)$ becomes a subalgebra of $H^\infty(D)$. Our argument at this point is motivated by Behrens' discussion of $\mathscr{M}(\Delta \times N)$ in [1]. As Behrens notes, Carleson's theorem shows that $\Delta \times N$ is dense in $\mathscr{M}(\Delta \times N)$.

Let $\varphi \in \mathscr{M}(D)$, and let Z be the function in $H^\infty(D)$ defined by $Z(\lambda, n) = \lambda$. We will find a net in D converging to φ, and for this we consider two cases.

First, suppose that $|\varphi(Z| > r$. The restriction $\tilde{\varphi}$ of φ to $H^\infty(\Delta \times N)$ belongs to $\mathscr{M}(\Delta \times N)$, so there is a net $(\lambda_\alpha, k_\alpha)$ in $\Delta \times N$ such that $(\lambda_\alpha, k_\alpha) \to \tilde{\varphi}$ in $\mathscr{M}(\Delta \times N)$. In other words, $f(\lambda_\alpha, k_\alpha) \to \varphi(f)$ for all $f \in H^\infty(D)$ which extend to be analytic on each slice $\Delta \times \{k\}$, $k \geqq 1$. In particular, $\lambda_\alpha = Z(\lambda_\alpha, k_\alpha) \to \varphi(Z)$, so that $r < |\lambda_\alpha| < 1$ and $(\lambda_\alpha, k_\alpha) \in D$ eventually. If $F \in H^\infty(D)$ is arbitrary, we expand F in a Laurent series, writing $F = F_0 + F_1$, where $F_0(\lambda, k)$ is analytic on $\Delta \times \{k\}$, $F_1(\lambda, k)$ is analytic on $E_n = D_n \cup \{|\lambda| \geqq 1\}$, and $F_1(\infty, k) = 0$. Note that F_0 and F_1 belong to $H^\infty(D)$, because the annuli we are splitting across have the same widths. In fact, $F_0 \in H^\infty(\Delta \times N)$. Now $F_1(\lambda, k) = F_1(\varphi(Z), k) + (\lambda - \varphi(Z))H(\lambda, k)$, where $H(\cdot, k)$ is analytic on E_k. Since the distance from the boundaries of the E_k to $\varphi(Z)$ always exceeds $|\varphi(Z)| - r$, we find that $H \in H^\infty(D)$. Hence

$$F = G + (Z - \varphi(Z))H ,$$

where $G \in H^\infty(\Delta \times N)$, and $H \in H^\infty(D)$. Now $F(\lambda_\alpha, k_\alpha) = G(\lambda_\alpha, k_\alpha) + (\lambda_\alpha - \varphi(Z))H(\lambda_\alpha, k_\alpha)$ converges to $\varphi(G) = \varphi(F)$. So $(\lambda_\alpha, k_\alpha) \to \varphi$ in $\mathscr{M}(D)$, and φ is in the closure of D in $\mathscr{M}(D)$.

Next, suppose that $|\varphi(Z)| \leqq r$. The E_k defined above are circle domains with $\leqq m - 1$ holes. The induction assumption shows that if E is the disjoint union of the E_k, then E is dense in $\mathscr{M}(E)$. Hence there is a net $(\lambda_\alpha, k_\alpha) \in E_{k_\alpha} \times \{k_\alpha\}$ such that $f(\lambda_\alpha, k_\alpha) \to \varphi(f)$ for all $f \in H^\infty(D)$ which extend to be analytic on E. The Laurent series argument again shows that eventually $(\lambda_\alpha, k_\alpha) \in D$, and $(\lambda_\alpha, k_\alpha) \to \varphi$ in $\mathscr{M}(D)$. Again φ lies in the closure of D.

It follows that D is dense in $\mathscr{M}(D)$, contradicting our previous assertion. That completes the proof of the theorem.

Now for $m, n \geqq 1$ and $\delta > 0$, let $C_m(n, \delta)$ denote the best possible constant for which property (*) is valid for domains whose complements have $\leqq m$ components. The $C_m(n, \delta)$ increase with m. If $\sup_m C_m(n, \delta) = C(n, \delta)$ is finite for all $n \geqq 1$ and $\delta > 0$, then every open subset D of the complex plane has property (*), with constants $C(n, \delta)$. This can be seen by approximating each component of D by finitely connected domains, and using a normal families argument. If this is the case, then D is dense in $\mathscr{M}(D)$ for every planar open set D. On the other hand, we have the following.

THEOREM 3.4. *If there exist $n > 1$ and $\delta > 0$ such that $\sup_m C_m(n, \delta) = \infty$, then there is a domain (=connected open set) D such that D is not dense in $\mathscr{M}(D)$.*

Proof. Suppose that for some integer $n \geqq 1$ and some $\delta > 0$, there is a finitely connected domain D_k such that property (*) fails, with constant $C(n, \delta) = k$. We can assume that D_k is contained in the rectangle $\{-1 < \mathrm{Re}\,(z) < 1, 2^{-k-1} < \mathrm{Im}\,(z) < 2^{-k}\}$, and that ∂D_k meets both vertical sides of the rectangle. As in 3.1, $\cup D_k$ is not dense in $\mathscr{M}(\cup D_k)$. Hence there is a point $\lambda \in \partial(\cup D_k)$ such that $\mathscr{M}_\lambda(\cup D_k)$ is not contained in the closure of $\cup D_k$. We can assume that $\mathrm{Re}\,(\lambda) \geqq 0$. Let E be the union of $\cup D_k$ and the rectangle $\{-1 < \mathrm{Re}\,(z) < -1/2, 0 < \mathrm{Im}\,(z) < 1\}$. Then E is connected, and $\mathscr{M}_\lambda(E) = \mathscr{M}_\lambda(\cup D_k)$. By 2.4, E is not dense in $\mathscr{M}(E)$. That proves the theorem.

4. An example of Bishop. Here we present an example of a one-dimensional analytic variety W which is not dense in $\mathscr{M}(W)$. The example has been in circulation for some time, being originally discovered by E. Bishop some years ago, but the example has never appeared in print.

To construct the example, let S be the shell $\{(z, w): 1/2 < \max(|z|, |w|) < 1\}$ in C^2. For each integer n, let V_n be the set of $(z, w) \in S$ such that either $2^n z$ or $2^n w$ is a Gaussian integer. The V_n form an increasing sequence of connected one-dimensional analytic subvarieties of S, whose union is dense in S.

Suppose f is a bounded function on $\cup V_n$ which is analytic on each V_n. From Schwarz's lemma it is easy to see that f is uniformly continuous, so that f extends to be continuous and analytic on S. By Hartogs' theorem, f extends to be analytic on the unit polydisc in C^2.

LEMMA 4.1. *There fails to exist a constant $C > 0$ with the following property: For each n, there are $f_n, g_n \in H^\infty(V_n)$ satisfying $zf_n + wg_n = 1$ and $|f_n| \leqq C$, $|g_n| \leqq C$.*

Proof. Suppose there is such a constant. A normal families argument produces bounded functions f and g on $\cup V_n$ such that $zf + wg = 1$, and f and g are analytic on each V_n. By the remarks preceding the lemma, f and g extend analytically to the unit polydisc, and the extensions satisfy $zf + wg = 1$. Substituting $z = w = 0$, we obtain a contradiction, thereby establishing the lemma.

THEOREM 4.2. *There is a connected one-dimensional analytic variety W such that $H^\infty(W)$ separates the points of W, while W is not dense in the maximal ideal space of $H^\infty(W)$.*

Proof. Let W be the variety obtained from the disjoint (!) union of the V_n, $n \geqq 2$, by identifying some prescribed point p_n of V_n to the point of V_{n+1} with the same z and w coordinates, so that distinct identified pairs have distinct coordinates. Then W is a connected variety, the coordinate functions z and w remain defined on W, and they satisfy $|z| + |w| > 1/2$ on W. By 4.1, there fail to exist functions $f, g \in H^\infty(W)$ satisfying $zf + wg = 1$, so that W is not dense in $\mathscr{M}(W)$.

5. **Extension to Riemann surfaces.** It is easy to extend Lemmas 1.1 and 1.2, which allow one to localize the fibers and fiber algebras, to domains on a finite bordered Riemann surface. More specifically, we can easily handle the following situation.

Let D be an open set on a Riemann surface S, let $\lambda \in \partial D$, and let U be an open coordinate disc centered at λ. Suppose there is a function h meromorphic on $D \cup U$ such that $h(\lambda) = 0$, $h^{-1}(h(U)) = U$, and H is a one-to-one covering of U over $h(U)$. If $f \in H^\infty(D \cap U)$, then $f \circ h^{-1} \in H^\infty(h(D) \cap h(U))$. By 1.1, there is a function $G \in H^\infty(h(D))$ such that $G - f \circ h^{-1}$ is analytic at 0 and vanishes there. Then $G \circ h = F \in H^\infty(D)$, and $F - f$ is analytic at λ and vanishes there. So Lemma 1.1 is valid. Also, Lemma 1.2 is valid. If the fiber $\mathscr{M}_\lambda(D)$ is defined as in §2, then 2.2 and the localization Theorem 2.3 are true.

Now suppose D is a domain on a finite bordered Riemann surface. It is easy to see, using meromorphic functions, that 2.1 is valid, that is, that $\mathscr{M}(D)$ can be partitioned into disjoint closed "fibers" $\mathscr{M}_\lambda(D)$ over points $\lambda \in \bar{D}$. In this case, the required function h always exists, for any point $\lambda \in \partial D$, so that the fibers are local. In particular, if D is a finite bordered Riemann surface, then D is dense in $\mathscr{M}(D)$, and the fibers and fiber algebras associated with points of ∂D are identical to those associated with the disc algebra $H^\infty(\varDelta)$. This latter theorem has been proved in a variety of ways in the literature. For one of the simplest proofs, see [7].

If D is an open set lying on a compact Riemann surface, such that $H^\infty(D)$ contains a nonconstant analytic function, and if the fibers $\mathscr{M}_\lambda(D)$ are defined as in §2, then again the $\mathscr{M}_\lambda(D)$, $\lambda \in \bar{D}$, form a partition of $\mathscr{M}(D)$ into disjoint closed subsets, and the localization Theorem 2.3 is valid. The details of the proofs are left.

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SPHERE TRANSITIVE STRUCTURES AND THE TRIALITY AUTOMORPHISM

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Let G be a compact connected Lie group which acts transitively and effectively on a sphere S^{n-1}. A manifold M is said to have a sphere transitive structure if the structure group of the tangent bundle of M can be reduced from $O(n)$ to G. The study of the existence of such structures is a generalization of the well-known problem of the existence of almost complex structures. We completely solve the question of existence of sphere transitive structures on spheres.

For our study of sphere transitive structures we need to know some facts about the triality automorphism λ of Spin (8). We completely determine the cohomology homomorphism induced by λ on the cohomology of the classifying space of Spin (8).

Berger [1] has classified the holonomy groups of manifolds having an affine connection with zero torsion. Either from this classification or directly from Simons [11], it follows that the holonomy group of an irreducible Riemannian manifold which is not a symmetric space acts transitively on a sphere.

On the other hand we have the following elementary fact: if the holonomy group of a Riemannian manifold M is G, then the structure group of the tangent bundle of M can be reduced to G. Therefore a more fundamental question than whether or not a Riemannian manifold M has a given Lie group G as its holonomy group is the question of the reduction of the structure group of the tangent bundle of M to G. In this paper we consider the latter question and give some necessary conditions and some sufficient conditions in terms of characteristic classes. From the remarks above it suffices to consider the case when G is a connected Lie group which acts transitively and effectively on a sphere.

We introduce the following notions.

DEFINITIONS. Let $\xi = (E, M, p, F)$ be a vector bundle where M is a CW-complex and dim $F = n$. Then a *sphere transitive reduction* is a reduction of the structure group $O(n)$ of ξ to a connected Lie subgroup G of $O(n)$ which acts transitively and effectively on the sphere S^{n-1}. In the special case when ξ is the tangent bundle of M we call the reduction a *sphere transitive structure* on M.

According to [10] the connected Lie groups G which act effectively and transitively on spheres are the following: $SO(n)$, $U(n)$,

$SU(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot SO(2)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$, and $\mathrm{Spin}(9)$. We have

$$\begin{aligned} SO(n)/SO(n-1) &= S^{n-1},\ U(n)/U(n-1) = SU(n)/SU(n-1) = S^{2n-1}\,, \\ \mathrm{Sp}(n)/\mathrm{Sp}(n-1) &= \mathrm{Sp}(n)\cdot SO(2)/\mathrm{Sp}(n-1)\cdot SO(2) \\ &= \mathrm{Sp}(n)\cdot\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\cdot\mathrm{Sp}(1) = S^{4n-1}\,, \\ G_2/SU(3) = S^6\,,\quad & \mathrm{Spin}(7)/G_2 = S^7\,,\quad \mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}\,. \end{aligned}$$

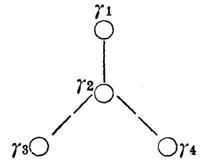
In § 2 we discuss the triality automorphism λ of Spin (8) and the cohomology of the self homeomorphism of the classifying space induced by λ. The results of § 2 are then used in § 3 to determine the cohomology of the classifying space $B\,\mathrm{Spin}(n)$ ($n = 7, 8, 9$) and a good deal of the cohomology of BG_2. Then we determine some necessary conditions for sphere transitive reductions for the cases $G = G_2$, Spin (7), Spin (9). In § 4 we discuss the existence of sphere transitive structures on certain homogeneous spaces. In particular we completely solve the problem of the existence of sphere transitive structures on spheres.

2. The cohomology of the triality automorphism. Spin (8) is the simply connected compact Lie group whose Lie algebra is of type D_4. Now D_4 is the unique simple Lie algebra with an outer automorphism of order 3. In fact, if $\mathrm{Aut}(D_4)$ (resp. $\mathrm{Inn}(D_4)$) denotes the group of all (resp. inner) automorphisms of D_4, then the factor group $\mathrm{Aut}(D_4)/\mathrm{Inn}(D_4)$ is isomorphic to the symmetric group on 3 letters. Let $\kappa, \lambda \in \mathrm{Aut}(D_4)$ be such that their images in $\mathrm{Aut}(D_4)/\mathrm{Inn}(D_4)$ generate this group and satisfy the relations $\lambda^3 = 1$, $\kappa^2 = 1$, $\kappa\lambda\kappa = \lambda^2$.

According to [7] it is possible to choose κ and λ so that the *principle of triality* holds. This means the following. Let V be the 8-dimensional algebra of Cayley numbers and denote the product of $x, y \in V$ by xy. Then for $A \in D_4$, $x, y \in V$ we have

$$(Ax)y + x(\lambda(A)y) = ((\lambda\kappa)(A))(xy)\,.$$

The Dynkin diagram of D_4 is



where $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is a simple system of roots of D_4. Since κ and λ are outer, they give rise to symmetries of the Dynkin diagram of D_4. It may be checked that $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ may be chosen so that

$\kappa(\gamma_1)=\gamma_1$, $\kappa(\gamma_2)=\gamma_2$, $\kappa(\gamma_3)=\gamma_4$, $\kappa(\gamma_4)=\gamma_3$, $\lambda(\gamma_1)=\gamma_3$, $\lambda(\gamma_2)=\gamma_2$, $\lambda(\gamma_3)=\gamma_4$, $\lambda(\gamma_4)=\gamma_1$. Henceforth we assume that the principal of triality holds and that the above choice of simple roots has been made.

Since Spin (8) is simply connected, λ and κ induce outer automorphisms of Spin (8); these in turn induce homeomorphisms of $B\,\mathrm{Spin}\,(8)$, which we continue to denote by λ and κ. In order to determine the cohomology of λ and κ, it will be convenient to use some cohomology classes introduced by Thomas [12]. Let p: $B\,\mathrm{Spin}\,(n)\rightarrow BSO(n)$ be the map defined by the covering homomorphism of Spin (n) over $SO(n)$. Denote by w_i the universal Stiefel-Whitney classes, by P_i the universal Pontryagin classes, and by X the Euler class of $BSO(8)$. Then $H^*(BSO(8),\ \boldsymbol{Z})=\boldsymbol{Z}[P_1,\ P_2,\ P_3,\ X]+$ 2-torsion and $H^*(BSO(8),\ \boldsymbol{Z}_2)=\boldsymbol{Z}_2[w_2,\ \cdots,\ w_8]$. According to Thomas [12] there exist cohomology classes $Q_i\in H^*(B\,\mathrm{Spin},\ \boldsymbol{Z})$ $(i=1,\ 2,\ 3,\ 4)$ and $w_i^*\in H^*(B\,\mathrm{Spin},\ \boldsymbol{Z}_2)$ $(i=4,\ 6,\ 7,\ 8)$ (where Spin denotes the stable Spin group) such that

$$\begin{array}{ll} p^*(P_1)=2Q_1 & p^*(w_i)=w_i^*\ \ (i=4,\ 6,\ 7,\ 8)\\ p^*(P_2)=2Q_2+Q_1^2 & p^*(w_i)=0\quad (i=2,\ 3,\ 5)\\ p^*(P_3)=Q_3 & \rho_2(Q_1)=w_4^*,\ \rho_2(Q_2)=w_8^*\\ p^*(P_4)=2Q_4+Q_2^2 & \rho_2(Q_3)=w_6^{*2},\ \rho_4(Q_4)=w_{16}^*\ . \end{array}$$

The cohomology classes Q_1, Q_2, Q_3, w_4^*, w_6^*, w_7^*, w_8^* give rise to the cohomology classes in $H^*(B\,\mathrm{Spin}\,(8),\ \boldsymbol{Z})$ and $H^*(B\,\mathrm{Spin}\,(8),\ \boldsymbol{Z}_2)$ which we denote by the same letters.

THEOREM 2.1. (i) *There exist*

$$Y\in H^8(B\,\mathrm{Spin}\,(8),\boldsymbol{Z})\quad \textit{and}\quad \omega\in H^8(B\,\mathrm{Spin}\,(8),\boldsymbol{Z}_2)$$

such that

$$\begin{aligned} H^*(B\,\mathrm{Spin}\,(8),\ \boldsymbol{Z}) &= \boldsymbol{Z}[Q_1,\ Q_2,\ Q_3,\ Y]+\text{2-torsion}\\ H^*(B\,\mathrm{Spin}\,(8),\ \boldsymbol{Z}_2) &= \boldsymbol{Z}_2[w_4^*,\ w_6^*,\ w_7^*,\ w_8^*,\ \omega]\ . \end{aligned}$$

Furthermore Y *and* ω *can be chosen so that* $p^*(X)=2Y-Q_2$ *and* $\rho_2(Y)=\omega$.

(ii) *The cohomology homomorphisms* λ^* *and* κ^* *are given as follows*:

$$\begin{array}{ll} \lambda^*(Q_1)=Q_1\ , & \lambda^*(w_i^*)=w_i^*\ \ (i=4,\ 6,\ 7)\ ,\\ \lambda^*(Q_2)=3Y-2Q_2\ , & \lambda^*(w_8^*)=\omega\ ,\\ \lambda^*(Y)=Y-Q_2\ , & \lambda^*(\omega)=w_8^*+\omega\ ,\\ \lambda^*(Q_3)=Q_3+2Q_1Y-2Q_1Q_2\ , & \kappa^*(w_i^*)=w_i^*\ \ (i=4,\ 6,\ 7,\ 8)\ ,\\ \kappa^*(Q_i)=Q_i\ \ (i=1,\ 2,\ 3)\ , & \kappa^*(\omega)=w_8^*+\omega\ ,\\ \kappa^*(Y)=-Y+Q_2\ . & \end{array}$$

Before proving this theorem we state without proof a lemma which we shall need.

LEMMA 2.2. *Let* $s: K \to L$ *be a* p^n*-fold covering of a compact connected Lie group where* p *is a prime, and denote by*

$$s^*: H^*(BL, \boldsymbol{Z}) \longrightarrow H^*(BK, \boldsymbol{Z})$$

the corresponding cohomology homomorphism of classifying spaces. Let S *be a subset of* $H^*(BK, \boldsymbol{Z})$ *such that* S *generates* $s^*(H^*(BL, \boldsymbol{Z}))$ *as a group (ring) and* $\rho_p(S)$ *generates* $\rho_p(H^*(BK), \boldsymbol{Z}) \subsetneqq H^*(BK, \boldsymbol{Z}_p)$ *as a group (ring).* (ρ_p *denotes reduction* mod p.) *Then* S *generates* $H^*(BK, \boldsymbol{Z})$ *as a group (ring).*

Proof of Theorem 2.1. Using a result of Borel [2] it is not hard to see that w_4^*, w_6^*, w_7^*, and w_8^* are generators of

$$H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2)\ .$$

Furthermore if $\rho_0: \boldsymbol{Z} \to \boldsymbol{R}_0$ denotes the inclusion, where $\boldsymbol{R}_0$ is the rationals, then it is obvious that

$$H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{R}_0) = \boldsymbol{R}_0[\rho_0(Q_1), \rho_0(Q_2), \rho_0(Q_3), \rho_0(p^*(X))]\ .$$

We first establish part of (ii). The automorphism κ of Spin (8) gives rise to an outer automorphism $\tilde{\kappa}$ of $SO(8)$; this is the ordinary orientation reversing automorphism of $SO(8)$. The induced homomorphism $\tilde{\kappa}^*$ is the identity on $H^*(BSO(8), \boldsymbol{Z}_2)$ and satisfies $\tilde{\kappa}^*(P_i) = P_i$ $(i = 1, 2, 3)$, $\tilde{\kappa}^*(X) = -X$. Hence $\kappa^*(w_i^*) = w_i^*$ $(i = 4, 6, 7, 8)$, and $\kappa^*(Q_i) = Q_i$ $(i = 1, 2, 3)$. It is also easy to see that $\lambda^*(Q_1) = Q_1$ and $\lambda^*(w_i^*) = w_i^*$ for $i = 4, 6, 7$.

We may write

$$\lambda^*(P^*(\rho_0(X))) = a\rho_0(X) + b\rho_0(Q_2) + c\rho_0(Q_1^2)\ ,$$
$$\lambda^*(\rho_0(Q_2)) = d\rho_0(X) + e\rho_0(Q_2) + f\rho_0(Q_1^2)\ ,$$

where a, b, c, d, e, f are rational numbers. Using the facts that $\lambda^*(Q_1^2) = Q_1^2$, $\lambda^3 = 1$, $\kappa\lambda\kappa = \lambda^2$, and the knowledge of κ^*, we calculate that $c = f = 0$, $a = e = -1/2$, and $bd = -3/4$.

To compute b, d, and $\lambda^*(\rho_0(Q_3))$ we must resort to some calculations with roots. Let $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, and $\tilde{X}$ denote the real cohomology classes corresponding to Q_1, Q_2, Q_3, and $p^*(X)$. Then we may regard $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$ and $\tilde{X}$ as polynominals on the Lie algebra of a maximal torus of Spin (8), i.e., polynomials in the roots of Spin (8). A calculation shows in fact that (if we write $\gamma_0 = -\gamma_1 - 2\gamma_2 - \gamma_3 - \gamma_4$),

$$\begin{aligned}
\tilde{Q}_1 &= -2\varepsilon(\gamma_0^2 + \gamma_1^2 + \gamma_3^2 + \gamma_4^2)\,, \\
\tilde{Q}_2 &= \varepsilon^2(-\gamma_0^2\gamma_1^2 - 2\gamma_3^2\gamma_4^2 + \gamma_0^2\gamma_3^2 + \gamma_0^2\gamma_4^2 + \gamma_1^2\gamma_3^2 + \gamma_1^2\gamma_4^2)\,, \\
\tilde{X} &= \varepsilon^2(-\gamma_0^2\gamma_3^2 - \gamma_1^2\gamma_4^2 + \gamma_0^2\gamma_4^2 + \gamma_1^2\gamma_3^2)\,, \\
\tilde{Q}_3 &= -2\varepsilon^3(\gamma_0^4\gamma_3^2 + \gamma_0^4\gamma_4^2 + \gamma_1^4\gamma_3^2 + \gamma_1^4\gamma_4^2 + \gamma_3^4\gamma_0^2 + \gamma_3^4\gamma_1^2 + \gamma_4^4\gamma_0^2 + \gamma_4^2\gamma_1^2 \\
&\quad -2\gamma_0^2\gamma_1^2\gamma_3^2 - 2\gamma_0^2\gamma_1^2\gamma_4^2 - 2\gamma_0^2\gamma_3^2\gamma_4^2 - 2\gamma_1^2\gamma_3^2\gamma_4^2)\,.
\end{aligned}$$

Thus we obtain

$$(*)\qquad \begin{cases}
\lambda^*(\rho_0(X)) = -\dfrac{1}{2}\rho_0(X) - \dfrac{1}{2}\rho_0(Q_2)\,, \\
\lambda^*(\rho_0(Q_2)) = \dfrac{3}{2}\rho_0(X) - \dfrac{1}{2}\rho_0(Q_2)\,, \\
\lambda^*(\rho_0(Q_3)) = \rho_0(Q_3) + \rho_0(Q_1X) - \rho_0(Q_1Q_2)\,.
\end{cases}$$

Define $Y = -\lambda^*(p^*(X))$ and $\omega = \rho_2(Y)$. Then $\lambda^*(w_8^*) = \omega$. From this, equations (*), and the fact that $H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z})$ has only 2-torsion, we obtain the rest of (ii).

From (ii) and Borel [2] we see that ω may be taken to be the remaining generator of $H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{Z}_2)$. This fact together with (ii) and Lemma 2.2 imply (i).

3. The cohomology of $B\,\mathrm{Spin}\,(7)$, $B\,\mathrm{Spin}\,(9)$, and BG_2. We first compute the cohomology of $B\,\mathrm{Spin}\,(7)$ and its inclusion in $BSO(8)$. Actually there are two natural 8-dimensional representations of Spin (7) according to [8]. These are equivalent in $O(8)$ but not in $SO(8)$. Denote these representations by j_+ and j_-. In the terminology of [8] j_+ and j_- give rise to the two distinct 3-fold vector cross products on R^8. Let i: Spin (7) $\to$ Spin (8) be the natural inclusion. The following lemma [8], [13] will be necessary.

LEMMA 3.1. *We have the following commutative diagrams*

$$\begin{array}{ccc}
\mathrm{Spin}\,(8) & \xrightarrow{\lambda} & \mathrm{Spin}\,(8) \\
{\scriptstyle i}\uparrow & & \downarrow{\scriptstyle p} \\
\mathrm{Spin}\,(7) & \xrightarrow[j_+]{} & SO\,(8)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Spin}\,(8) & \xrightarrow{\lambda^2} & \mathrm{Spin}\,(8) \\
{\scriptstyle i}\uparrow & & \downarrow{\scriptstyle p} \\
\mathrm{Spin}\,(7) & \xrightarrow[j_-]{} & SO(8)\,.
\end{array}$$

Where it is convenient we write $j_\pm$ to mean either j_+ or j_-. Let i^*: $H^*(B\,\mathrm{Spin}\,(8)) \to H^*(B\,\mathrm{Spin}\,(7))$ and

$$j_\pm^*\colon H^*(BSO\,(8)) \to H^*(B\,\mathrm{Spin}\,(7))$$

be the induced cohomology homomorphisms of i and $j_\pm$ on classifying spaces.

THEOREM 3.2. *Identify* $i^*(w_i^*)$ *with* w_i^* $(i = 4, 6, 7)$, $i^*(\omega)$ *with* ω, $i^*(Q_i)$ *with* Q_i $(i = 1, 3)$ *and* $i^*(Y)$ *with* Y. *Then we have*

(i) $H^*(B\,\mathrm{Spin}\,(7), \boldsymbol{Z}) = \boldsymbol{Z}[Q_1, Q_3, Y] +$ *2-torsion,*
$H^*(B\,\mathrm{Spin}\,(7), \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_4^*, w_6^*, w_7^*, \omega]$;

(ii) $i^*(w_8^*) = 0$ *and* $i^*(Q_2) = 2Y$;

(iii) $j_\pm^*(P_1) = 2Q_1$, $\quad j_\pm^*(w_i) = w_i^*$ $(i = 4, 6, 7)$
$j_\pm^*(P_2) = -2Y + Q_1^2$, $\quad j_\pm^*(w_8^*) = \omega$;
$j_\pm^*(P_3) = Q_3 - 2Q_1Y$,
$j_\pm^*(X) = \mp Y$,

(iv) *The kernel of* $j_\pm^*$ *on integral cohomology is the ideal generated by* $4P_2 - P_1^2 \mp 8X$.

Proof. Since i: Spin (7) $\to$ Spin (8) covers the ordinary inclusion of $SO(7)$ in $SO(8)$, we have $(i^*\circ p^*)(X) = 0$. Thus $i^*(Q_2) = 2Y$. From this fact, Theorem 2.1 and Lemma 2.2 we obtain (i) and (ii). Furthermore (iii) follows from (i), (ii), and Lemma 3.1; finally (iv) is an easy calculation from (iii).

Let M be a CW-complex and let ξ be an oriented vector bundle over M with fiber dimension 8. Denote by $f\colon M \to BSO(8)$ the classifying map determined by ξ. We shall say that ξ admits a *nontransitive* Spin (7) reduction if $f = p\circ i\circ g$ for some $g\colon M \to B\,\mathrm{Spin}\,(7)$:

$$\begin{array}{ccc} B\,\mathrm{Spin}\,(7) & \xrightarrow{i} & B\,\mathrm{Spin}\,(8) \\ {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[f]{} & BSO\,(8)\,. \end{array}$$

(Here i and p denote the maps induced by the maps Spin (7) $\to$ Spin (8) and Spin (8) $\to SO(8)$ which we also designate by i and p.) On the other hand by Lemma 3.1, M admits a sphere transitive Spin (7) reduction in the sense of this paper if and only if for some $g\colon M \to B\,\mathrm{Spin}\,(7)$ we have $f = p\circ\lambda\circ i\circ g$ or $f = p\circ\lambda^2\circ i\circ g$. Therefore we have the following lemma.

LEMMA 3.3. *Assume* $w_2(\xi) = 0$. *Then* ξ *has a transitive* Spin (7) *reduction (that is a reduction of* $SO(8)$ *to* $j_\pm$ (Spin (7))) *if and only if* $\lambda^{\mp 1}(\xi)$ *has a nontransitive* Spin (7) *reduction.*

Next we determine the primary and secondary obstructions to the existence of sphere transitive Spin (7) structures.

THEOREM 3.4. *Let* M *be a* CW*-complex and let* ξ *be an oriented vector bundle over* ξ *with fiber dimension* 8. *Denote by* $c^2(\xi)$ *and*

$c^8(\xi)$ *the primary and secondary obstructions to the existence of a transitive* $j_\pm(\mathrm{Spin}\,(7))$ *structure. Then* $c^2(\xi) \in H^2(M, Z_2)$, $c^8(\xi) \in H^8(M, Z)$, *and we have*

$$c^2(\xi) = w_2(\xi) ,$$
$$16c^8(\xi) = 4P_2(\xi) - P_1^2(\xi) \pm 8X(\xi) .$$

Proof. We first note that $SO(8)/\mathrm{Spin}\,(7)$ is diffeomorphic to real projective space P^7. Hence $c^2(\xi) \in H^2(M, \pi_1(P^7)) = H^2(M, Z_2)$ and $c^8(\xi) \in H^8(M, \pi_7(P^7)) = H^8(M, Z)$. A transgression argument given in [8] shows that $w_2(\xi) = c^2(\xi)$.

Assume that $w_2(\xi) = 0$. By Lemma 3.3, ξ has a sphere transitive $j_\pm(\mathrm{Spin}\,(7))$ structure if and only if $\lambda^{\mp 1}(\xi)$ has a nontransitive Spin (7) structure. The first obstruction to the latter is $X(\lambda^{\mp 1}(\xi))$, as is well-known. On the other hand by Theorem 2.1 and 3.2 we have

$$16X^{\pm 1}(\lambda(\xi)) = 4P_2(\xi) - P_1^2(\xi) \mp X(\xi) .$$

Hence the theorem follows.

COROLLARY 3.5. *Let ξ be an oriented vector bundle with fiber dimension* 8 *over a CW-complex M. Assume that* $\dim M \leqq 8$ *and that* $H_8(M, Z)$ *has no 2-torsion. Then ξ has a sphere transitive* $j_\pm(\mathrm{Spin}\,(7))$ *structure if and only if* $w_2(\xi) = 0$ *and*

$$4P_2(\xi) - P_1^2(\xi) \pm X(\xi) = 0 .$$

Theorems 2.1 and 3.4 and Corollary 3.5 correct an error in [8].
We now turn to Spin (9). First we need a lemma.

LEMMA 3.6. *We have the following commutative diagram:*

$$\begin{tikzcd}
 & \mathrm{Spin}\,(8) \times \mathrm{Spin}\,(8) \arrow[r, "\Delta"] & \mathrm{Spin}\,(16) \arrow[d, "p"] \\
\mathrm{Spin}\,(8) \arrow[ur, dashed, "\lambda\times\lambda^2"] \arrow[r, "k"'] & \mathrm{Spin}\,(9) \arrow[r, "l"'] & SO(16)
\end{tikzcd}$$

where Δ is the standard map of Spin (8) $\times$ Spin (8) *into* Spin (16), *p is the covering projection, k is the standard inclusion of* Spin (8) *in* Spin (9), *and l is the sphere transitive* 16*-dimensional representation of* Spin (9).

Proof. Let F_4 denote the automorphism group of the exceptional Jordan algebra of 3×3 Hermitian matrices of Cayley numbers. Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The subgroup H_i of F_4 which leaves E_i fixed is isomorphic to Spin (9) (see [7]). On the other hand Spin (8) is isomorphic to $H_1 \cap H_2 \cap H_3$. Let

$$V_1 = \text{matrices of the form } \begin{pmatrix} 0 & z & w \\ \bar{z} & 0 & 0 \\ \bar{w} & 0 & 0 \end{pmatrix},$$

$$V_2 = \text{matrices of the form } \begin{pmatrix} 0 & z & 0 \\ \bar{z} & 0 & w \\ 0 & \bar{w} & 0 \end{pmatrix},$$

$$V_3 = \text{matrices of the form } \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & w \\ \bar{z} & \bar{w} & 0 \end{pmatrix}.$$

Then V_i is an irreducible representation space for H_i. Since there is only one irreducible 16-dimensional representation of Spin (9), each representation of H_i on V_i is just l. Now the representation of Spin (8) on V_1 is $\lambda \times 1$, on V_2 is $\lambda \times \lambda^2$, and on V_3 is $1 \times \lambda^2$. Hence we get the commutative diagram

$$\begin{tikzcd} & \mathrm{Spin}\,(8) \times \mathrm{Spin}\,(8) \arrow[rr] & & \mathrm{Spin}\,(16) \arrow[d, "p"] \\ \mathrm{Spin}\,(8) \arrow[ur, "\lambda^{i-1} \times \lambda^i"] \arrow[rr, "k_i"'] & & \mathrm{Spin}\,(9) \arrow[r, "l"'] & SO(16)\,. \end{tikzcd}$$

We claim that k_2 is the standard inclusion of Spin (8) in Spin (9) while k_1 and k_3 are not. This may be proved by showing that $k_i^*(H^*(B\,\mathrm{Spin}\,(9), \boldsymbol{R}_0))$ is $\boldsymbol{R}_0[P_1, P_2, P_3, X^2] \subseteq H^*(B\,\mathrm{Spin}\,(8), \boldsymbol{R}_0)$ for $i = 2$, but not for $i = 1$ or 3. (See the proof of the next theorem.) This completes the proof of the lemma.

THEOREM 3.7. (i) *There exist cohomology classes*

$$Z \in H^{16}(B\,\mathrm{Spin}\,(9), \boldsymbol{Z})$$

and $\phi \in H^{16}(B\,\mathrm{Spin}\,(9), \boldsymbol{Z}_2)$ *such that*

$$H^*(B\,\mathrm{Spin}\,(9), \boldsymbol{Z}) = \boldsymbol{Z}[Q_1, Q_2, Q_3, Z] + 2\text{-}torsion$$
$$H^*(B\,\mathrm{Spin}\,(9), \boldsymbol{Z}_2) = \boldsymbol{Z}_2[w_4, w_6, w_7, w_8, \phi]\,.$$

Here $k_2^*(Q_i) = Q_i$ $(i = 1, 2, 3)$, $k_2^*(4Z) = p^*(X^2 - P_2^2)$, $k_2^*(w_i^*) = w_i^*$ $(i = 4, 6, 7, 8)$, *and* $k_2^*(\phi) = \omega^2 + \omega w_8^*$.

(ii) *We have (modulo elements of order* 2)

$$
\begin{aligned}
l^*(P_1) &= 4Q_1\\
l^*(P_2) &= -2Q_2 + 6Q_1^2\\
l^*(P_3) &= 2Q_3 - 6Q_1Q_2 + 4Q_1^2\\
l^*(P_4) &= -34Z - 7Q_2^2 + 4Q_1Q_3 - 6Q_1^2Q_2 + Q_1^4\\
l^*(P_5) &= 28Q_1Z - 2Q_2Q_3 + 2Q_1^2Q_3 + 10Q_1Q_2^2 - 2Q_1^3Q_2\\
l^*(P_6) &= 22Q_2Z - 2Q_1^2Z + Q_3^2 - 2Q_1Q_2Q_3 + 5Q_2^3 + Q_1^2Q_2^2\\
l^*(P_7) &= 2Q_3Z - 10Q_1Q_2Z + Q_2^2Q_3 - 3Q_1Q_2^3\\
l^*(X) &= Z\,.
\end{aligned}
$$

and

$$
\begin{aligned}
l^*(w_i) &= 0 \text{ for } i = 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13\\
l^*(w_8) &= w_4^{*2} + w_8^*\\
l^*(w_{12}) &= w_6^{*2} + w_4^*w_8^*\\
l^*(w_{14}) &= w_7^{*2} + w_6^*w_8^*\\
l^*(w_{15}) &= w_7^*w_8^*\\
l^*(w_{16}) &= \phi\,.
\end{aligned}
$$

Proof. Let ξ be an 8-dimensional vector bundle with $w_2(\xi) = 0$ and set $\nu = \lambda(\xi) \oplus \lambda^2(\xi)$. Then the Pontryagin, Euler, and Stiefel-classes of ν may be computed by means of the Whitney sum formula together with Theorem 2.1. On the other hand any maximal torus (maximal 2-subgroup) of Spin (8) is also a maximal torus (maximal 2-subgroup) of Spin (9). Therefore the formulas for above mentioned characteristic classes are the most general possible.

Set $Z = l^*(X)$ and $\phi = l^*(w_{16})$. Then we obtain (ii). Finally (i) follows from (ii) and Lemma 2.2.

Theoretically the kernel of l^* can be determined from Theorem 3.7 (ii). This yields some necessary conditions that a 16-dimensional vector bundle have a transitive Spin (9) reduction. However, we omit the details. In the only example we consider in § 4, namely the Cayley plane, it is simpler to use Theorem 3.7 itself.

We conclude this section by noting a few facts about the cohomology of BG_2 and its inclusion in $B\,\mathrm{Spin}\,(7)$.

LEMMA 3.8. *Let g be the standard inclusion of G_2 in $SO(7)$, and denote by h the lifting of g into* Spin (7):

$$\begin{tikzcd} & \mathrm{Spin}\,(7) \arrow[r, "i"] \arrow[d] & \mathrm{Spin}\,(8) \\ G_2 \arrow[ur, "h"] \arrow[r, "g"'] & SO(7)\,. & \end{tikzcd}$$

If i denotes the standard inclusion of Spin (7) *in* Spin (8), *then we have*

$$\lambda \circ i \circ h = i \circ h\ .$$

Proof. This follows from the fact that G_2 is the fixed point set of λ.

THEOREM 3.9. (i) *We have*

$$\begin{aligned} H^*(BG_2, \boldsymbol{R}_0) &= \boldsymbol{R}_0[g^*(P_1), g^*(P_3)] \\ &= \boldsymbol{R}_0[h^*(Q_1), h^*(Q_3)]\ , \end{aligned}$$

where g^* and h^* are induced by g and h defined in the previous lemma and $\boldsymbol{R}_0$ denotes the rationals.

(ii) *In integral cohomology, the kernel of g^* is the ideal generated by $4P_2 - P_1^2$ and the kernel of h^* is the ideal generated by Y.*

Proof. The proof of (i) and the fact that $g^*(4P_2 - P_1^2) = 0$ consists of identifying the Pontryagin classes with polynomials in the roots of $SO(7)$, computing the images of these polynomials under g^*, and using the fact that there are two generators of $M^*(BG_2, \boldsymbol{R}_0)$, one 4-dimensional, and the other 12-dimensional. We omit the details. From Lemma 3.8, Theorem 2.1 and Theorem 3.2, we have $h^*(Y) = 0$ and $h^*(\omega) = 0$. An easy calculation shows that $g^*(4P_2 - P_1^2) = 0$. That Y and $4P_2 - P_1^2$ generate the kernels of h^* and g^* follows from (i).

4. Sphere transitive structures on spheres and other homogeneous spaces. The study of the existence of almost complex structures on spheres is a well-known problem in algebraic topology; it was solved by Borel and Serre [4]. Thus the results of this section can be viewed as a generalization of this problem. Many of the results we present are not new. However, we give them in order that we may write down in an organized fashion the complete solution to the problem of the existence of sphere-transitive structures on spheres.

We shall need two preliminary results.

LEMMA 4.1. *Let G act transitively and linearly on S^{2n-1} with* *isotropy subgroup H. Then if the tangent bundle of S^{2n} can be reduced to G, the subgroup of elements of $\pi_{2n-2}(H)$ which are inessential in G has order at most 2.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} H \arrow[r, "k"] & G \arrow[r, "h"] \arrow[d, "p_1"'] & SO(2n) \arrow[r, "j"] \arrow[dl, "p_2"] & SO(2n+1) \arrow[d, "p_3"] \\ & S^{2n-1} & & S^{2n} . \end{tikzcd}$$

Here the p_i are evaluation maps, j and k denote the inclusion of the respective isotropy subgroups, and h denotes the representation of G arising from the action on S^{2n-1}. Let ∂_1 be the boundary operator in the homotopy sequence of the fibration $H \xrightarrow{k} G \xrightarrow{p} S^{2n-1}$ and ∂_2 the boundary operator in the homotopy sequence of fibration $SO(2n) \to SO(2n+1) \to S^{2n}$. Let $\iota_k \in \pi_k(S^k)$ denote the homotopy class of the identity map of S^k. A reduction of the structure group of the tangent bundle of S^{2n} to G is equivalent to the existence of an element $\alpha \in \pi_{2n-1}(G)$ such that $h_*(\alpha) = \partial_2(\iota_{2n})$. Then $p_{1*}(\alpha) = p_{2*}h_*(\alpha) = p_{2*}\partial_2(\iota_{2n}) = 2\iota_{2n-1}$ and so $\partial_1(2\iota_{2n-1}) = \partial_1(p_{1*}\alpha) = 0$. Hence $\partial_1(\pi_{2n-1}(S^{2n-1}) \subseteqq \pi_{2n-2}(H)$ has order at most 2. By the exactness of the homotopy sequence this subgroup is equal to $\ker(k: \pi_{2n-2}(H) \to \pi_{2n-2}(G))$.

LEMMA 4.2. *We have $\pi_{4n-2}(\mathrm{Sp}(n)) = 0$ and $(2n-1)!$ divides the order of $\pi_{4n-2}(\mathrm{Sp}(n-1))$ for $n \geqq 2$.*

Proof. $\pi_{4n-2}(\mathrm{Sp}(n))$ is in the stable range and is 0 by Bott periodicity. To prove the other assertion we consider the homomorphism of homotopy sequences of fibrations induced by the commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(n-1) \arrow[r] \arrow[d, "\iota"] & \mathrm{Sp}(n) \arrow[r] \arrow[d] & S^{4n-1} \arrow[d, "\text{identity}"] \\ U(2n-1) \arrow[r] & U(2n) \arrow[r] & S^{4n-1} \end{tikzcd}$$

where the horizontal lines are fibrations. Let ∂_1 and ∂_2 be the boundary maps of the homotopy sequences of the upper and lower lines, respectively. Then $\iota_* \circ \partial_1 = \partial_2$; hence the order of $\mathrm{Im}(\partial_1)$ is a multiple of the order of $\mathrm{Im}(\partial_2)$. But $Z_{(2n-1)!} = \pi_{4n-2}(U(2n-1)) \subset \mathrm{Im}\,\partial_2$ see [5]. Hence $(2n-1)!$ divides the order of $\pi_{4n-2}(\mathrm{Sp}(n-1))$.

THEOREM 4.3. *Let $\tau(S^n)$ denote the tangent bundle of S^n. The following is a complete list of sphere transitive structures on* *spheres*:

(i) $SO(n)$ *on* $\tau(S^n)$,
(ii) $U(3)$ *on* $\tau(S^6)$,
(iii) $SU(3)$ *on* $\tau(S^6)$,
(iv) G_2 *on* $\tau(S^7)$.

Proof. We have (i) because S^n is orientable and (iv) because S^7 is parallelizable. (ii) is a consequence of the fact that S^6 has an almost complex structure. Actually, however, it turns out that structure group of the tangent bundle $\tau(S^6)$ can be reduced to $SU(3)$ (see [8]) so that (iii) holds.

Next we show that there are no other sphere transitive structures. We do this case by case.

$U(n)$: Borel and Serre proved that for $n \neq 1, 3$ $\tau(S^{2n})$ cannot have a $U(n)$ structure.

$SU(n)$: Since $\tau(S^{2n})$ $(n \neq 1, 3)$ cannot have a $U(n)$ structure, it cannot have an $SU(n)$ structure because $SU(n) \subsetneqq U(n)$.

$\mathrm{Sp}\,(n)$: Since $\tau(S^{4n})$ $(n \neq 1)$ cannot have a $U(2n)$ structure and $\mathrm{Sp}\,(n) \subsetneqq U(2n)$, $\tau(S^{4n})$ cannot have a $\mathrm{Sp}\,(n)$ structure.

$\mathrm{Sp}\,(n)\cdot SO(2)$: We have $\mathrm{Sp}\,(n)\cdot SO(2) \subsetneqq U(2n)$. Thus the argument for $\mathrm{Sp}\,(n)$ applies in this case also.

$\mathrm{Sp}\,(n)\cdot\mathrm{Sp}\,(1)$: For $n \geqq 1$, $\mathrm{Sp}\,(n)\cdot\mathrm{Sp}\,(1)$ is covered by

$$\mathrm{Sp}\,(n) \times \mathrm{Sp}\,(1) = \mathrm{Sp}\,(n) \times S^3 .$$

We have $\pi_k(\mathrm{Sp}\,(n)\cdot\mathrm{Sp}\,(1)) \cong \pi_k\,\mathrm{Sp}\,(n) \oplus \pi_k(S^3)$ for $k > 1$. By the second part of Lemma 4.2, it follows that for $n \geqq 2$, $\pi_{4n-2}\,(\mathrm{Sp}\,(n)\cdot\mathrm{Sp}\,(1)) = \pi_{4n-2}(S^3)$ and $\pi_{4n-2}\,(\mathrm{Sp}\,(n-1)\cdot\mathrm{Sp}\,(1))$ is the direct sum of $\pi_{4n-2}(S^3)$ with a group of order at least $(2n-1)!$. Since $\pi_{4n-2}(S^3)$ is finite, it follows that the necessary condition for a $\mathrm{Sp}\,(n)\cdot\mathrm{Sp}\,(1)$-structure on S^{4n} given by Lemma 4.1 fails, for $n > 1$.

$\mathrm{Spin}\,(7)$: According to Theorem 3.2 (iv) a necessary condition that an 8-dimensional vector bundle ξ have a transitive $\mathrm{Spin}\,(7)$ reduction is that $4P_2(\xi) - P_1^2(\xi) \mp 8X(\xi) = 0$. The tangent bundle of S^8 (or its negative) does not satisfy this condition.

$\mathrm{Spin}\,(9)$: Suppose the tangent bundle $\tau = \tau(S^{16})$ had a transitive $\mathrm{Spin}\,(9)$ structure. We have $P_i(\tau) = 0$ $(i = 1, \cdots, 7)$, $X(\tau) = 2$. Hence by Theorem 3.7 (ii), $Q_i(\tau) = 0$ $(i = 1, \cdots, 7)$ and $Z(\tau) = 0$ (at least with rational coefficients). This contradicts the fact that we must have $X(\tau) = Z(\tau)$. The same argument shows that $-\tau$ cannot have a transitive $\mathrm{Spin}\,(9)$ reduction.

We conclude with some brief remarks about the existence of sphere transitive structures on various simply connected compact homogeneous spaces other than spheres. Denote by $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{Q})$

complex and quaternionic projective spaces of real dimension $2n$ and $4n$, respectively. Also let $\bar{Q}_n$ denote the space of all nonoriented 2-planes in R^{n+2}.

THEOREM 4.4. *The homogeneous spaces* $S^6 \times S^2$, $S^4 \times S^4$, $S^4 \times S^2 \times S^2$, $(S^2)^4$, $P^4(C)$, $P^3(C) \times S^2$, $P^2(C) \times P^2(C)$, $P^2(C) \times S^4$, $P^2(C) \times S^2 \times S^2$, $P^1(Q) \times P^2(C)$, $P^1(Q) \times S^4$, $P^1(Q) \times S^2 \times S^2$, $\bar{Q}_2 \times P^1(Q)$, $\bar{Q}_2 \times P^2(C)$, $\bar{Q}_2 \times S^4$, $\bar{Q}_2 \times S^2 \times S^2$ *do not possess sphere transitive* Spin (7) *structures.*

Proof. For each case one computes (see [3]) the Pontryagin and Euler classes and verifies that they do not satisfy $P_2 - 4P_1^2 \pm 8X = 0$.

In contrast to Theorem 4.4 we have the following result.

THEOREM 4.5. *Either orientation of the spaces* $P^2(Q)$, $\bar{Q}_4$, *and* $G_2/SO(4)$ *possesses a sphere transitive* Spin (7) *structure.*

Proof. According to [3] each of these spaces has integral cohomology $Z[u]/(u^4)$ where u is a 4-dimensional generator. Furthermore $P_1 = 2u$, $P_2 = 7u^2$, and $X = \pm 3u^2$ for each of these spaces (with the proper choice of u). Theorem (4.5) now follows from Theorem 3.4.

It would be interesting to construct explicitly a sphere transitive Spin (7) structure (i.e., a 3-fold vector cross product) on $P^2(Q)$.

Finally we have the following theorem.

THEOREM 4.6. *Let* $C = F_4/\text{Spin}(9)$ *denote the Cayley plane with the canonical orientation. Then* C *does not possess a sphere transitive* Spin (9) *structure, but* $-C$ *does.*

Proof. We have $H^*(C, Z) = Z[u]/(u^4)$ where u is an 8-dimensional generator. With the proper choice of u we have by [3] that for the Cayley plane, $P_2 = 6u$, $P_4 = 39u^2$, $P_1 = P_3 = 0$, and $X = \pm 3u^2$. It is well known that at least one orientation of C possesses a sphere transitive Spin (9) structure. It is not hard to verify that $-C$ satisfies the conclusions of Theorem 3.7 while C does not. Hence we get Theorem 4.6.

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ON GENERALIZED FORMS OF APOSYNDESIS

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If a point set is both connected and closed it is called a continuum. The structure of a nonlocally connected continuum can be described in terms of its aposyndetic properties. In this paper various forms of continuum aposyndesis, that is, aposyndesis with respect to subcontinua, are considered. It is shown that the presence of any of these forms of aposyndesis in a compact metric continuum which is totally nonconnected im kleinen (not connected im kleinen at any point) insures nonsemi-local-connectedness on a dense open subset of the continuum and the set of weak cut points in each open subset of the continuum has cardinality at least c.[1] A weak cut point theorem for compact plane continua is established. An example is given which indicates that this result does not hold in Euclidean 3-space. Near aposyndesis, a generalization of aposyndesis, is introduced. It is shown that the presence of this property in a totally nonaposyndetic, separable, metric continuum implies the existence of uncountably many weak cut points.

DEFINITION. Let x, y, and z be distinct points of a continuum M. If every subcontinuum of M which contains x and y also contains z, then z is said to *cut M weakly* between x and y. A point z of M is said to be a *weak cut point* of M if there exist two points x and y in M such that z cuts M weakly between x and y.

DEFINITION. Let S be a subset of a continum M and let x be a point of $M - S$. If M contains a continuum H and an open set U such that $x \in U \subset H \subset M - S$, then M is said to be *aposyndetic* at x with respect to S. Note that if M is a regular Hausdorff continuum, M being aposyndetic at a point p with respect to every closed set in $M - \{p\}$ is equivalent to M being connected im kleinen at p. Let x be a point of a continuum M; if for each point y of $M - \{x\}$, M is aposyndetic at x with respect to y, then M is said to be *aposyndetic* at x.

Let S is a subset of a continuum M. If x is a point of $M - S$ and M is not aposyndetic at x with respect to S, then M is said to be *nonaposyndetic* at x with respect to S.

2. Continuum aposyndesis. In the introduction it is pointed

[1] For a related result see [5, Th. 15]. For definitions of unfamiliar terms and phrases see [7] and [9].

out that connectedness im kleinen at a point of a regular Hausdorff continuum can be thought of in terms of aposyndesis at that point with respect to closed sets which do not contain the point. This concept can be generalized by considering aposyndesis at a point with respect to closed connected sets (i.e., continua) which do not contain the point.

DEFINITION. Let M be a continuum and p and q be two distinct points in M. If, for each continuum K in $M - \{p\}$ which contains q, M is aposyndetic at p with respect to K, then M is said to have *property A* at p with respect to q. If, for every point x in $M - \{p\}$, M has property A at p with respect to x, then M is said to have *property A* at p.

Obviously, if a regular Hausdorff continuum M is connected im kleinen at a point p in M, then M has property A at p. An example due to F. B. Jones indicates that the converse of this statement is false [6, Example 3]. The compact plane continuum described by Jones has property A at a point y and is not connected im kleinen at y. The point y in this continuum is a weak cut point. The following theorem indicates that a compact plane continuum has these properties (property A and nonconnectedness im kleinen) at a point only if the point is a weak cut point of the continuum.

LEMMA. *If a compact plane continuum M is not connected im kleinen at a point x, then for each open set U in the plane which contains x, there exists a pair of points $\{y, z\}$ in $U \cap M$ such that M is nonaposyndetic at x which respect to $\{y, z\}$.*[2]

Proof. Assume that there is an open set U containing x such that M is aposyndetic at x with respect to every pair of points in $U \cap M$. Since M is not connected im kleinen at x, there exists a circular region G such that $\mathrm{Cl}\, G$ (the closure of G) is contained in U and a sequence $K_1, K_2, K_3, \cdots$ of distinct components of $M \cap \mathrm{Cl}\, G$ such that (1) for each positive integer i, K_i contains the point y_i of a sequence $y_1, y_2, y_3, \cdots$ of points of J (the boundary of G) converging to the point y and (2) x is in the limit inferior of $K_1, K_2, K_3, \cdots$.

Since M is aposyndetic at x with respect to any pair of points of $U \cap M$, M is aposyndetic at x with respect to y. Therefore, there exists a continuum H in $M - \{y\}$ such that x is contained in $\mathrm{Int}\, H$ (the interior of H). Each component of $H \cap G$ has a limit point in J. Hence there exists a subsequence $K'_1, K'_2, K'_3, \cdots$ of $K_1, K_2, K_3, \cdots$

[2] This lemma is stated in [6]. The proof does not appear in the literature.

such that for each i, K_i' contains the point z_i of a sequence $z_1, z_2, z_3, \cdots$ of points of J converging to the point z of $J - \{y\}$.

By assumption M is aposyndetic at x with respect to $\{y, z\}$. Hence there is a continuum L such that $x \in \operatorname{Int} L \subset L \subset M - \{y, z\}$. Let A and B denote disjoint subarcs of J containing y and z respectively {as nonendpoints) such that $(A \cup B) \cap L = \emptyset$. Since every component of $L \cap G$ has a limit point in J, there exist three positive integers, i, j, and k, such that each of K_i', K_j', and K_k' intersects each of A, B, and $J - (A \cup B)$. Since $J - (A \cup B)$ has exactly two components, some two of these three continua must intersect the same component of $J - (A \cup B)$. This leads to a contradiction of Theorem 28 of [7, p. 156].

THEOREM 1. *If a compact plane continuum M has property A at a point x and is not connected im kleinen at x, then x is a weak cut point of M.*

Proof. By the preceding lemma, there exists a pair of points $\{y, z\}$ in $M - \{x\}$ such that M is nonaposyndetic at x with respect to $\{y, z\}$. M must be aposyndetic at x with respect to each continuum in $M - \{x\}$ since M has property A at x. Therefore no subcontinuum of M in $M - \{x\}$ contains both y and z.

EXAMPLE 1. A compact continuum M in Euclidean 3-space which has property A at a point p, and is not connected im kleinen at p, may fail to be cut weakly by p. To see this define $A_i = \{(0, 0, 1/n) \mid n = i, i+1, i+2, \cdots\}$. Let C_1 be the join of $\operatorname{Cl} A_1$ with the point $(1, 0, 0)$. For $i = 2, 3, \cdots$, define C_i to be the join of $\operatorname{Cl} A_i$ with the point $(1, 1/i, 0)$. Let $M = \bigcup_{i=1}^{\infty} C_i$. See Figure 1. Let p be the point $(0, 0, 0)$. Any subcontinuum in the complement of p intersect only finitely many of the C_i's. It follows that M has property A at p. Clearly M is not connected im kleinen at p and p does not cut M weakly between any two points in $M - \{p\}$. Note that M is not semi-locally-connected at p.

THEOREM 2. *If a regular Hausdorff continuum M is semi-locally-connected at a point p and has property A at p, then M is connected im kleinen at p.*[3]

Proof. Assume that M is not connected im kleinen at p. First it will be shown that under this assumption p must be a weak cut

[3] This has been previously observed for compact Hausdorff continua [**1**, Th. 4].

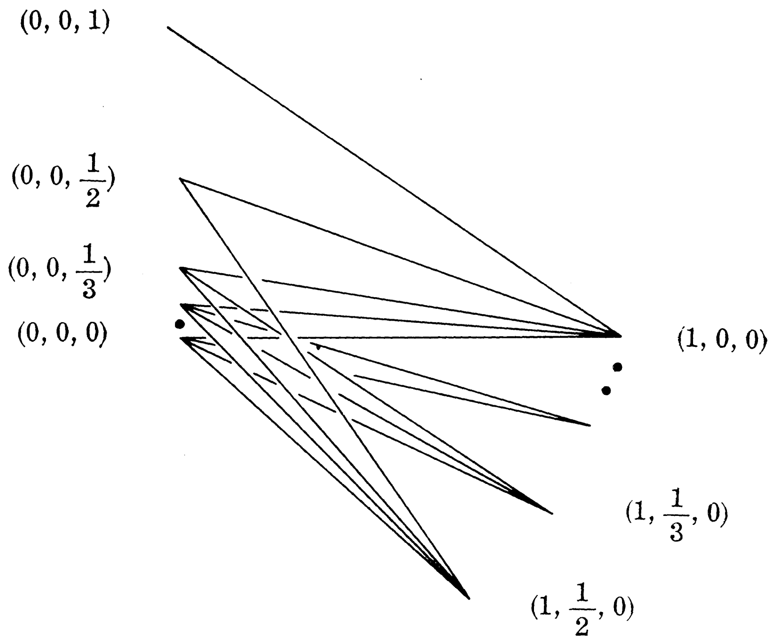

FIGURE 1.

point of M.

Since M is not connected im kleinen at p and is semi-locally-connected at p, there exists an open set U in M containing p such that M is nonaposyndetic at p with respect to $M - U$, and $M - U$ has only a finite number of components. If p is not a weak cut point then the components of $M - U$ can be joined together by a finite number of continua in $M - \{p\}$. Let L be the union of these continua. Every continuum containing p in its interior must meet the continuum $(M - U) \cup L$. But this contradicts the fact that M has property A at p. Therefore p must be a weak cut point.

Since M is semi-locally-connected at p and p is a weak cut point, p must separate M [8, Th. 6.2]. Suppose that for each component C of $M - \{p\}$ which meets $M - U$, the set $C \cup \{p\}$ is a connected subspace of M which is aposyndetic at p with respect to $(C \cup \{p\}) - U$. Then in each of these subspaces there is a continuum which contains p in its interior, relative to the subspace, which does not meet $M - U$. The sum of these continua and the components of $M - \{p\}$ which do not meet $M - U$ form a continuum in M which contains p in its interior and misses $M - U$. This contradicts the choice of U. Therefore there is a component C in $M - \{p\}$ such that the subspace $C \cup \{p\}$ is nonaposyndetic at p with respect to $(C \cup \{p\}) - U$. It follows that the subspace $C \cup \{p\}$ is not connected im kleinen at p. Let H be the subspace $C \cup \{p\}$. Note that H is semi-locally-connected at p, since

M is semi-locally-connected at p.

The subspace H has property A at p. To see this let Q denote a subcontinuum of $H - \{p\}$. Because M has property A at p, and Q is a subcontinuum of M, there exists a subcontinuum K of $M - Q$ which contains p in its interior. Since p is a separating point, the set $H \cap K$ is a subcontinuum of H and p is in the interior of $H \cap K$ relative to H. Therefore H has property A at p.

By applying to the subspace H the argument presented in the second paragraph of this proof, one can conclude that p must cut H weakly and therefore separate H. But this is impossible since H consists of the point p and a component of $M - \{p\}$. Hence M is connected im kleinen at p.

In bicompact T_1 continua, property A and local connectedness are equivalent as global properties, since if a bicompact T_1 continuum has property A everywhere, then it is aposyndetic at each of its points and therefore semi-locally-connected [5, Th. 0], and from Theorem 2 it follows that the continuum is everywhere connected im kleinen and therefore locally connected.

It is clear that a T_1 continuum M has property A at a point p with respect to a point q if and only if for each open set G in $M - \{q\}$ which contains p, M is aposyndetic at p with respect to the q-component of $M - G$. From this point of view one can generalize this property as follows.

Definition. Let p and q be distinct points of a continuum M. If for each open set G in $M - \{q\}$ which contains p, there exists a point r in G such that M is aposyndetic at r with respect to the q-component of $M - G$, then M is said to have *property* B at p with respect to q. If for each point x in $M - \{p\}$, M has property B at p with respect to x, then M is said to have *property* B at p. If M has property B at each point of M, then M is said to have *property* B.

Obviously, if a continuum M contains a dense subset D such that M has property A at each point of D, then M has property B. The following example indicates that property B is considerably weaker than property A.

Example 2. There exists a compact plane continuum M which has property B and is totally nonaposyndetic (not aposyndetic at any of its points) hence does not have property A at any point.

Let $M_1, M_2, \cdots$ be a sequence of closed plane point sets defined by induction as follows. Let S be the unit disk and let M_1 be the closure of the union of the (topological) disk sequence $D_1, D_2, \cdots$

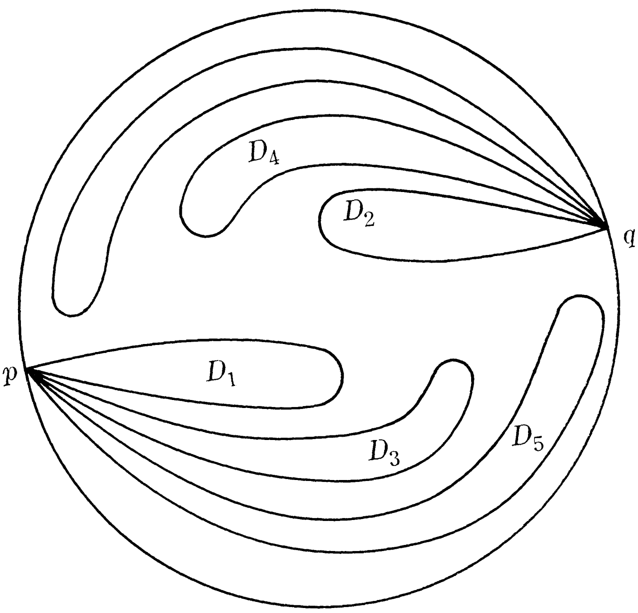

FIGURE 2.

indicated in Figure 2. The boundary of S is the limiting set of D_1, $D_2, \cdots$. The diameter of each of $D_1, D_2, \cdots$ is greater than 3/4. Let p and q be the two separating points in M_1. Note that M_1 is nonaposyndetic at each point of the boundary of S with respect to one of p and q.

Assume M_n to be defined and let $D_1', D_2', \cdots$ be a counting of the disks in M_n. Let f_i be a homeomorphism of S onto D_i' such that the distance from $f_i(p)$ to $f_i(q)$ is greater than $(1/4) + (1/n + 3)$. Let $M_{n+1} = \mathrm{Cl}\,[\bigcup_{i=1}^{\infty} f_i(M_1)]$. The homeomorphisms of S onto the disks of M_n are chosen in such a way that the disks in M_{n+1} will be of diameter greater than $(1/2) + (1/n + 3)$ and the set of separating points in M_{n+1} is $1/n + 1$ dense in M_n. Define $M = \bigcap_{n=1}^{\infty} M_n$.

M has property B. To see this let x and y be two points in M and let G be an open set in $M - \{y\}$ containing x. Let C be the y-component of $M - G$. Since the point set consisting of homeomorphic images of $\{p, q\}$ is dense in M, there is a point s in $M \cap G$ which is a separating point in M_n (for some n). There exists a disk D in M_n such that y is not in D and s separates $D - \{s\}$ from $M_n - D$ in M_n. It follows that $M \cap D$ is a continuum in $M - C$ which has an interior point in G.

It is clear that M is nonaposyndetic at each point which is on the boundary of some defining continuum M_n in S. Let x be a point of M which is in the interior of M_n for every positive integer n. There exist a nest of disks $d_1, d_2, \cdots$ and two sequences of points, $s_1, s_2, \cdots$

and $t_1, t_2, \cdots$, such that (1) for each positive integer n, the point x is contained in d_n; (2) d_n is a maximal disk in M_n; (3) the point s_n separates $d_{n+1} - \{s_n\}$ from $(d_n - d_{n+1}) \cap M_{n+1}$ in M_{n+1}; and (4) t_n is the other point which separates $d_n \cap M_{n+1}$. Note that for each n, M is nonaposyndetic at s_n with respect to t_n. If x is a limit point of $s_1, s_2, \cdots$, then there exists a point t in M, distinct from x, which is a limit point of the sequence $t_1, t_2, \cdots$, such that M is nonaposyndetic at x with respect to t [5, Th. 1]. If x is not a limit point of $s_1, s_2, \cdots$, then there exists a point s in M, distinct from x, which is a limit point of this sequence. It follows that M is nonaposyndetic at x with respect to s. Therefore M is totally nonaposyndetic.

DEFINITION. Let p be a point of a continuum M. If there exists a point q in $M - \{p\}$ such that M has property B at p with respect to q, then M is said to have *property* C at p.

Obviously, if a continuum has property B at a point p, then it has property C at p. One can see that property C is weaker than property B by considering the Cantor Cone. This continuum has property C at each point which is in the interior of an arc, but it has property B only at the vertex.

THEOREM 3. *If a compact metric continuum M has property C at each point of a dense G_δ subset of M and is totally nonconnected im kleinen on a dense G_δ subset of M, then M is totally nonsemi-locally-connected on a dense open subset of M.*

Proof. Let U be an open set in M. If one can show that there exists an open subset G of U such that M is not semi-locally-connected at any point of G, then the existence of the dense open subset of M with the desired condition with follow immediately.

The open set U contains an open set V, no component of which contains an open set [3, Th. 2]. It follows that for each point x in V, M is nonaposyndetic at x with respect to $M - V$. Define $D_i = \{x \in V \mid$ for some y in $M - S(x, 1/i)$ ($S(x, 1/i)$ is the circular open set in M with the point x as center and with radius $1/i$), M has property B at x with respect to $y\}$. Since $\bigcup_{i=1}^{\infty} D_i$ is a second category subset of V, for some positive integer n, the set $\mathrm{Cl}\, D_n$ contains an open set in V. It follows that there exists an open set W in V such that W contains a dense subset D with the condition that for each point x in D, there exists a point y in $M - W$ such that M has property B at x with respect to y.

If M is totally nonsemi-locally-connected on W, then W has the required conditions. Assume that this is not the case. That is, there exists a point p in W such that M is semi-locally-connected at p. There exists an open set Q containing p in W such that $M - Q$ has only a finite number of components. Suppose that p is not a weak cut point. Join together the components of $M - Q$ with a finite number of continua in $M - \{p\}$. Let L be the union of these continua and let K be the continuum $(M - Q) \cup L$. The set $M - K$ is open and contained in Q. Since D is dense in W and $M - W$ is contained in K there exist a point x in $D \cap (M - K)$ and a point y in K such that M has property B at x with respect to y. But this is impossible since M is nonaposyndetic at each point of $M - K$ with respect to K. Therefore p is a weak cut point in M.

Since M is semi-locally-connected at p, p is a separating point in M and each component of $M - \{p\}$ is both open and closed relative to $M - \{p\}$. Let X be a component of $M - \{p\}$ which meets $M - Q$. The set $S = X \cup \{p\}$ is a connected subspace of M. S is semi-locally-connected at p. The point p is not a weak cut point in S for if it were it would also separate S which is clearly impossible. $S - Q$ has only a finite number of components. Join there components together with a finite number of continua in X. Let F denote the union of these continua with $S - Q$. The set F is a continuum in $S - \{p\}$ which contains $S - Q$. The set $G = X - F$ is open in M. Let x be a point in G and assume that M is semi-locally-connected at x. Each open subset of G which contains x cuts M weakly between p and F (i.e., each continuum in M which meets both F and p must also meet each open subset of G which contains x). To see this suppose that there exist an open set R containing x in G and there exists a continuum H in $M - R$ which meets both p and F. It follows that $M - Q$ is contained in one component of $M - R$. But for some point s in R there exists a point t in $M - W$ such that M has property B at s with respect to t. Therefore there exists a point r in R such that M is aposyndetic at r with respect to the t-component of $M - R$. This is a contradiction since $M - Q$ is contained in the t-component of $M - R$ and M is nonaposyndetic at r with respect to $M - Q$. It follows that x cuts M weakly between p and F. Since M is semi-locally-connected at x, M is separated by x between p and F. $S - \{x\}$ can be written as the union of two mutually separated sets P and E such that p is contained in P and F is contained in E. $P \cup \{x\}$ is a subcontinuum of M which has a nonvoid interior and is contained in Q. This contradicts the fact that no component of Q contains an open set. Therefore M is totally nonsemi-locally-connected on G.

THEOREM 4. *If a compact metric continuum M is nonsemi-locally-connected at each point of a G_δ subset which is dense in M, then the* *set of weak cut points in each open subset of M has cardinality at least c.*

Proof. Let U be an open subset of M. Define the set $D = \{x \in M \mid$ for each open set V containing x, there exists an open subset W of V containing x such that M is aposyndetic at x with respect to each point of the boundary of $W\}$. If D is dense in U, then there exists a dense G_δ subset J of U such that each point of J is a weak cut point in M [4, Th. 4].

Suppose that D is not dense in U. There exists an open set G in $U - D$. Since G is second category, there exist a point x in G and a point r in M such that if y is a point of M and M is nonaposyndetic at x with respect to y, then y cuts M weakly between x and r [2, Th. 4]. There exists an open set Q containing x in G such that for each open subset R of Q there is a point y in the boundary of R such that M is nonaposyndetic at x with respect to y. There are c open sets in Q which contain x and have mutually disjoint boundaries. It follows that there are c points in Q which cut M weakly between x and r.

THEOREM 5. *If a compact metric continuum M has property C at each point of a dense G_δ subset of M and is totally nonconnected im kleinen on a dense G_δ subset of M, then each open subset of M contains a set of weak cut points of M which has cardinality c.*

Proof. M is totally nonsemi-locally-connected on a dense open subset of M (Theorem 3). The conclusion follows from Theorem 4.

COROLLARY. *If a compact metric continuum M has property B and is totally nonconnected im kleinen on a dense G_δ subset of M, then each open subset of M contains a set of weak cut points of M which has cardinality c.*

Note that if a compact metric continuum M is totally nonconnected im kleinen on a dense G_δ subset of M and contains a dense subset D such that M has property A at each point of D, then M contains a dense G_δ set of weak cut points of M [4, Th. 4]. However, the existence of such continua is still an open question.

3. Near aposyndesis.

In § 2 property B is introduced as a weaker form of continuum aposyndesis. In this section aposyndesis (aposyndesis with respect to points) is generalized in a similar fashion.

DEFINITION. A continuum M is said to be *nearly aposyndetic* at a point p in M with respect to a point q in M if each open set in M containing p contains a point r such that M is aposyndetic at r with respect to q. Let p be a point of M; if for each point q in $M - \{p\}$, M is nearly aposyndetic at p with respect to q, then M is said to be *nearly aposyndetic* at p.

It is easily seen that if a continuum M has property C at a point p and is not nearly aposyndetic at p, then p must be a weak cut point in M. Note that the Cantor Cone has these two properties at each point of a dense open set.

THEOREM 6. *A compact metric continuum M is not nearly aposyndetic at a point p with respect to a point s if and only if there exists an open set G in M containing p, such that if U is a nonvoid open subset of G, then s cuts M weakly between some two points in U.*

Proof. If M is not nearly aposyndetic at p with respect to s, then there exists an open set G containing p such that M is nonaposyndetic at each point of G with respect to s. This open set G has the desired property [2, Th. 2].

To see that the condition is sufficient, assume that M is nearly aposyndetic at p with respect to s. Let G be an open set in M containing p. There exists a point x in G such that M is aposyndetic at x with respect to s. Therefore there is a continuum K and an open set U such that $x \in U \subset K \subset M - \{s\}$. It follows that s does not cut M weakly between any two points of the open set $G \cap U$ in G.

If a continuum M has property B, then M is nearly aposyndetic (that is, M is nearly aposyndetic at each of its points). It follows that the totally nonaposyndetic continuum M in Example 2 is nearly aposyndetic. One can see from this example that near aposyndesis is considerably weaker than aposyndesis. M in Example 2 is totally nonsemi-locally-connected. The following example indicates that this is not necessarily the case for totally nonaposyndetic continua which are nearly aposyndetic.

EXAMPLE 3. There exists a compact nearly aposyndetic, totally nonaposyndetic continuum M in E^3 (Euclidean 3-space) which is semi-locally-connected on a dense open subset of M.

Let C be the Cantor set and its image on the interval $[-1, 0]$. For each point z of C define the set

$$S_z = \{(x, y, z) \in E^3 \mid x = 0 \text{ or } x = 1 \text{ and } 0 \leqq y \leqq 1, \text{ or } \\ y = 0 \text{ or } y = 1 \text{ and } 0 \leqq x \leqq 1\}.$$

Let $S = \bigcup_{z \in C} S_z$. Define the continuum M to be the decomposition of S obtained as follows. For each positive real number z in C, identify the point $(z, 0, z)$ with the point $(z, 0, 0)$. For each negative real number z in C, identify the point $(-z, 1, z)$ with the point $(-z, 1, 0)$. See Figure 3. M is semi-locally-connected at each point which is not in the XY-plane. Note that M contains a Cantor set of weak cut points.

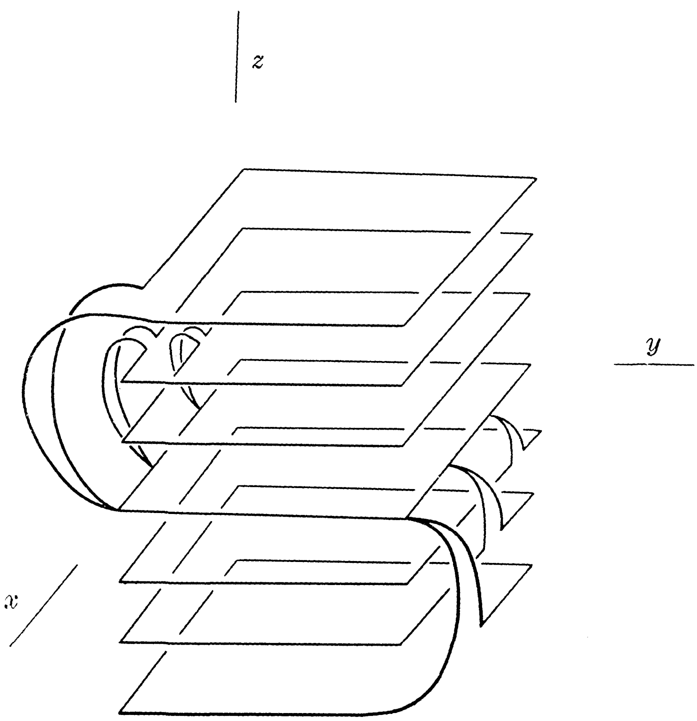

FIGURE 3.

It is possible for a compact totally nonaposyndetic metric continuum to have only one weak cut point [5, Example 1]. However, if the continuum is also nearly aposyndetic then one is assured of the existence of more than countably many weak cut points.

THEOREM 7. *If a compact metric continuum M is nearly aposyndetic and totally nonaposyndetic, then M has uncountably many weak cut points.*

Proof. Assume that M has only countably many weak cut points. Let $s_1, s_2, \cdots$ be a counting of these points. Let Q denote a countable dense subset of M. Since M is totally nonaposyndetic, M contains a dense G_δ subset I such that if x is a point in I and M is nonaposyndetic at x with respect to a point y, then y cuts x weakly from each point of $Q - \{y\}$ in M [2, Corollary 1]. For each positive integer i, define

$$D_i = \{x \in I \mid s_i \text{ cuts } x \text{ from each point of } Q - \{s_i\}\} .$$

$\bigcup_{i=1}^{\infty} D_i = I$. Since I is second category, there is a positive integer n such that D_n is somewhere dense. Let G be an open set in $\mathrm{Cl}\, D_n$ which does not contain s_n. Note that G has the property described in Theorem 6. It follows that M is not nearly aposyndetic at any point of G with respect to s_n. But this is a contradiction. Therefore M must contain uncountably many weak cut points.

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ON SUBGROUPS OF A PSEUDO LATTICE ORDERED GROUP

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The purpose of this note is to investigate some problems raised in a recent paper of Conrad and Teller concerning o-ideals and p-subgroups in an abelian pseudo lattice ordered group.

The concept of a pseudo lattice ordered group ("p-group") has been introduced by Conrad [1]. In recent papers by Teller [5] and Conrad and Teller [2] there is developped a systematic theory of p-groups. Let G be an abelian p-group. In § 3 it is proved that if M is a subgroup of G such that $\{a, b\} \cap M \neq \varnothing$ for any pair of p-disjoint elements $a, b \in G$, then M contains a prime o-ideal; this generalizes a result from [2]. In § 4 we prove that the intersection of two p-subgroups of a p-group G need not be a p-subgroup of G. Moreover, if Δ is a partially ordered set and for each $\delta \in \Delta$ $H_\delta \neq \{0\}$ is a linearly ordered group, then for the mixed product $G = V(\Delta, H_\delta)$ the following conditions are equivalent: (i) for any two p-subgroups A, B of G their intersection $A \cap B$ is a p-subgroup of G as well; (ii) G is an l-group. If A is an o-ideal of a p-group G and B is a p-subgroup of G, then $A + B$ is a p-subgroup of G.

2. Preliminaries. Let G be a partially ordered group. G is a Riesz group (cf. Fuchs [3], [4]) if it is directed and if from a_i, $b_j \in G$, $a_i \leqq b_j$ $(i, j = 1, 2)$ it follows that there exists $c \in G$ satisfying $a_i \leqq c \leqq b_j$ $(i, j = 1, 2)$. G is a p-group (cf. [1] and [5]) if it is Riesz and if each $g \in G$ has a representation $g = a - b$ such that $a, b \in G$, $a \geqq 0$, $b \geqq 0$ and

$$(*) \qquad x \in G,\ x \leqq a,\ x \leqq b \Longrightarrow nx \leqq a,\ nx \leqq b$$

for any positive integer n.

Throughout the paper G denotes an abelian p-group. Elements $a, b \in G$, $a \geqq 0$, $b \geqq 0$ satisfying $(*)$ are called p-disjoint. A subgroup M of G is a p-subgroup, if for each $m \in M$ there are elements $a, b \in M$ such that a, b are p-disjoint in G and $m = a - b$. A subgroup C of G is an o-ideal, if it is directed and if $0 \leqq g \leqq c \in C$, $g \in G$ implies $g \in C$. Let $O(G)$ be the system of all o-ideals of G (partially ordered by the set inclusion). An o-ideal C of G is called prime, if G/C is a linearly ordered group. For any pair a, b of p-disjoint elements $H(a, b)$ denotes the subgroup of G generated by the set

$$\{0 \leqq m \in G \mid m \leqq a,\ m \leqq b\}\ .$$

Then $H(a, b) \in O(G)$ (cf. [2]).

Let Δ be a partially ordered set and let $H_\delta \neq \{0\}$ be a linearly ordered group for each $\delta \in \Delta$. Let $V = V(\Delta, H_\delta)$ be the set of all Δ-vectors $v = (\cdots, v_\delta, \cdots)$ where $v_\delta \in H_\delta$, for which the support $S(v) = \{\delta \in \Delta \mid v_\delta \neq 0\}$ contains no infinite ascending chain. An element $v \in V$, $v \neq 0$ is defined to be positive if $v_\delta > 0$ for each maximal element $\delta \in S(v)$. Then ([2], Th. 5.1) V is a p-group; V is an 1-group if and only if Δ is a root system (i.e., $\{\delta \in \Delta \mid \delta \geqq \gamma\}$ is a chain for each $\gamma \in \Delta$).

3. Subgroups containing a prime o-ideal. The following assertion has been proved in [2] (Proposition 4.3):

(A) For $M \in O(G)$, the following are equivalent: (1) M is prime; (2) the o-ideals of G that contain M form a chain; (3) if a and b are p-disjoint in G, then $a \in M$ or $b \in M$.

Further it is remarked in [2] that each subgroup M of G fulfilling (3) is a p-subgroup and any subgroup containing a prime o-ideal satisfies (3); then it is asked whether a subgroup M of a p-group G satisfies (3) if and only if it contains a prime o-ideal (a similar assertion is known to be valid for lattice ordered groups). We shall prove that the answer is positive.

We need the following propositions (cf. [2] and [5]):

(B) Let $g = a - b \in G$ where a and b be p-disjoint. Then $g = x - y$, where x and y are p-disjoint, if and only if $x = a + m$ and $y = b + m$ for some $m \in H(a, b)$.

(C) If a and b are p-disjoint, then na and nb are p-disjoint for any positive integer n and $H(a, b) = H(na, nb)$ ([2], Proposition 3.1).

LEMMA 1. *Let M be a subgroup of G fulfilling* (3) *and let a, b be p-disjoint elements in G. Then $H(a, b) \subset M$.*

Proof. Let $h \in H(a, b)$. According to (3) we may assume without loss of generality that $a \in M$. Suppose (by way of contradiction) that $h \notin M$. Then $a + h \notin M$, hence by (B) $b + h \in M$, and analogously $b - h \in M$, thus $2b \in M$. Further $2a + h \notin M$ and therefore according to (C) and (B) $2b + h \in M$, which implies $h \in M$.

LEMMA 2. *Let M be a subgroup of G satisfying* (3) *and let $X = \{X_i\}$ be the system of all o-ideals of G such that $X_i \subset M$. Then the system X has a largest element.*

Proof. Let Y be the subgroup of G generated by the set $\bigcup X_i$.

Then $Y \subset M$ and Y is the supremum of the system $\{X_i\}$ in the lattice $\mathscr{G}$ of all subgroups of G. Since $O(G)$ is a complete sublattice of $\mathscr{G}$ ([2], Th. 2.1), $Y \in O(G)$ and thus $Y \in X$.

Let H be the subgroup of G generated by the set $\bigcup H(a, b)$ where a, b runs over the system of all p-disjoint pairs of elements in G. Since each set $H(a, b)$ is an o-ideal ([2]), $H = \bigvee H(a, b)$ (a and b p-disjoint in G) where $\bigvee$ denotes the supremum in the lattice $O(G)$. According to Lemma 1 $H \subset M$ whenever the subgroup M of G satisfies (3).

For any $u, v \in G$, $u \leqq v$, the interval $[u, v]$ is the set

$$\{x \in G \ u \leqq x \leqq v\} .$$

LEMMA 3. *Let M be a subgroup of G satisfying* (3) *and let N be the largest o-ideal of G that is contained in M. Let $g \in G$, $g > 0$. Then*

$$[0, g] \subset M \Longrightarrow g \in N .$$

Proof. According to Lemma 2 the largest o-ideal N in M exists. Assume that $g \in G$, $g > 0$, $[0, g] \subset M$. The set

$$Z = \bigcup_{n=1}^{\infty} [-ng, ng]$$

is clearly an o-ideal in G. Let $z \in Z$, hence $z \in [-ng, ng]$ for a positive integer n. This implies $0 \leqq y \leqq 2ng$ where $y = z + ng$. Since G is a Riesz group, according to [3, p. 158, Th. 27] there are elements $g_1, \cdots, g_{2n} \in G$, $0 \leqq g_i \leqq g$ such that $y = g_1 + \cdots + g_{2n}$. Thus $g_i \in M$, therefore $y \in M$ and $Z \subset M$. Now we have $Z \subset N$ and so $g \in N$.

LEMMA 4. *Let M be a subgroup of G fulfilling* (3) *and let N be the largest o-ideal of G contained in M. Then G/N is a linearly ordered group.*

Proof. Assume (by way of contradiction) than G/N is not linearly ordered. According to Lemma 1 $H \subset N$, hence by [2], Theorem 4.1 G/N is a lattice ordered group. Thus there exist elements $X, Y \in G/N$ such that $X \wedge Y = \bar{0}$, $X > \bar{0}$, $Y > \bar{0}$ ($\bar{0}$ being the neutral element of G/N). From [2] (Proposition 2.2, (ii)) it follows that there are elements $x \in X$, $y \in Y$ such that x and y are p-disjoint in G and hence $x \in M$ or $y \in M$. Clearly $x \notin N$, $y \notin N$ and thus according to Lemma 3 there exist elements $x_1, y_1 \in G$ such that

$$0 < x_1 \leqq x,\ 0 < y_1 \leqq y,\ x_1 \notin M,\ y_1 \notin M .$$

Then in G/N we have $\bar{0} < x_1 + N \leqq x + N = X$, $\bar{0} < y_1 + N \leqq y + N = Y$, whence

$$(x_1 + N) \wedge (y_1 + N) = \bar{0} .$$

Thus by using repeateadly [2], Proposition 2.2, we can choose elements $x_2 \in x_1 + N$, $y_2 \in y_1 + N$ such that x_2 and y_2 are p-disjoint in G. Therefore (without loss of generality) we may assume $x_2 \in M$ and this implies $x_1 \in x_1 + N = x_2 + N \subset M$, a contradiction. The proof is complete.

THEOREM 1. *Let M be a subgroup of a p-group G. Then* (3) $\Rightarrow$ (2) *and the condition* (3) *is equivalent to* (1′) *M contains a prime o-ideal.*

Proof. According to Lemma 4 (3) $\Rightarrow$ (1′). By [2] (1′) $\Rightarrow$ (3). Assume that M is a subgroup of G fulfilling (3). Let K_1, K_2 be o-ideals of G such that $M \subset K_1 \cap K_2$. Let N have the same meaning as in Lemma 4. Since $N \subset M$,

$$K_1 \subset K_2 \Longleftrightarrow K_1/N \subset K_2/N .$$

K_1/N and K_2/N are o-ideals of G/N and G/N is linearly ordered, hence $K_1/N \subset K_2/N$ or $K_2/N \subset K_1/N$; therefore (2) holds.

If M is an o-ideal of G satisfying (3), then by Theorem 1 M contains a prime o-ideal N; according to [2] (Corollary 1 to the Induced Homomorphism Theorem) G/M is isomorphic to $(G/N)/(M/N)$ and hence (G/N being linearly ordered) G/M is a linearly ordered group and M is prime. Thus it follows from Theorem 1 that (3) $\Rightarrow$ (1) for $M \in O(G)$ (cf. (A)).

Let us remark that if M is a subgroup of G fulfilling (3) then M need not contain any nonzero o-ideal that is a lattice; further (3) is not implied by (2).

EXAMPLE 1. Let B be an infinite Boolean algebra that has no atoms and put $\Delta = \{b \in B \mid b \neq 0\}$. For each $\delta \in \Delta$ let $H_\delta = E$ where E is the additive group of all integers with the natural order, $G = V(\Delta, H_\delta)$. Let $M = \{v \in G \mid v_1 = 0\}$ (by 1 we denote the greatest element of B). Then M is a prime o-ideal of G, hence M satisfies (3) and M contains no lattice ordered o-ideal different from $\{0\}$.

EXAMPLE 2. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 < \delta_3$, $\delta_2 < \delta_3$ and δ_1, δ_2 are incomparable. Put $H_{\delta_i} = E (i = 1, 2, 3)$, $G = V(\Delta, H_\delta)$, $M = (v \in G \mid v_{\delta_1} = v_{\delta_2} = 0\}$. Then the only o-ideal that contains M is G, thus (2) holds. Let $a, b \in G$ such that $a_{\delta_1} = 1$, $a_{\delta_2} = a_{\delta_3} = 0$, $b_{\delta_2} = 1$, $b_{\delta_1} = b_{\delta_3} = 0$. The elements a and b are p-disjoint in G and $a \notin M$, $b \notin M$, hence M does not fulfil (3).

4. Intersections and sums of two p-subgroups. Another problem formulated in [2] is whether the intersection of two p-subgroups of a p-group G must be a p-subgroup of G; there is remarked in [2] that this conjecture seems rather dubious. The answer to this problem is negative.

EXAMPLE 3. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 > \delta_3$, $\delta_2 > \delta_3$ and δ_1, δ_2 are incomparable. Let $H_{\delta_i} = E(i = 1, 2, 3)$, $G = V(\Delta, H_\delta)$. We write $v(\delta_i)$ instead of v_{δ_i}. Let $c_i \neq 0$ $(i = 1, 2)$ be positive integers, $c_1 \neq c_2$. Denote

$$A_i = \{v \in G \mid v(\delta_3) = c_i[v(\delta_1) + v(\delta_2)]\}$$

$(i = 1, 2)$. Let $i \in \{1, 2\}$ be fixed. For proving that A_i is a p-subgroup of G we have to verify that to each $v \in A_i$ we can choose $a, b \in A_i$, $a \geqq 0$, $b \geqq 0$ such that $(*)$ holds and $v = a - b$. It is easy to verify that it suffices to consider the case when 0 and v are uncomparable, hence we may assume $v(\delta_1) > 0$, $v(\delta_2) < 0$ (the case $v(\delta_1) < 0$, $v(\delta_2) > 0$ being analogous). Let $a, b \in G$,

$$a(\delta_1) = v(\delta_1),\ a(\delta_2) = 0,\ a(\delta_3) = c_i a(\delta_1)\ ,$$
$$b(\delta_1) = 0,\ b(\delta_2) = -v(\delta_2),\ b(\delta_3) = -c_i v(\delta_2)\ .$$

Then a and b have the desired properties, hence A_i is a p-subgroup of G. Denote $C = A_1 \cap A_2$. If $v \in C$, we have

$$c_1[v(\delta_1) + v(\delta_2)] = v(\delta_3) = c_2[v(\delta_1) + v(\delta_2)]$$

and thus (since $c_1 \neq c_2$)$v(\delta_3) = 0$, $v(\delta_2) = -v(\delta_1)$. Therefore any element $v \in C$, $v \neq 0$ is incomparable with 0 and C is not a p-subgroup of G.

The method used in this example can be employed for proving the following theorem:

THEOREM 2. *Let Δ be a partially ordered set and for each $\delta \in \Delta$ let $H_\delta \neq \{0\}$ be a linearly ordered group, $V = V(\Delta, H_\delta)$. If V is not lattice ordered, then V contains infinitely many pairs of p-subgroups A_1, A_2 such that $A_1 \cap A_2$ is not a p-subgroup of V.*

Proof. Assume that V is not lattice ordered. Then Δ is no root system, hence there exist elements $\delta_1, \delta_2, \delta_3$ such that $\delta_1 > \delta_3$, $\delta_2 > \delta_3$ and δ_1, δ_2 are incomparable. Choose $e_i \in H_{\delta_i}$, $e_i > 0$ and let c_1, c_2 be positive integers, $c_1 \neq c_2$. Let $V_1 = \{v \in V \mid v_\delta = 0$ for each $\delta \notin \{\delta_1, \delta_2, \delta_3\}\}$,

$$A_i = \{v \in V_1 \mid v(\delta_1) = n_1 e_1,\ v(\delta_2) = n_2 e_2,\ v(\delta_3) = c_i(n_1 + n_2)e_3\}$$

where n_1 and n_2 run over the set of all integers $(i = 1, 2)$. Analogously as in Example 3 we can verify that A_1 and A_2 are p-subgroups of V. Let $v \in C = A_1 \cap A_2$. Then $c_1(n_1 + n_2) = c_2(n_1 + n_2)$, thus $n_2 = -n_1$ and $v(\delta_3) = 0$. Therefore no element of C is strictly positive and C is no p-subgroup of G. Since the positive integers $c_1 \neq c_2$ are arbitrary there exist enfinitely many such pairs A_1, A_2.

As a corollary, we obtain:

PROPOSITION 1. *Let* $V = V(\Delta, H_\delta)$, *where each* H_δ *is linearly ordered. Then the following conditions are equivalent*: (i) V *is lattice ordered*; (ii) *if* A *and* B *are* p-*subgroups of* V, *then* $A \cap B$ *is a* p-*subgroup of* V *as well.*

Proof. By Theorem 2 (ii) implies (i). Let V be lattice ordered. Then a subgroup A of V is a p-subgroup of V if and only if it is an 1-subgroup of V; since the intersection of two 1-subgroups is an 1-subgroup, (ii) is valid.

PROPOSITION 2. *Let* Δ *be a partially ordered set and for any* $\delta \in \Delta$ *let* $H_\delta \neq \{0\}$ *be a linearly ordered group. Assume that there exist* $\delta_1, \delta_2, \delta_3 \in \Delta$ *such that* $\delta_1 < \delta_3$, $\delta_2 < \delta_3$ *and* δ_1, δ_2 *are incomparable,* $V = V(\Delta, H_\delta)$. *Then there are infinitely many* p-*subgroups* A, B *of* V *such that* $A + B$ *is not a* p-*subgroup of* V.

Proof. Denote $V_1 = \{v \in V \mid v(\delta) = 0 \text{ for each } \delta \notin \{\delta_1, \delta_2, \delta_3\}\}$ and let c be a fixed positive integer, $e_i \in H_{\delta_i}$, $e_i > 0$ $(i = 1, 2, 3)$. Put

$$A = \{v \in V_1 \mid v(\delta_1) = ne_1,\ v(\delta_2) = -cne_2,\ v(\delta_3) = ne_3\}\ ,$$
$$B = \{v \in V_1 \mid v(\delta_1) = v(\delta_2) = 0,\ v(\delta_3) = ne_3\}$$

where n runs over the set of all integers. A and B are linearly ordered subgroups of V, hence they are p-subgroups of V. The set $C = A + B$ is the system of all elements $v \in V_1$ such that

$$v(\delta_1) = n_1 e_1\ ,\quad v(\delta_2) = -cn_1 e_2\ ,\quad v(\delta_3) = n_2 e_3$$

where n_1, n_2 are arbitrary integers. Hence there is $g \in C$ satisfying

$$g(\delta_1) = e_1\ ,\quad g(\delta_2) = -ce_2\ ,\quad g(\delta_3) = 0\ .$$

If $g = a - b$, $a \in C$, $b \in C$, $a \geqq 0$, $b \geqq 0$, then $a \neq 0 \neq b$ (since $g \not> 0$, $g \not< 0$), thus $a(\delta_3) = b(\delta_3) \geqq e_3$. There exists $v \in V_1$ such that $v(\delta_3) = a(\delta_3)$, $v(\delta_1) < a(\delta_1)$ and $b(\delta_1)$, $v(\delta_2) < a(\delta_2)$ and $b(\delta_2)$. Thus $v < a$, $v < b$, but $2v \not< a$, $2v \not< b$. Therefore a and b are not p-disjoint in G and C is no p-subgroup of G.

One of the problems raised in [2] is affirmatively solved by

THEOREM 3. *Let A be an o-ideal of G and let B be a p-subgroup of G. Then $A + B$ is a p-subgroup of G.*

Proof. Let us denote $G/A = \bar{G}$ and for any $t \in G$ write $t + A = \bar{t}$. Let $A + B = X$, $x \in X$. There are elements $a \in A$, $b \in B$ such that $x = a + b$ and since B is a p-subgroup there exist $b_1, b_2 \in B$ such that $b = b_1 - b_2$ and b_1, b_2 are p-disjoint in G. Further $x = u - v$, u, $v \in G$, where u and v are p-disjoint in G. According to [2] $\bar{G}$ is a p-group and by [2], Proposition 2.2, $\bar{b}_1$ and $\bar{b}_2$ ($\bar{u}$ and $\bar{v}$) are p-disjoint in G. Further we have

$$\bar{x} = \bar{b}_1 - \bar{b}_2 = \bar{u} - \bar{v} ,$$

hence if we apply (B) (§ 3) to the p-group $\bar{G}$ it follows that there exists $\bar{m} \in H(\bar{u}, \bar{v})$ fulfilling

$$\bar{b}_1 = \bar{u} + \bar{m} , \quad \bar{b}_2 = \bar{v} + \bar{m} .$$

Again, by Proposition 2.2 of [2], there is $m_1 \in \bar{m}$ such that $m_1 \in H(u, v)$. Thus according to (B) the elements $u_1 = u + m_1$ and $v_1 = v + m_1$ are p-disjoint in G and $x = u_1 - v_1$. Since

$$u_1 \in \bar{u}_1 = \bar{u} + \bar{m}_1 = \bar{u} + \bar{m} = \bar{b}_1 = b_1 + A \subset A + B = X$$

and analogously $v_1 \in X$, the set X is a p-subgroup of G.

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ON UNIFORM CONVERGENCE FOR WALSH-FOURIER SERIES

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In 1940 R. Salem formulated a sufficient condition for a continuous and periodic function to have a trigonometric Fourier series which converges uniformly to the function. In this paper we will formulate a similar condition, which implies that the Walsh-Fourier series of such a function has this property. Furthermore we show that our result is stronger than certain classical results, and that it also implies the uniform convergence of the Walsh-Fourier series of certain classes of continuous functions of generalized bounded variation. The latter is analogous to results obtained by L. C. Young and R. Salem for trigonometric Fourier series.

Let $\{\varphi_n(x)\}$ be the sequence of Rademacher functions, i.e.,

$$\varphi_0(x) = +1 \left(0 \leqq x < \frac{1}{2}\right), \qquad \varphi_0(x) = -1 \left(\frac{1}{2} \leqq x < 1\right),$$
$$\varphi_0(x+1) = \varphi_0(x) .$$

$\varphi_n(x) = \varphi_0(2^n x)$, $(n = 1, 2, 3, \cdots)$. In [3] R. E. A. C. Paley gave the following definition for the Walsh functions $\{\psi_n(x)\}$: $\psi_0(x) \equiv 1$, and, if $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, with $n_1 > n_2 > \cdots > n_r$, then $\psi_n(x) = \varphi_{n_1}(x)\varphi_{n_2}(x) \cdots \varphi_{n_r}(x)$. J. L. Walsh [6] proved that the system $\{\psi_n(x)\}$ is a complete orthonormal system. For every Lebesgue-integrable function $f(x)$ of period 1 there is a corresponding Walsh-Fourier series (WFS):

$$f(x) \sim \sum_{k=0}^{\infty} c_k \psi_k(x), \quad \text{with} \quad c_k = \int_0^1 f(t)\psi_k(t)dt .$$

As in the case of trigonometric Fourier series (TFS), we can find a simple expression for the partial sums of a WFS,

$$S_n(f, x) = \sum_{k=0}^{n-1} c_k \psi_k(x) = \int_0^1 f(x \dotplus t) D_n(t)dt ,$$

where $D_n(t) = \sum_{k=0}^{n-1} \psi_k(t)$. For the meaning of $\dotplus$ and for further notations, definitions and properties of the WFS we refer to [2].

2. In [4], Chapter VI, R. Salem proved the following theorem: Let $f(x)$ be a continuous function of period 2π. For odd n, let

$$T_n(x) = \sum_{p=0}^{(n-1)/2} (p+1)^{-1}[f(x + 2p\pi/n) - f(x + (2p+1)\pi/n)]$$

and let $Q_n(x)$ be obtained from $T_n(x)$ by changing π into $-\pi$. Then, if $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} Q_n(x) = 0$ uniformly in x, the TFS of $f(x)$ converges uniformly to $f(x)$. R. Salem also showed that this theorem implies both the Dini-Lipschitz test for continuous functions with modulus of continuity $\omega(f, \delta) = o(\log \delta^{-1})^{-1}$ as $\delta \to 0$, and Jordan's theorem on continuous functions of bounded variation. Finally, he extended this last theorem to certain classes of continuous functions of generalized bounded variation. For a proof of Salem's results, see also [1], Chapter IV, § 5.

3. Our main result about WFS can be stated as follows:

THEOREM. *Let $f(x)$ be a continuous function of period* 1. *Let*

$$U_n(x) = \sum_{p=1}^{2^n-1} p^{-1} \,|\, f(x + 2p/2^{n+1}) - f(x + (2p+1)/2^{n+1}) \,| \,.$$

Then, $\lim_{n\to\infty} U_n(x) = 0$ *uniformly in x implies that* $\lim_{k\to\infty} S_k(f, x) = f(x)$ *uniformly in x.*

Proof. For each natural number k we have

$$S_k(f, x) - f(x) = \int_0^1 D_k(t)[f(x+t) - f(x)]dt \,.$$

Let $k = 2^n + k'$, with $0 \leqq k' < 2^n$, then, according to [2], p. 386, we have $D_k(t) = D_{2^n}(t) + \psi_{2^n}(t) \cdot D_{k'}(t)$, where

$$D_{2^n}(t) = \begin{cases} 2^n & \text{on } [0, 2^{-n}) \\ 0 & \text{on } [2^{-n}, 1) \end{cases}, \quad D_{k'}(t) = k' \text{ on } [0, 2^{-n}) \,,$$

and

$$\psi_{2^n}(t) = \begin{cases} +1 & \text{on } [2p/2^{n+1}, (2p+1)/2^{n+1}) \\ -1 & \text{on } [(2p+1)/2^{n+1}, (2p+2)/2^{n+1}) \end{cases} \quad \text{for } p = 0, 1, \cdots, 2^n - 1$$

Therefore,

$$\begin{aligned} |\, S_k(f, x) - f(x) \,| \leqq & \left| \int_0^1 D_{2^n}(t) \,[f(x+t) - f(x)]dt \right| \\ & + \left| \int_0^1 \psi_{2^n}(t) D_{k'}(t)[f(x+t) - f(x)]dt \right| = A + B \,. \end{aligned}$$

For the first term of this sum we have

$$A \leqq 2^n \int_0^{2^{-n}} |f(x \dotplus t) - f(x)|\, dt \leqq \omega(f, 2^{-n}) .$$

For the second term we have

$$\begin{aligned} B &= \left| \sum_{p=0}^{2^n-1} \left(\int_{2p/2^{n+1}}^{(2p+1)/2^{n+1}} D_{k'}(t)[f(x \dotplus t) - f(x)]dt \right.\right. \\ &\quad \left.\left. - \int_{(2p+1)/2^{n+1}}^{(2p+2)/2^{n+1}} D_{k'}(t)[f(x \dotplus t) - f(x)]dt \right) \right| \\ &= \left| \sum_{p=0}^{2^n-1} \int_{2p/2^{n+1}}^{(2p+1)/2^{n+1}} (D_{k'}(t)[f(x \dotplus t) - f(x)] \right. \\ &\quad \left. - D_{k'}(t + 2^{-n-1})[f(x \dotplus (t + 2^{-n-1})) - f(x)])dt \right| . \end{aligned}$$

Now we observe that, since $k' < 2^n$, $D_{k'}(t)$ is a sum of functions $\psi_i(t)$ with $i < 2^n$. Each of these functions is constant on the intervals $[k/2^n, (k+1)/2^n)$, $(k = 0, 1, \cdots, 2^n - 1)$. Therefore, if $t \in [2p/2^{n+1}, (2p+1)/2^{n+1})$, then $D_{k'}(t) = D_{k'}(t + 2^{-n-1}) = D_{k'}(2p/2^{n+1})$. Thus we have

$$\begin{aligned} B &= \left| \sum_{p=0}^{2^n-1} \int_{2p/2^{n+1}}^{(2p+1)/2^{n+1}} D_{k'}(p/2^n)[f(x \dotplus t) - f(x \dotplus (t + 2^{-n-1}))]dt \right| \\ &= \left| \sum_{p=0}^{2^n-1} \int_0^{2^{-n-1}} D_{k'}(p/2^n)[f(x \dotplus (t + 2p/2^{n+1})) \right. \\ &\quad \left. - f(x \dotplus (t + (2p+1)/2^{n+1}))]dt \right| \\ &= \left| \sum_{p=0}^{2^n-1} 2^{-n-1} \int_0^1 D_{k'}(p/2^n)[f(x \dotplus (t + 2p)/2^{n+1}) \right. \\ &\quad \left. - f(x \dotplus (t + 2p + 1)/2^{n+1})]dt \right| \\ &\leqq \left| 2^{-n-1} \int_0^1 D_{k'}(0)[f(x \dotplus t/2^{n+1}) - f(x \dotplus (t+1)/2^{n+1})]dt \right| \\ &\quad + \left| \sum_{p=1}^{2^n-1} 2^{-n-1} \int_0^1 \cdots\, dt \right| \\ &\leqq 2^{-n-1} \cdot k' \cdot \omega(f, 2^{-n-1}) + \left| \int_0^1 \left(2^{-n-1} \sum_{p=1}^{2^n-1} \cdots \right) dt \right| = B_1 + B_2 . \end{aligned}$$

Using the fact that for $u \in (0, 1)$, $|D_k(u)| < 2u^{-1}$, [2], Lemma 1, we obtain the following inequality for the integrand, I, of B_2:

$$\begin{aligned} |I| \leqq \sum_{p=1}^{2^n-1} 2^{-n-1} \cdot 2^{n+1} \cdot p^{-1} | f(x \dotplus (t + 2p)/2^{n+1}) \\ - f(x \dotplus (t + 2p + 1)/2^{n+1}) | . \end{aligned}$$

Now we observe that for every $t \in [0, 1)$ there is an $\tilde{x} \in [0, 1)$, $\tilde{x} = \tilde{x}(t)$, such that $x \dotplus (t + q)/2^{n+1} = \tilde{x} \dotplus q/2^{n+1}$ for all $q = 1, 2, \cdots, 2^{n+1} - 1$. Therefore

$$|I| \leqq \sum_{p=1}^{2^n-1} p^{-1} |f(\tilde{x} \dotplus 2p/2^{n+1}) - f(\tilde{x} \dotplus (2p+1)/2^{n+1})| = U_n(\tilde{x}) .$$

Under the hypothesis of our theorem $U_n(\tilde{x}) \to 0$ uniformly in $\tilde{x}$ as $n \to \infty$. This implies that $B_2 \to 0$ uniformly in x as $n \to \infty$, and so, $\lim_{k\to\infty} (S_k(f, x) - f(x)) = 0$ uniformly in x.

4. In this section we will show that our main theorem implies two classical results for WFS. The first is the Dini-Lipschitz test for WFS, which was first proved in [2], Th. XIII. A generalization of it can be found in [5], § (3.5).

COROLLARY 1. *Let $f(x)$ be a continuous function of period* 1 *and let $\omega(f, \delta) = o(\log \delta^{-1})^{-1}$ as $\delta \to 0$. Then the* WFS *of $f(x)$ converges uniformly to $f(x)$.*

Proof. We see immediately that

$$| U_n(x) | \leqq \sum_{p=1}^{2^n-1} p^{-1}\omega(f, 2^{-n-1}) \leqq \omega(f, 2^{-n-1})C \log 2^n$$

for some constant C. Thus $\lim_{n\to\infty} U_n(x) = 0$ uniformly in x.

The next corollary is Jordan's test for WFS, which was first proved in [6], Th. IV.

COROLLARY 2. *Let $f(x)$ be a continuous function of period* 1. *If $f(x)$ is of bounded variation on* [0, 1], *then its* WFS *converges uniformly to $f(x)$.*

Proof. We can find a nondecreasing sequence of natural numbers $\{m(n)\}$ such that (a) $m(n) < 2^n - 1$ for all n, (b) $m(n) \to \infty$ as $n \to \infty$, (c) $\omega(f, 2^{-n-1}) \log m(n) \to 0$ as $n \to \infty$. Then,

$$\begin{aligned} U_n(x) | &\leqq \omega(f, 2^{-n-1})\left[1 + \frac{1}{2} + \cdots + \frac{1}{m(n)}\right] \\ &\quad + \sum_{p=m(n)+1}^{2^n-1} p^{-1} |f(x \dotplus 2p/2^{n+1}) - f(x \dotplus (2p+1)/2^{n+1})| \\ &\leqq C\omega(f, 2^{-n-1}) \log m(n) + (m(n)+1)^{-1} \operatorname{Var}(f) . \end{aligned}$$

Thus $\lim_{n\to\infty} U_n(x) = 0$ uniformly in x.

Finally we will prove a theorem for WFS analogous to certain results of L. C. Young [7] and R. Salem [4] for TFS, and which is an extension of Jordan's theorem. First we will give a definition of bounded Φ-variation.

Let $\varphi(u)$ be a continuous, strictly increasing function defined for $u \geqq 0$, such that $\varphi(0) = 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$. Let ψ be the inverse of φ. Next, let $\Phi(u) = \int_0^u \varphi(t)dt$ and $\Psi(u) = \int_0^u \psi(t)dt$. Functions so obtained, are called complementary in the sense of W. H. Young, and they satisfy the following inequality, due to W. H. Young: if $a, b \geqq 0$, then $ab \leqq \Phi(a) + \Psi(b)$, see [8], *p.* 16.

DEFINITION. A function $f(x)$ on $[0, 1)$ is said to be of bounded Φ-variation if there is an $M < \infty$ such that for each finite partition $0 \leqq x_1 < x_2 \cdots < x_n \leqq 1$ we have $\sum_{i=1}^{n-1} \Phi(|f(x_{i+1}) - f(x_i)|) < M$.

We can prove the following

COROLLARY 3. *Let $\Phi(x)$ and $\Psi(x)$ be functions complementary in the sense of W. H. Young and let $\sum_{k=1}^{\infty} \Psi(k^{-1}) < \infty$. Let $f(x)$ be a continuous function of period* 1 *and of bounded Φ-variation. Then $\lim_{n\to\infty} S_n(f, x) = f(x)$ uniformly in x.*

Proof. Since $\sum_{k=1}^{\infty} \Psi(k^{-1}) < \infty$, we can find a sequence $\{\varepsilon(k)\}$ of positive numbers, decreasing to 0 as $k \to \infty$, and for which

$$\sum_{k=1}^{\infty} \Psi(k\varepsilon(k))^{-1}) < \infty .$$

Let

$$|f(x + 2p/2^{n+1}) - f(x + (2p + 1)/2^{n+1})| = \Delta_p .$$

Then, according to Young's inequality, we have

$$\Delta_p \cdot (p\varepsilon(p))^{-1} \leqq \Phi(\Delta_p) + \Psi((p\varepsilon(p))^{-1}) .$$

From our hypothesis it follows that there is a constant $N < \infty$ such that for each m

$$\sum_{p=m}^{2^n-1} \Delta_p(p\varepsilon(p))^{-1} \leqq \sum_{p=m}^{2^n-1} \Phi(\Delta_p) + \sum_{p=m}^{2^n-1} \Psi((p\varepsilon(p))^{-1}) < N .$$

Therefore,

$$\sum_{p=m}^{2^n-1} \Delta_p p^{-1} < N\varepsilon(m) .$$

Choosing $\{m(n)\}$ as in the proof of Corollary 2, we have

$$|U_n(x)| \leqq \omega(f, 2^{-n-1})\left[1 + \frac{1}{2} + \cdots + \frac{1}{m(n)}\right] + N\varepsilon(m(n) + 1) ,$$

i.e., $U_n(x) \to 0$ uniformly in x as $n \to \infty$.

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ON CERTAIN TOEPLITZ OPERATORS IN TWO VARIABLES

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The problem of inverting and/or factoring Weiner-Hopf operators in two variables is one of the basic unsolved problems in classical analysis. In this paper we shall consider operators which are a perturbation of a product of operators in one variable, the perturbation differing from such simple operators by an operator in one variable. The principal tools used are the spectral mapping theorem combined with the known results on operators in one variable.

2. **Preliminaries.** Consider the space l_2 of sequences of complex numbers

$$\xi = \{\xi_{ij}\}_{i,j}^{\infty} c_{ij} = 0$$

with

$$\|\xi\| = \Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\xi_{kj}|^2\Big)^{1/2} < \infty \ . \tag{2.1}$$

Let

$$a = \{a_j\}_{j=-\infty}^{\infty} \qquad b = \{b_j\}_{j=-\infty}^{\infty} \qquad c = \{c_j\}_{j=-\infty}^{\infty} \tag{2.2}$$

be absolutely convergent sequences of complex numbers. Define

$$\begin{aligned} g &= \{g_{ij}\}_{i,j=-\infty}^{\infty} \\ g_{ij} &= a_i b_j + c_j\delta(j) \end{aligned} \tag{2.3}$$

for

$$\delta(j) = 0 \text{ if } j \neq 0 \ , \quad \delta(0) = 1 \ .$$

It is clear that

$$\sum_{j=-\infty}^{\infty}\ \sum_{i=-\infty}^{\infty} |g_{ij}| < \infty$$

We are concerned with the operation

$$T_g \colon l_2 \to l_2 \ ,$$

$$\begin{aligned} (T_g\xi)_{ij} &= \sum_{L=0}^{\infty}\ \sum_{k=0}^{\infty} g_{j-k,j-L}\xi_{kL} \\ [g_{ij} &= g_{i,j}] \ . \end{aligned} \tag{2.4}$$

Our techniques and results are exactly the same in the two variable Weiner-Hopf integral analogue.

Define

$$G(e^{i\theta}, e^{i\varphi}) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_{kj} e^{-ij\varphi - ik\theta}$$

$$A(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\theta}$$

(2.5)
$$B(e^{i\varphi}) = \sum_{k=-\infty}^{\infty} b_k e^{-ik\varphi}$$

$$C(e^{i\varphi}) = \sum_{j=-\infty}^{\infty} c_j e^{-ij\varphi} .$$

So

(2.6)
$$G(e^{i\theta}, e^{i\varphi}) = A(e^{i\theta})B(e^{i\varphi}) + C(e^{i\varphi}) .$$

Let $\hat{l}_2$ denote the space of doubly infinite sequences in the second subscript, singly infinite in the first.

$$\xi \in \tilde{l}_2 , \qquad \xi = \{\xi_{ij}\}_{j=-\infty, i=0}^{\infty}$$

(2.7)
$$\| \xi \|_{\sim} = \Big(\sum_{j=-\infty}^{\infty} \sum_{i=0}^{\infty} | \xi_{ij} |^2 \Big)^{1/2} .$$

Let us Fourier transform l_2 and $\tilde{l}_2$ with respect to the second subscript, i.e.,

(2.8)
$$\hat{\xi} = \{\xi_k(e^{i\varphi})\}_{k=0}^{\infty} \in E_2$$

where

$$\xi_k(e^{i\varphi}) = \sum_{j=0}^{\infty} \xi_{kj} e^{ij\varphi} .$$

We shall obtain on E_2 a transformed Toeplitz operator whose elements are themselves Toeplitz operators

Define

(2.9)
$$P_+^{(\varphi)} \sum_{j=-\infty}^{\infty} \xi_{kj} e^{ij\varphi} = \sum_{j=0}^{\infty} \xi_{kj} e^{ij\varphi} ,$$

and on the space of singly semi-infinite sequences, define the Toeplitz map

(2.10)
$$(A\eta)_i = \sum_{j=0}^{\infty} a_{i-j} \eta_j .$$

Thus, the total operator transforms to the compound operator

(2.11)
$$P_+^{(\varphi)} B(e^{i\varphi}) A \hat{\xi} + P_+^{(\varphi)} C(e^{i\varphi}) \hat{\xi} = P_+^{(\varphi)} L(e^{i\varphi}) \hat{\xi} .$$

3. Main results.

MAIN THEOREM. (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C)

(A) (1) $G(e^{i\theta}, e^{i\varphi}) \neq 0$ real θ, φ.

(2) The change in argument of $G(e^{i\theta}, e^{i\varphi})$ as θ goes from 0 to 2π is 0 for any real φ.

(3) The change in argument of $G(e^{i\theta}, e^{i\varphi})$ as φ goes from 0 to 2π is 0 for any real θ.

(B) $L(e^{i\varphi})$ can be factored

$L(e^{i\varphi}) = L_-(e^{i\varphi})L_+(e^{i\varphi})$ for $0 \leqq \varphi \leqq 2\pi$ where

$L_-(e^{i\varphi})$, $L_+(e^{i\varphi})$ commute and are continuous in φ and bounded for each φ. Moreover $L_-(e^{i\varphi})$ has an analytic operator valued extension to $|z| > 1$ which is invertible for these z, $L_+(e^{i\varphi})$ has an analytic operator valued extension to $|z| < 1$ invertible for these z. This factorization is unique if $L_-(\infty) = I$.

(C) T_g is invertible.

Proof. Assume (A). Consider

$$\mu B(e^{i\varphi}) + C(e^{i\varphi}) \tag{3.1}$$

where first $\mu = A(e^{i\theta_0})$ for some real θ_0. Conditions (1) and (3) and the results of [1] guarantee that a factorization

$$\mu B(e^{i\varphi}) + C(e^{i\varphi}) = D_-(\mu, e^{i\varphi})D_+(\mu, e^{i\varphi}) \tag{3.2}$$

exists for each such μ where the factors D_- and D_+ have the same properties as functions as $L_-(e^{i\varphi})$ and $L_+(e^{i\varphi})$ have as operators. Since property 3 is a homotopic invariant, such a factorization fails to exist for some μ in the spectrum of A if and only if $\exists \mu_0$ in the spectrum of A and some real φ_0 with

$$\mu_0 B(e^{i\varphi_0}) + C(e^{i\varphi_0}) = 0\,. \tag{3.3}$$

If $B(e^{i\varphi_0}) \neq 0$, then $C(e^{i\varphi_0}) \neq 0$ by condition (1). Thus,

$$\mu_0 = -\frac{C(e^{i\varphi_0})}{B(e^{i\varphi_0})}\,. \tag{3.4}$$

But by condition (2) the change in argument as θ goes from 0 to 2π of $[a(e^{i\theta}) - \mu_0] = 0$, thus μ_0 does not belong to the spectrum of A.

Thus the factorization (3.4) exists for all μ in the spectrum of A. It is clear from the construction involved in [1] that the factors are locally analytic in μ for μ in the spectrum of A.

We normalize so that $D_-(\mu, \infty) = 1$. Then the equation (for any μ in the spectrum of A)

$$(3.5) \qquad P_+^{(\varphi)}[\mu B(e^{i\varphi}) + C(e^{i\varphi})]h_\mu(e^{i\varphi}) = 1 ,$$

on the space of Fourier transforms of semi-infinite sequences with one subscript, has the unique solution

$$h_\mu(e^{i\varphi}) = D_+^{-1}(\mu, e^{i\varphi}) .$$

Single-valuedness of the factors is now immediate. Thus each factor is analytic separately in μ on the spectrum of A. Moreover, for such μ, $D_-(\mu, e^{i\varphi})$ has an analytic extension to $|z| > 1$, invertible for $|z| \geqq 1$. Thus, the operator $D_-(A, e^{i\varphi})$ has the same properties, by the spectral mapping theorem. We may make the analogous statements about $D_+(\mu, e^{i\varphi})$.

Thus, by the spectral mapping theorem we may replace μ by A in $D_-(\mu, e^{i\varphi})$, $D_+(\mu, e^{i\varphi})$ and obtain $L_-(e^{i\varphi})$, $L_+(e^{i\varphi})$ with all the appropriate properties of analyticity in z and invertibility.

Next, suppose

$$(3.6) \qquad M_-(e^{i\varphi})M_+(e^{i\varphi}) = L_-(e^{i\varphi})L_+(e^{i\varphi}),$$

M_-, M_+ having the same properties, then

$$(3.7) \qquad L_-^{-1}(e^{i\varphi})M_-(e^{i\varphi}) = L_+(e^{i\varphi})M_+^{-1}(e^{i\varphi})$$

and hence they are both analytic in the whole plane and equal to the identity at ∞, or

$$(3.8) \qquad L_-(e^{i\varphi}) = M_-(e^{i\varphi}) , \qquad L_+(e^{i\varphi}) = M_+(e^{i\varphi})$$

or the factorization is unique.

Thus $A \Rightarrow B$.

Now, let us assume B. We wish to solve:

$$(3.9) \qquad P_+^{(\varphi)}L(e^{i\varphi})\hat{\xi} = \hat{\eta} .$$

Consider the operator $P_+^{(\varphi)}L_-^{-1}(e^{i\varphi})$.

By the isometry of the Fourier transform

$$(3.10) \qquad \| P_+^{(\varphi)}L_-^{-1}(e^{i\varphi}) \| \leqq \sup_{0 \leqq \varphi \leqq 2\pi} \||\, L_-^{-1}(e^{i\varphi}) \,|\|$$

where $\||\ \ |\|$ denotes operator norm on the space of semi infinite sequences with one subscript. Similarly

$$(3.11) \qquad \| L_+^{-1}(e^{i\varphi}) \| \leqq \sup_{0 \leqq \varphi \leqq 2\pi} \||\, L_+^{-1}(e^{i\varphi}) \,|\| .$$

Let

$$\hat{\xi} = L_+^{-1}(e^{i\varphi})P_+^{(\varphi)}L_-^{-1}(e^{i\varphi})\hat{\eta} .$$

Then

$$P_+L(e^{i\varphi})\hat{\xi} = \hat{\eta} + P_+^{(\varphi)}L_-(e^{i\varphi})(P_+^{(\varphi)}-I)L^{-1}(e^{i\varphi})\hat{\eta} ,$$

but by the anti-analyticity of $L_-(e^{i\varphi})$ and the definition of $P+^{(\varphi)}$ we have

$$P_+^{(\varphi)}L(e^{i\varphi})\hat{\xi} = \hat{\eta} .$$

Thus T_g has a right inverse. This right inverse is easily shown to be a left inverse using the anti-analyticity of $L_-^{-1}(e^{i\varphi})$.

Next we assume C. Suppose $G(e^{i\theta_0}, e^{i\varphi_0}) = 0$.

Then, if T_g is invertible, so is $T_{g_M} + \delta I$, for all M large enough and all $|\delta|$ small enough and:

$$G_M(e^{i\theta}, e^{i\varphi}) = \sum_{j=-M}^{M} a_j e^{-ij\theta} \sum_{k=-M}^{M} b_k e^{-ik\varphi} + \sum_{k=-M}^{M} b_k e^{-ik\varphi} \tag{3.12}$$

Moreover, we may choose M_0, φ_1, θ_1, δ_0 such that $|\delta_0| < \delta$ and $M_0 \geqq M$ and

$$G_{M_0}(e^{i\theta_1}, e^{i\varphi_1}) + \delta_0 = 0 . \tag{3.13}$$

Next consider the sequence of vectors ξ^N, where

$$\begin{aligned} &\xi_{jk}^N = \frac{1}{\sqrt{N+1}} e^{i(j\theta_1 + k\varphi_1)} \text{ if } 0 \leqq j,\ k \leqq N \\ &\xi_{jk}^N = 0 \text{ otherwise} . \end{aligned} \tag{3.14}$$

Clearly

$$\|\xi^N\| = 1 \text{ while } \lim_{N\to\infty} (T_{g_{M_0}} + \delta_0 I)\xi^N = 0 .$$

Contradiction.

Now suppose the change in argument in condition 2 is $2\pi\eta_\theta \neq 0$. (This number is obviously independent of φ). Thus, for M large, $G_M(e^{i\theta}, e^{i\varphi})$ has the same η_θ for each φ_0. If $\eta_\theta < 0$, then $L_M(e^{i\varphi_0})$ annihilates some vector

$$K = \{k_0, k_1, \cdots\} , \qquad |||K||| = 1 .$$

But then the sequence

$$\begin{aligned} &\xi_{kj}^N = \frac{1}{\sqrt{N+1}} k_i e^{ij\varphi_0} \text{ if } 0 \leqq j \leqq N , \\ &\xi_{ij}^N = 0 \text{ if } N < j, \text{ has the property} \end{aligned} \tag{3.15}$$

$$\|\xi^N\| = 1, \text{ but } \lim_{N\to\infty} T_{g_M}\xi^N = 0 \,. \tag{3.16}$$

Thus T_{g_M} is not invertible, hence neither is T_g. If $\eta_\theta > 0$, we merely consider T_g^*. Finally, we assume that the change in argument is $2\pi\ \eta_\varphi \neq 0$ in condition (3). Then consider

$$e^{-i\eta_\varphi\varphi}G(e^{i\theta}, e^{i\varphi}) = H(e^{i\theta}, e^{i\varphi}) \,.$$

This function obeys the conditions of (A), hence it is factorable and T_h is invertible. However, if $\eta_\varphi > 0$, then

$$S^{*\eta_\varphi} = S_\varphi^{*\eta_\varphi} T_g(T_g)^{-1} = T_h(T_g)^{-1} \tag{3.17}$$

where S_φ is the right shift operator on the j subscript. This is impossible since the two operators on the right are invertible. If $\eta_\varphi < 0$, we merely consider the adjoint.

Thus (C) $\Rightarrow$ (A), and we are finished.

4. Example. Let $B(e^{i\varphi})$ and $C(e^{i\varphi})$ have finite expansions

$$\begin{aligned} B &= \sum_{j=-N}^{M} b_j e^{-ij\varphi} \\ C &= \sum_{j=-N}^{M} c_j e^{-ij\varphi} \end{aligned} \tag{4.1}$$

and suppose

$$\mu b_{-N} + c_{-N} \neq 0 \quad \text{for} \quad \mu \text{ in the spectrum of } A \,. \tag{4.2}$$

Assume conditions 1, 2, 3. Then we may factor

$$\mu B(z) + C(z) = (\mu b_{-N} + c_{-N}) \prod_{i=1}^{N} (z - x_i(\mu)) \prod_{j=1}^{M} \left(1 - \frac{y_j^{(\mu)}}{z}\right) \tag{4.3}$$

for all μ in the spectrum of A and each $|x_i(\mu)| > 1$, $|y_j(\mu)| < 1$. See [3]. Then it follows that

$$L(z) = L_-(z)L_+(z) \tag{4.4}$$

with

$$L_-(z) = \prod_{j=1}^{M} \left(1 - \frac{y_j^M(A)}{z}\right) \tag{4.5}$$

$$L_+(z) = (Ab_{-N} + c_{-N}) \prod_{i=1}^{N} (z - x_i(A)) \,. \tag{4.6}$$

We expect this factorization to play an important role in the study of difference equations arising from hyperbolic systems in regions in space having corners.

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ON THE MEASURABILITY OF PERRON INTEGRABLE FUNCTIONS

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By means of majorants and minorants a Perron-like integral can be defined in an arbitrary topological space. Although for its definition only a finitely additive set function is used, it turns out that if the underlying topological space is Hausdorff and locally compact, then the integral itself gives rise to a regular measure. The natural question, whether every integrable function is measurable with respect to this measure, is the subject of our paper.

In § 2 some sufficient conditions for measurability of integrable functions are given and the connection of our measure with the original set function is described. The results of this section are then applied to integration with respect to the natural and monotone convergences. The natural convergence, which can be used in any topological space is discussed in § 3. In § 4 some elementary properties of the monotone convergence are derived. This convergence can be used in any locally pseudo-metrizable space and it seems to be the most important convergence for the definition of an integral over a differentiable manifold. A proof that for the monotone convergence every integrable function is measurable is given in § 5. Finally, § 6 contains a few illustrative examples.

Throughout, P is a topological space which is always assumed to be Hausdorff and locally compact. The reader can, however, easily detect those parts of the paper which remain correct in an arbitrary topological space P. By $P^{\sim} = P \cup (\infty)$ we denote a one point compactification of P. If $A \subset P$, A^- and $A^{\sim}$ stand for closure of A in P and $P^{\sim}$, respectively. The interior of a set $A \subset P$ is denoted by A^0. For $x \in P^{\sim}$, Γ_x is a local base at x in $P^{\sim}$ (see [3], p. 50). We shall always assume that $U \subset P$ and U^- is compact for all $U \in \Gamma_x$ with $x \in P$. If σ is a pre-algebra of subsets of P (see [5], 1.1) such that $\{U \cap P : U \in \Gamma_x\} \subset \sigma$ for every $x \in P^{\sim}$, we call the pair $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ a *net structure* in P.

If $\delta \subset \sigma$ and $A \subset P^{\sim}$ we let $\delta_A = \{B \in \delta : B \subset A\}$. A system $\delta \subset \sigma$ is said to be *semihereditary* if and only if $\sigma_0 \cap \delta \neq \emptyset$ for every finite disjoint collection $\sigma_0 \subset \sigma$ whose union belongs to δ. A system $\delta \subset \sigma$ is said to be *stable* if and only if $\emptyset \notin \delta$ and for every $A \in \delta$ and every $x \in P^{\sim}$ there is a $U \in \Gamma_x$ such that $\delta_{A-U} \neq \emptyset$.

A convergence[1)] in a net structure $\langle \sigma, \Gamma_x \rangle$ is a function κ which

1) What we call a convergence is sometimes called a *derivation basis* (see [**1**], 1.1).

to every $x \in P^{\sim}$ associates a family κ_x of nets $\{B_U, U \in \Gamma, \subset\} \subset \sigma$ where Γ is a cofinal subset of Γ_x. For $\delta \subset \sigma$ and $x \in P^{\sim}$, $\kappa_x(\delta) = \{\{B_U\} \in \kappa_x : \{B_U\} \subset \delta\}$ and $\delta^* = \{x \in P^{\sim} : \kappa_x(\delta) \neq \emptyset\}$.

A convergence κ is called *admissible* if and only if the following conditions are satisfied:

$\mathscr{K}_1$. For every $x \in P^{\sim}$, $\{U \cap P, U \in \Gamma_x, \subset\} \in \kappa_x$.

$\mathscr{K}_2$. If $x \in P^{\sim}$ and $\{B_U, U \in \Gamma, \subset\} \in \kappa_x$ then for every $V \in \Gamma_x$ there is a $U_V \in \Gamma$ such that $B_U \subset V$ for all $U \in \Gamma$ for which $U \subset U_V$.

$\mathscr{K}_3$. If $x \in P^{\sim}$, $\{B_U\}_{U \in \Gamma} \in \kappa_x$, and Γ' is a cofinal subset of Γ, then also $\{B_U\}_{U \in \Gamma'} \in \kappa_x$.

$\mathscr{K}_4$. If $x \in P^{\sim}$, $\{B_U\} \in \kappa_x$, and $A \in \sigma$, then also $\{B_U \cap A\} \in \kappa_x$.

$\mathscr{K}_5$. If $\delta \subset \sigma$ is a nonempty semihereditary system, then δ^* is nonempty.

$\mathscr{K}_6$. If $\delta \subset \sigma$ is a nonempty semihereditary stable system, then δ^* is uncountable.

A triple $\mathfrak{J} = \langle \mathfrak{N}, \kappa, G\rangle$ is called an *integration base* in P if and only if $\mathfrak{N} = \langle \sigma, \Gamma_x\rangle$ is a net structure in P, κ is an admissible convergence in $\mathfrak{N}$, and G is a nonnegative finitely additive function[2] on σ such that $G(A) < +\infty$ for every $A \in \sigma$ with A^- compact.

It was shown in [6] that integration bases exist in P and that for each of them we can define a nonabsolutely convergent integral I which is closely related to the Lebesgue integral. For the reader's convenience we shall summarize the basic definitions.

Let $x \in P^{\sim}$, $A \subset P$, and let F be a function on σ_A. We call the number ${}_{\#}F(x, A) = \inf\{\liminf F(B_\alpha) : \{B_\alpha\} \in \kappa_x(\sigma_A)\}$ the *lower limit* of F at x relative to A and the number ${}_{*}F(x, A) = {}_{\#}(F/G)(x, A)$[3] the *lower derivate* of F at x relative to A and it is denoted by $I(f, A)$.

Let $A \in \sigma$ and let f be a function on A^-. A superadditive function M on σ_A is said to be a *majorant* of f on A if and only if there is a countable set $Z_M \subset A^-$ such that ${}_{\#}(-G)(x, A) \geqq 0$ for all $x \in Z_M$, ${}_{\#}M(x, A) \geqq 0$ for all $x \in Z_M \cup (\infty)$, and $-\infty \neq {}_{*}M(x, A) \geqq f(x)$ for all $x \in A^- - Z_M$. The number $I_u(f, A) = \inf M(A)$ where the infimum is taken over all majorants of f on A is called the *upper integral* of f over A. If $I_u(f, A) = -I_u(-f, A) \neq \pm\infty$ this common value is called the *integral* of f over A.

If $A \in \sigma$ and f is a function on A^-, we denote by $\mathfrak{M}(f, A)$ the family of all majorants of f on A. The family of all functions integrable over $A \in \sigma$ is denoted by $\mathfrak{P}(A)$.

For $A \subset P$, χ_A denotes the characteristic function of A in P. By $\mathfrak{C}$ and $\mathfrak{U}$ we denote the families of all compact and open subsets of P, respectively. Using the integral I, we shall define measure spaces

2) Unless specified otherwise, by a function we always mean an extended real-valued function.

3) We let $a/0 = +\infty$ for $a \geqq 0$, $a/0 = -\infty$ for $a < 0$, and $a/(\pm\infty) = 0$.

$(P, \mathfrak{T}, \tau)$ and $(P, \mathfrak{T}_0, \tau_0)$ as follows:

(i) $\mathfrak{T}$ is the family of all sets $A \subset P$ such that $\chi_{A \cap C} \in \mathfrak{P}(P)$ for every $C \in \mathfrak{C}$; and for $A \in \mathfrak{T}$, $\tau(A) = I_u(\chi_A, P)$.

(ii) For $A \subset P$,

$$\tau_0(A) = \inf\{\tau(U) : U \in \mathfrak{U} \text{ and } A \subset U\}$$

and $\mathfrak{T}_0$ is the family of all τ_0-measurable subsets of P.

These measure spaces will play an essential part in our paper. Some of their important properties can be found in [7], §'s 3 and 4; e.g., there is a proof that they actually are measure spaces. We just recall here that $\mathfrak{U} \subset \mathfrak{T}_0 \subset \mathfrak{T}$ and that the measure τ_0 is regular.

2. **Measurability in general.** In this section we shall prove a few general theorems concerning the measurability of integrable functions. Throughout we shall assume that there is given an integration base $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa, G \rangle$ in P.

PROPOSITION 2.1. *If the lower derivate of every superadditive function on σ is $\mathfrak{T}_0$-measurable, then $\mathfrak{T} = \mathfrak{T}_0$ and every function from $\mathfrak{P}(P)$ is $\mathfrak{T}_0$-measurable.*

Proof. Let $A \in \mathfrak{T}$ with $A^- \in \mathfrak{C}$. Then by [7], 2.7 there are narrow majorants $M_n \in \mathfrak{M}^\wedge(\chi_A, P)$ (see [7], 2.5) for which $M_n(P) - I(\chi_A, P) < 1/n$, $n = 1, 2, \cdots$. If $B_n = \{x \in P : {}_*M_n(x, P) \geqq 1\}$ then by our assumption $B_n \in \mathfrak{T}_0$. Letting $B = A^- \cap (\bigcap_{n=1}^\infty B_n)$, we have $B \in \mathfrak{T}_0$, $B^- \in \mathfrak{C}$, and $A \subset B$. Because

$$\tau(B_n) - \tau(A) \leqq M_n(P) - I(\chi_A, P) < 1/n$$

for $n = 1, 2, \cdots$, it follows that $\tau(B - A) = 0$. Now replacing A by $B - A$ and repeating the previous construction, we obtain a set $C \in \mathfrak{T}_0$ for which $C^- \in \mathfrak{C}$, $B - A \subset C$, and

$$\tau(C) = \tau(B - A) + \tau(C - [B - A]) = 0\,.$$

By [7], 4.7 also $\tau_0(C) = 0$ and since τ_0 is a complete measure, $A = B - (B - A)$ belongs to $\mathfrak{T}_0$.

If $A \in \mathfrak{T}$ is arbitrary, then $(A \cap C)^- \in \mathfrak{C}$ for every $C \in \mathfrak{C}$. Thus $A \cap C \in \mathfrak{T}_0$ for every $C \in \mathfrak{C}$ and it follows from [7], 4.7 that $A \in \mathfrak{T}_0$.

The last part of the proposition is now a direct consequence of [7], 4.3.

The previous proposition and Proposition 4.3 in [7] indicate the importance of the following:

PROPOSITION 2.2. *Let M be a function on σ_A where $A \in \sigma$, and* *let c be a real number. If* $\sigma_A(M, c) = \{B \in \sigma_A : M(B)/G(B) < c\}$, *then* $\bigcap_{n=1}^{\infty} \sigma_A^*(M, c + 1/n) = \{x \in A^\sim : {}_*M(x, A) \leqq c\}$.

Proof. If ${}_*M(x) \leqq c$ and n is a positive integer, then there is a net $\{B_U\}_{U \in \Gamma} \in \kappa_x(\sigma_A)$ such that

$$\liminf \, [M(B_U)/G(B_U)] < c + 1/n \, .$$

Hence there is a cofinal subset Γ' of Γ such that $\{B_U\}_{U \in \Gamma'} \subset \sigma_A(M, c + 1/n)$. It follows from $\mathscr{K}_3$ that $x \in \sigma_A^*(M, c + 1/n)$. On the other hand if $x \in \sigma_A^*(M, c + 1/n)$ then it follows that ${}_*M(x) \leqq c + 1/n$, $n = 1, 2, \cdots$.

DEFINITION 2.3. An integration base $\mathfrak{J}$ in P is said to be *measurable* if and only if every function from $\mathfrak{P}(P)$ is $\mathfrak{T}_0$-measurable. It is said to be *strongly measurable* if and only if $\delta^* - (\infty)$ belongs to $\mathfrak{T}_0$ for every $\delta \subset \sigma$.

It follows at once from 2.1 and 2.2 that every strongly measurable integration base is measurable. On the other hand, Example 6.1 shows that a measurable integration base need not be strongly measurable. From [7], 4.7 we see that if $\mathfrak{J}$ is measurable, then $\mathfrak{T} = \mathfrak{T}_0$.

REMARK 2.4. Let c be a real number and let $A \in \sigma$. It is easy to see that $\sigma_A(M, c)$ is semihereditary whenever M is a superadditive function on σ_A. Furthermore, it can be shown that if ${}_\#(-G)(x, A) \geqq 0$ for every $x \in A^\sim$, then $\sigma_A(M, c)$ is semihereditrary and stable whenever M is a majorant for some function on A^- (see [6], (4.4)). However, Example 6.1 indicates that there is no link between the semihereditariness or stability of $\delta \subset \sigma$ and the $\mathfrak{T}_0$-measurability of $\delta^* - (\infty)$.

THEOREM 2.5. *If* $C \in \mathfrak{C}$, *then*

$$\tau_0(C) = \inf \sum_{i=1}^{n} G(A_i)$$

where the infimum is taken over all finite families $\{A_i\}_{i=1}^n \subset \sigma$ *for which* $C \subset (\bigcup_{i=1}^n A_i)^0$.

Proof. Let $C \in \mathfrak{C}$ and let $A_1, \cdots, A_n$ be sets from σ for which $C \subset (\bigcup_{i=1}^n A_i)^0$. If we set $M(B) = \sum_{i=1}^n G(B \cap A_i)$ for $B \in \sigma$, then $M \in \mathfrak{M}(\chi_C, P)$ and so

$$\tau_0(C) = \tau(C) \leqq M(P) = \sum_{i=1}^{n} G(A_i) \, .$$

On the other hand, given $C \in \mathfrak{C}$ and $\varepsilon > 0$, there is a $U \in \mathfrak{U}$ such that

$C \subset U$, $U^{-} \in \mathfrak{C}$, and $\tau_0(U) < \tau_0(C) + \varepsilon$. Using [5], (1.1), we can find disjoint sets $B_1, \cdots, B_m$ from σ for which

$$C \subset \left(\bigcup_{i=1}^{m} B_i\right)^0 \subset \bigcup_{i=1}^{m} B_i^{\sim} \subset U\,.$$

If we set $N(B) = -\sum_{i=1}^{m} G(B \cap B_i)$ for $B \in \sigma$, then $N \in \mathfrak{M}(-\chi_U, P)$ and thus

$$\begin{aligned}\tau_0(C) + \varepsilon > \tau_0(U) = \tau(U) &= -I(-\chi_U, P)\\ &\geqq -N(P) = \sum_{i=1}^{m} G(B_i)\,.\end{aligned}$$

Using the regularity of τ_0 (see [7], 4.7), we obtain the following:

COROLLARY 2.6. *Let $\langle \sigma, \Gamma_x, \kappa, G \rangle$ and $\langle \sigma', \Gamma'_x, \kappa', G' \rangle$ be two integration bases in P. If $\sigma \cap \sigma'$ is a pre-algebra which contains a topological base of P and if $G = G'$ on $\sigma \cap \sigma'$,* then $(\mathfrak{T}_0, \tau_0) = (\mathfrak{T}'_0, \tau'_0)$.

Proof. Suppose $\tau'_0(C) > \tau_0(C)$ for some $C \in \mathfrak{C}$. Then by [5], (1.1) there is a disjoint finite family $\{A_i\} \subset \sigma$ such that $C \subset (\cup A_i)^0$ and $\sum G(A_i) < \tau'(C)$. Since C is compact, using again [5], (1.1), we can find a disjoint finite family $\{B_j\} \subset \sigma \cap \sigma'$ such that $C \subset \cup B_j^0 \subset \cup B_j \subset (\cup A_i)^0$. Hence

$$\begin{aligned}\tau'_0(C) &\leqq \sum_j G'(B_j) = \sum_j G(B_j)\\ &= \sum_{i,j} G(A_i \cap B_j) \leqq \sum_i G(A_i)\end{aligned}$$

which is a contradiction. By symmetry $\tau_0(C) = \tau'_0(C)$ for every $C \in \mathfrak{C}$. Now the corollary follows from [7], 4.7.

The previous corollary is the main reason why we are discussing $\mathfrak{T}_0$-measurability rather than $\mathfrak{T}$-measurability.

Let $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa, G \rangle$ and $\mathfrak{J}' = \langle \sigma', \Gamma'_x, \kappa', G' \rangle$ be two integration bases in P. If $\sigma' \subset \sigma$, G' is the restriction of G to σ', and for every $x \in P^{\sim}$, $\Gamma'_x \subset \Gamma_x$ and $\kappa'_x \subset \kappa_x$, we say that $\mathfrak{J}$ is *larger* than $\mathfrak{J}'$ and write $\mathfrak{J}' \prec \mathfrak{J}$. Obviously, the relation $\prec$ is a partial ordering in the family of all integration bases in P. Imitating the proof of Theorem 31 in [4], one can easily see that it $\mathfrak{J}' \prec \mathfrak{J}$ then $\mathfrak{P}(A) \subset \mathfrak{P}'(A)$ for every $A \in \sigma'$; here $\mathfrak{P}(A)$ and $\mathfrak{P}'(A)$ are the families associated with $\mathfrak{J}$ and $\mathfrak{J}'$, respectively. From this and Corollary 2.6 it follows that if $\mathfrak{J}$ is a measurable base in P so is $\mathfrak{J}'$ for every $\mathfrak{J}' \succ \mathfrak{J}$.

The next difinition and proposition will be used in § 5.

DEFINITION 2.7. An integration base $\mathfrak{J}$ in P is said to be *locally* *strongly measurable* if and only if for every $x \in P$ there is an integration base $\mathfrak{J}'$ in P (generally depending on x) which satisfies the following conditions:

(i) $(\mathfrak{T}_0, \tau_0) = (\mathfrak{T}'_0, \tau'_0)$.

(ii) There is a neighborhood $U \in \sigma \cap \sigma'$ of x such that $\mathfrak{P}(U) \subset \mathfrak{P}'(U)$ and $\delta^{*\prime} - (\infty) \in \mathfrak{T}'_0$ whenever $\delta \subset \sigma'_U$.

PROPOSITION 2.8. *If $\mathfrak{J}$ is locally strongly measurable then $\mathfrak{J}$ is measurable.*

Proof. Let $f \in \mathfrak{P}(P)$. We shall show that every point $x \in P$ has a neighborhood V such that f restricted to V^- is $\mathfrak{T}_0$-measurable. It will follow that f restricted to any compact subset of P is $\mathfrak{T}_0$-measurable and hence by [7], 4.7 also f itself is $\mathfrak{T}_0$-measurable.

Choose $x \in P$ and let $\mathfrak{J}'$ and U have the same meaning as in Definition 2.7. Then by [6], 6.8, $f \subset \mathfrak{P}'(U)$ and we can choose majorants $M_n \in \mathfrak{M}'(f, U)$ such that $M_n(U) - I'(f, U) < 1/n$, $n = 1, 2, \cdots$. By the definition of majorant (see [6], 3.2) with each M_n there is associated a certain countable set $Z_{M_n} \subset U^-$. For $x \in P$ let $h_n(x) = {}_{*'}M_n(x, U)$ if $x \in U^- - Z_{M_n}$ and $h_n(x) = +\infty$ otherwise; here, of course, ${}_{*'}M_n(x, U)$ denotes the lower derivate computed in $\mathfrak{J}'$. By 2.2, the h_n are $\mathfrak{T}_0$-measurable, and so is $h = \inf h_n$. Set $r(x) = h(x) - f(x)$ if this difference has meaning and $r(x) = 0$ otherwise. Since $h \geqq f$, $r \geqq 0$ and

$$\begin{aligned} 0 \leqq I'_u(r, U) &\leqq \inf I'_u(h_n, U) - I'(f, U) \\ &\leqq \inf [M_n(U) - I'(f, U)] = 0 \end{aligned}$$

(see [6], 6.4). Now choose $V \in \Gamma'_x$ such that $V^- \subset U^0$ and let $r_1 = r\chi_{V^-}$ (we define $(\pm\infty) .0 = 0$). Since $0 \leqq r_1 \leqq r$, $I'_u(r_1, U) = 0$ and $r_1 \in \mathfrak{P}'(U)$. Exactly as before we can define a $\mathfrak{T}_0$-measurable function $g \geqq r_1$ such that if we set $s(x) = g(x) - r_1(x)$ whenever this difference has meaning and $s(x) = 0$ otherwise, then $I'(s, U) = 0$. Letting $g_1 = g\chi_{V^-}$ and $s_1 = s\chi_{V^-}$, we obtain $I'(g_1, P) = I'(g_1, U) = I'(s_1, U) + I'(r_1, U) = 0$; for $g_1 = 0$ on $(P - U)^-$ and $0 \leqq s_1 \leqq s$. Since g_1 is nonnegative, $\mathfrak{T}_0$-measurable, and has a compact support it follows from [7] 4.2 and 4.7 that $\int_P g_1 d\tau_0 = 0$. Because τ_0 is a complete measure and $g_1 \geqq r_1 \geqq 0$, also r_1 is $\mathfrak{T}_0$-measurable. Therefore f restricted to V^-, which is equal to $h + r_1$ restricted to V^-, is $\mathfrak{T}_0$-measurable too.

Let $\mathfrak{J}$ be measurable or strongly measurable and let $A \in \sigma$ be different from P. Then, in general, we do not know whether the functions from $\mathfrak{P}(A)$ are $\mathfrak{T}_0$-measurable over A^-. This fact, e.g., caused the main difficulty in proving Proposition 2.8. The following proposition is a contribution to this problem.

PROPOSITION 2.9. *Let* $\mathfrak{J}$ *be measurable, let* $\sigma \subset \mathfrak{T}_0$, *and let* $G(A) = \tau_0(A)$ *for every* $A \in \sigma$ *for which* A^- *is compact. If* $A \in \sigma$ *is such that* $A \cap (P - A)^-$ *is* τ_0-σ-*finite, then every function from* $\mathfrak{P}(A)$ *is* $\mathfrak{T}_0$-*measurable.*

Proof. If $f \in \mathfrak{P}(A)$ let $f^\wedge(x) = f(x)$ for $x \in A^-$ and $f^\wedge(x) = 0$ for $x \in P - A^-$. According to [7], 4.14, $f^\wedge \in \mathfrak{P}(P)$ and the proposition follows.

3. Some remarks on the natural convergence. Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a net structure in P and let κ be a convergence in $\mathfrak{N}$. If for every $x \in P^\sim$, κ_x consists of all nets $\{B_U\}$ which satisfy the condition $\mathscr{K}_2$, then κ is called the *natural convergence* and it is denoted by κ^0. According to [6], 4.3 the natural convergence κ^0 is admissible.

Hence assume that there is given an integration base $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa^0, G \rangle$ where κ^0 is the natural convergence. It is easy to see that for $\delta \subset \sigma$, δ^* is closed in $P^\sim$ (see [5], 2.1) and so $\mathfrak{J}$ is strongly measurable. In fact we have more precise information.

LEMMA 3.1. *Let* $A \in \sigma$ *and let* M *be a function on* σ_A. *If* ${}_*M(x, A) > -\infty$ *for all* $x \in P$, *then the function* ${}_*M(\cdot, A)$ *is lower semicontinuous.*[4]

Proof. If c is a real number, then

$$\{x \in P : {}_*M(x, A) > c\} = (P - A^-) \cup \{x \in A^- : {}_*M(x, A) > c\} = P - \{x \in A^\sim : {}_*M(x, A) \leqq c\}\,.$$

By 2.2, $\{x \in A^\sim : {}_*M(x, A) \leqq c\}$ is closed in $P^\sim$ and the lemma follows.

THEOREM 3.2. *The measure* τ *is regular.*

Proof. Since we already know that τ is inner regular on $\mathfrak{U}$ and finite on $\mathfrak{C}$ (see [7], 3.13), it remains to show that τ is outer regular on $\mathfrak{T}$. Hence choose $A \in \mathfrak{T}$ and $\varepsilon \in (0, 1)$. Since everything is trivial if $\tau(A) = +\infty$, we may assume that $\tau(A) < +\infty$. By [7], 3.10, $\chi_A \in \mathfrak{P}_0(P)$ and so there is a narrow majorant $M \in \mathfrak{M}(\chi_A, P)$ such that $M(P) - \tau(A) < \varepsilon$ (see [7], 2.5, 2.7). By Lemma 3.1, ${}_*M(\cdot, P)$ is lower semicontinuous and hence the set $U = \{x \in P : {}_*M(x, P) > 1 - \varepsilon\}$ is open. Clearly $A \subset U$ and $M/(1 - \varepsilon)$ is a majorant of χ_U. Therefore

[4] See [3], Chapter 3, Problem F, p. 101.

$$\tau(A) \leqq \tau(U) \leqq M(P)/(1-\varepsilon) < [\tau(A)+\varepsilon]/(1-\varepsilon)$$

and the outer regularity of τ at A follows from the arbitrariness of ε.

COROLLARY 3.3. $(\mathfrak{T}, \tau) = (\mathfrak{T}_0, \tau_0)$.

Proof. By a rather standard procedure it follows from [7], 4.7 that the measure τ_0 has no proper regular extension. Hence $\mathfrak{T} = \mathfrak{T}_0$ and because both τ and τ_0 are regular, also $\tau = \tau_0$.

4. **The monotone convergence.** Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a given net structure in P and let κ be a convergence in $\mathfrak{N}$. If for every $x \in P^{\sim}$, κ_x consists of all nets $\{B_U, U \in \Gamma, \subset\} \in \kappa_x^0$ such that $B_U \subset B_V$ whenever $U \subset V$, then κ is called the *monotone convergence* and it is denoted by κ^1. The following proposition indicates the essential difference between κ^0 and κ^1.

PROPOSITION 4.1. *If* $x \in P^{\sim}$ and $\{B_U, U \in \Gamma, \subset\} \in \kappa_x^1$, *then either* $x \in \bigcap_{U \in \Gamma} B_U^{\sim}$ *or there is a* $V \in \Gamma$ *such that* $B_U = \emptyset$ *for all* $U \in \Gamma$ *for which* $U \subset V$.

Proof. If $x \notin \bigcap_{U \in \Gamma} B_U^{\sim}$ then $x \notin B_{U_0}^{\sim}$ for some $U_0 \in \Gamma$ and hence there is a $U_1 \in \Gamma_x$ such that $U_1 \cap B_{U_0} = \emptyset$. To U_1 we can assign a $U_2 \in \Gamma$ such that $U \in \Gamma$ and $U \subset U_2$ implies $B_U \subset U_1$. On the other hand, $U \in \Gamma$ and $U \subset U_0$ implies $B_U \subset B_{U_0}$ and thus V can be any element of Γ for which $V \subset U_0 \cap U_2$.

REMARK 4.2. Let P be the set of all real numbers with the usual topology. Let $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa, G \rangle$ be an integration base in P defined as follows: σ is the pre-algebra generated by all one-side-open intervals, for $x \in P^{\sim}$, $\Gamma_x \subset \sigma$ is an arbitrary local base at x in $P^{\sim}$, and G is the Lebesgue measure on σ. Using the previous proposition, we see rather easily that if $\kappa = \kappa^1$ is the monotone convergence, then $\mathfrak{J}$ gives precisely the classical *Perron integral* (see [10], Chapter VI, § 8).

We also note that a *singularization* of a monotone convergence is again a monotone convergence (see [8], § 2).

PROPOSITION 4.3. *Let* $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ *be a net structure in P. If the space P is locally metrizable, then the monotone convergence in* $\mathfrak{N}$ *is admissible.*

Proof. Conditions $\mathscr{K}_1 - \mathscr{K}_4$ are satisfied obviously. To show that also $\mathscr{K}_5$ and $\mathscr{K}_6$ are satisfied we can repeat verbatim the proofs of Proposition 3.1 and Theorem 3.2 in [5], respectively.

We note that in an arbitrary topological space the monotone convergence still satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$; however, we do no know whether it also satisfies conditions $\mathscr{K}_5$ and $\mathscr{K}_6$. An example of a net structure in a nonlocally metrizable space in which the monotone convergence is still admissible will be given in 6.2.

We shall close this section with a proposition which will show how conditions $\mathscr{K}_1 - \mathscr{K}_6$ are related to each other.

PROPOSITION 4.4. *Conditions $\mathscr{K}_1 - \mathscr{K}_5$ are independent and they do not imply $\mathscr{K}_6$. Conditions $\mathscr{K}_1 - \mathscr{K}_4$ and $\mathscr{K}_6$ are independent and they imply $\mathscr{K}_5$.*

Proof. Examples 6.3 and 6.4 show that $\mathscr{K}_1 - \mathscr{K}_4$ do not imply $\mathscr{K}_5$ and that $\mathscr{K}_1 - \mathscr{K}_5$ do not imply $\mathscr{K}_6$, respectively. The remaining examples which are needed to prove the independence are quite simple and their construction will be left to the reader. We shall complete the proof by showing that $\mathscr{K}_1$, $\mathscr{K}_4$, and $\mathscr{K}_6$ imply $\mathscr{K}_5$.

Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a net structure in P, let κ be a convergence in $\mathfrak{N}$ satisfying conditions $\mathscr{K}_1$, $\mathscr{K}_4$, and $\mathscr{K}_6$, and let $\delta \subset \sigma$ be a nonempty semi-hereditary system. If δ is stable, then by $\mathscr{K}_6$, δ^* is uncountable and so nonempty. Hence suppose that δ is not stable. Then either $\emptyset \in \delta^* = P^\sim$ [see [6], (4.1)] or there is an $A \in \delta$ and an $x \in P^\sim$ such that $\delta_{A-U} = \emptyset$ for all $U \in \Gamma_x$. Choose $U \in \Gamma_x$. Since $U \cap A \in \sigma$, $A = (U \cap A) \cup (\bigcup_{i=1}^n B_i)$ where $B_1, \cdots, B_n$ are disjoint sets from σ_{A-U}. Therefore $B_1, \cdots, B_n$ do not belong to δ and because δ is semi-hereditary, we conclude that $A \cap U \in \delta$. Now it follows from $\mathscr{K}_1$ and $\mathscr{K}_4$ that $x \in \delta^*$ and thus again δ^* is nonempty.

COROLLARY 4.5. *Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a net structure in P and let σ contain no nonempty semihereditary stable system. Then every convergence in $\mathfrak{N}$ which satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$ is admissible.*

The assumption of this corollary is always satisfied if P is countable. It is also satisfied if P is the set of all ordinals less than a given ordinal α topologized by the order topology (see [5], 1.4).

COROLLARY 4.6. *Let $\mathfrak{N}$ be a net structure in P and let κ and κ' be two convergences in $\mathfrak{N}$ satisfying conditions $\mathscr{K}_1 - \mathscr{K}_4$. If $\kappa_x = \kappa'_x$ for all but countably many $x \in P^\sim$, then κ is admissible if and only if κ' is admissible.*

Proof. Let S be the countable set of those $x \in P^\sim$ for which

$\kappa'_x \neq \kappa_x$. Since $\delta^* - S = \delta^{*\prime} - S$ for every $\delta \subset \sigma$, it follows that κ satisfies condition $\mathscr{K}_6$ if and only if κ' does.

According to Proposition 4.4 condition $\mathscr{K}_5$ is superfluous for the admissibility of a convergence. Nevertheless, for a given convergence, establishing $\mathscr{K}_5$ is usually the first step in establishing $\mathscr{K}_6$ (see [5] and [9]). It should be also noted that a convergence which satisfies only conditions $\mathscr{K}_1 - \mathscr{K}_5$ is still adequate for the definition of the narrow integral (see [7], 2.5).

5. **Measurability with respect to the monotone convergence.** Throughout this section we shall assume that the space P, in addition to Hausdorff and locally compact, is also locally metrizable. We shall assume that there is given an integration base $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa^1, G \rangle$ where κ^1 is the monotone convergence and we shall prove that $\mathfrak{J}$ is measurable. We begin with a simple but useful remark.

REMARK 5.1. Let $x \in P$ and let $\{B_U, U \in \Gamma, \subset\} \in \kappa^1_x$. Since P, being locally metrizable, is first countable, Γ has a linearly ordered countable cofinal subset $\Gamma' = \{U_n\}$. Hence there is a sequence $\{C_n\} \in \kappa^1_x(\{B_U\})$ such that $C_{n+1} \subset C_n$ for $n = 1, 2, \cdots$; for it suffices to set $C_n = B_{U_n}$. The sequence $\{C_n\}$ may consist only of a single element if x is an isolated point of P.

LEMMA 5.2. *Let $A \in \sigma$ and let $\delta \subset \sigma_A$. If σ_A is countable then $\delta^* - (\infty)$ is $\mathfrak{T}_0$-measurable.*

Proof. Since $\Gamma_x \subset \sigma$ for all $x \in P$, it follows from the countability of σ_A that A is paracompact and hence metrizable by [2], Th. 2–28, p. 81. Choose a metric on A and if $B \subset A$ denote by $d(B)$ the diameter of B with respect to this metric. Because $\emptyset \in \delta$ implies $\delta^* - (\infty) = P$ which is $\mathfrak{T}_0$-measurable, we shall assume that $\emptyset \notin \delta$. Let $\{B_k\}_k$ be an enumeration of the family $\{B \in \delta : d(B) < 1\}$. If $B_{k_1 \cdots k_n}$, where n and $k_1, \cdots, k_n$ are positive integers, has been already defined we let $\{B_{k_1 \cdots k_n k}\}_k$ be an enumeration of the family $\{B \in \delta_{B_{k_1 \cdots k_n}} : d(B) < 1/(n+1)\}$. Setting $B_{k_1 \cdots k_n} = \emptyset$ for those groups $(k_1 \cdots k_n)$ of positive integers for which $B_{k_1 \cdots k_n}$ was not previously defined, we obtain a determining system $\{B^-_{k_1 \cdots k_n}\}$ of $\mathfrak{T}_0$-measurable sets (see [10], Chapter II, § 5). By [10], Chapter II, Th. (5.5), p. 50, its uncleus

$$N = \bigcup_{k_1 k_2 \cdots} \bigcap_{n=1}^{\infty} B^-_{k_1 \cdots k_n}$$

is also $\mathfrak{T}_0$-measurable. On the other hand, using 5.1 it is easy to see that $N = \delta^* - (\infty)$.

COROLLARY 5.3. *If σ is countable then $\mathfrak{J}$ is strongly measurable.*

LEMMA 5.4. *Let σ be a pre-algebra of subsets of P, $\delta \subset \sigma$, and let $A \in \delta$ be such that $\{A \cap B : B \in \delta\}$ is countable. Then there is a pre-algebra $\sigma' \subset \sigma$ containing δ and for which σ'_A is countable.*

Proof. Let δ^0 consist of all finite intersections of elements from δ. For $B, B' \in \delta^0_A$ we let $(B, B') = \{C_1, \cdots, C_n\}$ where $C_1, \cdots, C_n$ are disjoint sets from σ for which $B - B' = \bigcup_{i=1}^n C_i$. For $B, B' \in \delta^0$ we let $[B, B'] = \{D_1, \cdots, D_m\}$ where $D_1, \cdots, D_m$ are disjoint sets from σ for which $(B - B') - A = \bigcup_{j=1}^m D_j$. Set $\alpha = \cup \{(B, B') : B, B' \in \delta^0_A\}$, $\beta = \cup \{[B, B'] : B, B' \in \delta^0\}$, and $\delta^1 = \delta^0 \cup \alpha \cup \beta$. Then $\delta \subset \delta^1 \subset \sigma$, $A \in \delta^1$, and $\{A \cap B : B \in \delta^1\} = \{A \cap B : B \in \delta^0 \cup \alpha\}$ is countable. If $B, B' \in \delta$, then $B \cap B \in \delta^1$ and

$$B - B' = (B \cap A - B' \cap A) \cup [(B - B') - A]$$
$$= \left(\bigcup_{i=1}^{n} C_1\right) \cup \left(\bigcup_{j=1}^{m} D_j\right)$$

where the last term is a disjoint union of sets from δ^1. Note also that $\emptyset \in \delta^1$ and $P \in \delta^1$, for $\emptyset = A - A$ and P is the empty intersection of sets from δ. Let $\sigma_1 = \delta$ and assuming that σ_n has been already defined let $\sigma_{n+1} = \sigma^1_n$, $n = 1, 2, \cdots$. The system $\sigma' = \bigcup_{n=1}^\infty \sigma_n$ has now all the desired properties.

THEOREM 5.5. *The integration base $\mathfrak{J}$ is locally strongly measurable.*

Proof. Choose $x_0 \in P$ and $U \in \Gamma_{x_0}$ whose closure U^- is compact and contained in some open metrizable neighborhood of x_0. Then for each $x \in P^\sim$ we can define a local base $\Gamma'_x \subset \Gamma_x$ such that $\bigcup_{x \in U^-} \Gamma'_x$ is countable and $U \cap V = \emptyset$ for every $V \in \Gamma'_x$ with $x \in P^\sim - U^-$. Setting $\delta = \{U\} \cup (\bigcup_{x \in P^\sim}\{V \cap P : V \in \Gamma'_x\})$, we have $\delta \subset \sigma$, $U \in \delta$, and $\{U \cap V : V \in \delta\}$ is countable. Let σ' be a prealgebra from Lemma 5.4 and let G' be the restriction of G to σ'. Then $\mathfrak{J}' = \langle \sigma', \Gamma'_x, \kappa^1, G' \rangle$ is an integration base and by 2.6, $(\mathfrak{T}_0, \tau_0) = (\mathfrak{T}'_0, \tau'_0)$. Since $\mathfrak{J}' \prec \mathfrak{J}$ (see § 2), $\mathfrak{P}(U) \subset \mathfrak{P}'(U)$ and the theorem follows from 5.2.

COROLLARY 5.6. *If P is metrizable then $(\mathfrak{T}, \tau) = (\mathfrak{T}_0, \tau_0)$.*

This corollary follows from [3], Chapter V, Corollary 35, p. 160 and [7], 4.9.

6. Examples. Four examples illustrating the previous sections will be given here.

EXAMPLE[5] 6.1. For $i = 1, 2, \cdots$ let $P_i = \{0, 1\}$ be the two point set with the discrete topology and let μ_i be the measure in P_i defined by $\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2$. We set $P = \prod_{i=1}^{\infty} P_i$, $\mu = \prod_{i=1}^{\infty} \mu_i$, and define σ as the family of all μ-measurable subsets of P. Then P is a compact metrizable space whose points are sequences $\{x_i\}_{i=1}^{\infty}$ of zeroes and ones, μ is a regular measure in P, and σ is a σ-algebra containing all rectangles. If $x = \{x_i\} \in P$ we let $\Gamma_x = \{U_n\}_{n=1}^{\infty}$ where $U_n = \{\{y_i\} \in P : y_i = x_i,\ i = 1, 2, \cdots, n\}$, $n = 1, 2, \cdots$. It follows from 5.5 and 2.8 that $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa^1, \mu \rangle$, where κ^1 is the monotone convergence in $\langle \sigma, \Gamma_x \rangle$, is a measurable integration base. We shall show, however, that there is a nonempty semihereditary stable system $\delta \subset \sigma$ for which δ^* is not τ_0-measurable. Thus, in particular, the integration base $\mathfrak{J}$ is not strongly measurable.

For $x = \{x_i\}$ let $f(x) = \{f_i(x)\}$ where $f_{2i}(x) = x_{2i}$ and $f_{2i-1}(x) = 0$, $i = 1, 2, \cdots$. Then $f : P \to P$ is a continuous map and we denote by Q its image. The sets $Q_x = f^{-1}(x)$ with $x \in Q$ are disjoint, nonempty and prefect, and their union is equal to P. If $x = \{x_i\} \in Q$ and $n \geqq 1$ is an integer, let $Q_x^n = \{\{y_i\} \in P : y_{2i} = x_{2i},\ i = 1, 2, \cdots, n\}$. Then $\mu(Q_x^n) = 2^{-n}$ and $\bigcap_{n=1}^{\infty} Q_x^n = Q_x$. Hence $\mu(Q_x) = 0$ for all $x \in Q$. Let $A \subset P$ be closed and let $\mu(A) > 0$. By the compactness of P, $f(A)$ is also closed and so it is either countable or its cardinality is the continuum. Since $A \subset \bigcup_{x \in A} Q_{f(x)} = \bigcup_{y \in f(A)} Q_y$ and $\mu(A) > 0$, it follows that the cardinality of $f(A)$ is the continuum. Plainly $Q_y \cap A \neq \emptyset$ for all $y \in f(A)$.

Let γ be the least ordinal whose cardinality is the continuum and let $\{A_\alpha : 0 \leqq \alpha < \gamma\}$ be a well-ordering of all closed subsets of P with positive measure. By the previous paragraph there are $x_0, x_0' \in Q$, $x_0 \neq x_0'$, such that $Q_{x_0} \cap A_0 \neq \emptyset$ and $Q_{x_0'} \cap A_0 \neq \emptyset$. Let β be an ordinal less than γ and assume that for all ordinals α less than β we have already defined distinct elements $x_\alpha, x_\alpha' \in Q$ such that $Q_{x_\alpha} \cap A_\alpha \neq \emptyset$ and $Q_{x_\alpha'} \cap A_\alpha \neq \emptyset$. Since the cardinality of $Q' = \{x_\alpha, x_\alpha' : 0 \leqq \alpha < \beta\}$ is less than the continuum and the cardinality of $\{x \in Q : Q_x \cap A_\beta \neq \emptyset\}$ is equal to the continuum, we can choose $x_\beta, x_\beta' \in Q - Q'$, $x_\beta \neq x_\beta'$, such that $Q_{x_\beta} \cap A_\beta \neq \emptyset$ and $Q_{x_\beta'} \cap A_\beta \neq \emptyset$. Letting $B = \cup \{Q_{x_\alpha} : 0 \leqq \alpha < \gamma\}$ and $B' = \cup \{Q_{x_\alpha'} : 0 \leqq \alpha < \gamma\}$, we have $B \cap B' = \emptyset$ and $A \cap B \neq \emptyset$, $A \cap B' \neq \emptyset$ for every closed set $A \subset P$ for which $\mu(A) > 0$. Therefore every closed subset of B or $P - B$ has measure equal to zero. If $B \in \sigma$, then by the regularity of μ, $\mu(B) = \mu(P - B) = 0$ which is impossible for $\mu(P) = 1$. Hence B and similarly B' are not μ-measurable.

Now let δ consist of all uncountable subsets of Q_{x_α}, $0 \leqq \alpha < \gamma$. Then δ is a nonempty semihereditary stable subsystem of σ and δ^*

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[5] This example is due to K. Prikry.

computed by κ^1 is equal to B. Since the measure μ has no regular extension, it follows from [7], 4.12 that $\mu = \tau_0$ and so δ^* is not $\mathfrak{T}_0$-measurable.

Note that the hypothesis of the continuum was not used in this example.

EXAMPLE 6.2. Let P be a compact Boolean space (see [3], Chapter 5, Problem S, p. 168), let σ be the algebra of all compact-open subsets of P, and let $\Gamma_x = \{U \in \sigma : x \in U\}$ for all $x \in P$. If κ^1 is the monotone convergence in $\langle \sigma, \Gamma_x \rangle$, then by 4.1, every net from κ_x^1 has the form $\{U, U \in \Gamma, \subset\}$ where Γ is a cofinal subset of Γ_x. It follows from [9], 4.3 that κ^1 is admissible. Since, e.g., the Tychonoff product of any family of finite discrete spaces is a compact Boolean space, we see that the space P need not be locally metrizable.

EXAMPLE 6.3. Let $P = [0, 1)$ together with the usual topology and let σ be the pre-algebra consisting of all half-open intervals $[a, b) \subset P$. We shall identity $P^\sim$ with $[0, 1]$ and for every $x \in P^\sim$ we shall let $\Gamma_x = \{[x - 1/n, x + 1/n) \cap [P \cup (x)]\}_{n=1}^\infty$. If $x \in P^\sim$ then let κ_x consist of all sequences $\{[x - 1/n_k, x + 1/n_k) \cap B\}_{k=1}^\infty$ where $B \in \sigma$ and $\{n_k\}$ is an increasing sequence of positive integers. Thus defined the convergence $\kappa = \{\kappa_x : x \in P^\sim\}$ clearly satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$ and if $\{B_n\} \in \kappa_x$, then for all sufficiently large n, B_n is a half-open interval of rational length (which may be zero). Hence if δ consists of all intervals of irrational length, then $\delta^* = \varnothing$. However, it is easy to see that $\delta \subset \sigma$ is a nonempty semihereditary stable system (see [9], 4.2) and so κ does not satisfy conditions $\mathscr{K}_5$ and $\mathscr{K}_6$.

EXAMPLE 6.4. Let P, $P^\sim$, and σ be the same as in Example 6.3. For $x \in P^\sim$, $\Gamma_x = \{[x - 1/n, x_n) \cap [P \cap (x)]\}_{n=1}^\infty$ where $\{x_n\}_{n=1}^\infty$ is a decreasing sequence of irrational numbers converging to x. Denote by Q the set of all rational numbers in $[0, 1]$. If $x \in P^\sim$ then let κ_x consist of all sequences $\{[a_n, b_n)\}_{n=1}^\infty \subset \sigma$ such that $\lim a_n = \lim b_n = x$ and for all sufficiently large n, either $b_n \in P - Q$ or $b_n = b_{n+1}$. It is easy to see that thus defined the convergence $\kappa = \{\kappa_x : x \in P^\sim\}$ satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$. Let $\delta \subset \sigma$ be a nonempty semihereditary system and let $[a, b) \in \delta$. If $a = b$ then $\delta^* = P^\sim$. Hence assume that $a < b$ and choose an $x_1 \in (a, b) - Q$ such that $\max(x_1 - a, b - x_1) \leqq \frac{2}{3}(b - a)$. By the semihereditariness of δ, e.g., $[a, x_1) \in \delta$. Now choose $x_2 \in (a, x_1) - Q$ such that $\max(x_2 - a, x_1 - x_2) \leqq \frac{2}{3}(x_1 - a)$ and select an interval from $[a, x_2)$, $[x_2, x_1)$ which belongs to δ. Inductively, we obtain a decreasing sequence $\{B_n\}_{n=1} \subset \delta$ for which $\bigcap_{n=1}^\infty B_n^\sim = (x)$. Obviously, $x \in \delta^*$ and so κ satisfies also condition $\mathscr{K}_5$. However, κ does not satisfy condition $\mathscr{K}_6$. To see this, let δ_0 consist of all intervals $[a, b) \in \sigma$ such that $b - a > 0$ and $b \in Q$. Then $\delta_0 \subset \sigma$ is a nonempty semihereditary stable system and $\delta_0^* = Q$ is countable.

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ON THE CONFORMAL MAPPING OF VARIABLE REGIONS

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We establish an estimate for the functional

$$I(f, g; \rho) = \int_{C_\rho} |f(t) - g(t)| \cdot |dt| ,$$

C_ρ **is the circumference** $|t| = \rho, 0 \leqq \rho < 1$. **Here** f **and** g **are normalized conformal mapping functions of** $|z| < 1$ **onto a pair of bounded, open, simply connected, origin containing domains in the** w **plane whose boundaries are near each other in some sense. In the second part of the paper we establish an estimate for the functional** $I(f', g'; \rho)$ **in case the boundaries are additionally assumed to be rectifiable.**

We are motivated by the fact that if one of the domains is a disc we get the case of "nearly circular" domains which has been much studied.

Aside from an absolute constant our estimates are geometric in nature, being expressed in terms of numbers which are derived from properties of the boundaries of the mapped domains. They are of interest to us because they hold uniformly for all ρ, $0 \leqq \rho < 1$ and because they approach zero when one of the domains converges to the other as described in the paper.

1. DEFINITION 1. Let D_f and D_g denote a pair of open, bounded, simply connected sets in the w plane both of which contain the origin. Let Γ_f and Γ_g denote their respective boundaries. Let Δ denote the component of $D_f \cap D_g$ which contains the origin and let Γ denote the boundary of Δ. Let λ_f be the radius of the largest disk lying in the complement of Γ_f and having its center on Γ (if no such disk exists, write $\lambda_f = 0$). Let λ_g be analogously defined. The inner distance is defined by the formula

$$\varepsilon = \varepsilon(\Gamma_f, \Gamma_g) = \mathrm{Max}\,(\lambda_f, \lambda_g) .$$

The statement '$\phi(z)$ is a normalized mapping function' means that $\phi(z)$ is the conformal mapping function of one bounded, simply connected, origin containing domain onto another and that $\phi(0) = 0$, and $\phi'(0)$ is positive.

The symbol C_ρ will always be used to denote the locus $|t| = \rho$, $0 \leqq \rho < 1$.

Let R_1 and R_2 denote the radii of two circles with centers at $w = 0$

which are such that the boundaries Γ_f and Γ_g lie in the ring

$$0 < R_1 \leqq |w| \leqq R_2 .$$

THEOREM 1. *If $f(z)$ and $g(z)$ are the normalized mapping functions of $|z| < 1$ onto D_f and D_g respectively, if $0 < \varepsilon(\Gamma_f, \Gamma_g) < R_1$, then*

$$I(f, g; \rho) = \int_{C_\rho} |f(t) - g(t)| \cdot |dt| \leqq K_1 R_2 \left(\frac{\varepsilon}{R_1}\right)^{1/6} .$$

The number K_1 is an absolute constant, and the inequality holds uniformly for all ρ, $0 \leqq \rho < 1$.

Before proving Theorem 1 we state some results which are used in the proof.

LEMMA A. ([4], p. 349.) *Let D be a bounded, simply connected domain which contains the origin and let $z = \psi(w)$ be the normalized mapping function of D onto the disk $|z| < 1$ in the z plane. If w is a point of D at a distance δ from the boundary of D, then*

$$1 - |\psi(w)| \leqq 4\sqrt{\delta \psi'(0)} .$$

LEMMA B. ([3], p. 563.) *Let $w = \phi(z)$ be the normalized mapping function of $|z| < 1$ onto the domain whose boundary D lies in the ring $1 - \sigma \leqq |w| \leqq 1$, $0 < \sigma < 1$. Then*

$$\int_{C_\rho} |\phi(t) - t|^2 \cdot |dt| \leqq K_2 \sigma^2 .$$

The number K_2 is an absolute constant, and inequality holds uniformly for all ρ, $0 \leqq \rho < 1$.

LEMMA C. ([1], p. 165.) *If $F(z)$ and $\Theta(z)$ are regular in $|z| < 1$ if $\Theta(0) = 0$ and $|\Theta(z)| < 1$ in $|z| < 1$, then*

$$\int_{C_\rho} |F(\Theta(t))|^2 \cdot |dt| \leqq \int_{C_\rho} |F(t)|^2 \cdot |dt| ,$$

uniformly valid for all ρ, $0 \leqq \rho < 1$.

2. *Proof of Theorem* 1. (a) From Definition 1, each point of Γ will have distance at most ε from Γ_f. The inverse of $f(z)$ maps Δ onto a domain E which lies in $|z| < 1$. Let E_1 denote the boundary of E. From Lemma A, the set E_1 will lie in the ring

$$1 - 4\sqrt{\frac{\varepsilon}{f'(0)}} \leqq |z| \leqq 1 .$$

Since

$$f'(0) \geqq \inf_{|z|<1} \left| \frac{f(z)}{z} \right| \geqq R_1 ,$$

the set E_1 will lie in the ring

$$1 - 4\sqrt{\frac{\varepsilon}{R_1}} \leqq |z| \leqq 1 .$$

The above inequality fails to define a ring if $\varepsilon/R_1 \geqq 1/16$. We treat the two cases separately. Let $\omega(z)$ be the normalized mapping function of $|z| < 1$ onto E. If $\varepsilon/R_1 < 1/16$, we have from Lemma B,

$$J(\rho) = \int_{C_\rho} |\omega(t) - t|^2 |dt| \leqq 16K_2 \frac{\varepsilon}{R_1} .$$

For the case $1/16 \leqq \varepsilon/R_1 < 1$, we have trivially,

$$J(\rho) \leqq 4 \cdot 2\pi\rho \leqq 128\pi \cdot \frac{\varepsilon}{R_1} .$$

Thus, if $K_3 = \text{Max}\,[128\pi, 16K_2]$, then

$$J(\rho) \leqq K_3 \frac{\varepsilon}{R_1},\ 0 < \varepsilon < R_1 . \tag{1}$$

(b) For $0 \leqq r \leqq 1, |z| < 1$ let

$$B_r(z) = f(z) - f(rz) .$$

Then

$$f(z) - f(\omega(z)) = B_r(z) - B_r(\omega(z)) + f(rz) - f(r\omega(z)) .$$

Hence

$$\begin{aligned} &\int_{C_\rho} |f(t) - f(\omega(t))| \cdot |dt| \\ &\leqq \int_{C_\rho} |B_r(t)| \cdot |dt| + \int_{C_\rho} |B_r(\omega(t))| \cdot |dt| \\ &\quad + \int_{C_\rho} |f(rt) - f(r\omega(t))| \cdot |dt| \equiv I_1 + I_2 + I_3 . \end{aligned} \tag{2}$$

If $f(z) = \sum_1^\infty a_k z^k$ then

$$\begin{aligned} I_1^2 &\leqq 2\pi\rho \cdot \int_{C_\rho} |B_r(t)|^2 \cdot |dt| = 2\pi\rho \Sigma\, |a_k|^2 \cdot \rho^{2k} \cdot (1 - r^k)^2 \cdot 2\pi\rho \\ &\leqq 4\pi^2 \Sigma\, |a_k|^2 (1 - r^k) \\ &= 4\pi^2 \Sigma\, |a_k|^2 (1 - r)(1 + r + r^2 + \cdots + r^{k-1}) \end{aligned}$$

$$\leqq 4\pi^2(1-r)\Sigma(|a_k|^2\cdot k) = 4\pi(1-r)\cdot(\text{area of } D_f)$$
$$\leqq 4\pi(1-r)\cdot\pi R_2^2 .$$

Thus, if $K_4^2 = 4\pi^2$,

$$I_1 \leqq K_4 R_2\sqrt{1-r},\ 0 \leqq r \leqq 1 . \tag{3}$$

From Lemma C, the same bound is valid for I_2:

$$I_2 \leqq K_4 R_2\sqrt{1-r},\ 0 \leqq r \leqq 1 . \tag{4}$$

(c) If $0 < r < \alpha < 1$, we have for the integrand of I_3:

$$\begin{aligned}
|f(rt) - f(r\omega(t))| &\leqq \frac{1}{2\pi}\int_{C_\alpha} |f(\gamma)|\cdot\left|\frac{1}{\gamma - rt} - \frac{1}{\gamma - r\omega}\right|\cdot|d\gamma| \\
&\leqq \frac{1}{2\pi}\int_{C_\alpha} |f(\gamma)|\cdot\left|\frac{r\omega - rt}{(\gamma - rt)(\gamma - r\omega)}\right|\cdot|d\gamma| \\
&\leqq \frac{\sup|f|\cdot r|\omega - t|}{2\pi}\int_{C_\alpha}\frac{|d\gamma|}{|\gamma - rt|\cdot|\gamma - r\omega|} \\
&\leqq \frac{R_2|\omega - t|}{2\pi}\left[\int_{C_\alpha}\frac{|d\gamma|}{|\gamma - rt|^2}\right]^{1/2}\cdot\left[\int_{C_\alpha}\frac{|d\gamma|}{|\gamma - r\omega|^2}\right]^{1/2} \\
&\leqq \frac{R_2|\omega - t|}{2\pi}\left[\frac{2\pi\alpha}{\alpha^2 - |rt|^2}\right]^{1/2}\cdot\left[\frac{2\pi\alpha}{\alpha^2 - |r\omega|^2}\right]^{1/2} .
\end{aligned}$$

Let $\alpha \to 1$ and we obtain

$$|f(rt) - f(r\omega(t))| \leqq \frac{R_2|\omega(t) - t|}{1 - r},\ 0 < r < 1 .$$

Hence, from (1)

$$\begin{aligned}
\int_{C_\rho} |f(rt) - f(r\omega(t))|\cdot|dt| &\leqq \frac{R_2}{1-r}\int_{C_\rho}|\omega(t) - t|\,|dt| \\
&\leqq \frac{R_2}{1-r}\left[\int_{C_\rho}|\omega - t|^2\,|dt|\right]^{1/2}\cdot\sqrt{2\pi\rho} \\
&\leqq \frac{R_2}{1-r}\left[2\pi K_3\left(\frac{\varepsilon}{R_1}\right)\right]^{1/2} .
\end{aligned} \tag{5}$$

If we combine (2), (3), (4) and (5), we obtain the estimate

$$\begin{aligned}
&\int_{C_\rho} |f(t) - f(\omega(t))|\cdot|dt| \\
&\qquad \leqq 2K_4R_2\sqrt{1-r} + \frac{R_2}{1-r}\left[2\pi K_3\left(\frac{\varepsilon}{R_1}\right)\right]^{1/2},\ 0 < r < 1 .
\end{aligned} \tag{6}$$

(d) The whole argument can be repeated with $g(z)$ in place of $f(z)$. In this case we shall have an estimate analogous to (6):

$$
\begin{aligned}
(6')\qquad & \int_{C_\rho} |g(t) - g(\omega_1(t))| \cdot |dt| \\
& \leqq 2K_4R_2\sqrt{1-r} + \frac{R_2}{1-r}\left[2\pi K_3\left(\frac{\varepsilon}{R_1}\right)\right]^{1/2},\ 0 < r < 1 .
\end{aligned}
$$

The function $\omega_1(z)$ is the normalized mapping function of $|z| < 1$ onto the image of Δ under the inverse of $g(z)$. Since $f(\omega(z))$ and $g(\omega_1(z))$ are both normalized mapping functions from $|z| < 1$ onto Δ it follows from the uniqueness that

$$
(7)\qquad f(\omega(z)) \equiv g(\omega_1(z)),\ |z| < 1 .
$$

If we combine (6), (6′), (7) and choose r so that $1 - r = (\varepsilon/R_1)^{1/3}$, the conclusion of the theorem is established.

Throughout the remainder of the paper we shall assume the situation of Theorem 1 with the added hypothesis that Γ_f and Γ_g are rectifiable Jordan curves of lengths L_f and L_g. In this case it is well-known that $\bar{D}_f$ is the continuous image of $|z| \leqq 1$ and that if $f'(z)$ is defined at the boundary by

$$
f'(e^{i\theta}) = \operatorname*{Lim}_{z\to e^{i\theta}} \frac{f(z) - f(e^{i\theta})}{z - e^{i\theta}},\ |z| \leqq 1 ,
$$

then $f'(e^{i\theta})$ exists almost everywhere, is Lebesgue summable, and

$$
L_f = \int_0^{2\pi} |f'(e^{i\theta})|\, d\theta .
$$

3. The following definition ([4], p. 337) and lemma ([4], p. 337) are useful.

DEFINITION α. Let c denote a crosscut of D_f which does not pass through $w = 0$. Let T denote that subregion of D_f determined by c which does not contain $w = 0$. Let λ denote the diameter of c and let Λ denote the diameter of T. For any $\delta > 0$ consider all possible crosscuts c for which $\lambda \leqq \delta$. The crosscut modulus is defined is defined to be

$$
\eta_f(\delta) = \sup_{\lambda\leqq\delta} \Lambda .
$$

The crosscut modulus is monotonic and has the property:

$$
\eta_f(\delta) \to 0 \quad \text{as} \quad \delta \to 0 .
$$

LEMMA D. *Let A_f denote the area of D_f. Let z_0 be any point on $|z| = 1$ and k_s the part of the circle $|z - z_0| = s$ which lies in*

$|z| < 1$. *Then for every* $s, 0 < s < 1$, *there exists a* σ, $s \leqq \sigma \leqq s^{1/2}$ *such that the image of* k_σ *is a crosscut of length*

$$l_\sigma \leqq \left(\frac{2\pi A_f}{\log \frac{1}{s}}\right)^{1/2} .$$

We introduce the abbreviation:

$$\nu_f(\delta) = \eta_f\left(\left(\frac{2\pi A_f}{\log \frac{1}{\delta}}\right)^{1/2}\right), 0 < \delta < 1 . \tag{8}$$

An immediate consequence of Lemma D is

LEMMA 1.

$$h_f(r) = \operatorname*{Sup}_{|z|=1} |f(z) - f(rz)| \leqq \nu_f(1 - r), 0 < r < 1 .$$

4. DEFINITION 2. For $m \geqq 2$, let $\{w_1, w_2, w_3, \cdots, w_m\}$ be any set of m distinct points taken in cyclic order on Γ_f and so distributed that Γ_f is partitioned into m subarcs of equal length, each subarc having length $L_{f/m}$. Let l_λ be the length of the perimeter of the cyclically determined polygon, and let λ, the norm of the partition, be defined by

$$\lambda = \operatorname{Max}[|w_1 - w_m|, |w_2 - w_1|, |w_3 - w_2|, \cdots, |w_m - w_{m-1}|] .$$

The number l_λ can be written as

$$l_\lambda = |w_1 - w_m| + \sum_{k=1}^{m-1} |w_{k+1} - w_k| .$$

For any $\delta > 0$ consider all partitions for which $\lambda \leqq \delta$. Let

$$U_f(\delta) = \operatorname*{Inf}_{\lambda \leqq \delta} l_\lambda .$$

It is easily shown that $\operatorname{Sup} U_f(\delta) = L_f$. We define the modulus of rectifiability to be

$$\zeta_f(\delta) = L_f - U_f(\delta) .$$

The modulus $\zeta_f(\delta)$ is monotonic and has the property: $\zeta_f(\delta) \to 0$ as $\delta \to 0$.

LEMMA 2. *If* $L_f(r)$ *is the length of the level curve in* D_f *which is the image of* $|z| = r$, *then*

$$L_f - L_f(r) \leqq \zeta_f(\sqrt{\nu(1 - r)}) + 2L_f\sqrt{\nu_f(1 - r)} + 4\nu_f(1 - r), 0 < r < 1 .$$

Proof. Let the positive integer m be defined by

$$m = \left[\frac{L_f}{\sqrt{\nu_f}(1-r)}\right] + 2 . \tag{9}$$

Let $w_1, w_2, \cdots, w_m$ be a set of points in cyclic order Γ_f, so arranged that Γ_f is partitioned into m equal subarcs, each subarc having length L_f/m. Clearly the norm of the partition does not exceed L_f/m and if l_m is the length of the perimeter of the polygon, then

$$L_f - l_m \leqq \zeta_f\left(\frac{L_f}{m}\right) . \tag{10}$$

We define the points $z_k, \tilde{w}_k$ by $w_k = f(z_k)$, $\tilde{w}_k = f(rz_k)$. The set $\tilde{w}_k$ determines a polygon inscribed in the level curve in D_f which is the image of $|z| = r$. Comparing corresponding sides of the polygons, we have from Lemma 1,

$$\begin{aligned} |w_{k+1} - w_k| &\leqq |w_{k+1} - \tilde{w}_{k+1}| + |\tilde{w}_{k+1} - \tilde{w}_k| + |\tilde{w}_k - w_k| \\ &\leqq h_f(r) + |\tilde{w}_{k+1} - \tilde{w}_k| + h_f(r) \\ &\leqq 2\nu_f(1-r) + |\tilde{w}_{k+1} - \tilde{w}_k| . \end{aligned}$$

Similarly,

$$|\tilde{w}_{k+1} - \tilde{w}_k| \leqq 2\nu_f(1-r) + |w_{k+1} - w_k| .$$

Thus, if l'_m is the length of the perimeter of the level curve polygon,

$$|l'_m - l_m| \leqq 2m\nu_f(1-r) . \tag{11}$$

Noting that $l'_m \leqq L_f(r)$, we have from (10) and (11)

$$\begin{aligned} L_f - L_f(r) &\leqq L_f - l'_m \leqq L_f - l_m + |l_m - l'_m| . \\ &\leqq \zeta_f\left(\frac{L_f}{m}\right) + 2m\nu_f(1-r) . \end{aligned} \tag{12}$$

From (9)

$$\frac{L_f}{\sqrt{\nu_f}(1-r)} \leqq m \leqq \frac{L_f}{\sqrt{\nu_f}(1-r)} + 2 .$$

The conclusion follows from (10), (11) and (12).

In the estimate of Lemma 2, it would appear that the first term should dominate the others and this will be so if ζ_f is sufficiently weak. However, it is possible (e.g., if D_f is a disk) for the term $2L_f\sqrt{\nu_f}$ to be dominant. For purpose of final estimate we introduce the boundary functional

$$\beta_f(\delta) = \zeta_f(\sqrt{\nu_f(\delta)}) + 2L_f\sqrt{\nu_f(\delta)} + 4\nu_f(\delta),\ 0 < \delta < 1 . \tag{13}$$

LEMMA 3.

$$\int_{C_\rho} |f'(t) - f'(rt)|\cdot|dt| \leqq 2\sqrt{L_f \beta_f(1-r)}\,,$$
$$0 < r < 1\,, \quad \text{for all} \quad \rho, 0 \leqq \rho < 1\,.$$

Proof. The function $\sqrt{f'(z)}$ (i.e., the branch which is positive at the origin) is regular in $|z| < 1$. If $\sqrt{f'(z)} = \sum_0^\infty c_k z^k$, it is well known that $\Sigma |c_k|^2$ is convergent and

$$L_f = \int_0^{2\pi} \sqrt{f'(e^{i\theta})}\overline{\sqrt{f'(e^{i\theta})}}d\Theta = 2\pi\Sigma\,|c_k|^2\,,$$
$$L_f(r) = \int_0^{2\pi} \sqrt{f'(re^{i\theta})}\overline{\sqrt{f'(re^{i\theta})}}rd\Theta = 2\pi r\cdot\Sigma\,|c_k|^2 r^{2k},\ 0 < r < 1\,.$$

We write

$$\left[\int_{C_\rho} |f'(t) - f'(rt)|\cdot|dt|\right]^2$$
$$\leqq \int_{C_\rho} |\sqrt{f'(t)} - \sqrt{f'(rt)}|^2\cdot|dt|\cdot\int_{C_\rho} |\sqrt{f'(t)} + \sqrt{f'(rt)}|^2\cdot|dt|$$
$$= I_1\cdot I_2\,,$$
$$I_1 = 2\pi\rho\cdot\Sigma\,|c_k|^2\rho^{2k}(1 - 2r^k + r^{2k}) \leqq 2\pi\Sigma\,|c_k|^2(1 - r^{2k})$$
$$= L_f - L_f(r)\cdot\frac{1}{r} \leqq L_f - L_f(r)\,,$$
$$I_2 = 2\pi\rho\,\Sigma\,|c_k|^2\,\rho^{2k}(1 + 2r^k + r^{2k}) \leqq 2\pi\Sigma(|c_k|^2\cdot 4) = 4L_f\,.$$

From these inequalities and Lemma 2, the conclusion is apparent.

5. Final estimates. We assert:

THEOREM 2. *If Γ_f and Γ_g are rectifiable Jordan curves of lengths L_f and L_g, if $0 < \varepsilon/R_1 < 1$, then*

$$I(f', g'; \rho) \leqq 2[\sqrt{L_f} + \sqrt{L_g} + M/R_1^{1/2}]\sqrt{\beta_f(\sigma)} + 2\sqrt{L_g\mu}\,,$$
uniformly for all $\rho, 0 \leqq \rho < 1$,

where $\sigma = (\varepsilon/R_1)^{1/24}$, $\mu = |L_f - L_g|$, $M = \text{Max}\,[K_1R_2, 2\sqrt{L_gK_1R_2}]$.

Proof. Write

$$I(f', g'; \rho) \leqq \int_{C_\rho} |f'(t) - f'(rt)|\cdot|dt| + \int_{C_\rho} |f'(rt) - g'(rt)|\cdot|dt|$$
$$+ \int_{C_\rho} |g'(rt) - g'(t)|\cdot|dt| = I_1 + I_2 + I_3\,.$$

Choose $1 - r = \sigma$, from Lemma 3,

$$I_1 \leqq 2\sqrt{L_f \beta_f(\sigma)} .$$

Let $0 < \rho < \alpha < 1$, then, from Theorem 1

$$I_2 \leqq \int_{C_\rho} \left[\frac{1}{2\pi} \int_{C_\alpha} \left| \frac{f(\gamma) - g(\gamma)}{(\gamma - rt)^2} \right| \cdot | d\gamma | \right] \cdot | dt | \leqq \frac{K_1 R_2 (\varepsilon / R_1)^{1/6}}{(1 - r)^2}$$
$$= K_1 R_2 \sigma^2 \leqq K_1 R_2 \sigma .$$

From the proof of Lemma 3 (with g in place of f)

$$I_3 \leqq 2\sqrt{L_g} (L_g - L_g(r))^{1/2} ,$$

and

$$L_g - L_g(r) \leqq | L_g - L_f | + L_f - L_f(r) + | L_f(r) - L_g(r) |$$
$$= \mu + A + B .$$

From Lemma 2, $A \leqq \beta_f(\sigma)$, and

$$B = \left| \int_{C_\rho} [| f'(rt) | - | g'(rt) |] \, | dt | \leqq \int_{C_\rho} | f'(rt) - g'(rt) | \cdot | dt | \right|$$
$$= I_2 \leqq K_1 R_2 \sigma^2 .$$

Thus,

$$I_3 \leqq 2\sqrt{L_g} (\mu + A + B)^{1/2} \leqq 2\sqrt{L_g} (\mu^{1/2} + A^{1/2} + B^{1/2}) .$$

Combining estimates we have

$$I(f', g'; \rho) \leqq 2(\sqrt{L_f} + \sqrt{L_g}) \sqrt{\beta_f(\sigma)} + 2\sqrt{L_g \mu} + (2\sqrt{L_g K_1 R_2} + K_1 R_2) \sigma . \tag{14}$$

From (8) and (13) and the definition of η_f,

$$\sqrt{\beta_f(\sigma)} \geqq 2(\nu_f(\sigma))^{1/2} \geqq \eta_f \left(\left(\frac{2\pi A_f}{\log \dfrac{1}{\sigma}} \right)^{1/2} \right)^{1/2} \geqq \left(\frac{2\pi A_f}{\log \dfrac{1}{\sigma}} \right)^{1/4}$$
$$\geqq ((2\pi^2 R_1^2)^{1/4} \cdot \sigma) \geqq R_1^{1/2} \sigma .$$

Hence

$$(2\sqrt{L_g K_1 R_2} + K_1 R_2) \sigma \leqq \frac{2M\sqrt{\beta_f(\sigma)}}{R_1^{1/2}} \tag{15}$$

the conclusion follows from (14) and (15).

LEMMA 4. *If* $\mu = | L_f - L_g |$ *and if*

$$I^* = \operatorname*{Sup}_{\rho} I(f', g'; \rho), \ 0 \leqq \rho < 1 , \quad \textit{then} \quad \mu \leqq I^* .$$

Proof. We have

$$\begin{aligned}|L_f(\rho) - L_g(\rho)| &= \left|\int_{C_\rho} |f'(t)|\cdot|dt| - \int_{C_\rho} |g'(t)|\cdot|dt|\right| \\ &\leqq \left|\int_{C_\rho} |f'(t) - g'(t)|\cdot|dt|\right| = I(f', g'; \rho) \leqq I^* .\end{aligned}$$

Let $\rho \to 1$ on the left and the lemma is proved.

LEMMA 5.

$$|f(e^{i\theta}) - g(e^{i\theta})| \leqq I^* .$$

Proof. The Fejer-Riesz inequality asserts that

$$A = \int_{-1}^{1} |H(x)|^p\, dx \leqq \frac{1}{2}\int_0^{2\pi} |H(e^{i\alpha})|^p\, d\alpha = B ,$$

$$p > 0 \text{ and } x \text{ is real} .$$

Here $H(z)$ which is regular in $|z| < 1$ belongs to the Hardy class H^p in $|z| \leqq 1$. Let $p = 1$ and we make the choice $H(z) = \rho e^{i\theta}(f'(z\rho e^{i\theta}) - g'(z\rho e^{i\theta}))$. Noting that $A \geqq \left|\int_0^1 H(x)dx\right|$, that $2B = I(f', g'; \rho) \leqq I^*$, we let $\rho \to 1$ and we get the conclusion of the lemma.

We are now able to state our convergence theorem as

THEOREM 3. *If the f boundary is held fixed and the g boundary is allowed to vary, a necessary and sufficient condition that $I(f', g'; \rho) \to 0$ uniformly for all ρ, $0 \leqq \rho < 1$, is that $\mu + \sigma \to 0$.*

Proof. We get the sufficiency from Theorem 2. From Lemma 4 we see that $I^* \to 0$ implies that $\mu \to 0$ which is one part of the necessity. From Lemma 5, we see that if I^* is arbitrarily small the boundary point $f(e^{i\theta})$ will be arbitrarily close to the g boundary and vice versa. So we have $I^* \to 0$ implies $\varepsilon \to 0$ implies that $\inf R_1 > 0$ so that $I^* \to 0$ implies that $\sigma \to 0$. This completes the proof of Theorem 3.

Without estimate, S. E. Warschawski [2] established a result that is similar to Theorem 3.

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SUFFICIENT CONDITIONS FOR A RIEMANNIAN MANIFOLD TO BE LOCALLY SYMMETRIC

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In a locally symmetric Riemannian manifold the scalar curvature is constant and each k-th covariant derivative of the Riemannian curvature tensor vanishes. In this note, we show that if the covariant derivatives of the Riemannian curvature tensor satisfy some algebraic conditions at each point, then the Riemannian manifold is locally symmetric.

Let R be the Riemannian curvature tensor of a Riemannian manifold M^m with a positive-definite metric tensor g. Manifolds and tensors are assumed to be of class C^∞ unless otherwise stated. We denote by ∇ the Riemannian connection defined by g. For tangent vectors X and Y, we consider $R(X, Y)$ as a derivation of the tensor algebra at each point. A conjecture by K. Nomizu [4] is that $R(X, Y)\cdot R = 0$ on a complete and irreducible manifold $M^m(m \geqq 3)$ implies $\nabla R = 0$, that is, M^m is locally symmetric. Here we consider some additional conditions.

For an integer k and tangent vectors $V_k, \cdots, V_1$ at a point p of M^m, we adopt a notation:

$$\begin{aligned}(\nabla_V^k R) &= (V_k, V_{k-1}, \cdots, V_1; \nabla^k R)\\ &= (V_k^t V_{k-1}^s \cdots V_1^r \nabla_t \nabla_s \cdots \nabla_r R^a_{bcd}),\end{aligned}$$

where V_k^t, etc., are components of V_k, etc., and $\nabla_t\nabla_s \cdots \nabla_r R^a_{bcd}$ are components of the k-th covariant derivative $\nabla^k R$ of R in local coordinates.

PROPOSITION 1. *Let $M^m(m \geqq 3)$ be a real analytic Riemannian manifold. Assume that*
(1.0) *the restricted holonomy group is irreducible,*
(1.1) $R(X, Y)\cdot R = 0$,
(1.2) $R(X, Y)\cdot(\nabla_V^k R) = 0$ *for* $k = 1, 2, \cdots$.
Then M^m is locally symmetric.

Here we note that condition (1.0) means that it holds at some, hence every, point and condition (1.1), and (1.2), mean that for any point p and for any tangent vectors X, Y, $V_k, \cdots, V_1$ at p, they hold.

PROPOSITION 2. *Let $M^m(m \geqq 3)$ be a Riemannian manifold. Assume* (1.1) *and* (1.2) *and that*

(1.0)′ *the infinitesimal holonomy group is irreducible at every point. Then M^m is locally symmetric.*

Propositions 1 and 2 are essentially related to the following results.

PROPOSITION 3. *Let $M^m (m \geqq 3)$ be a Riemannian manifold. Assume that the restricted holonomy group H^0 (the infinitesimal holonomy group H', resp.) is irreducible, and R is invariant by H^0 (H', resp.). Then M^m is locally symmetric.*

PROPOSITION 3′. (*J. Simons* [5], p. 233) *Let M^m $(m \geqq 3)$ be an irreducible Riemannian manifold. Assume that R is invariant by the holonomy group H. Then M^m is locally symmetric.*

Proposition 3 is a generalization of a result by A. Lichnerowicz ([2], p. 11), which contains an assumption of compactness. We remark here that condition (1.2) is equivalent to

$$(1.2)' \qquad R(X, Y)\cdot(\nabla_{V_k}\nabla_{V_{k-1}}\cdots\nabla_{V_1}R) = 0 \text{ for } k = 1, 2, \cdots ,$$

where $X, Y, V_k, \cdots, V_1$ are vector fields on M^m.

With respect to Nomizu's conjecture and the above propositions we have

THEOREM 4. *Let M^m $(m \geqq 3)$ be a Riemannian manifold. Assume that*

(i) *the scalar curvature S is constant,*

(ii) $R(X, Y)\cdot R = 0$,

(iii) $R(X, Y)\cdot\nabla_V R = 0$,

(iv) $R(X, Y)\cdot(X, V; \nabla^2 R) = 0$,

(or (iv)′ $R(X, Y)\cdot\nabla_X\nabla_V R = 0$ *for vector fields*).

Then M^m is locally symmetric.

THEOREM 5. *Let M^m $(m \geqq 3)$ be a Riemannian manifold. Assume that*

(i) *the Ricci curvature tensor R_1 is parallel;* $\nabla R_1 = 0$,

(ii) $R(X, Y)\cdot R = 0$,

(iii) $R(X, Y)\cdot\nabla_V R = 0$.

Then M^m is locally symmetric.

In Theorems 4 and 5, if $m = 2$, then $\nabla R_1 = 0$ implies $\nabla R = 0$.

In Theorem 5, if M^m is compact, (iii) can be dropped (A. Lichnerowicz [2], or K. Yano [6], p. 222).

In §2 we reduce proofs of Propositions 1 and 2 to that of Proposition 3, and next we reduce proofs of Propositions 3 and 3′ to that of Theorem 4. In §3 we prove Theorems 4 and 5.

2. Holonomy algebras.

Conditions (1.1) and (1.2) imply that

$$[R(X, Y), (\nabla_V^k R)(A, B)] = (\nabla_V^k R)(R(X, Y)A, B) + (\nabla_V^k R)(A, R(X, Y)B) \tag{2.1}$$

for $k = 0, 1, \cdots$, where $\nabla^0 R$ means R, and $[T, T']$ for linear transformations T, T' means $TT' - T'T$.

Now we show

LEMMA 2.1. *The condition* (2.1) *implies*

$$[(\nabla_W^j R)(X, Y), (\nabla_V^k R)(A, B)] = (\nabla_V^k R)((\nabla_W^j R)(X, Y)A, B) + (\nabla_V^k R)(A, (\nabla_W^j R)(X, Y)B) \tag{2.2}$$

for $j, k = 0, 1, 2, \cdots$. *And* (2.1) *is equivalent to*

$$[(\nabla_W^j R)(X, Y), R(A, B)] = R((\nabla_W^j R)(X, Y)A, B) + R(A, (\nabla_W^j R)(X, Y)B) \tag{2.3}$$

for $j = 0, 1, 2, \cdots$.

Proof. We prove (2.2) by induction in j and by tensor calculus in local coordinates. By (2.1), (2.2) holds for $(j, k) = (0, k)$, $k = 0, 1, 2, \cdots$. Assume that (2.2) holds for $(j-1, k), (j-2, k), \cdots, (0, k)$, $k = 0, 1, 2, \cdots$. Then, denoting by $\nabla_t \nabla_s \cdots \nabla_r R^p_{uxy}$ the j-th covariant derivative of R and by $\nabla_f \cdots \nabla_e R^u_{qab}$ the k-th covariant derivative of R, we show

$$\nabla_t \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_f \cdots \nabla_e R^u_{qab} - \nabla_f \cdots \nabla_e R^p_{uab} \nabla_t \nabla_s \cdots \nabla_r R^u_{qxy} = \nabla_f \cdots \nabla_e R^p_{qvb} \nabla_t \nabla_s \cdots \nabla_r R^v_{axy} + \nabla_f \cdots \nabla_e R^p_{qav} \nabla_t \nabla_s \cdots \nabla_r R^v_{bxy} \,. \tag{2.4}$$

In fact, we have

$$\begin{aligned}
&\nabla_t \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_f \cdots \nabla_e R^u_{qab} - \nabla_f \cdots \nabla_e R^p_{uab} \nabla_t \nabla_s \cdots \nabla_r R^u_{qxy} \\
&\quad = \nabla_t(\nabla_s \cdots \nabla_r R^p_{uxy} \nabla_f \cdots \nabla_e R^u_{qab}) \\
&\qquad - \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_t \nabla_f \cdots \nabla_e R^u_{qab} \\
&\qquad - \nabla_t(\nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy}) \\
&\qquad + \nabla_t \nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy} \\
&\quad = \nabla_t(\nabla_f \cdots \nabla_e R^p_{qvb} \nabla_s \cdots \nabla_r R^v_{axy} + \nabla_f \cdots \nabla_e R^p_{qav} \nabla_s \cdots \nabla_r R^v_{bxy}) \\
&\qquad - \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_t \nabla_f \cdots \nabla_e R^u_{qab} \qquad \text{(by (2.2) for } (j-1, k)) \\
&\qquad + \nabla_t \nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy} \\
&\quad = \nabla_f \cdots \nabla_e R^p_{qvb} \nabla_t \nabla_s \cdots \nabla_r R^v_{axy} + \nabla_f \cdots \nabla_e R^p_{qav} \nabla_t \nabla_s \cdots \nabla_r R^v_{bxy}
\end{aligned}$$

$$+ (\nabla_t\nabla_f \cdots \nabla_e R^p_{uab}\nabla_s \cdots \nabla_r R^u_{qxy} - \nabla_s \cdots \nabla_r R^p_{uxy}\nabla_t\nabla_f \cdots \nabla_e R^u_{qab})$$
$$+ (\nabla_t\nabla_f \cdots \nabla_e R^p_{qub}\nabla_s \cdots \nabla_r R^v_{axy} + \nabla_t\nabla_f \cdots \nabla_e R^p_{qav}\nabla_s \cdots \nabla_r R^v_{bxy}) .$$

The second and third terms vanish by (2.2) for $(j-1, k+1)$. Therefore we have (2.4).

Similarly we can show that (2.3) implies (2.2), including (2.1).

By the theory of holonomy groups (cf. A. Nijenhuis [3]), the set of linear transformations

$$(2.5) \qquad R(X, Y), (\nabla_W R)(X, Y), (\nabla^2_W R)(X, Y), \cdots$$

for $X, Y, W_1, \cdots \in M_p$, the tangent space to M at p of M, spans a Lie algebra h'_p called the infinitesimal holonomy algebra at p. h'_p generates the infinitesimal holonomy group H'_p which is a subgroup of the local holonomy group $H^*_p = H^0_p(U)$. Clearly H^*_p is a subgroup of the restricted holonomy group H^0_p. If a Riemannian manifold is real analytic we have $H' = H^* = H^0$.

The condition (2.3) implies that

$$(2.6) \qquad [T, R(A, B)] = R(TA, B) + R(A, TB)$$

for any $T \in h'_p$. This says that R is invariant by T. Therefore, for any element $\alpha \in H'_p$ we have

$$(2.7) \qquad \alpha R(A, B)C = R(\alpha A, \alpha B)\alpha C \qquad \text{for } A, B, C \in M_p .$$

Thus, we have reduced proofs of Propositions 1 and 2 to proof of Proposition 3.

Since (2.7) or (2.6) is equivalent to (2.1), condition (2.7) implies conditions (ii), (iii) and (iv) of Theorem 4. Consequently, if we show that, under the conditions in Proposition 3 (3′, resp.), the scalar curvature S is constant, then Proposition 3 (3′, resp.) will follow from Theorem 4.

Let E_i, $1 \leqq i \leqq m$, be an orthonormal basis at p. Then the Ricci curvature tensor R_1 is given by

$$R_1(X, Y) = \sum_i g(R(X, E_i)Y, E_i) .$$

Since R is invariant by H' or H^0 or H, we have $R_1(X, Y) = R_1(\alpha X, \alpha Y)$ for any $\alpha \in H'$, or H^0 or H. Since H' or H^0 or H is irreducible, we have some real number λ so that $R_1 = \lambda g$ at p. Because p is an arbitrary point of M and $m \geqq 3$, λ is constant on M, and hence $S = m\lambda$ is constant.

3. Proofs of Theorems 4 and 5. To prove theorems it suffices to show two propositions below.

PROPOSITION 3.1. *On M^m $(m \geqq 3)$ assume that*
(i) *the scalar curvature S is constant,*
(ii) $(R(X, Y)\cdot R)(X, V) = 0$,
(iii) $(R(X, Y)\cdot \nabla_V R)(X, Y)V = 0$,
(iv) $(R(X, Y)\cdot \nabla_V R_1)(V, X) = 0$,
(v) $(R(X, Y)\cdot (X, V; \nabla^2 R_1)(V, Y) = 0$,
(or (v)′ $(R(X, Y)\cdot \nabla_X \nabla_V R_1)(V, Y) = 0$ *for vector fields*).
Then we have $\nabla R = 0$.

Proof. Let $\{E_i\}$ be an orthonormal basis at p of M. Put $X = E_x$, $Y = E_y$, $V = E_v$ in (iii) and take a sum on x, y, v. Then we have

$$R^{irxy}\nabla_v R_r{}^v{}_{xy} - R^{rvxy}\nabla_v R^i{}_{rxy} - R^{rx}{}_{xy}\nabla_v R^{iv}{}_r{}^y - R^{ry}{}_{xy}\nabla_v R^{ivx}{}_r = 0 .$$

The third and fourth terms vanish. We apply the second Bianchi identity to the first two terms;

$$\begin{aligned} R^{irxy}(-\nabla_x R_r{}^v{}_{yv} - \nabla_y R_r{}^v{}_{vx}) &= -2R^{irxy}\nabla_y R_{rx} , \\ -R^{rvxy}(-\nabla_i R_{rvxy} - \nabla_r R_{vixy}) &= R^{rvxy}\nabla_i R_{rvxy} + R^{rvxy}\nabla_r R_{vixy} \\ &= R^{rvxy}\nabla_i R_{rvxy} + R^{rvxy}\nabla_v R_{irxy} . \end{aligned}$$

Therefore, we have

$$-4R^{irxy}\nabla_y R_{rx} + R^{rvxy}\nabla^i R_{rvxy} = 0 . \tag{3.1}$$

Likewise, (iv) implies that

$$R^{rvx}{}_y\nabla_v R_{rx} + R^{rx}{}_{xy}\nabla_v R^v_r = 0 . \tag{3.2}$$

And (v) implies that

$$R^{rvxy}\nabla_x\nabla_v R_{ry} + R^r{}_y{}^{xy}\nabla_x\nabla_v R^v_r = 0 . \tag{3.3}$$

For (v)′ we assume that E_i are local vector fields such that $(\nabla E_i)_p = 0$ and $\{E_i\}$ forms an orthonormal basis at p. Then we have the same (3.3).

Since $\nabla_v R^v_r = (1/2)\nabla_r S = 0$, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} R^{rvxy}\nabla_x\nabla_v R_{ry} &= 0 , \\ R^{rvxy}\nabla_i R_{rvxy} &= 0 . \end{aligned}$$

On the other hand, in a Riemannian manifold generally we have

$$\begin{aligned} \nabla^h\nabla_h(R_{ijkl}R^{ijkl}) &= 2(\nabla_h R_{ijkl}\nabla^h R^{ijkl}) \\ &\quad + 8R^{ijkl}\nabla_i\nabla_k R_{jl} + 4R^{ijkl}B^h_{jkl,hi} , \end{aligned} \tag{3.4}$$

where $B^i_{jkl,ab}X^aY^b$ are components of $R(X, Y)\cdot R$ (A. Lichnerowicz [2], p. 10). Since (ii) is equivalent to $B^h_{jkl,hi} = 0$, we have $\nabla_h R_{ijkl} = 0$.

PROPOSITION 3.2. *On M^m $(m \geqq 3)$ assume that*

(i) $\nabla R_1 = 0$,
(ii) $(R(X, Y)\cdot R)(X, V) = 0$,
(iii) $(R(X, Y)\cdot\nabla_V R)(X, Y) = 0$.

Then we have $\nabla R = 0$.

Proof. We have (3.1) by (iii). Then we have $\nabla_h(R_{ijkl}R^{ijkl}) = 0$. Therefore, (ii) and (3.4) show $\nabla R = 0$.

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LOCALLY COMPACT CLIFFORD SEMIGROUPS

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Let S be a locally compact Hausdorff semigroup which is a disjoint union of subgroups one of which is dense. If S the disjoint union of exactly two groups one of which is compact, then S has been completely described by K. H. Hofmann, and if S is the disjoint union of two subgroups where the dense subgroup G has the added property that it is abelian and G/G_0 is a union of compact groups, then S has been described in a previous paper of the author.

It is the purpose of this paper to consider S when each subgroup of S is a topological group when given the relative topology and G (the dense subgroup) has the added property that it is abelian and G/G_0 is a union of compact groups. In particular, we show how to reduce such a semigroup to a semigroup which is a union of real vector groups (§3). In §4 we give the structure of S under the added assumption that $E(S)$ is isomorphic to $E[(R^x)^n]$, where $(R^x)^n$ denotes the n-fold product of the nonnegative real numbers under multiplication.

2. Definitions and notations. If G is a topological group, G_0 will denote the identity component. Let $\mathscr{C}$ denote the full subcategory of the category of locally compact abelian groups whose objects G have the property that G/G_0 is a union of compact subgroups. Let $\mathscr{C}_c$ denote the full subcategory of $\mathscr{C}$ whose objects G_c have the property that G_c is a union of compact subgroups. If $G \in \mathscr{C}$, then by the structure theorem for locally compact abelian groups [2, p. 389] there is a real vector subgroup W of G such that $G/W \in \mathscr{C}_c$. If $W \cong R^n$, then $n = \dim G$ will be called the dimension of G. We will use the following properties of $\mathscr{C}$ and $\mathscr{C}_c$: P_1; for each G in $\mathscr{C}$ there is a unique subgroup $G_c \in \mathscr{C}_c$ such that G/G_c is a real vector group. P_2 [7]; if $\alpha\colon G \to W$ is a morphism in $\mathscr{C}$ with $\alpha(G)$ dense in W and if W is a real vector group, then there is morphism $\beta\colon W \to G$ in $\mathscr{C}$ such that $\alpha\beta = I_W$ (the identity morphism on W). P_3 [7]; if $\alpha\colon G \to H$ is a morphism in the category of locally compact abelian groups with $\alpha(G)$ dense in H and $G \in \mathscr{C}$, then $H \in \mathscr{C}$. Also, if G/G_0 is compact, then H/H_0 is compact.

Let $\mathscr{S}$ denote the category whose objects S are locally compact Hausdorff semigroups satisfying (i) S is a disjoint union of subgroups one of which is dense and (ii) each maximal subgroup of S is a member of $\mathscr{C}$, and whose morphisms are the continuous identity preserving homomorphisms. Let $\mathscr{R}$ denote the full subcategory of $\mathscr{S}$ whose objects R have the properties that (i) each maximal subgroup of R

is a real vector group and (ii) the minimial ideal of R exists and is compact (thus a zero for R).

Let $S \in \mathscr{S}$. Then we will use 1 to denote the identity for S. For each x in S let $H(x) = \{y \in S \mid yS = xS\}$. Since S is an abelian Clifford semigroup, each $H(x)$ is a maximal subgroup of S. Let $\delta: S \to E(S)$ be the function defined by $\delta(s)$ is the idempotent of S such that $H(s) = H(\delta(s))$. If $A \subseteqq S$, then $\bar{A}$ will denote the closure of A. Partially order $E(S)$ by $e \leqq f$ if and only if $ef = e$, and for each e and f in $E(S)$ let $(e, f) = \{a \in E(S) \mid e < a < f\}$. Let $Z = \{0, 1\}$ under multiplication, and let Z^n denote the n-fold product of n copies of Z. Finally, for a semigroup T we use $K(T)$ to denote the minimial ideal when it exists.

3. The purpose of this section is two fold. First we prove that each S in $\mathscr{S}$ splits into the direct product of two closed subsemigroups V and $\bar{W}$, where V is a real vector group and where $\bar{W} \in \mathscr{S}$ with the added property that $K(\bar{W}) \in \mathscr{C}_c$ (Proposition 3.5). Second we prove that there is a congruence ρ on S such that S/ρ is a locally compact Clifford semigroup with each H-class a real vector group and with $E(S) \cong E(S/\rho)$ (Theorem 3.11).

Throughout this section S will represent a fixed member of $\mathscr{S}$, and $E(S)^*$ will denote $E(S)\backslash\{1\}$.

LEMMA 3.1. *Let $e \in E(S)^*$. Then $H(e)$ is open in $S\backslash H(1)$ if and only if* $\dim H(e) = \dim H(1) - 1$.

Proof. By [7], if $H(e)$ is open in $S\backslash H(1)$, then $\dim H(e) = \dim H(1) - 1$.

Let $e \in E(S)$ with $\dim H(e) = \dim H(1) - 1$. Again by [7], if $f \in E(S)$ such that $e < f$, then $\dim H(e) < \dim H(f)$. Thus, since $\dim H(f) < \dim H(1)$ for all f in $E(S)^*$ [7], $(e, 1) = \varnothing$. Let $\psi: S \to eS$ be the morphism defined by $\psi(s) = es$. Since $H(e)$ is a topological group, $H(e)$ is open in $\overline{H(e)}$ [8] which is eS. Since ψ is continuous and since $H(e) = (S\backslash H(1)) \cap \psi^{-1}(H(e))$, it follows that $H(e)$ is open in $S\backslash H(1)$.

COROLLARY 3.2. *If $e \in E(S)^*$, then there is an f in $E(S)$ with $e < f$ and* $\dim H(e) = \dim H(f) - 1$.

Proof. Let $f \in E(S)$ with $e < f$ and $(e, f) = \varnothing$. Then $H(e) \subseteqq \overline{H(f)}$. Let $\psi: \overline{H(f)} \to e\overline{H(f)}$ morphism defined by $\psi(s) = es$. Since $(e, f) = \varnothing$, $H(e) = (\overline{H(f)}\backslash H(f)) \cap (\psi^{-1}(H(e)))$, and it follows that $H(e)$ is open in $\overline{H(f)}\backslash H(f)$. Thus, by Lemma 3.1, $\dim H(e) = \dim H(f) - 1$.

LEMMA 3.3. *A subgroup $H \in \mathscr{C}_c$ of S is closed in S.*

Proof. Let $g \in \bar{H}$. Since $H \subseteqq H(e)_c$ for some e in $E(S)$ and $g \in \overline{\delta(g)H}$, it follows that $g \in H(g)_c$. Thus there is a compact subgroup C of $H(g)_c$ with $g \in C$. Since $\{g^n\}_{n=1}^{\infty} \subseteqq C$ and C is compact, $\delta(g) \in \overline{\{g^n\}_{n=1}^{\infty}}$ [4, p. 15] which is a subset of $\bar{H}$; thus $\delta(g) \in \bar{H}$. By [7], there are no maximal subgroups of $\bar{H}$ which are topological other than $\bar{H}$; thus $\delta(g) = e$, and $\bar{H} \subseteqq H(e)$. Thus we need only show that H is a closed subgroup of $H(e)$, but this follows since H is a locally compact subgroup of a locally compact topological group.

PROPOSITION 3.4. *Let $e \in E(S)$, and let ψ be the map from S onto eS defined by $\psi(s) = es$. Then there are closed subgroups V and W of $H(1)$ with the following properties:*

(a) $\bar{W} = \psi^{-1}(H(e)_c)$,

(b) *V is a real vector group, and*

(c) *The morphism $m: V \times \bar{W} \to \psi^{-1}(H(e))$ defined by $m(v, w) = v \cdot w$ is an isomorphism.*

Proof. Let α be the natural map from $H(e)$ onto $H(e)/H(e)_c$, let Q be the corestriction of $\psi|_{H(1)}$ to $H(e)$, and let $\beta: H(e)/H(e)_c \to H(1)$ be a morphism in $\mathscr{C}$ such that $(\alpha Q)\beta$ is the identity map on $H(e)/H(e)_c$ [P_2]. Let $V = \beta(H(e)/H(e)_c)$, and let $W = Q^{-1}(H(e)_c)$. Then V and W are the desired closed subgroups of $H(1)$. The inverse of m is given by $s \mapsto ((\beta\alpha\psi)(s), [(\beta\alpha\psi)(s)]^{-1}s)$ which is clearly continuous. The theorem now follows.

PROPOSITION 3.5. *There are closed subgroups V and W of $H(1)$ with the following properties:*

(a) V is a real vector group,

(b) *$K(\bar{W}) \in \mathscr{C}_c$, and*

(c) *The morphism $m: V \times \bar{W} \to S$ defined by $m(v, w) = v \cdot w$ is an isomorphism.*

Proof. Again by [7], if $e \in E(S)^*$, then $\dim H(e) < \dim H(1)$. Thus there is an f in $E(S)$ with $\dim H(f) \leqq \dim H(e)$ for all e in $E(S)$. Since $\dim H(ef) \leqq \min \{\dim H(e), \dim H(f)\}$ with equality holding only for $e < f$ or $f \leqq e$, f is unique. The proposition now follows from Proposition 3.4 along with the observation that $S = \psi^{-1}(H(f))$ where $\psi: S \to fS$ is the morphism defined by $\psi(s) = sf$ for all s in S.

PROPOSITION 3.6. *If there is a s_0 in S with $H(s_0)_c$ compact, then $H(s)_c$ is compact for all s in S.*

Proof. From the structure theorem for locally compact abelian groups [2, p. 389] one can get that if $G \in \mathscr{C}_c$, then G_0 is compact. Thus for any s in S we have that $H(s)_c$ is compact if and only if $H(s)_c/(H(s)_c)_0$ is compact. But $H(s)_c/H(s)_c)_0$ is compact if and only if $H(s)/H(s)_0$ is compact. Therefore, by P_3 and since $\overline{H(1)} = S$, the theorem will follow if we can prove that $H(1)/H(1)_0$ is compact.

We do this by contradiction. That is, assume $H(1)/H(1)_0$ is not compact, and let $e \in E(S)$ satisfying the following:

(i) $H(e)/H(e)_0$ is compact,

(ii) $\delta(s_0) \leqq e$, and

(iii) if $f \in E(S)$ with $e < f$, then $H(f)/(f)_0$ is not compact.

By Corollary 3.2 and since $e \neq 1$, there is an f in $E(S)$ with $e < f$ and $\dim H(e) = \dim H(f) - 1$. Let $T = \overline{H(f)}$, and let $\psi\colon T \to eT$ be the morphism defined by $\psi(s) = se$. By Proposition 3.4, there is a real vector subgroup V of $H(f)$, a closed subgroup W of $H(f)$ with $\psi^{-1}(H(e)_c) = \bar{W}$, and a morphism $m\colon V \times \bar{W} \to \psi^{-1}(H(e))$ which is an isomorphism. Since $\bar{W}\backslash W = H(e)_c$ which is compact and by [3], W contains a compact subgroup C such that W/C is a real vector group. Thus $H_W(f)_c$ is compact. Since the corestriction of $m|_{V\times W}\colon V\times W \to H(f)$ is an isomorphism and V is a real vector group, it now follows that $H(f)_c$ is compact. This is the desired contradiction and the proof now follows.

SUBLEMMA. *Let e and f be elements of $E(S)$ with* $\dim H(e) = \dim H(f) + 1$ *and with $f < e$. If H is a subgroup of $H(e)$ with $H \in \mathscr{C}_c$, then fH is a closed subgroup of S.*

Proof. Let $g \in \overline{fH} \cap H(f)$. Since $H \in \mathscr{C}_c$, $fH \subseteqq H(f)_c$, and thus there is a compact subgroup C of $H(f)$ which is open relative to $H(f)_c$ and with $g \in C$. Let $\psi\colon \overline{H(e)} \to \overline{fH(e)}$ be the morphism defined by $\psi(s) = fs$. It follows from Proposition 3.4 and the fact that $H(f)$ is open in $\overline{H(e)}\backslash H(e)$ that $\psi^{-1}(C)$ is a locally compact semigroup which contains a dense group $\psi^{-1}(C) \cap H(e)$ whose complement C is compact. By [3], there is a unique compact subgroup C_1 of $\psi^{-1}(C) \cap H(e)$ and a one-parameter subgroup M of $\psi^{-1}(C) \cap H(e)$ such that $\psi^{-1}(C) = \bar{M}\cdot C_1$. Let $\{g_\alpha\}_{\alpha \in A}$ be a net in fH which converges to g. Since C is open in $H(f)_c$, there is a $\beta \in A$ such that if $\alpha \geqq \beta$, then $g_\alpha \in C$. For each $\alpha \in A$ with $\alpha \geqq \beta$ there is an $h_\alpha \in H$ with $g_\alpha = fh_\alpha$. It follows that each $h_\alpha \in C_1$, and therefore there is an h in $C_1 \cap H$ such that $fh = g$. Thus $\overline{fH} \subseteqq fH \subseteqq \overline{fH}$. We now have fH is a closed subgroup of $H(f)_c$, and therefore $fH \in \mathscr{C}_c$. The sublemma now follows by Lemma 3.3.

LEMMA 3.7. *If H is a subgroup of S with $H \in \mathscr{C}_c$ and if $f \in E(S)$, then fH is closed.*

Proof. Let $h \in H$; then $\delta(h)\cdot f \leqq \delta(h)$. If $\delta(h)f = \delta(h)$, then $fH = f\delta(h)H = \delta(h)H = H$ which is closed by Lemma 3.3. If $\delta(h)\cdot f < \delta(h)$, then there is a chain of idempotents $e_1 \cdots, e_{q+1}$ which is maximal with respect to the properties: (i) $e_1 = \delta(h)f$ and (ii) $e_{q+1} = \delta(h)$. Observe that since $e_1, \cdots, e_{q+1}$ is maximal, $\dim H(e_i) = \dim H(e_{i+1}) - 1$ for $i = 1, 2, \cdots, q$. If fH is not closed, then there is an integer p, $1 \leqq p \leqq q$ such that e_pH is not closed and $e_{p+1}H$ is closed. Since $e_pH = (e_p \cdot e_{p+1})H = e_p(e_{p+1}H)$ and since $e_{p+1}H \in \mathscr{C}_c$, e_pH is closed (sublemma). Thus e_pH is both closed and not closed which is impossible; thus it follows that fH must be closed.

Now that one has Lemma 3.7 it is easy to prove the following corollary.

COROLLARY 3.8. (i) *For each x in S, $xH(1)_c$ is closed.*

(ii) *If U is a nonempty compact subset of S, then $U\cdot H(1)_c$ is closed.*

THEOREM 3.9. *Let $R = \{(x, y) \in S \times S \mid xH(1)_c = yH(1)_c\}$. Then R is a congruence, and S/R is a locally compact semigroup with the following properties*:

(i) *If θ is the natural map from S onto S/R, then θ is an open map and $\theta\ (H(s)) \cong H(s)/(\delta(s)H(1)_c)$ for all s in S.*

(ii) *The corestriction of $\theta|_{E(S)}$ to $E(S/R)$ is an isomorphism.*

Proof. Clearly R is a congruence. Since $H(1)$ acts as a group of homeomorphisms on S and since $\theta^{-1}(\theta(A)) = A\cdot H(1)_c$ for all $A \neq \varnothing$, it follows that θ is an open map. Since θ is an open map, S/R is locally compact and also multiplication is continuous. We now show S/R is Hausdoff. Let $x, y \in S$ with $xH(1)_c \neq yH(1)_c$. Since $yH(1)_c$ is closed (Corollary 3.8) and since S is a locally compact (thus regular) Hausdorff space, there is a compact neighborhood N_x of x with $N_x \cap yH(1)_c = \varnothing$. Thus $y \notin N_x\cdot H(1)_c$ which is closed by Corollary 3.8, and using the fact that S is regular we obtain a compact neighborhood N_y of y with $N_y \cap (N_x\cdot H(1))_c = \varnothing$. It follows that $(N_y\cdot H(1)_c) \cap (N_x\cdot H(1)_c) = \varnothing$, and thus S/R is Hausdorff. This completes the proof.

REMARK. We wish to point out that each maximal subgroup of S/R is connected, and thus $H(\theta(s))_c$ is compact for each s in S.

LEMMA 3.10. *Let $T \in \mathscr{S}$ with $K(T)$ compact. Then for each nonnegative integer n there is a T_n in $\mathscr{S}$ and a surmorphism α_n: $T \to T_n$ in $\mathscr{S}$ satisfying*:

(a) *The corestriction of $\alpha_n|_{E(S)}$ to $E(T_n)$ is an isomorphism.*

(b) *If $x \in T$ with $\dim H(x) \leqq n$, then $\alpha_n(H(x)) = H(\alpha_n(x)) \cong H(x)/H(x)_c$.*

(c) *If* $x \in T$ *with* $\dim H(x) > n$, *then the corestriction of* $\alpha|_{H(x)}$ *to* $H(\alpha(x))$ *is an isomorphism.*

Proof. The proof is by induction. Let $R_0 = \{(x, y) \mid x = y$ or $x \in K(T)$ and $y \in K(T)\}$. Clearly R_0 is a congruence, and since $K(T)$ is compact, it follows that T/R_0 is a locally compact semigroup. Let α_0 be the natural map from T onto $T/R_0 = T_0$. Then, clearly, α_0 and T_0 satisfy (a)-(c) for $n = 0$.

Let k be a nonnegative integer such that there is a $T_k \in \mathscr{S}$ and a surmorphism $\alpha_k\colon T \longrightarrow T_k$ satisfying (a)-(c). If $k \geqq \dim H(1)$, then let $T_{k+1} = T_k$ and $\alpha_{k+1} = \alpha_k$. Then T_{k+1} and α_{k+1} satisfy (a)-(c). If $k < \dim H(1)$, let $A = \{e \in E(T_k) \mid \dim H(e) = k + 1\}$, and let $\hat{T}_k = \{x \in T_k \mid x \in \overline{H(e)}$ for some e in $A\}$. For each e in A let $\psi_e\colon S \longrightarrow eS$ be the morphism defined by $\psi_e(s) = es$. Then $\psi_e^{-1}(H(e)) \cap \hat{T}_k = H(e)$, and thus each $H(e)$ is open relative to $\hat{T}_k$. Let $R_{k+1} = \{(x, y) \in T_k \times T_k \mid x = y$ or $\delta(x) = \delta(y) \in A$ and $x \in yH(\delta(y))_c\}$. It is easy to show that R_{k+1} is a congruence. By Proposition 3.6 and since $K(T_k) = \{0\}$, each $H(e)_c$ is compact. Since each $H(e)_c$ is compact and since each $H(e)$ with $e \in A$ is open in $\hat{T}_{k+1}$, it follows that T_k/R_{k+1} is a locally compact semigroup. Let $T_{k+1} = T_k/R_{k+1}$ and $\alpha_{k+1} = \eta\alpha_k$, where η is the natural map from T_k onto T_k/R_{k+1}. Then $T_{k+1} \in \mathscr{S}$ and $\alpha_{k+1}\colon T \longrightarrow T_{k+1}$ is a surmorphism satisfying (a)-(c) for $n = k + 1$. The theorem now follows by induction.

THEOREM 3.11. *Let* $S \in \mathscr{S}$. *Then there is a* $T \in \mathscr{S}$ *and a surmorphism* $\alpha\colon S \longrightarrow T$ *in* $\mathscr{S}$ *satisfying*:

(i) *The corestriction of* $\alpha|_{E(S)}$ *onto* $E(T)$ *is an isomorphism.*

(ii) *Each H-class of* T *is a real vector group.*

Proof. By Proposition 3.5, there is an isomorphism $\beta\colon S \longrightarrow V \times \bar{T}$ where V is a real vector group and where $\bar{T} \in \mathscr{S}$ with $K(\bar{T}) \in \mathscr{C}_c$. By first applying Theorem 3.9 and then Lemma 3.10 for $n = \dim H(1)$ one can obtain a surjective morphism $\beta_1\colon \bar{T} \longrightarrow T_n$ which preserves the structure of $E(\bar{T})$ and where the H-class of T_n are real vector groups. Let $T = V \times T_n$ and $\alpha\colon S \longrightarrow V \times T_n$ be the map defined by $\alpha(s) = (p_{r_1}(\beta(s)), \beta_1(p_{r_2}(\beta(s)))$. Then clearly T and $\alpha\colon S \longrightarrow T$ satisfy the conditions of the theorem.

4. Let $\mathscr{S}_1$ denote the full subcategory of $\mathscr{S}$ whose objects S have the property that $E(S) \cong Z^q$ for some nonnegative integer q. In this section we characterize the objects in $\mathscr{S}_1$. The fact that there are objects in $\mathscr{S}$ that are not in $\mathscr{S}_1$ is demonstrated by J. G. Horne, Jr., in [6]. However, if $S \in \mathscr{S}$ with $\dim H(1) \leqq 2$, then it is shown

that $S \in \mathscr{S}_1$.

Let R_+ denote the multiplicative group of positive real numbers, and recall that R^x denotes the multiplicative semigroup of nonnegative real numbers.

LEMMA 4.1. *Let E be a Hausdoff topological space which is the disjoint union of $R_+ \times R^x$ and a singleton set $\{w\}$, where $R_+ \times R^x$ has the product topology. If $\{w\} \cup (R_+ \times \{0\})$ is homeomorphic to R^x with $w \in \overline{(0, 1] \times \{0\}}$, then E is not locally compact at w.*

Proof. We assume E is locally compact at w and show that this assumption leads to the conclusion that R^x is compact. Let U be an open neighborhood of w with $\bar{U}$ compact. Then $\bar{U}\backslash U$ is a compact subset of $R_+ \times R^x$. Since $w \cup (R_+ \times \{0\})$ is homeomorphic to R^x with $\overline{(0, 1] \times \{0\}} = ((0, 1] \times \{0\}) \cup \{w\}$, there is an a in R_+ with $\{(x, 0) \mid 0 < x < a\} \subseteqq U$. For each b in R_+ with $0 < b < a$ either $\{b\} \times R^x \subseteqq U$ or $(\{b\} \times R^x) \cap (\bar{U}\backslash U) \neq \varnothing$. To see this, assume $(\{b\} \times R^x) \cap (\bar{U}\backslash U) = \varnothing$. Then $\{b\} \times R^x$ is the disjoint union of the two relatively open sets $(E/\bar{U}) \cap (\{b\} \times R^x)$ and $U \cap (\{b\} \times R^x)$. Since $\{b\} \times R^x$ is connected and $\{b\} \times R^x \cap U \neq \varnothing$, $(E\backslash \bar{U}) \cap (\{b\} \times R^x) = \varnothing$ and hence $\{b\} \times R^x \subseteqq U$.

We now prove there is a $r_0 < a$ in R_+ satisfying; if $b \in R_+$ and $b \leqq r_0$, then $\{b\} \times R^x \subseteqq U$. If this were not the case, then by the above there would exists a sequence $\{b_n\}_{n=1}^{\infty}$ in R_+ such that $\{b_n, 0)\}_{n=1}^{\infty}$ converges to w, and each $(\{b_n\} \times R^x) \cap (\bar{U}\backslash U) \neq \varnothing$. For each positive integer n let x_n be an element of R^x such that $(b_n, x_n) \in \bar{U}\backslash U$. Since $\bar{U}\backslash U$ is a compact subset of $R_+ \times R^x$, the sequence $\{(b_n, x_n)\}_{n=1}^{\infty}$ has a cluster point (b, x). Thus $\{(b_n, 0)\}_{n=1}^{\infty}$ converges to w and clusters to $(b, 0)$ which is impossible. Thus we now can conclude that there is a r_0 in R_+ such that if $b \in R_+$ with $b \leqq r_0$, then $\{b\} \times R^x \subseteqq U$. We point out at this point that if $b \in R_+$ and $b \leqq r_0$, then $\overline{\{b\} \times R^x} = \{w\} \cup \{b\} \times R^x$.

For each l in R_x, $\{(r, l) \mid r_0 \leqq r\}$ is connected, and $(r_0, l) \in U$. Thus a similar argument to the one above proves there is an l_0 in R^x such that if $l \geqq l_0$ then $\{(r, l) \mid r_0 \leqq r\} \subseteqq U$. Similarly, there is a $t_0 \in R_+$ with $t_0 \geqq r_0$ and such that if $t \in R_+$ with $t \geqq t_0$, then $\{(t, l) \mid 0 \leqq l \leqq l_0\} \subseteqq U$. Let $B = [r_0, t_0] \times [0, l_0]$ which is a compact subset of $R_+ \times R^x$. It is easy to show that $E\backslash B \subseteqq U$ and thus $E = \bar{U} \cup B$ and is compact. In particular, $(R \times \{0\}) \cup \{w\}$ is compact and homeomorphic to R^x. This is the desired contradiction.

THEOREM 4.2. *If S is a member of $\mathscr{R}$ with* $\dim H(1) = 2$, *then* $E(S) \cong Z^2$.

Proof. Since $S \in \mathscr{R}$, S has a zero. By Corollary 3.2, there is an element f in $E(S)$ with $\dim H(f) = 1$.

Case 1. There is only one f in $E(S)$ with $\dim H(f) = 1$. That is, $E(S) = \{0, e, 1\}$. By [7], $S\backslash\{0\} \cong R_+ \times R^x$. By [5] and since $\overline{R_+ \times \{0\}} = (R_+ \times \{0\}) \cup \{0\}$, $\overline{R \times \{0\}}$ is homeomorphic to R^x. By applying Lemma 4.1 we have S is not locally compact at $\{0\}$. Thus Case 1 is impossible.

Case 2. There are exactly two idempotents e_1 and e_2 with $\dim H(e_1) = \dim H(e_2) = 1$. Clearly in this case $E(S) \cong Z^2$.

Case 3. There are at least three idempotents e_1, e_2, e_3 with $\dim H(e_i) = 1$. Let P_1, and P_2 be one-parameter subgroups of $H(1)$ with $\bar{P}_i = P_i \cup \{e_i\}$ (Proposition 3.5). Let $\{s_\alpha\}$ be a net in $H(1)$ which converges to e_3. Since $S\backslash H(1)$ is an ideal, $\{s_\alpha^{-1}\}$ does not have a cluster point. Since $H(1) = P_1 \cdot P_2$, there are nets $\{s_{1\alpha}\} \subseteqq P_1$ and $\{s_{2\alpha}\} \subseteqq P_2$ such that $s_{1\alpha} \cdot s_{2\alpha} = s_\alpha$ for all α. By [3] and since $\{s_\alpha^{-1}\}$ does not have a cluster point, either $\{s_{1\alpha}\}$ clusters to e_1 and $\{s_{2\alpha}^{-1}\}$ clusters to e_2 or $\{s_{1\alpha}^{-1}\}$ clusters to e_1 and $\{s_{2\alpha}\}$ clusters to e_2. But the former implies $e_1 = e_3 \cdot e_2$, and the latter implies $e_2 = e_1 \cdot e_3$. Since $e_3 \cdot e_2 = 0$ and $e_1 \cdot e_3 = 0$, either $e_1 = 0$ or $e_2 = 0$. This is the desired contradiction. Thus Case 3 is impossible.

LEMMA 4.3. *Let S be a member of $\mathscr{R}$ with* $\dim H_S(1) \geqq 2$, *and let $e \in E(S)$ with $H(e) \cong R_+$. Then there is an f in $E(S)$, such that* $\dim H(f) = \dim H_S(1) - 1$ *and that* $ef = 0$.

Proof. By Corollary 3.2 and since $\dim H_S(1) \geqq 2$, there is an idempotent e_1 in S such that $e < e_1$ and that $\dim H(e_1) = \dim H(e) + 1 = 2$. Let $T = \overline{H(e_1)}$, then T is a member of $\mathscr{R}$ and $\dim H_T(1) = 2$. Thus, by applying Theorem 4.2 to T one observes that there is an f in $E(T)^*$ (and thus in $E(S)^*$) such that $f \neq 0$ and that $ef = 0$. Let f be a maximal such idempotent with respect to $f \neq 0$ and $ef = 0$.

Claim. $\dim H(f) = \dim H_S(1) - 1$. If this were not the case, then applying Corollary 3.2 two time we observe there are idempotents f_1 and f_2 such that $f < f_1 < f_2$ and $\dim H(f) = \dim H(f_1) - 1 = \dim H(f_2) - 2$. By applying Proposition 3.4 to $\overline{H(f_2)}$ we observe there is a subsemigroup $R \subseteqq \overline{H(f_2)}$ such that $K(R) = \{f\}$ and $\dim H_R(1) = 2$. By Theorem 4.2, there is an idempotent f_3 in $E(R)^*$ (and thus $E(S)^*$) such that $f_3 \neq f$ and $f_3 \cdot f_1 = f$. But since f_1 and f_3 are elements of $E(S)^*$ which are larger that f, $ef_1 \neq 0$ and $ef_3 \neq 0$. In fact, since $ef_1 \leqq e$ and $ef_3 \leqq e$, $ef_1 = ef_2 = e$. However, $0 = ef = e(f_1 f_3) = ef_3 = e$, and this is the desired contradiction. Therefore, f is maximal in $E(S)^*$. From the proof of Lemma 3.1, we have that f maximal in $E(S)^*$ implies $\dim H(f) = \dim H(1) - 1$. For the remainder of this paper we will use the following notation. If $S \in \mathscr{S}$ and $e \in E(S)$, then $\psi_e \colon S \to eS$ is the morphism defined by $\psi_e(s) = es$ for all s in S.

We omit the proof of the next lemma since the proof is straight forward.

LEMMA 4.4. (i) *If f and e are element of $E((R^x)^n)$ with* $\dim H(f) = 1, \dim H(e) = n - 1$ *and if* $\psi_f^{-1}(f) \cap \psi_e^{-1}(e) = \{1\}$, *then the morphism* $m: \psi_f^{-1}(f) \times \psi_e^{-1}(e) \rightarrow (R^x)^n$, *defined by* $m(s, t) = st$, *is an isomorphism.*

(ii) *If* $e \in E(S)$ *with* $\dim H(e) = p$, *then* (a) $\psi_e^{-1}(e) \cong (R^x)^{n-p}$ *and* (b) $\psi_e[(R^x)^n] \cong (R^x)^p$.

LEMMA 4.5. *If* $\alpha: (R^x)^n \rightarrow (R^x)^n \in \mathscr{R}$ *is a surmorphism with* $\alpha(E(R^x)^n) = E((R^x)^n)$, *then α is an isomorphism.*

Proof. The proof is by induction on $\dim H(1)$. The lemma is trivially true for $n = 0$. If $n = 1$, then $\alpha(R_+)$ is a dense connected subgroup of R^x and thus $\alpha(R_+) = R_+$. By [2, p. 84], $\alpha|_{R_+}: R_+ \rightarrow R_+$ is an isomorphism, and thus it follows that α is bijective. We show α is a closed map. Let A be a closed subset of R^x. If $A \subseteqq R_+$, then there is an r in R_+ with $[0, r] \cap A = \emptyset$. Thus $\alpha(A)$ is closed in R_+ and $[0, f(r)) \cap \alpha(A) \subseteqq [0, f(r)] \cap \alpha(A) = \emptyset$. Since $[0, f(r))$ is open in R^x, $0 \notin \overline{\alpha(A)}$, and thus it follows that $\overline{\alpha(A)} = \alpha(A)$. If $0 \in A$, then either $A = R^x$ or there is an r in R_+ with $r \notin A$. If $A = R^x$, then clearly $\alpha(A)$ is closed. If there is an r in R_+ with $r \notin A$, then $A = ([0, r] \cap A) \cup ([r, \infty) \cap A)$. We now have

$$\begin{aligned}\alpha(A) &= \alpha([0, r] \cap A) \cup ([r, \infty) \cap A)] \\ &= \alpha([0, r] \cap A) \cup \alpha([r, \infty) \cap A)\ .\end{aligned}$$

Since $[0, r] \cap A$ is compact, $\alpha([0, r] \cap A)$ is compact, thus closed, and by the first case $\alpha([r, \infty) \cap A)$ is closed. We now have α is a closed bijection and thus an isomorphism.

Let n be an integer larger than 1 such that the lemma is true for all nonnegative integers less than n. Let S denote $(R^x)^n$, and define $\hat{\alpha}: E(S) \rightarrow E(S)$ by $\hat{\alpha}(e) = \alpha(e)$ for all e in $E(S)$. Since $\hat{\alpha}$ is bijective and since $E(S)$ is finite, $\hat{\alpha}$ is an isomorphism. For each e in $E(S)$ define $\psi_e: S \rightarrow eS$ by $\psi_e(s) = es$ for all s in S. Let $e_1 = (0, 1, 1, \cdots, 1)$ and $e_2 = (1, 0, 0, \cdots, 0)$, and let $A = \psi_{e_1}^{-1}(e_1)$ and $B = \psi_{e_2}^{-1}(e_2)$. Then $A \cong R^x$, $B \cong (R^x)^{n-1}$ and $e_1 \cdot e_2 = 0$. Define $F: A \times B \rightarrow S$ by $F(a, b) = ab$; then, by Lemma 4.4i, F is an isomorphism. Let $f_1 = \alpha(e_1)$ and $f_2 = \alpha(e_2)$. We now show $\alpha(A) = \psi_{e_1}^{-1}(f_1) \cong R^x$, $\alpha(B) = \psi_{f_2}^{-1}(f_2) \cong (R^x)^{n-1}$, and $\alpha(A) \cap \alpha(B) = \{1\}$. From which it will follow by Lemma 4.4i that the morphism $G: \alpha(A) \times \alpha(B) \rightarrow S$, defined by $G(a, b) = ab$, is an isomorphism. Let $A_1 = \psi_{f_1}^{-1}(f_1)$ and $A_2 = \psi_{f_2}^{-1}(f_2)$. Since $\hat{\alpha}$ is an isomorphism, $\hat{\alpha}$ preserves the less than order on $E(S)$; thus $\dim H(f_1) = \dim H(e_1) = n - 1$

and $\dim H(f_2) = \dim H(e_2) = 1$. Therefore, $A_1 \cong R^x$ and $A_2 \cong (R^x)^{n-1}$ (Lemma 4.4iia). If $A_1 \cap A_2 \neq \{1\}$, then either $f_1 \in A_1 \cap A_2$ or there is an element $g \in H(1) \cap A_1 \cap A_2$ with $g \neq 1$. Since $f_1 \cdot f_2 = \alpha(e_1) \cdot \alpha(e_2) = \alpha(e_1 e_2) = \alpha(0) = 0$, $f_1 \notin A_2$, and thus there is a $g \in H(1) \cap A_1 \cap A_2$ with $g \neq 1$. Since $A_1 \cong R^x$ either $\{g^n\}_{n=1}^{\infty}$ converges to f_1 or $\{(g^{-1})^n\}_{n=1}^{\infty}$ converges to f_1 [3]. But both imply $f_1 \in A_2$ which is impossible by the above. Thus $A_1 \cap A_2 = \{1\}$. Clearly, $\alpha(A) \subseteq A_1$. Let $t \in A_1$. Since $\alpha(S) = \alpha(A \cdot B) = \alpha(A) \cdot \alpha(B)$, there is an element $a \in \alpha(A)$ and $b \in \alpha(B)$ such that $t = ab$. It follows that $f_1 = f_1 t = f_1 a \cdot b = f_1 b$ which implies $b \in A_1$. But $\alpha(B) \subseteq B_1$ and $B_1 \cap A_1 = \{1\}$; thus $b = \{1\}$. The proof that $\alpha(B) = B_1$ is similar and will therefore be omitted. We now have the following commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "F^{-1}"'] & S \\ A \times B \arrow[r, "\alpha|_A \times \alpha|_B"] & \alpha(A) \times \alpha(B) \arrow[u, "G"'] \end{tikzcd} .$$

By the inductive hypothesis, $\alpha|_A: A \to \alpha(A)$ and $\alpha|_B: B \to \alpha(B)$ are isomorphisms. The lemma now follows.

LEMMA 4.6. *Let X, Y and Z be Hausdorff spaces and assume $F: X \times Y \to Z$ is a continuous surjection. If there are continuous surjections $\alpha: Z \to X$ and $\beta: Z \to Y$ such that the diagram*

$$\begin{tikzcd} & X \times Y \arrow[dl, "P_{r_1}"'] \arrow[d, "F"] \arrow[dr, "P_{r_2}"] & \\ X & Z \arrow[l, "\alpha"'] \arrow[r, "\beta"'] & Y \end{tikzcd}$$

is commutative, then F is a homeomorphism.

Proof. The inverse of F is given by $z \mapsto (\alpha(z), \beta(z))$ which is clearly continuous.

THEOREM 4.7. *If S is an object in both $\mathscr{R}$ and $\mathscr{S}_1$, then $S \cong (R^x)^n$ where $n = \dim H_S(1)$.*

Proof. The proof is by induction on $\dim H(1)$. The claim for $\dim H(1) = 1$ is proven in [5]. Let n be an integer larger than 1 such that the claim is true for all positive integers less than n. Let e be an idempotent with $e > 0$ and $eS \cong R^x$ (Corollary 3.2). By Lemma 4.3 there is an idempotent f with $f \neq 0$, $\dim H(f) = n - 1$ and $ef = 0$. Let $A = \psi_f^{-1}(f)$ and $B = \psi_e^{-1}(e)$. Then by the inductive hypothesis,

$A \cong R^x$ and $B \cong (R^x)^{n-1}$. Also $\psi_e^{-1}(H(e)) \cong H(e) \times B \cong R_+ \times B$ (Proposition 3.4). Now define a morphism $F: A \times B \to S$ by $F(a, b) = ab$. Observe that $\psi_e(F(a, b)) = eab = ea$ and $\psi_f(F(a, b)) = fb$. We now show $S = A \cdot B$. Since $E(S) \cong Z^n$ it follows that $E(S) = E(A) \cdot E(B)$. Let $s \in S$; then $\delta(s) = e_1 \cdot f_1$ for some $e_1 \in E(A)$ and $f_1 \in E(B)$. Also, $s = \delta(s) \cdot g$ for some $g \in H_S(1)$. Since $H_A(1) \cap H_B(1) = \{1\}$ (see proof that $A_1 \cap B_1 = \{1\}$ in Lemma 4.5), $g = a \cdot b$ for some $a \in H_A(1)$ and $b \in H_B(1)$. Thus $s = \delta(s)g = \delta(s)ab = e_1 f_1 ab = (e_1 a)(f_1 b) \in A \cdot B$. Clearly, $\psi_e(A) = eA \subseteqq eS$. Let $t \in eS$; then $t = ea \cdot b$ for some $a \in A$ and $b \in B$. Thus $t = eab = eb \cdot a = ea$ and hence $eA = eS$. By Lemma 4.5, $\psi_e|_A: A \to eS$ is an isomorphism. Similarly it can be shown that $fB = fS$ and thus, by Lemma 4.5, $\psi_f|_B: B \to fS$ is an isomorphism. We now have the following diagram

$$\begin{tikzcd}
R^x \cong A \arrow[d, "\psi_e|_A"] & A \times B \arrow[l, "P_{r_1}"'] \arrow[r, "P_{r_2}"] \arrow[d] & B \cong (R^x)^{n-1} \arrow[d, "\psi_f|_B"] \\
eS & S \arrow[l, "\psi_e"'] \arrow[r, "\psi_f"] & fS
\end{tikzcd}$$

which can be reduced to

$$\begin{tikzcd}
& A \times B \arrow[dl, "P_{r_1}"'] \arrow[d, "F"'] \arrow[dr, "P_{r_2}"] & \\
A & S \arrow[l, "(\psi_e^{-1}|_A)\circ\psi_e"] \arrow[r, "(\psi_f^{-1}|_B)\circ\psi_f"'] & B\,.
\end{tikzcd}$$

Thus by Lemma 4.6, F is an isomorphism, and the theorem now follows by induction.

DEFINITION. An object S in $\mathscr{S}$ is an *H-semigroup* if (i) $H_S(1) \cong R_+$ and (ii) $K(S)$ is compact.

LEMMA 4.8. *Let S be a object in $\mathscr{S}_1$ having the added properties that* (i) *$H_S(1)$ is a real vector group of dimension n and* (ii) *$K(S)$ is compact. Then there are subsemigroups $S_1, \cdots, S_n$ of S which are H-semigroups, the morphism m: $\times_{i=1}^n S_i \to S$ defined by $m((s_1, \cdots, s_n)) = s_1 \cdot s_2 \cdot \cdots \cdot s_n$ is a surmorphism which preserves the H-class structure of $\times_{i=1}^n S_i$,* and also *m induces an isomorphism on the groups of units. Further, for each i there is an idempotent e_i with* $\dim H(e_i) = n - 1$ *and $S_i = \psi_{e_i}^{-1}(H(e_i)_c)$.*

Proof. Since $E(S) \cong Z^n$, there are exactly n-idempotents $e_1, \cdots, e_n$ in S with $\dim H(e_i) = n - 1$. By Proposition 3.4 and since $H_S(1)$ is a real vector group, each $\psi_{e_i}^{-1}(H(e_i)_c)$, is an H-semigroup. Let $S_i = \psi_{e_i}^{-1}(H(e_i)_c)$, and let $F: S \to (R^x)^n$ be a surmorphism which preserves the H-class structure of S (Proposition 3.11 then Theorem 4.7). Since F preserves the H-class structure of S, $\dim H(e_i) = \dim H(F(e_i)) = n-1$ for $i = 1, 2, \cdots, n$ and, also, $F(S_i) = \psi_{e_i^1}^{-1}(H(e_i^1)) \cong R^x$ for $i = 1, 2, \cdots, n$, where $e_i^1 = F(e_i)$. Using the structure of $(R^x)^n$ we know $\psi_{e_i^1}^{-1}(F(e_i^1)) \cong R^x$ if and only if there is an integer $j(i)$, $1 \leqq j(i) \leqq n$ such that

$$P_{r_{j(i)}} \mid \psi_{e_i^1}^{-1}(F(e_i^1)) \colon \psi_{e_i^1}^{-1}(F(e_i^1)) \twoheadrightarrow R^x$$

is an isomorphism. For each i, $i = 1, 2, \cdots, n$ let $\pi_i\colon S_i \to S_i/K(S_i)$ be the natural map where $S/K(S_i)$ denotes the Rees quotient semigroup. Since each $K(S_i)$ is compact [3], π_i is a closed map. Thus for each i there is a bijective morphism $\beta_i\colon S_i \twoheadrightarrow R^x$ such that the following diagram commutes

$$\begin{tikzcd} S_i \arrow[r, "P_{r_{j(i)}} \circ F|S_i"] \arrow[d, "\pi i"'] & R^x \\ S/K(S_i) \arrow[ur, "\beta i"'] & \end{tikzcd} .$$

By Lemma 4.5 each β_i is an isomorphism. Since each $K(S_i)$ is compact, it is easy to show that a net $\{g_\alpha\}_{\alpha \in A} \subseteqq S_i$ has a cluster point if and only if $\{\pi_i(g_\alpha)\}_{\alpha \in A}$ has a cluster point. Thus it follows that $\{g_\alpha\}_{\alpha \in A} \subseteqq S_i$ has a cluster point if and only if $\{P_{r_{j(i)}}(F(g_\alpha))\}_{\alpha \in A}$ has a cluster point.

Let $x \in S$ and let $\{g_\alpha\}_{\alpha \in A}$ be a net in $H_S(1)$ which converges to x. Then for each α there are elements $g_i(\alpha) \in S_i$ $i = 1, 2, \cdots, n$ such that $g_\alpha = g_1(\alpha) \cdot g_2(\alpha) \cdot \cdots \cdot g_n(\alpha)$. Since $P_{r_{j(i)}} F(g_i(\alpha)) = P_{r_{j(i)}}(F(g_\alpha))$ for $i = 1, 2, 3, \cdots, n$ and since $P_{r_{j(i)}}(F(g_\alpha))$ has a cluster point and by the above, each $\{g_i(\alpha)\}_{\alpha \in A}$ has a cluster point. Clearly, we can choose a subnet $\{g_\alpha\}_{\alpha \in B}$ such that each $\{g_i(\alpha)\}_{\alpha \in B}$ converges. It now follows that $x \in m(\times_{i=1}^n S_i)$. Clearly, m induces an isomorphism on the groups of units.

THEOREM 4.9. *Let $S \in \mathscr{S}_1$. Then $S \cong T \times \boldsymbol{R}^n$ for a suitable n and where T is an object in $\mathscr{S}_1$ satisfying the following: There are subsemigroups $S_1, \cdots, S_n$ of T with each S_i an H-semigroup and a surmorphism $m\colon H_T(1)_c \times (\times_{i=1}^n S_i) \twoheadrightarrow T$ which preserves the H-class structure and which induces an isomorphism on the groups of units. Further, there are surmorphisms $G_1\colon S \twoheadrightarrow (R^x)^n$* and *$G_2\colon H_T(1)_c \times (\times_{i=1}^n S_i) \twoheadrightarrow (R^x)^n$ such that the following diagram is commutative*

$$\begin{tikzcd} H_T(1)_c \times (\times_{i=1}^m S_i) \arrow[r, "m"] \arrow[dr, "G_2"'] & T \arrow[d, "G_1"] \\ & (R^x)^n \end{tikzcd}$$

Proof. By Proposition 3.5, $S \cong T \times \boldsymbol{R}^m$ for a suitable choice of m, where $T \in \mathscr{S}$ with $K(T) \in \mathscr{C}_c$. Since $E(S) \cong Z^n$ for some n and since $E(S) \cong E(T)$, $T \in \mathscr{S}_1$. Using Lemma 3.1 and Corollary 3.2, it is easy to see that $\dim H_T(1) = n$. Since $E(S) \cong Z^n$, there are exactly n idempotents $e_1, \cdots, e_n$ such that $\dim H(E_i) = n - 1$. For each e_i let C_i be a compact subgroup of $H(e_i)_c$ which is open relative to $H(e_i)_c$. It follows from Proposition 3.4 and the fact that each $H(e_i)$ is open in $T \backslash H_T(1)$, that each $\psi_{e_i}^{-1}(C_i)$ is a locally compact semigroup which contains a dense group whose complement is compact. Since each $\psi_{e_i}^{-1}(C_i) \in \mathscr{S}$ and by [7], there is a one-parameter subgroup $P_i \subseteqq \psi_{e_i}^{-1}(C_i) \cap H_T(1)$ such that $\bar{P}_i \cap C_i \neq \emptyset$. For each i let $S_i = \bar{P}_i$; then each S_i an H-semigroup. Let m: $H_T(1)_c \times (\times_{i=1}^n S_i) \rightarrow T$ be a morphism defined by $m(g, s_1, \cdots, s_n) = g \cdot s_1 \cdot s_2 \cdot \cdots \cdot s_n$ and let m_1: $\times_{i=1}^n S_i \rightarrow T$ be the morphism defined by $m_1(s) = m(1, s)$ for all s in $\times_{i=1}^n S_i$.

Let T/R be the semigroup constructed as in Theorem 3.9 and let F: $T \rightarrow T/R$ be the natural map. Since F preserves the H-class structure, $\dim H(F(e_i)) = n - 1$ for each i. Since for each i $F(K(S_i))$ is a compact ideal for $F(P_i)$, $\overline{F(P_i)} = F(P_i) \cup F(K/S_i))$ [5]; thus $F(S_i) = \overline{F(P_i)}$. Also, $H(F(e_i))_c$ is a compact ideal for $F(P_i)$; thus $F(S_i) = \overline{F(P_i)} = F(P_i) \cup H(F(e_i))_c$. It now follows from Lemma 4.8 that $F(m_1(\times_{i=1}^n S_i)) = T/R$ and thus $m_i(\times_{i=1}^n S_i) \cdot H_T(1) = T$. Therefore, m is a surmorphism.

Let $T_1 = \overline{m_1(\times_{i=1}^n S_i)}$. Since $E(T) = E(m_i(\times_{i=1}^n S_i)) \cong Z^n$, $E(T) \cong Z^n$, and thus it follows that $\dim H_{T_1}(1) = n$. Let F_1: $T_1 \rightarrow T_1/R_1$ be the natural map where T_1/R_1 is the semigroup guaranteed by Theorem 3.9. Let $H_1 = H_{T_1}/R_1(1)$. Then H_1 is an n-dimensional vector group with $\overline{F_1(m_1(\times_{i=1}^n P_i))} = H_1$. Thus by P_2 there is a morphism β: $H_1 \rightarrow \times_{i=1}^n P_i$ such that $F_1 m_1 \beta = I_{T_1/R_1}$. It follows that the inverse of $F_1|_{m_1}(\times_{i=1}^n P_i)$ is the corestriction of $m_1\beta$ to $m_1(\times_{i=1}^n P_i)$. Thus $m_1(\times_{i=1}^n P_i)$ is a locally compact subgroup $H_{T_1}(1)$ and thus closed. Therefore, it follows that the corestriction of $m_1 | \times_{i=1}^n P_i$: $\times_{i=1}^n P_i \rightarrow \times_{i=1}^n P_i$ is an isomorphism. Since $H_T(1) = m_i(\times_{i=1}^n P_i) \cap H_T(1)_c$ and $m_1(\times_{i=1}^n P_i) \cap H_T(1)_c = \{1\}$, it now easily follows that m induces an isomorphism on the group of units.

The remainder of the proof follows directly from Theorem 3.11 and Theorem 4.7.

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FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

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In this paper the Lie algebra analogues to groups with property E of Bechtell are investigated. Let $\mathfrak{X}$ be the class of solvable Lie algebras with the following property: if H is a subalgebra of L, then $\phi(H) \subseteqq \phi(L)$ where $\phi(L)$ denotes the Frattini subalgebra of L; that is, $\phi(L)$ is the intersection of all maximal subalgebras of L. Groups with the analogous property are called E-groups by Bechtell. The class $\mathfrak{X}$ is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal N of $L \in \mathfrak{X}$ be the Frattini subalgebra of L. Only solvable Lie algebras of finite dimension are considered here.

The following notation will be used. We let $N(L)$ be the nil radical of L and $S(L)$ be the socle of L; that is, $S(L)$ is the union of all minimal ideals of L. If A and B are subalgebras of L, let $Z_B(A)$ be the centralizer of A in B. The center of A will be denoted by $Z(A)$. If $[B, A] \subseteqq A$, we let $\mathrm{Ad}_A(B) = \{\mathrm{ad}\, b$ restricted to A; for all $b \in B\}$. L' will be the derived algebra of L and $L'' = (L')'$.

PROPOSITION 1. *Let L be a Lie algebra such that L' is nilpotent. Then the following are equivalent*:

(1) $\phi(L) = 0$.

(2) *$N(L) = S(L)$ and $N(L)$ is complemented by a subalgebra.*

(3) *L' is abelian, is a semi-simple L-module and is complemented by a subalgebra.*

Under these conditions, Cartan subalgebras of L are exactly those subalgebras complementary to L'.

Proof. Assume (1) holds. Nilpotency of L' implies $\phi(L) \supseteqq L''$, so L' is abelian and may be regarded as an L/L'-module. We may assume $L' = \sum \oplus V_\rho$, V_ρ indecomposable L/L'-submodules. If M is a maximal subalgebra of L and if $V_\rho \not\subseteqq M$, then $M \cap V_\rho$ is an ideal of L. If S is an L/L'-submodule of V_ρ properly contained between $M \cap V_\rho$ and V_ρ, then $M + S$ is a subalgebra of L properly contained between M and L, contradicting the maximality of M. Therefore M contains all maximal submodules of V_ρ for each ρ. Then $\phi(L) = 0$ implies the intersection of all maximal submodules of V_ρ is zero for each ρ. If $V_1, \cdots, V_s$ are maximal submodules of V_ρ with $V_1 \cap \cdots \cap V_s = 0$ and are minimal with respect to this property, we have $V = V_2 \cap \cdots V_s \neq 0$

and $V \cap V_1 = 0$ so that $V \oplus V_1 = V_\rho$, contradicting indecomposability. Therefore each V_ρ is irreducible and L' is a completely reducible L/L'-module and is also a completely reducible L-module. Since L is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let H be a Cartan subalgebra of L and let L_0 and L_1 be the Fitting null and one component of L with respect to H. Then $L = L_0 + L_1 = H + L_1 \subseteqq H + L'$ shows $L = H + L'$. We claim that $H \cap L' = 0$. If $H \cap L' \neq 0$, then, since L' is abelian, H is nilpotent and L' is a completely reducible L-module, L' is a sum of irreducible H-modules, $U_1, \cdots, U_\sigma$, such that for each U_i $\underbrace{[\cdots[U_i, H]\cdots H]}_{k}=0$ for some k, hence $[U_i, H]=0$. Thus $[H, L' \cap H] = 0$. One sees that each U_i is a central minimal ideal of L, and since $\phi(L) = 0$, U_i is complemented by a maximal subalgebra M. Therefore U_i is a one-dimensional direct summand of L, contradicting $U_i \subseteqq L'$. Hence $L' \cap H = 0$ and H is a complement to L' in L. Since $[H, H] \subseteqq H \cap L' = 0$, H is abelian. Any minimal ideal not in L' satisfies $[L, A] \subseteqq A \cap L' = 0$, so is central. Therefore $S(L) = L' + Z(L)$ and, since H is a Cartan subalgebra, $Z(L) \subseteqq H$. Let H_0 be a complementary subspace to $Z(L)$ in H. One sees that $N(L) = L' + Z(L) + (N(L) \cap H_0) = S(L) + (N(L) \cap H_0)$. If h is a nonzero element in $N(L) \cap H_0$, ad h is nilpotent but not zero which implies $[V_\rho, h] = V_\rho$ for some $V_\rho \subseteqq L'$ and $\overbrace{[\cdots[V_\rho, h]\cdots h]}^{k} = 0$ for some k, a contradiction. Thus $S(L) = N(L)$ and H_0 is a complement. Consequently (1) implies (2).

Assume (2) holds and proceed by induction on the dimension of L. Since $L' \subseteqq N(L) = S(L)$ and minimal ideals are abelian, L' is abelian. If every minimal ideal of L is contained in L', then $S(L) = L'$ and (3) follows. Therefore let A be a minimal ideal of L such that $A \nsubseteqq L'$. Hence $A \nsubseteqq \phi(L)$ and there exists a maximal subalgebra M of L such that $L = M + A$. Since $[L, A] \subseteqq A \cap L' = 0$, A is central, hence one-dimensional. It follows that L is the Lie algebra direct sum of M and A. Since M inherits the condition (2), M satisfies (3) by induction. It now follows that L also satisfies (3).

Assume (3) holds. Then L' is a sum of minimal ideals of L, which we denote by $A_1, \cdots, A_k$, and $L = L' + H$, H a subalgebra of L. Since $H' \subseteqq H \cap L' = 0$, H is abelian. One sees that $L' = [L', H]$ and, consequently, $A_i = [A_i, H]$ for all i. Since $Z_{A_i}(H)$ is central in L, $Z_{A_i}(H)$ is an ideal in L contained in A_i. Since $Z_{A_i}(H) \neq A_i$, $Z_{A_i}(H) = 0$. It follows that H is its own normalizer, hence is a Cartan subalgebra of L. Now $H + A_1 + \cdots + \hat{A}_i + \cdots + A_k$ is a maximal subalgebra of L since any containing algebra has a nonzero projection on A_i which is ad H stable, hence equal to A_i. Therefore $\phi(L) \subseteqq H$ and $\phi(L) \subseteqq H \cap L' = 0$. Hence (1) holds.

That complements to L' are Cartan subalgebras is shown in (3) implies (1). That Cartan subalgebras are complements to L' is shown in (1) implies (2). This completes the proof of Proposition 1.

THEOREM 1. *Let L be a Lie algebra such that L' is nilpotent and $\phi(L) = 0$. Then, for any subalgebra M of L, $\phi(M) = 0$.*

Proof. Suppose $L' \subseteqq M$. Let H be a complement to L' in L, so $H \cap M$ is a complement to L' in M. Since L acts completely reducibly on L' and L' is abelian, H acts completely reducibly on L'. Then, since H is abelian, $H \cap M$ acts completely reducibly on L', hence so does M. Therefore $L' = M' \oplus A$ for some ideal A in M where M acts completely reducibly on M' and $A + (H \cap M)$ is a complementary subalgebra of M' in M. Thus by Proposition 1, $\phi(M) = 0$.

Suppose $L' \not\subseteqq M$. Since $M + L'$ falls in the preceeding case, we may assume $M + L' = L$. Since L' is abelian, $L' \cap M$ is an ideal in L, $M/(L' \cap M)$ complements $L'/(L' \cap M) = (L/L' \cap M)'$ in $L/(L' \cap M)$ and $M/(L' \cap M)$ acts completely reducibly in $L'/(L' \cap M)$, $M/(L' \cap M)$ is a Cartan subalgebra of $L/(L' \cap M)$. Let C be a Cartan subalgebra of M. By Lemma 4 of [1], C is a Cartan subalgebra of L. Thus C is a complement to L' and $C + (L' \cap M) = M$ since $C \subseteqq M$. Hence C is a complement to $L' \cap M$ in M. Since M acts completely reducibly on $L' \cap M$ and $M' \subseteqq L' \cap M$, M acts completely reducibly on M', $L' \cap M = M' \oplus (L' \cap Z(M))$ and, since $Z(M) \subseteqq C$, $Z(M) \cap L' \subseteqq C \cap L = 0$. Therefore $C = M' = M$ and $C \cap M' = 0$. Now M satisfies part (3) of Proposition 1, hence $\phi(M) = 0$.

If L is a solvable Lie algebra it has been shown in [2] that $\phi(L)$ is an ideal of L. We look for a condition on the subalgebras of $L/\phi(L)$ which are necessary and sufficient that $L \in \mathfrak{X}$. In order to do this the following concept is introduced.

We shall say that a Lie algebra L is the reduced partial sum of an ideal A and a subalgebra B if $L = A + B$ and for any subalgebra C of L such that $L = A + C$ and $C \subseteqq B$ then $C = B$. It is noted that if $A \not\subseteqq \phi(L)$, then there exists a $B \neq L$ such that L is the reduced partial sum of A and B. On the other hand, if $A \subseteqq \phi(L)$ and L is the reduced partial sum of A and B, then $B = L$.

LEMMA 1. *Let L be the reduced partial sum of A and B. Then $A \cap B \subseteqq \phi(B)$.*

Proof. Suppose $C = A \cap B \not\subseteqq \phi(B)$. Then B contains a subalgebra D such that $C + D = B$. Then $L = A + B = A + C + D = A + D$. This contradicts the minimality of B.

LEMMA 2. *Let L be the reduced partial sum of A and B. Then* $\phi(L/A) \simeq A + \phi(B)/A$.

Proof. Since $A \cap B \subseteqq \phi(B)$, $A \cap \phi(B) = A \cap B$. Since $L/A \simeq A + B/A \simeq B/A \cap B$, $\phi(L/A) \simeq \phi(B/A \cap B) \simeq \phi(B)/A \cap B = \phi(B)/A \cap \phi(B) \simeq A + \phi(B)/A$.

PROPOSITION 2. *The following are equivalent for the Lie algebra* L:

(1) $L \in \mathfrak{X}$.

(2) *For any subalgebra H of $L/\phi(L)$, $\phi(H) = 0$.*

Proof. Let L satisfy (1) and let $\pi: L \rightarrow L/\phi(L)$ be the natural homomorphism. Then $\phi(\pi(L)) = \pi(\phi(L)) = 0$. Let $\bar{W}$ be a subalgebra of $L/\phi(L)$ and let W be the subalgebra of L which contains $\phi(L)$ and corresponds to $\bar{W}$. Since L satisfies (1), $\phi(W) \subseteqq \phi(L)$. If $\phi(W) = \phi(L)$, then $\phi(\pi(W)) = \pi(\phi(W)) = \pi(\phi(L)) = 0$. Suppose then that $\phi(W) \subset \phi(L)$. Then W can be represented as a reduced partial sum $W = \phi(L) + K$. Let T be a subalgebra of W such that $T/\phi(L) \simeq \phi(W/\phi(L))$. If $T/\phi(L) \neq 0$, then $T = T \cap (\phi(L) + K) = (T \cap \phi(L)) + (T \cap K) = \phi(L) + (T \cap K)$. Consequently there exists an $x \in T \cap K$, $x \notin \phi(L)$. Since $\phi(K) \subseteqq \phi(L)$, $x \notin \phi(K)$ and there exists a maximum subalgebra S of K such that $x \notin S$. We claim that either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in W. Suppose $\phi(L) + S \neq W$ and let J be a subalgebra of W which contains $\phi(L) + S$. Then $S \subseteqq J \cap K$, so, by the maximality of S, either $J \cap K = S$ or $J \cap K = K$. If $J \cap K = S$, then $\phi(L) + S = \phi(L) + (J \cap K) = J \cap (\phi(L) + K) = J \cap W = J$. If $J \cap K = K$, then $J \supseteqq K$ and, since $J \supseteqq \phi(L)$, $J \supseteqq \phi(L) + K = W$, hence $J = W$. Consequently there exist no subalgebras of W properly contained between $\phi(L) + S$ and W, hence either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in W. If $\phi(L) + S = W$, then $\phi(L) + K$ is not a reduced partial sum which is a contradiction. If $\phi(L) + S$ is maximal in W, then $\phi(L) + S/\phi(L) \supseteqq \phi(W/\phi(L)) \simeq T/\phi(L)$. Hence $T \subseteqq \phi(L) + S$. Since $S \subseteqq \phi(L) + S$ and $x \in T \cap K \subset T \subseteqq \phi(L) + S$, $K = \{S, x\} \subseteqq \phi(L) + S$. Then $W = \phi(L) + K \subseteqq \phi(L) + S \subseteqq W$ implies $\phi(L) + K$ is not a reduced partial sum, a contradiction. Hence $\phi(\bar{W}) = T/\phi(L) = 0$ and (2) is satisfied.

If $L/\phi(L)$ satisfies (2), then $\pi(\phi(H)) \subseteqq \phi(\pi(H)) = 0$ for every subalgebra H of L. Then $\phi(H) \subseteqq \phi(L)$ for every subalgebra H of L.

Combining Proposition 2 and Theorem 1 we have

THEOREM 2. *Let L be a Lie algebra such that L' is nilpotent. Then* $L \in \mathfrak{X}$.

THEOREM 3. *Let $L \in \mathfrak{X}$ and let T be a Lie homomorphism of L.*

Then $T(\phi(L)) = \phi(T(L))$.

Proof. $T(\phi(L))$ is always contained in $\phi(T(L))$ by Proposition 1 in [6]. If $N =$ kernel $T \subseteqq \phi(L)$, then equality holds by Proposition 2 in [6]. Suppose $N \not\subseteq \phi(L)$. Let $L = N + K$ be a reduced partial sum. Using Lemma 2, $\phi(T(L)) = \phi(L/N) \simeq N + \phi(K)/N = T(\phi(K))$. Since $T(N + \phi(L)) = T(\phi(L)) \subseteqq \phi(T(L)) = \phi(L/N) = T(N + \phi(K))$, $N + \phi(L) \subseteqq N + \phi(K) \subseteqq N + \phi(L)$. Hence $N + \phi(L) = N + \phi(K)$ and $\phi(T(L)) = T(\phi(K)) = T(\phi(L))$.

THEOREM 4. *Let $L \in \mathfrak{X}$. Necessary conditions that an ideal N of L be the Frattini subalgebra of L are that*

(1) $\phi(\mathrm{Ad}_N(L)) = \mathrm{Ad}_N(\phi(L))$.

(2) *There exists a subalgebra M of L such that* $M/N \simeq \mathrm{Ad}_N(L)/\mathrm{Ad}_N(\phi(L))$.

Proof. (1) Let T be the mapping from L into the derivation algebra of N by $T(x) = \mathrm{ad}\, x$ restricted to N for all $x \in L$. Then $T(\phi(L)) = \mathrm{Ad}_N(\phi(L)) = \phi(T(L)) = \phi(\mathrm{Ad}_N(L))$.

(2) Let $M = Z_L(\phi(L))$. Suppose that $M \not\subseteq \phi(L)$ and let $F = L/\phi(L)$ and $A = (M + \phi(L))/\phi(L)$. Since $\mathrm{Ad}_{\phi(L)}(L) \simeq L/M$ and $\mathrm{Ad}_{\phi(L)}(\phi(L)) \simeq \phi(L)/Z(\phi(L)) = \phi(L)/M \cap \phi(L) = (M + \phi(L))/M$, $F/A \simeq (L/\phi(L))/(M + \phi(L)/\phi(L)) \simeq L/(M + \phi(L)) \simeq (L/M)/((M + \phi(L))/M) \simeq \mathrm{Ad}_{\phi(L)}(L)/\mathrm{Ad}_{\phi(L)}(\phi(L))$. Since $\phi(F) = 0$, there exists a subalgebra D in F such that F is the reduced partial sum of A and D. Using Proposition 2 and Lemma 1, $A \cap D \subseteqq \phi(D) = 0$, hence $A \cap D = 0$. Let E be the subalgebra of L which contains $\phi(L)$ and corresponds to D. Then $E/\phi(L) \simeq D \simeq F/A \simeq \mathrm{Ad}_{\phi(L)}(L)/\mathrm{Ad}_{\phi(L)}(\phi(L))$. If $M \subseteqq \phi(L)$, then $\mathrm{Ad}_{\phi(L)}(L)/\mathrm{Ad}_{\phi(L)}(\phi(L)) \simeq (L/M)/(\phi(L)/Z(\phi(L))) = (L/M)/(\phi(L)/M \cap \phi(L)) = (L/M)/(\phi(L)/M) \simeq L/\phi(L)$.

Related to part (1) of Theorem 4 are the following results.

THEOREM 5. *Let $L \in \mathfrak{X}$ and let K be an ideal of L containing $\phi(L)$. Then $\phi(\mathrm{Ad}_K(L)) \simeq \mathrm{Ad}_K(K)$ if and only if $K = \phi(L) + Z(K)$.*

Proof. Let T be the Lie homomorphism from L into the derivation algebra of K given by $T(x) = \mathrm{ad}\, x$ restricted to K for each $x \in L$. Then $\phi(\mathrm{Ad}_K(L)) = \phi(T(L)) = T(\phi(L)) = \mathrm{Ad}_K(\phi(L)) \simeq \phi(L)/Z_{\phi(L)}(K) = \phi(L)/(Z(K) \cap \phi(L)) \simeq (\phi(L) + Z(K))/Z(K)$. If $\phi(L) + Z(K) = K$, then $\mathrm{Ad}_K(K) \simeq K/Z(K) = (\phi(L) + Z(K))/Z(K) \simeq \phi(\mathrm{Ad}_K(L))$. If $\phi(L) + Z(K) \subset K$, then $\mathrm{Ad}_K(K) \simeq K/Z(K) \supset (\phi(L) + Z(K))/Z(K) = \phi(\mathrm{Ad}_K(L))$.

THEOREM 6. *Let $L \in \mathfrak{X}$ and let A be an ideal of L contained in $\phi(L)$. Then $\phi(\mathrm{Ad}_A(L)) \simeq \mathrm{Ad}_A(A)$ if and only if $\phi(L) = A + Z_{\phi(L)}(A)$.*

Proof. If $\phi(L) = A + Z_{\phi(L)}(A)$, then $\mathrm{Ad}_A(A) = \mathrm{Ad}_A(\phi(L)) = T(\phi(L)) = \phi(T(L)) = \phi(\mathrm{Ad}_A(L))$.

Conversely, $\mathrm{Ad}_A(L) \simeq L/Z_L(A)$ and $Z_L(A) + A/Z_L(A) \simeq A/Z(A) = \mathrm{Ad}_A(A)$. Then $L/Z_L(A) + A \simeq \mathrm{Ad}_A(L)/\mathrm{Ad}_A(A)$ and $\phi(L/Z_L(A) + A) \simeq \phi(\mathrm{Ad}_A(L)/\mathrm{Ad}_A(A)) = \phi(\mathrm{Ad}_A(L))/\mathrm{Ad}_A(A) = 0$. Hence $\phi(L) \subseteqq Z_L(A) + A$ and $\phi(L) = Z_{\phi(L)}(A) + A$.

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THE GROUP CHARACTER AND SPLIT GROUP ALGEBRAS

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G. J. Janusz defined a splitting ring R for a group G of order n invertible in R. Then, the Brauer splitting theorem was given by G. Szeto which proves the existence of a finitely generated projective and separable splitting ring for G. Let M be a RG-module and R_0 be a subring of R. Then we say that M is realizable in R_0 if and only if there exists a R_0G-module N such that $M \cong R \otimes_{R_0} N$ as left RG-modules. This paper gives a characterization of splitting rings in terms of the concept of realizability as in the field case. The other main results in this paper are the structure theorem for split group algebras and some properties of group characters.

Throughout this paper we assume that the ring R is a commutative ring with no idempotents except 0 and 1, that the group G has order n invertible in R, and that all RG-modules are unitary left RG-modules. We know that the order of G, n, is invertible in R if and only if RG is separable.

1. In this section we study splitting rings in two ways. That is, splitting rings can be characterized in terms of the concept of realizability and structure theorem for split group algebras will be given.

PROPOSITION 1. *Assume the ring R has no idempotents except* 0 *and* 1, *and P is a finitely generated and projective R-module. Then P is a faithful R-module.*

Proof. Because P is a finitely generated and projective R-module, $R = \alpha(P) + \mathrm{Tr}(P)$ where $\alpha(P)$ is the kernel of the operation of R on P and $\mathrm{Tr}(P)$ is the trace ideal of P in R ([3], Proposition A.3). Thus $\alpha(P)$ is a left direct summand of R ([3], Th. A.2(d)). But R has no idempotents except 0 and 1 so that $\alpha(P) = 0$. Therefore P is a faithful R-module.

Using the above proposition we can have the following definition given by G. J. Janusz.

DEFINITION 1. A ring R is a splitting ring for G if the group algebra RG is the direct sum of central separable R-algebras, each equivalent to R in the Brauer group of R; that is,

$$RG \cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i) ,$$

where $\{P_i\}$ are finitely generated and projective R-modules. The number of different conjugate classes in G is equal to s ([5], Definition 6).

DEFINITION 2. Let M be a RG-module and R_0 be a subring of R. Then we say that M is realizable in R_0 if and only if there exists a R_0G-module N such that $M \cong R \otimes_{R_0} N$ as left RG-modules.

THEOREM 2. *If R is strongly separable over R_0 and R is a splitting ring for G, $RG \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i)$; then P_i is realizable in R_0 for all i if and only if R_0 is a splitting ring for G.*

Proof. If R_0 is a splitting ring for G, that is, if

$$R_0G \cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_{R_0}(P_i, P_i) ,$$

then $P_i \cong R_0 \otimes_{R_0} P_i$. This means that P_i is realizable in R_0 for all i.

Conversely, if P_i is realizable in R_0 for all i, then there is R_0G-module M_i such that $P_i \cong R \otimes_{R_0} M_i$ for all i. Since R is a strongly separable R_0-algebra, $R_0 \cdot 1$ is a R_0-direct summand of R. By the definition of a split group algebra, P_i is a finitely generated and projective R-module for each i; so M_i is a finitely generated and projective R_0-module for each i. In fact, because $R \cong (R_0 \cdot 1 \oplus R_0')$ for some R_0-module R_0',

$$P_i \cong (R_0 \cdot 1 \oplus R_0') \otimes_{R_0} M_i \cong (R_0 \cdot 1 \otimes_{R_0} M_i) \oplus (R_0' \otimes_{R_0} M_i) .$$

Thus $M_i \cong R_0 \cdot 1 \otimes_{R_0} M_i$ is a R_0-direct summand of P_i. On the other hand, P_i is finitely generated and projective over R and R is finitely generated and projective over R_0; so P_i is finitely generated and projective over R_0. Therefore M_i is a finitely generated and projective R_0-module. We then have

$$\begin{aligned} RG &\cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i) \cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_R(R \otimes_{R_0} M_i, R \otimes_{R_0} M_i) \\ &\cong R \otimes_{R0} \left(\bigoplus\sum_{i=1}^{s} \mathrm{Hom}_{R_0}(M_i, M_i)\right) . \end{aligned}$$

Noting that M_i is a finitely generated projective and faithful R_0-module for each i by Proposition 1, we have that $\mathrm{Hom}_{R_0}(M_i, M_i)$ is a central separable R_0-algebra with a unique central idempotent in R_0G for each i ([2], Proposition 5.1). Therefore $R_0G \cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_{R_0}(M_i, M_i)$. This proves that R_0 is a splitting ring for G.

We are going to discuss the structure of a split group algebra over some kinds of rings, in particular, over a Dedekind ring.

THEOREM 3. *Let P denote a finitely generated and projective R-module.* (a) *If R is a Dedekind domain, then* $\mathrm{Hom}_R(P, P)$ *is free as a R-module. Consequently, a split group algebra is a free R-module.* (b) *If R is a local ring or a semi-local ring or a principal ideal Dedekind domain, then* $\mathrm{Hom}_R(P, P)$ *is a matrix ring over R.*

Proof. Because P is a finitely generated and projective R-module, $\mathrm{Hom}_R(P, P) \cong P \otimes_R \mathrm{Hom}_R(P, R)$. Let the rank of P be k. Then $P \cong \bigoplus\sum_{i=1}^{k-1} R \oplus I$, $\sum_{i=1}^{k-1} R$ are $k-1$ copies of R and I is in the class group of R. By substitution,

$$\begin{aligned}
P \otimes_R \mathrm{Hom}_R(P, R) &\cong \left(\oplus\sum_{i=1}^{k-1} R \oplus I\right) \otimes_R \mathrm{Hom}_R\left(\oplus\sum_{i=1}^{k-1} R \oplus I, R\right)\\
&\cong \left(\oplus\sum_{i=1}^{k-1} R \oplus I\right) \otimes_R \left(\oplus\sum_{i=1}^{k-1} \mathrm{Hom}_R(R, R) \oplus \mathrm{Hom}_R(I, R)\right)\\
&\cong \left(\oplus\sum_{i=1}^{k-1} R \oplus I\right) \otimes_R \left(\oplus\sum_{i=1}^{k-1} R \oplus I^{-1}\right)\\
&\cong \left(\oplus\sum_{i=1}^{(k-1)^2} R\right) \oplus \left(\oplus\sum_{i=1}^{k-1} R \otimes_R I^{-1}\right) \oplus \left(\oplus\sum_{i=1}^{k-1} R \otimes_R I\right) \oplus (I \otimes_R I^{-1})\\
&\cong \left(\oplus\sum_{i=1}^{(k-1)^2} R\right) \oplus \left(\oplus\sum_{i=1}^{k-1} I^{-1}\right) \oplus \left(\oplus\sum_{i=1}^{k-1} I\right) \oplus R\\
&\cong \left(\oplus\sum_{i=1}^{(k-1)^2+1} R\right) \oplus \left(\oplus\sum_{i=1}^{2k-2} R\right)\\
&\cong \left(\oplus\sum_{i=1}^{k^2}\right) R. \quad \text{This proves part (a).}
\end{aligned}$$

For part (b), because P is a free module of finite rank over each of these rings, $\mathrm{Hom}_R(P, P)$ is a matrix ring over R. For a local ring R, see Theorem 12 in Chapter 9 in [6]. For a semi-local ring R, see the remark on Theorem 3.6 in [2]. For a principal ideal Dedekind domain, see Exercises 22.5 and 56.6 in [4].

REMARK. There exist split group algebras over those rings in the above theorem from the proof of the Brauer splitting theorem ([8], Th. 2).

THEOREM 4. *Let R denote a Dedekind domain, P a finitely generated and projective R-module and $P(R)$ the class group of R. Then, for $P \cong \bigoplus\sum_{i=1}^{k-1} R \oplus J$, there is I in $P(R)$ such that $I^k = J^{-1}$ where $k = \mathrm{rank}\,(P)$ and J is in $P(R)$ if and only if* $\mathrm{Hom}_R(P, P)$ *is a matrix ring over R of order k by k.*

Proof. Because $\operatorname{Hom}_R(P, P)$ is a matrix ring over R if and only if there exists I in $P(R)$ such that $P \otimes_R I \cong \oplus \sum_{i=1}^{k} R$, a direct sum of k-copies of R (Lemma 9, [7]). But $P \cong \oplus \sum_{i=1}^{k-1} R \oplus J$ for some

$$J \text{ in } P(R); \text{ so } \left(\left(\oplus \sum_{i=1}^{k-1} R\right) \oplus J\right) \otimes_R I \cong \oplus \sum_{i=1}^{k} R\,,$$

$$\left(\oplus \sum_{i=1}^{k-1} I\right) \oplus (J \otimes_R I) \cong \oplus \sum_{i=1}^{k} R\,,$$

$$\left(\oplus \sum_{i=1}^{k-1} I\right) \oplus (J \cdot I) \cong \oplus \sum_{i=1}^{k} R$$

where we use the fact that $J \otimes_R I \cong J \cdot I$. But

$$\left(\oplus \sum_{i=1}^{k-1} I\right) \oplus (J \cdot I) \cong \oplus \sum_{i=1}^{k-1} R \oplus I^k \cdot J\,;$$

then $I^k \cdot J = R$. So, if we can prove the fact that $J \otimes_R I \cong J \cdot I$, the theorem is proved. In fact, because $J \cdot I$ is in $P(R)$ and $J \cdot I$ is projective and finitely generated, the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\pi) \longrightarrow J \otimes_R I \xrightarrow{\pi} J \cdot I \longrightarrow 0$$

splits. Thus $J \otimes_R I \cong \operatorname{Ker}(\pi) \oplus J \cdot I$. Let R_M denote the quotient ring with respect to a prime ideal M.

$$R_M \otimes_R (J \otimes_R I) \cong R_M \otimes_R \operatorname{Ker}(\pi) \oplus R_M \otimes {}_R(J \cdot I)\,,$$

that is, $R_M \cong R_M \otimes_R \operatorname{Ker}(\pi) \oplus R_M$. Hence $R_M \otimes_R \operatorname{Ker}(\pi) = 0$ for all prime ideals M. On the other hand, because $\operatorname{Ker}(\pi)$ is finitely generated, $\operatorname{Ker}(\pi) = 0$ by Nakayama's lemma. This proves that $J \otimes_R I \cong J \cdot I$. Therefore the theorem is completed.

COROLLARY 5. *Keep the same notations as Theorem* 4. *If the rank of P and the order of J are relative prime, then* $\operatorname{Hom}_R(P, P)$ *is a matrix ring over R.*

Proof. It suffices to prove that there exists I in $P(R)$ such that $J^{-1} = I^k$ by Theorem 4. Consider the subgroup generated by J^k. Because k, the rank of P and the order of J are relative prime, this subgroup is the same as the subgroup generated by J. Hence $J = J^{ik}$ for some i from 1 to the order of J minus 1. Thus $I = (J^{-1})^i$ is what we want. In fact, $I^k = (J^{-1})^{ik} = (J^{ik})^{-1} = J^{-1}$.

DEFINITION 3. The subgroup of $P(R)$, U, is called the $R - Z$ group for a finitely generated and projective R-module P if $U = \{I$ such that I is in $P(R)$ and $I \cdot P = P\}$. (For this group see Theorem 14 and Theorem 15 in [7]).

THEOREM 6. (a) *Let R be a Dedekind domain and $H = \{J$ such that $P \cong \bigoplus\sum_{i=1}^{k-1} R \oplus J$ and $\mathrm{Hom}_R(P, P)$ is a matrix ring over R where J is in $P(R)\}$. Then H is a subgroup of $P(R)$.* (b) *Assume the $R - Z$ group is equal to $P(R)$. Then, P is a free R-module if and only if $\mathrm{Hom}_R(P, P)$ is a matrix ring over R.*

Proof. For any J' and J'' in H, there are I' and I'' in $P(R)$ such that $J' \cdot (I')^k = R$ and $J'' \cdot (I'')^k = R$ by Theorem 4. We then have $J' \cdot J'' \cdot (I' \cdot I'')^k = (J' \cdot I'^k)(J'' \cdot I''^k) = R$. Thus $J' \cdot J''$ is in H. Also, for any J in H, there is I in $P(R)$ such that $J \cdot I^k = R$. We then have $J^{-1} \cdot (I^k)^{-1} = R$, that is, $J^{-1} \cdot (I^{-1})^k = R$. Thus J^{-1} is in H. Therefore H is a subgroup of $P(R)$. This proves part (a).

For part (b), one way is clear. If P is free, then $\mathrm{Hom}_R(P, P)$ is a matrix ring over R. Conversely, if $\mathrm{Hom}_R(P, P)$ is a matrix ring over R, $P \cong \bigoplus\sum_{i=1}^{k-1} R \oplus J$ with J in H by Theorem 4. But the $R - Z$ group is equal to $P(R)$; then $I^k = R$ for all I in $P(R)$. Thus $H = 0$. Therefore P is a free R-module.

REMARK. (a) Corollary 5 can be expressed in terms of the $R - Z$ group as following. If the exponent of the $R - Z$ group and the order of J is relative prime, then $\mathrm{Hom}_R(P, P)$ is a matrix ring over R.

(b) Theorem 4, Corollary 5, and Theorem 6 tell us the structure of $\mathrm{Hom}_R(P, P)$, any component of a split group algebra. We thus have the similar structure theorems for group algebras by considering $P_1, P_2, \cdots P_s$ and $J_1, J_2, \cdots J_s$ in the same time where P_i, $i = 1, 2, \cdots s$ are in the definition of a split group algebra RG with

$$P_i \cong \bigoplus\sum_{i=1}^{k_i-1} R \oplus J_i \text{ as in Theorem 4.}$$

2. Let us recall the group character of a finitely generated and projective RG-module.

DEFINITION 4. Let M be a finitely generated and projective RG-module with dual basis $\{F_1, F_2, \cdots F_u; X_1, X_2, \cdots X_u\}$. Then the group character $T_M: G \to R$ is defined by $T_M(g) = \sum_{i=1}^{u} F_i(gX_i)$ for any g in G ([8], § 2).

In this section some properties of group characters will be given. Let K be a field and $K(\varphi)$ be $K(\varphi(g_1), \varphi(g_2), \cdots \varphi(g_n))$ where φ is a group character for $G = \{g_1, g_2, \cdots g_n\}$. We know that $K(\varphi)$ is a separable extension over K. In the ring case, $R[T^i]$ can be proved as a strongly separable R-algebra where T^i is a group character for G. Finally, we point out the usual orthogonality relations on group characters in the ring case.

THEOREM 1. (a) *Let* T^i *be* T_{P_i} *where* P_i *is in the definition of a split group algebra* RG (see Definition 1). *Then* $T^i(g)$ *is a constant for all splitting rings* R *with the same prime ring* R_0 *for a given group* G, *where* g *is in* G. (b) $T^i(g)$ *is a sum of* n_i^{th}*-roots of* 1 *where* g *is in* G *and* $g^{n_i} = 1_G$, *the identity of* G.

Proof. Since R is a splitting ring for $G, RG \cong \bigoplus\sum_{i=1}^s \mathrm{Hom}_R(P_i, P_i)$. Setting $R' = R[\sqrt[m]{1}]$ where $\sqrt[m]{1}$ is a primitive m^{th}-root of 1 and m is the exponent of G, we have

$$R'G \cong R' \otimes_R RG \cong R' \otimes_R \left(\bigoplus\sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i)\right)$$
$$\cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_{R'}(R' \otimes_R P_i, R' \otimes_R P_i) .$$

By Lemma 1 in [8], R' is also a splitting ring for G. Clearly,

$$T_{R'\otimes_R P_i} = T^i \cdots (1) .$$

Next, consider $R'' = R_0[\sqrt[m]{1}]$. It is a splitting ring for G ([8], Th. 2); that is, $R''G \cong \bigoplus\sum_{i=1}^s \mathrm{Hom}_{R''}(P_i'', P_i'')$. We then have

$$R'G \cong R' \otimes_{R''} R''G \cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_{R'}(R' \otimes_{R''} P_i'', R' \otimes_{R''} P_i'') .$$

Thus $T_{P_i''} = T_{R'\otimes_{R''}P_i''} \cdots$ (2), and for each i

$$\mathrm{Hom}_{R'}(R' \otimes_{R''} P_i'', R' \otimes_{R''} P_i'') = \mathrm{Hom}_{R'}(R' \otimes_R P_i, R' \otimes_R P_i) .$$

The later implies that $R' \otimes_{R''} P_i'' \cong (R' \otimes_R P_i) \otimes_{R'} J$, where J is in the class group of R' ([7], Lemma 9). Consequently,

$$T_{R'\otimes_{R''}P_i''} = T_{R'\otimes_R P_i} \cdots (3) .$$

From (1), (2) and (3), $T^i = T_{P_i''}$. But R'' depends on R_0 and G only so that T^i is a constant for all splitting rings R with the same prime ring R_0 for a given group G, $i = 1, 2, \cdots, s$. This proves part (a).

The proof for part (b) divides into two cases. Case 1. Char(R) is equal to p^r where p is a prime integer and r is a positive integer. Then the prime ring of R is $Z/(p^r)$ where Z is the set of integers. Let $\sqrt[m]{1}$ be a primitive m^{th}-root of 1 where m is the exponent of G. Then $R' = Z/(p^r)[\sqrt[m]{1}]$ is a splitting ring for G ([8], Th. 2); that is, $R'G \cong \bigoplus\sum_{i=1}^s \mathrm{Hom}_{R'}(P_i, P_i)$. Since R' is a local ring (see the proof of Theorem 2 in [8]) and P_i is a finitely generated and projective R'-module for each i, P_i is a free R'-module for each i ([6], Th. 12 in Chapter 9). Therefore $T^i(g)$ is a sum of n_i^{th}-roots of 1 where g is in

G and $g^{n_i} = 1_G$, the identity of G.

Char (R) is equal to 0. Then the prime ring of R is $Z(n)$, the quotient ring of Z with respect to the multiplicative closed set $\{n, n^2, \cdots\}$. By the Brauer splitting theorem again, $R' = Z(n)[\sqrt[m]{1}]$ is a splitting ring for G; that is $R'G \cong \bigoplus\sum_{i=1}^{s} \mathrm{Hom}_{R'}(P_i, P_i)$. Since R' is a principal ideal Dedekind domain, P_i is a free R'-module for each i ([4], Exercises 22.5 and 56.6). Therefore $T^i(g)$ is a sum of n_i^{th}-roots of 1 as in Case 1.

THEOREM 2. *Let $R[T^i]$ denote $R[T^i(g_1), T^i(g_2), \cdots]$ where G is equal to $\{g_1, g_2, \cdots, g_n\}$. Then $R[T^i]$ is a strongly separable R-algebra for each i.*

Proof. As in the above theorem, R divides into two cases. Case 1. Char $(R) = 0$. Then the prime ring of R is $Z(n)$, the quotient ring of integers with respect to the multiplicative closed set $\{n, n^2, \cdots\}$. We know that the quotient field of $Z(n)[T^i(g)]$ is $Q(T^i(g))$ for each g in G and the quotient field of $Z(n)[\sqrt[m]{1}]$ is $Q(\sqrt[m]{1})$, where Q is the set of rationals. Because $Z(n)[\sqrt[m]{1}]$ is separable over $Z(n)$ by the Brauer splitting theorem, $Q(\sqrt[m]{1})$ is unramified over Q ([1], Th. 2.5). But $Q(T^i(g))$ is a subset of $Q(\sqrt[m]{1})$ and contains Q; so $Q(T^i(g))$ is unramified over Q ([9], Proposition 3.2.4). Thus $Z(n)[T^i(g)]$ is separable over $Z(n)$ by Theorem 2.5 in [1] again. This implies that $R \otimes_{Z(n)} Z(n)[T^i(g)]$ is a separable R-algebra ([2], Corollary 1.6); so $R[T^i(g)]$, the homomorphic image of $R \otimes_{Z(n)} Z(n)[T^i(g)]$, is also a separable R-algebra. On the other hand, because $T^i(g)$ is integral over R, $R[T^i(g)]$ is a strongly separable R-algebra. Therefore $R[T^i]$ is a strongly separable R-algebra.

Case 2. Char (R) is p^r for some prime integer p and a positive integer r. Then the prime ring of R is $Z/(p^r)$. We know that $Z/(p^r)[T^i(g)]$ is a local ring with the nilpotent maximal ideal $(p)/(p^r)[T^i(g)]$. Also, $Z/(p^r)[T^i(g)]$ is a Noetherian ring such that

$$(p)/(p^r)[T^i(g)] \cap Z/(p^r) = (p)/(p^r) .$$

Let M denote $(p)/(p^r)[T^i(g)]$. Then $(p)/(p^r)\cdot(Z/(p^r)[T^i(g)])_M$ is equal to $M\cdot(Z/(p^r)[T^i(g)])_M$ for $T^i(g)$ is in M, $(\ \ )_M$ is a local ring at M.

$$Z/(p^r)[T^i(g)]/(p)/(p^r)[T^i(g)] \cong Z/(p)(T^i(g))$$

is a separable $Z/(p)$ extension. Therefore $Z/(p^r)[T^i(g)]$ is a separable $Z/(p^r)$-algebra ([1], § 1). Then as in Case 1, $R[T^i]$ is a strongly separable R-algebra by the same arguments. This proves the theorem.

REMARK. We know that an element α in the separable closure of R is separable means that it satisfies a separable polynomial over R. This is also equivalent to that $R[\alpha]$ is a separable R-algebra ([5], Lemma 2.7). Then $T^i(g)$ is a separable element such that $T^i(g)$ is a sum of n_i^{th}-roots of 1. Because these roots satisfy the separable polynomial, $X^{n_i} - 1 = 0$, all roots are also separable elements. But it is not true that a sum of separable elements is separable. The following example is due to G. J. Janusz. Let R be $Z(2)$, the quotient ring of Z with respect to the multiplicative closed set $\{2, 2^2, \cdots\}$, S be $R[i]$ where $i^2 = -1$. Then S is strongly separable over R. An element $a + ib$ is a separable element if and only if $(a + ib) - (a - ib) = 2ib$ is invertible in S ([5], Lemma 2.1). Hence the separable elements are of the form $a + i2^j$ where a is in $Z(2)$ and $j = 0, 1, 2, \cdots$. Clearly, $1 + i$ and $1 + i2$ are separable elements but $(1 + i) + (1 + i2) = (2 + i3)$ is not.

We conclude this section by pointing out the usual orthogonality relations on group characters as in the field case.

THEOREM 3. *If* $T^i = T_{P_i}$, *for* $i = 1, 2, \cdots, s$, *then*

$$\sum_g T^i(g) T^j(g^{-1}) = n\delta_{ij} ,$$

where n *is the order of* G *and* δ_{ij} *is the Kronecker delta.*

Proof. Let E_i be the i^{th}-central primitive idempotent of RG,

$$E_i = \sum_g \frac{k_i T^i(g^{-1})}{n} g ,$$

where $k_i = \operatorname{rank}(P_i)$ ([8], Lemma 5). Taking the characters in both sides, we have the answer.

REMARK. By using the above theorem and standard methods, the other usual orthogonality relations on group characters can be proved (see § 31 in [4]).

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HOMOLOGICAL DIMENSION AND SPLITTING TORSION THEORIES

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The concept of a torsion theory $(\mathscr{T}, \mathscr{F})$ for left R-modules has been defined by S. E. Dickson. A torsion theory is called splitting if it has the property that the torsion submodule of every left R-module is a direct summand. Under restrictive hypotheses on the ring R, several specific splitting theories have previously been examined. This paper continues the investigation to more general classes of torsion theories. In the first section, comparisons are made between injective modules and torsion modules for a splitting theory, and the following results are obtained: (1) A torsion class $\mathscr{T}$ is closed under taking injective envelopes if and only if the maximal $\mathscr{T}$-torsion submodule of an injective module is injective. (2) If $(\mathscr{T}, \mathscr{F})$ is splitting and $R \in \mathscr{F}$, then inj dim $(T) \leqq 1$ for all $T \in \mathscr{T}$. (3) If $(\mathscr{T}, \mathscr{F})$ is splitting and hereditary and if $R \in \mathscr{F}$, then every homomorphic image of a $\mathscr{T}$-torsion injective module is injective. In § 2 it is shown that rings R, for which R has zero singular ideal and Goldie's torsion theory is splitting, have the property: l. gl. dim $R \leqq 2$. It is shown that the relative homological dimension arising from a hereditary torsion theory often gives information about splitting, especially when this dimension is zero. In the final sections, the zero-dimensionality of a hereditary torsion theory is discussed and related to results of J. P. Jans. The rings, all of whose hereditary torsion theories have dimension zero, are characterized as direct sums of finitely many right perfect rings, each of which has a unique maximal ideal.

In this paper, all rings R have identity, and all modules are unitary left R-modules. The category of left R-modules is denoted by ${}_R\mathscr{M}$.

A torsion theory of modules is a pair $(\mathscr{T}, \mathscr{F})$ of subclasses of ${}_R\mathscr{M}$ satisfying:

(1) $\mathscr{T} \cap \mathscr{F} = \{0\}$.

(2) $B \subseteqq A$ and $A \in \mathscr{T}$ implies $A/B \in \mathscr{T}$.

(3) $B \subseteqq A$ and $A \in \mathscr{F}$ implies $B \in \mathscr{F}$.

(4) For each $A \in {}_R\mathscr{M}$, there exists a (necessarily unique) exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

For this definition and the following results, the reader is referred to [5].

Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory for ${}_R\mathscr{M}$. Modules in $\mathscr{T}$ are called torsion, and those in $\mathscr{F}$ are called torsionfree. Each $A \in {}_R\mathscr{M}$ has a unique maximal torsion submodule, denoted by $\mathscr{T}(A)$. $\mathscr{T}$ is closed under taking direct sums, and $\mathscr{F}$ is closed under taking direct products. $\mathscr{T} = \{T \in {}_R\mathscr{M} \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } F \in \mathscr{F}\}$, and $\mathscr{F} = \{F \in {}_R\mathscr{M} \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } T \in \mathscr{T}\}$. A subclass $\mathscr{C}$ of ${}_R\mathscr{M}$ is closed under taking extensions if $A, B \in \mathscr{C}$ and $0 \to A \to X \to B \to 0$ is exact imply $X \in \mathscr{C}$. Both $\mathscr{T}$ and $\mathscr{F}$ are closed under taking extensions. A class $\mathscr{C}$ is closed under taking injective envelopes if $A \in \mathscr{C}$ implies $E(A) \in \mathscr{C}$, where $E(A)$ denotes the injective envelope of A. $\mathscr{T}$ is closed under taking submodules if and only if $\mathscr{F}$ is closed under taking injective envelopes. When $(\mathscr{T}, \mathscr{F})$ has this property, then $(\mathscr{T}, \mathscr{F})$ is called a hereditary torsion theory. In this case $\mathscr{T}$ is also a class of negligible modules in the sense of P. Gabriel [10], and hence there is a topologizing and idempotent filter $F(\mathscr{T})$ of left ideals associated with $\mathscr{T}$. For results concerning these filters, the reader is referred to [10] or [15].

For convenience $\mathrm{Ext}_R^n(A, B)$ will be written as $\mathrm{Ext}^n(A, B)$ throughout this paper. The following notations concerning homological dimensions are used for the ring R and the R-module M:

$$\begin{aligned} \mathrm{inj\,dim}\,(M) &= \inf\{n \mid \mathrm{Ext}^{n+1}(-, M) = 0\} \\ h.\dim(M) &= \inf\{n \mid \mathrm{Ext}^{n+1}(M, -) = 0\} \\ \mathrm{l.\,gl.\,dim}\,R &= \inf\{\mathrm{h.\,dim}\,(M) \mid M \in {}_R\mathscr{M}\}\,. \end{aligned}$$

1. Injectives and splitting. Let $(\mathscr{T}, \mathscr{F})$ be a splitting torsion theory for ${}_R\mathscr{M}$, i.e., $\mathscr{T}(M)$ is a summand of each $M \in {}_R\mathscr{M}$. Since an injective module is always a summand of any module containing it, it is natural to wonder how much a module in $\mathscr{T}$ must "resemble" an injective module. The first lemma examines the case of the maximal torsion submodule of an injective module. It shows that the splitting of $(\mathscr{T}, \mathscr{F})$ implies that $\mathscr{T}$ is closed under taking injective envelopes.

LEMMA 1.1. *Suppose $(\mathscr{T}, \mathscr{F})$ is a torsion theory for ${}_R\mathscr{M}$. Then $\mathscr{T}$ is closed under injective envelopes if and only if $\mathscr{T}(A)$ is injective for each injective module $A \in {}_R\mathscr{M}$.*

Proof. ($\Rightarrow$): Let A be injective. Then $E(\mathscr{T}(A)) \in \mathscr{T}$ by hypothesis. But then $E(\mathscr{T}(A))/\mathscr{T}(A) \in \mathscr{T}$ and

$$E(\mathscr{T}(A))/\mathscr{T}(A) \subseteq A/\mathscr{T}(A) \in \mathscr{F}\,.$$

Hence $E(\mathscr{T}(A))/\mathscr{T}(A) \in \mathscr{T} \cap \mathscr{F} = \{0\}$.

($\Leftarrow$): Let $T \in \mathscr{T}$. By hypothesis, $E(T) = \mathscr{T}(E(T)) \oplus F$, where $F \in \mathscr{F}$. Since $\mathscr{T}(E(T)) + T \in \mathscr{T}$ is contained in $E(T)$, then $T \subseteqq \mathscr{T}(E(T))$, and hence $F = 0$.

The following lemma is clear:

LEMMA 1.2. *The following are equivalent for a torsion theory* $(\mathscr{T}, \mathscr{F})$ *for* ${}_R\mathscr{M}$.

(1) $(\mathscr{T}, \mathscr{F})$ *is splitting.*

(2) $\operatorname{Ext}(F, T) = 0$ *for all* $F \in \mathscr{F}$, $T \in \mathscr{T}$.

THEOREM 1.3. *Let* $(\mathscr{T}, \mathscr{F})$ *be a splitting torsion theory for* ${}_R\mathscr{M}$. *If* $R \in \mathscr{F}$, *then* $\operatorname{inj\,dim}(T) \leqq 1$ *for all* $T \in \mathscr{T}$.

Proof. Since $R \in \mathscr{F}$, every submodule of a free R-module is in $\mathscr{F}$. So for each $M \in {}_R\mathscr{M}$, there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F projective and $K, F \in \mathscr{F}$. Hence by Lemma 1.2, the exact sequence

$$\operatorname{Ext}(K, T) \longrightarrow \operatorname{Ext}^2(M, T) \longrightarrow \operatorname{Ext}^2(F, T) = 0$$

yields $\operatorname{Ext}^2(M, T) = 0$ for all $T \in \mathscr{T}$.

Now suppose for induction that $\operatorname{Ext}^n(M, T) = 0$ for all $T \in \mathscr{T}$. If $T \in \mathscr{T}$, then $E(T) \in \mathscr{T}$ by Lemma 1.1, and hence $E(T)/T \in \mathscr{T}$. So, by the induction hypothesis, the exact sequence

$$\operatorname{Ext}^n(M, E(T)/T) \longrightarrow \operatorname{Ext}^{n+1}(M, T) \longrightarrow \operatorname{Ext}^{n+1}(M, E(T)) = 0$$

yields $\operatorname{Ext}^{n+1}(M, T) = 0$ for all $T \in \mathscr{T}$.

Hence the result follows by induction.

COROLLARY 1.4. *Let* $(\mathscr{T}, \mathscr{F})$ *be a splitting torsion theory for* ${}_R\mathscr{M}$. *Let A be an injective module and f a homomorphism of A. If* $R \in \mathscr{F}$ *and if the kernel of f is in* $\mathscr{T}$, *then the image of f is injective.*

Proof. Let K be the kernel of f, and let I be the image of f. Then Theorem 1.3 yields the following exact sequence for any $M \in {}_R\mathscr{M}$:

$$0 = \operatorname{Ext}^1(M, A) \longrightarrow \operatorname{Ext}^1(M, I) \longrightarrow \operatorname{Ext}^2(M, K) = 0\,.$$

Hence $\operatorname{Ext}^1(M, I) = 0$ by exactness, and so I is injective.

The following result is the special case of Corollary 1.4 for a hereditary torsion theory.

COROLLARY 1.5. *Let $(\mathscr{T}, \mathscr{F})$ be a splitting hereditary torsion theory for ${}_R\mathscr{M}$. If $R \in \mathscr{F}$, then every homomorphic image of a torsion injective module is injective.*

2. The Goldie theory. A submodule $A \subseteqq M$ is said to be essential in M if $A \cap B \neq 0$ for every nonzero submodule B of M. The singular submodule of $M \in {}_R\mathscr{M}$ is $Z(M) = \{x \in M \mid (0: x)$ is essential in $R\}$. If $Z(M) = 0$, then M is called nonsingular.

Goldie's torsion theory $(\mathscr{G}, \mathscr{N})$ is the torsion theory given by $\mathscr{N} = \{N \in {}_R\mathscr{M} \mid N$ is nonsingular$\}$ and $\mathscr{G} = \{G \in {}_R\mathscr{M} \mid Z(G)$ is essential in $G\}$. $(\mathscr{G}, \mathscr{N})$ is hereditary and has as its filter $F(\mathscr{G}) = \{I \mid I \subseteqq J$ essential in R, and $(I: x)$ is essential in R for all $x \in J\}$. This is the smallest topologizing and idempotent filter containing the essential left ideals. For other results on $(\mathscr{G}, \mathscr{N})$, the reader is referred to [1], [11] or [14].

V. Cateforis and F. Sandomierski [4] have studied the splitting of $(\mathscr{G}, \mathscr{N})$ for commutative rings with $Z(R) = 0$. $Z(R) = 0$ if and only if $Z(M) = \mathscr{G}(M)$ for all $M \in {}_R\mathscr{M}$. Hence saying $(\mathscr{G}, \mathscr{N})$ splits and $Z(R) = 0$ is equivalent to saying that the singular submodule always splits off. In [4] it is shown that whenever $(\mathscr{G}, \mathscr{N})$ is splitting, R is commutative, and $Z(R) = 0$, then l. gl. dim $R \leqq 1$. The results below show that this bound can be kept for modules in $\mathscr{N}$ (i.e., $h.$ dim $(N) \leqq 1$ for all $N \in \mathscr{N}$) when the commutative hypothesis on R is dropped. Moreover, if $(\mathscr{G}, \mathscr{N})$ splits and $Z(R) = 0$, then l. gl. dim $R \leqq 2$.

THEOREM 2.1. *If $(\mathscr{G}, \mathscr{N})$ splits and $R \in \mathscr{N}$, then $h.$ dim $(N) \leqq 1$ for all $N \in \mathscr{N}$.*

Proof. Let $N, F \in \mathscr{N}$. Then $E(N)/N \in \mathscr{G}$, so that

$$\operatorname{Ext}(F, E(N)/N) = 0$$

by Lemma 1.2. Then the exact sequence

$$0 = \operatorname{Ext}^1(F, E(N)/N) \longrightarrow \operatorname{Ext}^2(F, N) \longrightarrow \operatorname{Ext}^2(F, E(N)) = 0$$

yields $\operatorname{Ext}^2(F, N) = 0$ for all $F, N \in \mathscr{N}$. By Theorem 1.3,

$$\operatorname{Ext}^n(F, E(N)/N) = 0$$

for all $n \geqq 2$. So the exact sequence

$$0 = \operatorname{Ext}^n(F, E(N)/N) \longrightarrow \operatorname{Ext}^{n+1}(F, N) \longrightarrow \operatorname{Ext}^{n+1}(F, E(N)) = 0$$

yields $\operatorname{Ext}^{n+1}(F, N) = 0$ for all $F, N \in \mathscr{N}$ and all $n \geqq 2$.

Let $M \in {}_R\mathscr{M}$. By splitting $M \cong \mathscr{G}(M) \oplus M/\mathscr{G}(M)$. Hence

$$\mathrm{Ext}^n(F, M) \cong \mathrm{Ext}^n(F, \mathscr{G}(M)) \oplus \mathrm{Ext}^n(F, M/\mathscr{G}(M)) = 0$$

for all $n \geqq 2$ and all $F \in \mathscr{N}$, by Theorem 1.3 and the first part of the proof.

THEOREM 2.2. *If* $(\mathscr{G}, \mathscr{N})$ *splits and* $R \in \mathscr{N}$, *then* l. gl. dim $R \leqq 2$.

Proof. Let $F \in \mathscr{N}$ and $M \in {}_R\mathscr{M}$. By Theorem 1.3 there is an exact sequence

$$0 = \mathrm{Ext}^{n-1}(M, E(F)/F) \longrightarrow \mathrm{Ext}^n(M, F) \longrightarrow \mathrm{Ext}^n(M, E(F)) = 0$$

for all $n \geqq 3$. Thus $\mathrm{Ext}^n(M, F) = 0$ for all $n \geqq 3$.

Let $M, M_1 \in {}_R\mathscr{M}$. By splitting $M_1 \cong \mathscr{G}(M_1) \oplus M_1/\mathscr{G}(M_1)$. Hence, for $n \geqq 3$,

$$\mathrm{Ext}^n(M, M_1) = \mathrm{Ext}^n(M, \mathscr{G}(M_1)) \oplus \mathrm{Ext}^n(M, M_1/\mathscr{G}(M_1)) = 0$$

by Theorem 1.3 and the first part of the proof. Hence l. gl. dim $R \leqq 2$.

3. Relative homological algebra. In [6] the right derived functors of a torsion subfunctor of the identity were calculated. This leads to a relativized injective dimension of modules for each hereditary torsion theory, and hence to a global dimension of ${}_R\mathscr{M}$ depending on the hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ chosen. This global dimension is denoted by $\mathscr{T}$ gl. dim. R.

In [1] it is shown that if $\mathscr{G}$ gl. dim. $R = 0$, then $(\mathscr{G}, \mathscr{N})$ splits. S. E. Dickson has conjectured [7] that the simple theory $(\mathscr{S}, \mathscr{F})$ (i.e., the torsion theory whose torsion class is the smallest torsion class containing the simple R-modules) splits if and only if $\mathscr{S} = {}_R\mathscr{M}$. In this section it is shown that $\mathscr{S} = {}_R\mathscr{M}$ if and only if $\mathscr{S}$ gl. dim. $R = 0$. Moreover, for any hereditary torsion theory $(\mathscr{T}, \mathscr{F})$, Theorem 3.1 below shows that $\mathscr{T}$ gl. dim. $R = 0$ if and only if $\mathscr{F}$ is a TTF class in the sense of [13], i.e., a class closed under taking submodules, factor modules, direct products, and extensions.

The first right derived functor of $A \in {}_R\mathscr{M}$ relative to the hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ is

$$R_{\mathscr{T}}(A) = \mathscr{T}(E(A)/A) \Big/ \frac{\mathscr{T}(E(A)) + A}{A}.$$

Then $\mathscr{T}$ gl. dim. $R = 0$ if and only if $R_{\mathscr{T}}(A) = 0$ for all $A \in {}_R\mathscr{M}$.

Following [1], a module $F \in \mathscr{F}$ called $\mathscr{T}$-absolutely pure (relative to the hereditary torsion theory $(\mathscr{T}, \mathscr{F})$) if $L \supseteqq F$ and $L \in \mathscr{F}$ imply $L/F \in \mathscr{F}$. [1], Proposition 1.4 states that $F \in \mathscr{F}$ is $\mathscr{T}$-absolutely pure if and only if $E(F)/F \in \mathscr{F}$.

THEOREM 3.1. *For a hereditary torsion theory* $(\mathscr{T}, \mathscr{F})$, *the following are equivalent*:

(1) $T: M \to \mathscr{T}(M)$ $\forall M \in {}_R\mathscr{M}$ *is an exact functor.*

(2) *Every* $F \in \mathscr{F}$ *is* $\mathscr{T}$*-absolutely pure.*

(3) $\mathscr{F}$ *is closed under taking homomorphic images.*

(4) $\mathscr{T}$ gl. dim. $R = 0$.

Proof. (1) $\Rightarrow$ (2): Let $F, L \in \mathscr{F}$ and $L \supseteq F$. Then apply the exact functor T to the exact sequence $0 \to F \to L \to L/F \to 0$ to get $0 \to \mathscr{T}(F) \to \mathscr{T}(L) \to \mathscr{T}(L/F) \to 0$. Since $L \in \mathscr{F}$, then $\mathscr{T}(L/F) = 0$ by exactness, and hence $L/F \in \mathscr{F}$. Thus F is $\mathscr{T}$-absolutely pure.

(2) $\Rightarrow$ (3): Let $f: F \to M$ be an epimorphism of $F \in \mathscr{F}$, and let K be the kernel of f. Since $\mathscr{F}$ is closed under taking submodules, $K \in \mathscr{F}$, and hence $M \cong F/K \in \mathscr{F}$ by (2).

(3) $\Rightarrow$ (4): For any $M \in {}_R\mathscr{M}$, the exact sequence

$$0 \longrightarrow \mathscr{T}(M) \longrightarrow M \longrightarrow M/\mathscr{T}(M) \longrightarrow 0$$

induces the exact sequence

$$R_{\mathscr{T}}(\mathscr{T}(M)) \longrightarrow R_{\mathscr{T}}(M) \longrightarrow R_{\mathscr{T}}(M/\mathscr{T}(M)) \, .$$

By [6], Lemma 2, $R_{\mathscr{T}}(\mathscr{T}(M)) = 0$. Hence it is sufficient to show that $R_{\mathscr{T}}(F) = 0$ for all $F \in \mathscr{F}$. Since $\mathscr{F}$ is closed under injective envelopes, $\mathscr{T}(E(F)) = 0$ for all $F \in \mathscr{F}$. Hence the formula for $R_{\mathscr{T}}(F)$ reduces to $\mathscr{T}(E(F)/F)$ whenever $F \in \mathscr{F}$. But (3) and $E(F) \in \mathscr{F}$ imply $E(F)/F \in \mathscr{F}$, and hence $R_{\mathscr{T}}(F) = \mathscr{T}(E(F)/F) = 0$.

(4) $\Rightarrow$ (1): This is clear since T is always left exact.

The simple torsion theory $(\mathscr{S}, \mathscr{F})$ has $\mathscr{S}$ defined [5] by $T \in \mathscr{S}$ if and only if every nonzero homomorphic image of T has nonzero socle. Then $\mathscr{F}$ corresponding to $\mathscr{S}$ is the class of modules with zero socle.

COROLLARY 3.2. *The following are equivalent*:

(1) $\mathscr{S}$ gl. dim $R = 0$.

(2) *Nonzero modules have nonzero socles.*

Proof. (1) $\Rightarrow$ (2): Suppose $R \notin \mathscr{S}$, so that $\mathscr{S}(R)$ is a proper ideal of R. Let M be a maximal left ideal of R containing $\mathscr{S}(R)$. Then $R/M \in \mathscr{S}$ is a homomorphic image of $R/\mathscr{S}(R) \in \mathscr{F}$. But (1) and Theorem 3.1 (3) yield $R/M \in \mathscr{F}$, which contradicts $\mathscr{S} \cap \mathscr{F} = 0$. Hence $R \in \mathscr{S}$, and so $\mathscr{S} = {}_R\mathscr{M}$, i.e., (2) holds.

(2) $\Rightarrow$ (1): By (2), $\mathscr{S} = {}_R\mathscr{M}$ and hence $\mathscr{F} = \{0\}$. Thus $\mathscr{F}$ is trivially closed under homomorphic images, and hence (1) follows from Theorem 3.1.

Let $\mathscr{T}_S$ denote the smallest torsion class containing the simple R-module S. If each $T \in \mathscr{S} \subseteqq {}_R\mathscr{M}$ can be written as

$$T = \bigoplus \sum_{S \in \mathscr{C}} \mathscr{T}_S(T) ,$$

where $\mathscr{C}$ is a set of nonisomorphic simple R-modules, then R is said to have primary decomposition (PD) for $\mathscr{S}$. For further results on (PD), the reader is referred to [5] and [9].

In order to characterize rings for which every hereditary torsion theory has dimension zero, the following result of H. Bass [2] is needed:

THEOREM P. *The following are equivalent*:

(1) *R is right perfect.*

(2) *$R/J(R)$ is semi-simple Artinian and $J(R)$ is right T-nilpotent, where $J(R)$ denotes the Jacobson radical of R.*

(3) *R contains no infinite sets of orthogonal idempotents and nonzero left modules have nonzero socles.*

THEOREM 3.3. *Every hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ for ${}_R\mathscr{M}$ has $\mathscr{T}$ gl. dim $R = 0$ if and only if R is the direct sum of finitely many right perfect rings, each of which has a unique maximal twosided ideal.*

Proof. ($\Rightarrow$): By $\mathscr{S}$ gl. dim $R = 0$ and Corollary 3.2, nonzero modules have nonzero socles. From $\mathscr{T}_S$ gl. dim $R = 0$, Theorem 3.1, and [5], Theorem 5.3, it follows that R has (PD). Since each $\mathscr{T}_S(R)$ is a two sided ideal, then $R = R_1 \dotplus R_2 \dotplus \cdots \dotplus R_n$ (ring direct sum), where each $R_i = \mathscr{T}_S(R)$ for some simple module S. Then nonzero left R_i-modules have nonzero socles, and hence $J(R_i)$, the Jacobson radical of R, is right T-nilpotent by an argument of H. Bass [2].

It remains to show that $R_i/J(R_i)$ is a simple Artinian ring; for then the required properties of R_i follow from Theorem P. Let B be the inverse image in R_i of Soc $(R_i/J(R_i))$; then B is a two-sided ideal of R_i. If $B \neq R_i$ and M is any maximal left ideal of R_i containing B, then the following property holds: $R_i/M \cong R_i/M'$ implies $M' \supseteqq B$. Since nonzero R_i-modules have nonzero socles, then $B \neq J(R_i)$. So since $J(R_i)$ is the intersection of maximal left ideals of R_i, it follows that there exists a maximal left ideal M_1 such that $M_1 \not\supseteqq B$ and hence $R_i/M \not\cong R_i/M_1$. This contradicts the fact that R_i has only one simple R_i-module (up to isomorphism). Hence $B = R$, i.e., $R_i/J(R_i) = \mathrm{Soc}\,(R_i/J(R_i))$. Hence $R_i/J(R_i)$ is semi-simple Artinian. Since R_i has only one simple R_i-module up to isomorphism, then $R_i/J(R_i)$ is a simple ring.

($\Leftarrow$): Let $R = R_1 + R_2 + \cdots + R_n$ (ring direct sum), where each R_i is a right perfect ring with a unique maximal ideal. Then from Theorem P it follows that nonzero modules have nonzero socles. So for any hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ either $R_i \in \mathscr{T}$ or $R_i \in \mathscr{F}$ for $i = 1, 2, \cdots, n$. Then it is not hard to see that $\mathscr{F}$ is closed under homomorphic images, and hence $\mathscr{T}$ gl. dim $R = 0$ by Theorem 3.1.

A torsion theory $(\mathscr{T}, \mathscr{F})$ for ${}_R\mathscr{M}$ is said to be of simple type if it is hereditary and nonzero modules in $\mathscr{T}$ have nonzero socles. Then $(\mathscr{T}, \mathscr{F})$ is of simple type if and only if $\mathscr{T}$ is the smallest torsion class containing a given set of simple modules.

COROLLARY 3.4. *Suppose every hereditary torsion theory* $(\mathscr{T}, \mathscr{F})$ *for* ${}_R\mathscr{M}$ *has* $\mathscr{T}$ gl. dim $R = 0$. *Then the following are equivalent*:

(1) *Every torsion theory for* ${}_R\mathscr{M}$ *is of simple type.*

(2) $J(R)$ *is left T-nilpotent.*

(3) *Nonzero left R-modules have maximal submodules.*

Proof. (2) $\Leftrightarrow$ (3) is immediate from [12], Lemma 1 and Theorem 3.3.

(1) $\Rightarrow$ (3): Let $0 \neq A \in {}_R\mathscr{M}$ be a module with no maximal submodule. Define $\mathscr{F}$ by $\mathscr{F} = \{X \in {}_R\mathscr{M} \mid \operatorname{Hom}(A, X) = 0\}$. It is easily checked that $\mathscr{F}$ is closed under taking submodules, extensions, and direct products; hence $\mathscr{F}$ is a torsionfree class by [5], Theorem 2.3. Since all the simple left R-modules are in $\mathscr{F}$, this contradicts (1).

(3) $\Rightarrow$ (1): From Theorem 3.3 it follows that nonzero left modules have nonzero socles. Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory. It is sufficient to prove that for each $M \in \mathscr{T}$, $\operatorname{Soc}(M) \in \mathscr{T}$. If S is a simple submodule of $M \in \mathscr{T}$, then choose N maximal in the properties $N \subseteqq M$ and $N \cap S = 0$. Then S is isomorphic to an essential submodule of $M/N \in \mathscr{T}$. Since R has (PD), it follows that $M/N \in \mathscr{T}_S$, where $\mathscr{T}_S$ is the smallest torsion class containing S. Thus every maximal submodule T/N of M/N has the property $(M/N)/(T/N) \cong S$. (Such maximal submodules exist by (3).) Thus $M/N \in \mathscr{T}$ implies $S \in \mathscr{T}$.

COROLLARY 3.5. *Let R be commutative. Then the following are equivalent*:

(1) *Every hereditary torsion theory* $(\mathscr{T}, \mathscr{F})$ *has* $\mathscr{T}$ gl. dim. $R=0$.

(2) *R is a direct sum of finitely many local perfect rings.*

(3) $h.\dim(M) = 0$ *or* ∞ *for each* $M \in {}_R\mathscr{M}$.

(4) *Every torsion theory for* ${}_R\mathscr{M}$ *is splitting.*

(5) *R has* (PD) *and* $(\mathscr{S}, \mathscr{F})$ *is splitting.*

Proof. (1) $\Leftrightarrow$ (2) is Theorem 3.3; (2) $\Leftrightarrow$ (3) is a result of I. Kaplansky (see [2]); and (2) $\Leftrightarrow$ (5) is [9], Theorem 5.4.

(1) and (2) $\Rightarrow$ (4): By Corollary 3.4, every torsion theory is of simple type. (PD) follows from (2), and hence every torsion theory splits.

(4) $\Rightarrow$ (5): Suppose (PD) does not hold. Then there exists nonzero $M \in {}_R\mathscr{M}$ such that $\mathscr{S}(M) \neq \bigoplus\sum_{S \in A} \mathscr{T}_S(M)$, where A is a representative set of nonisomorphic simple modules. Let

$$S' \cong N\Big/\sum_{S \in A} \mathscr{T}_S(M) \subseteqq M\Big/\sum_{S \in A} \mathscr{T}_S(M) .$$

The $\mathscr{T}_{S'}$-torsion part of N is $\mathscr{T}_{S'}(M)$, by splitting $N = \mathscr{T}_{S'}(M) \oplus K$ and $\mathscr{T}_S(K) = K \cap \mathscr{T}_S(M) = \mathscr{T}_S(M)$. Since $K \neq \sum_{S \in A} \mathscr{T}_S(M)$, then

$$S' \cong K\Big/\sum_{S \in A-\{S'\}} \mathscr{T}_S(M) .$$

Since the smallest torsion theory containing the set $A - \{S'\}$ splits, then

$$K \cong \Big[\sum_{S \in A-\{S'\}} \mathscr{T}_S(M)\Big] \oplus S' ,$$

which is a contradiction to $K \cap \mathscr{T}_{S'}(M) = 0$.

4. Central splitting. A pair of torsion theories $(\mathscr{C}, \mathscr{T})$, $(\mathscr{T}, \mathscr{F})$ is called a torsion-torsionfree (TTF) theory. In this case $\mathscr{T}$ is both a torsion and a torsionfree class, and hence $\mathscr{T}$ is called a TTF class as in [13]. In § 3 it was pointed out that TTF theories are related to $\mathscr{C}$ gl. dim. $R = 0$, whenever $(\mathscr{C}, \mathscr{T})$ is hereditary. The splitting of TTF theories is studied in [13], and the following is the main result obtained:

THEOREM 4.1. ([13], Th. 2.4). *Suppose that* $(\mathscr{C}, \mathscr{T})$, $(\mathscr{T}, \mathscr{F})$ *is a* TTF *theory. Then the following are equivalent*:

(1) *For all* $M \in {}_R\mathscr{M}$, $M = \mathscr{C}(M) \oplus \mathscr{T}(M)$.

(2) $R = \mathscr{C}(R) \dotplus \mathscr{T}(R)$ *(ring direct sum)*.

(3) $\mathscr{F} = \mathscr{C}$.

(4) $\mathscr{T}(\mathscr{C}(M)) = 0$ *and* $\mathscr{C}(M/\mathscr{T}(M)) = M/\mathscr{T}(M)$ *for all* $M \in {}_R\mathscr{M}$.

The following questions concerning a TTF theory $(\mathscr{C}, \mathscr{T})$, $(\mathscr{T}, \mathscr{F})$ were raised in a conversation between R. L. Bernhardt and the author: (1) If $(\mathscr{T}, \mathscr{F})$ is splitting, is $(\mathscr{C}, \mathscr{T})$ also splitting? (2) In case $(\mathscr{C}, \mathscr{T})$ is splitting, when does $(\mathscr{C}, \mathscr{T})$ have the special type of splitting described in Theorem 4.1?

Examples are given to show that either one of $(\mathscr{C}, \mathscr{T})$ or $(\mathscr{T}, \mathscr{F})$ may be splitting without the other splitting. Conditions under which the splitting of one implies the splitting of the other are discussed.

If $(\mathscr{C}, \mathscr{T})$ satisfies the condition described in Theorem 4.1 (1), then $(\mathscr{C}, \mathscr{T})$ will be called central splitting (as in [3]). The following result ([13], Th. 2.1) may be useful to the reader in the sequel: A hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ for ${}_R\mathscr{M}$ has the property that $\mathscr{T}$ is closed under taking direct products if and only if the filter $F(\mathscr{T}) = \{K \mid R/K \in \mathscr{T},\ K \text{ is a left ideal of } R\}$ has a smallest element I. In this case $I = \mathscr{C}(R)$, where $(\mathscr{C}, \mathscr{T})$ is a torsion theory.

EXAMPLE 4.2. $\mathscr{G}$ is a TTF class and $(\mathscr{G}, \mathscr{N})$ is splitting, but $(\mathscr{C}, \mathscr{G})$ is not splitting. Let K be a field and A a countably infinite index set. Let $Q = \prod_{\alpha \in A} K^{(\alpha)}$, where $K^{(\alpha)} = K$. Then let

$$R = \sum_{\alpha \in A} K^{(\alpha)} + K \cdot 1 \subseteqq Q ,$$

where $1 \in Q$. It is shown in [4] that the Goldie torsion theory $(\mathscr{G}, \mathscr{N})$ is splitting. Since $Z(R) = 0$, then $F(\mathscr{G}) = \{R, \sum_{\alpha \in A} K^{(\alpha)}\}$, and hence $\mathscr{G}$ is closed under products. Finally, $(\mathscr{C}, \mathscr{G})$ is not splitting since $\mathscr{C}(R) = \sum_{\alpha \in A} K^{(\alpha)}$ is not a summand of R.

Before stating the first sufficient condition for the splitting of $(\mathscr{T}, \mathscr{F})$ to imply the splitting of $(\mathscr{C}, \mathscr{T})$, a lemma due to S. E. Dickson is needed. [7], Proposition 1 is a weaker form of this lemma, however, the proofs are almost identical.

LEMMA 4.3. *Let $I = \sum_{i=1}^{n} m_i R$ be a finitely generated right ideal of R. Then the class $\mathscr{D} = \{D \in {}_R\mathscr{M} \mid ID = D\}$ is closed under direct products.*

Proof. Let $D_\alpha \in \mathscr{D} (\alpha \in B)$. If $x \in \prod_{\alpha \in B} D_\alpha$, then for each $\alpha \in B$ there are $x_1^{(\alpha)}, x_2^{(\alpha)}, \cdots, x_n^{(\alpha)} \in D_\alpha$ such that

$$x_\alpha = m_1 x_1^{(\alpha)} + m_2 x_2^{(\alpha)} + \cdots + m_n x_n^{(\alpha)} .$$

Hence, if $x_1, x_2, \cdots, x_n$ are defined in the natural way, then

$$x = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \in I(\textstyle\prod_{\alpha \in B} D_\alpha) .$$

Hence $\mathscr{D}$ is closed under direct products.

THEOREM 4.4. *Let $(\mathscr{C}, \mathscr{T})$, $(\mathscr{T}, \mathscr{F})$ be a* TTF *theory such that $(\mathscr{T}, \mathscr{F})$ is splitting. Suppose the minimal ideal I in the filter $F(\mathscr{T})$ contains no nonzero nilpotent left ideals of R. Then $(\mathscr{C}, \mathscr{T})$ is central splitting if and only if I is finitely generated as a right ideal.*

Proof. $(\Leftarrow)$: Since $(\mathscr{T}, \mathscr{F})$ is splitting, $R = \mathscr{T}(R) \oplus F$ with

$F \in \mathscr{F}$. Then $R/F \in \mathscr{T}$, and hence $F \supseteq I$ by the definition of I. By Lemma 4.3 the class $\mathscr{D} = \{D \in {}_R\mathscr{M} \mid ID = D\}$ is closed under products.

Claim $\mathscr{C} = \mathscr{D}$. Suppose $D \in \mathscr{D}$ and $\varphi: D \to T \in \mathscr{T}$. Then $\varphi(D) = \varphi(ID) = I \cdot \varphi(D) \subseteq I \cdot T = 0$ and so it follows that $\mathrm{Hom}(D, T) = 0$ for all $T \in \mathscr{T}$. Thus $\mathscr{D} \subseteq \mathscr{C}$. Conversely, let $A \in \mathscr{C}$ and observe that $A/IA \in \mathscr{T}$ by the fact that $\mathscr{T} = \{M \in {}_R\mathscr{M} \mid IM = 0\}$. Since $\mathscr{C}$ is closed under homomorphic images and $\mathscr{C} \cap \mathscr{T} = 0$, it follows that $IA = A$. Thus $\mathscr{C} \subseteq \mathscr{D}$.

Next observe I is essential in F. For if K is a left ideal of R contained in F and $K \cap I = 0$, then $IK = 0$. Thus $K \subseteq \mathscr{T}(R) \cap F = 0$.

Claim $xI \neq 0$ for all $0 \neq x \in F$. For if not, let $yI = 0$ for $0 \neq y \in F$. Then $Ry \cap I \neq 0$ since I is essential in F. But $(Ry \cap I)^2 \subseteq RyI = 0$, which contradicts the hypothesis that I contains no nonzero nilpotent left ideals.

Hence F can be embedded as a left R-module in a product of copies of I in the usual way. Moreover, $\prod_{\alpha \in A} I_\alpha \in \mathscr{D}$ (where $I_\alpha = I$ and A is any index set) by Lemma 4.3 and the fact that $I^2 = I$. Since $(\mathscr{T}, \mathscr{F})$ splits, $\mathscr{T}$ is closed under taking injective envelopes by Lemma 1.1. So [5], Theorem 2.9, gives $\mathscr{C} = \mathscr{D}$ is closed under submodules; in particular, $F \in \mathscr{C}$ and $F = IF = I$. But $I = \mathscr{C}(R)$, and hence $R = \mathscr{T}(R) \oplus F = \mathscr{T}(R) \dotplus \mathscr{C}(R)$ (ring direct sum). Hence, $(\mathscr{C}, \mathscr{T})$ is central splitting by Theorem 4.1.

($\Rightarrow$): By Theorem 4.1, $R = \mathscr{C}(R) \dotplus \mathscr{T}(R)$ (ring direct sum) and hence $I = \mathscr{C}(R)$ is a principal right ideal.

PROPOSITION 4.5. *Let $(\mathscr{C}, \mathscr{T})$, $(\mathscr{T}, \mathscr{F})$ be a* TTF *theory such that $(\mathscr{T}, \mathscr{F})$ splits. Then the following are equivalent*:

(1) *$(\mathscr{C}, \mathscr{T})$ is central splitting.*

(2) *$(\mathscr{C}, \mathscr{T})$ is splitting.*

(3) *$\mathscr{C}$ is closed under taking injective envelopes.*

(4) *$\mathscr{C} \supseteq \mathscr{F}$.*

Proof. (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) follows from Lemma 1.1.

(3) $\Rightarrow$ (4): By $(\mathscr{T}, \mathscr{F})$ is splitting, Lemma 1.1, and [5], Theorem 2.9, $\mathscr{C}$ is closed under taking submodules. Let $F \in \mathscr{F}$ and note $\mathscr{C}(F) \subseteq F \subseteq E(\mathscr{C}(F))$: For if not, then there exists $0 \neq T \in \mathscr{T}$ such that $T \subseteq F$, which leads to a contradiction of $\mathscr{T} \cap \mathscr{F} = 0$. But (3) and $\mathscr{C}$ closed under submodules then yield $F \in \mathscr{C}$, and hence $\mathscr{C} \supseteq \mathscr{F}$.

(4) $\Rightarrow$ (1): Since $(\mathscr{T}, \mathscr{F})$ is splitting, write $R = \mathscr{T}(R) \oplus F$ with $F \in \mathscr{F}$. Since $R/F \in \mathscr{T}$, then $F \supseteq \mathscr{C}(R)$. But $F \in \mathscr{C}$ by (4), so $F = \mathscr{C}(R)$. Hence $R = \mathscr{T}(R) \dotplus \mathscr{C}(R)$ (ring direct sum), so that $(\mathscr{C}, \mathscr{T})$ is central splitting by Theorem 4.1.

EXAMPLE 4.6. $\mathscr{T}$ is a TTF class and $(\mathscr{C}, \mathscr{T})$ is splitting, but not central splitting. Let R be the ring of all 2×2 upper triangular matrices over the field Q of rational numbers. Let $I = \left\{\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| \, x, y \in Q\right\}$, so that I is a two-sided idempotent ideal of R. Define:

$$\begin{aligned} \mathscr{T} &= \{M \in {}_R\mathscr{M} \mid IM = 0\} \\ \mathscr{F} &= \{M \in {}_R\mathscr{M} \mid \operatorname{Hom}(T, M) = 0 \text{ for all } T \in \mathscr{T}\} \\ \mathscr{C} &= \{M \in {}_R\mathscr{M} \mid \operatorname{Hom}(M, T) = 0 \text{ for all } T \in \mathscr{T}\}\,. \end{aligned}$$

Then $\mathscr{T}$ is a TTF class, and $(\mathscr{C}, \mathscr{T})$ and $(\mathscr{T}, \mathscr{F})$ are torsion theories. Since R/I is a projective simple R-module, it follows that all modules in $\mathscr{T}$ are projective. Hence $(\mathscr{C}, \mathscr{T})$ is splitting. But $\mathscr{T}(R) = \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \middle| \, x, y \in Q\right\}$ is not a direct summand of R; so $(\mathscr{T}, \mathscr{F})$ is not splitting.

PROPOSITION 4.7. *Let $\mathscr{T}$ be a* TTF *class, and let $(\mathscr{C}, \mathscr{T})$ be a splitting torsion theory. Then the following are equivalent*:

(1) *$(\mathscr{C}, \mathscr{T})$ is central splitting*

(2) *$(\mathscr{C}, \mathscr{T})$ is hereditary*

(3) *$\mathscr{C}(R) \cap \mathscr{T}(R) = 0$*

(4) *$\mathscr{C}(R) \cap \mathscr{T}(R)$ contains no nonzero nilpotent left ideals of R.*

Proof. (1) $\Rightarrow$ (2) is immediate from Theorem 4.1 (3).

If (2) holds, then $\mathscr{C}(R) \cap \mathscr{T}(R) \in \mathscr{C} \cap \mathscr{T} = 0$, and hence (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) is trivial.

Suppose (4) holds. Since $(\mathscr{C}, \mathscr{T})$ is splitting, $R = \mathscr{C}(R) \oplus T$ with $T \in \mathscr{T}$. Hence $\mathscr{T}(R) \supseteqq T$. But then

$$[\mathscr{C}(R) \cap \mathscr{T}(R)]^2 \subseteqq \mathscr{C}(R) \cdot \mathscr{T}(R) = 0$$

implies $\mathscr{C}(R) \cap \mathscr{T}(R) = 0$ by (4). Hence $\mathrm{T} = \mathscr{T}(R)$, and thus (1) holds by Theorem 4.1 (2).

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FINITE LINEAR GROUPS OF DEGREE SEVEN II

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The determination of finite groups which can be represented as a group of 7×7 matrices irreducible over the complex numbers is finished in this paper. To simplify the cases, the matrices are assumed unimodular and the groups are primitive. The groups discussed here are essentially simple and have orders $7 \cdot 5^a \cdot 3^b \cdot 2^c$. The theory of groups with a prime to the first power in the group order and of course the representation of degree seven are used heavily in the determination.

This paper is the third in a series of papers discussing linear groups, the first two being [24, 25]. We shall prove the following result.

THEOREM I. *Suppose G has an irreducible complex representation X of degree* 7 *which is faithful, primitive and unimodular. Suppose, further, that G has an abelian 7-Sylow subgroup. Then, by* [5, 4A], *Z, the center of G, has order* 1 *or* 7 *and $G = G_1 \times Z$ for a subgroup G_1 of G. We prove that G_1 is one of the following groups. Let $|G_1|$ be the cardinality of G_1.*

I. $G_1 \cong PSL(2,13)$ $\quad |G_1| = 13 \cdot 7 \cdot 3 \cdot 2^2 = 1092$.
II. $G_1 \cong PSL(2,8)$ $\quad |G_1| = 7 \cdot 3^2 \cdot 2^3 = 504$.
III. $G_1 \cong A_8$ $\quad |G_1| = 8!/2 = 20160$.
IV. $G_1 \cong PSL(2,7)$ $\quad |G_1| = 7 \cdot 3 \cdot 2^3 = 168$.
V. $G_1 \cong U_3(3)$ $\quad |G_1| = 7 \cdot 3^3 \cdot 2^5 = 6048$.
VI. $G_1 \cong S_6(2)$ $\quad |G_1| = 7 \cdot 5 \cdot 3^4 \cdot 2^9 = 1451520$.
VII. *An extension of* III, IV, V *by an automorphism of order* 2 *or an extension of* II *by an automorphism of order* 3. *For* III *it is S_8; for* IV *it is induced by $PGL(2,7)$. For* V *and* II *it is induced by field automorphisms. For V it is $G_2(2)$.*

This result together with [25, Th. 4.1] determines the linear groups of degree 7. The proof is in several parts. Rather than use the notation G_1 we assume G is as stated in the theorem and assume $Z = e$. Set $|G| = g$, the order of G. By [5, 4A] we know $|G| = 7 \cdot g_0$ where $7 \nmid g_0$. Let $|G| = g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. We use the notation of [24, 25]. Thus χ is the character of X. As χ is of zero 7-defect, $\chi(\xi) = 0$ where ξ is an element of order 7 [7, Th. 1]. This means the eigenvalues are all distinct and so by [5, 3F] $C(P) = P$ where P

is a 7-Sylow group. The characters of G satisfy many properties described in [5, § 8]. These are different depending on the value $s = |N(P)/P|$. The possible values are 2, 3, or 6. The value $s = 1$ is impossible as by Burnside's theorem there would be a normal 7-complement contradicting the primitivity of X. The cases 2, 3, 6 are treated separately. The case $s = 6$ is by far the most difficult. It is treated first (§ 2–§ 5) because some of the ideas are used for the case $s = 3$ (§ 7). However much of the treatment for $s = 3$ (§ 7) and all of the treatment for $s = 2$ (§ 6) is independent of the earlier sections and can be read independently.

If there are primes higher than 7 occurring in $g = |G|$, G is case I by [17, 5, 2D]. In the remaining cases we assume $g = 7\cdot 5^a\cdot 3^b\cdot 2^c$. We know by [5, 3E and 25, 2.6] that $a \leqq 6$, $b \leqq 8$, $c \leqq 10$. A flow chart for the order of the elimination is given at the end.

As in [24 or 25], some notation is standard. Thus if K is a subset of G, $C(K)$ and $N(K)$ are the centralizer and normalizer of N. If $\gamma \in G$ we define $N(\gamma) = N(\langle\gamma\rangle)$. Also set $C(K) \cap K = Z(K)$. Let $|K|$ denote the cardinality of K. We have set $|G| = g$. We mention Theorem 2.1 of [25] which says that a 5-Sylow group of G is abelian. We label the principal p-block $B_0(p)$ for any prime p dividing g.

2. Preliminary properties of χ when $s = 6$. We first assume $s = 6$, $G = G'$. By [5, 8A], G is simple. As in [5, § 8] the results of [2] apply to G. There are seven characters χ_i, $i = 0, 1, \cdots, 6$, in $B_0(7)$ of G. Their degrees x_i, $i = 0, 1, \cdots, 6$ are congruent to $\pm 1 \pmod 7$. We set $\chi_i(\xi) = \delta_i$, $i = 0, 1, \cdots, 6$. Here $\delta_i = \pm 1$, ξ is an element of order 7. The degree equation for $B_0(7)$ is $\sum_{i=0}^{6} \delta_i x_i = 0$ [2, Th. 11 or 5, § 8]. We assume χ_0 is the identity character.

If χ is the character of degree 7 corresponding to X we have

$$\chi\bar{\chi} = \chi_0 + \sum_{i=1}^{6} a_i\chi_i + \eta \,.$$

Here η is a sum of irreducible characters of G of zero 7-defect. There must be some i with $a_i \neq 0$ $i = 1, 2, \cdots, 6$ for which $\delta_i = -1$. This is because $\chi(\xi) = 0$ and so $\chi\bar{\chi}(\xi) = 0$. The possible values of x_i are 6, 20, 27, and 48. This means G must have a character, say χ_i, of degree 6, 20, 27, or 48. We will consider each of these cases individually in this and later sections. The case $x_i = 6$ is easily eliminated by considering the restriction of χ to $N(\xi)$. As this analysis will be needed in later sections, we include more than is necessary here.

Let $N = N(\xi)$. We are assuming $|N| = 42 = 7\cdot 6$. Let τ be an element of order 6 in N. The character table is as follows where $\varepsilon = e^{2\pi i/3}$.

TABLE I. *Character table for $N = N(\zeta)$.*

element	1	ξ	τ	τ^2	τ^3	τ^4	τ^5
ψ_0	1	1	1	1	1	1	1
ψ_1	1	1	$-\varepsilon$	ε^2	-1	ε	$-\varepsilon^2$
ψ_2	1	1	ε^2	ε	1	ε^2	ε
ψ_3	1	1	-1	1	-1	1	-1
ψ_4	1	1	ε	ε^2	1	ε	ε^2
ψ_5	1	1	$-\varepsilon^2$	ε	-1	ε^2	$-\varepsilon$
ψ_6	6	-1	0	0	0	0	0.

If χ_j is a character of G of degree 6 $\chi_j \mid N = \psi_6$. However the eigenvalues of the representation corresponding to $\psi_6(\tau)$ are 1, -1, $-\varepsilon$, $-\varepsilon^2$, ε, ε^2. This means the determinant is -1. The representation X_j corresponding to χ_j cannot be unimodular and so $G \neq G'$. As we are assuming in this section that $G' = G$ we can assume there are no characters χ_j of degree 6.

For later use we require some further results. The restriction of χ to N must contain ψ_6 and ψ_j $0 \leqq j \leqq 5$ as constituents. In order that $X(\tau)$ be unimodular, $j = 3$. This gives

$$\chi \mid N = \psi_3 + \psi_6 \,. \tag{2.1}$$

Let $P_2(\chi)$ be the character corresponding to the symmetric tensors of rank 2 for X and $C_2(\chi)$ be the character corresponding to the skew symmetric tensors of rank 2. Their restrictions to N are as follows

$$C_2(\chi) \mid N = 3\psi_6 + \psi_1 + \psi_5 + \psi_3 \,, \tag{2.2}$$

$$P_2(\chi) \mid N = 4\psi_6 + 2\psi_0 + \psi_2 + \psi_4 \,. \tag{2.3}$$

We obtain similar results for a character χ_j of degree 8 or 20. Here $\chi_j \mid N$ will have two linear constituents. The character ψ_6 of degree six will appear once if χ_j has degree 8 and three times if χ_j has degree 20. As the representation corresponding to χ_j is unimodular there are three possibilities for the two linear constituents. These are $\psi_0 + \psi_3$, $\psi_1 + \psi_2$, $\psi_4 + \psi_5$. If χ_j has degree 8 this gives

$$\text{(2.4)}\quad \begin{array}{ll} \text{(i)} & \chi_j \mid N = \psi_6 + \psi_0 + \psi_3 \\ \text{(ii)} & \chi_j \mid N = \psi_6 + \psi_1 + \psi_2 \\ \text{(iii)} & \chi_j \mid N = \psi_6 + \psi_4 + \psi_5 \,. \end{array}$$

If χ_j has degree 20 the corresponding result is

$$
\begin{array}{lll}
 & \text{(i)} & \chi_j \mid N = 3\psi_6 + \psi_0 + \psi_3 \\
(2.5) & \text{(ii)} & \chi_j \mid N = 3\psi_6 + \psi_1 + \psi_2 \\
 & \text{(iii)} & \chi_j \mid N = 3\psi_6 + \psi_4 + \psi_5 .
\end{array}
$$

If χ_j is real the only possibilities are (2.4)(i) or (2.5)(i).
The following lemma will be needed several times.

LEMMA 2.1. *Let Q be a 5-Sylow group of G. If a character μ of G is not real and $\mu \mid Q$ is not rational, there are at least four distinct nonreal conjugates of μ.*

Proof. Let K be a splitting field for G containing $\lambda_1 = e^{2\pi i/25}$. Let K_1 be the subfield of K containing any r-th roots of 1 lying in K where r runs over the primes in g other than 5. We may pick $\sigma \in G(K/K_1)$ the Galois group of K over K_1 so that $\sigma(\lambda_1) = (\lambda_1)^2$. Suppose $\mu, \mu^\sigma, \bar{\mu}, \bar{\mu}^\sigma$ are not all distinct. As there are no elements of order 5^3 in $G[5, 3B]$, $\mu \mid Q \neq \mu^\sigma \mid Q$ by hypothesis. Also $\mu \neq \bar{\mu}$ by hypothesis. This means $\bar{\mu} = \mu^\sigma$. In particular $\mu^\sigma \mid Q = \bar{\mu} \mid Q$. Let $\eta = \sigma^{10}$. We have $\mu^\eta \mid Q = \bar{\mu} \mid Q = \mu^\sigma \mid Q$. This implies $\mu^\sigma \mid Q = \mu \mid Q$ a contradiction. We see $\mu, \mu^\sigma, \bar{\mu}, \bar{\mu}^\sigma$ are all distinct. Clearly none can be real. This proves the lemma.

Several times we will need to study the case in which Q is cyclic of order 5. That is $g = 7 \cdot 5 \cdot 3^b \cdot 2^c$. The results of [2] can be applied for $p = 5$. Let π be an element of order 5. By Burnside's theorem $C(\pi) = \langle \pi \rangle \times V$. As there are no elements of order $5 \cdot 7$, $\mid V \mid = 3^\beta 2^\gamma$. Let $\mid N(\pi)/C(\pi) \mid = w$. As G has no normal 5-complement w is 2 or 4 by Burnside's theorem. Each 5 block contains e nonexceptional characters and $4/e$ exceptional characters where e is 1, 2, or 4. Let $B_1(5)$ be the 5-block containing χ. If χ is nonexceptional there are two possibilities for $\chi \mid C(\pi)$ as can be seen from close inspection of [2, II, Th. 1]

$$\chi \mid \langle \pi \rangle \times V = \theta + \sum_{i=0}^{4} \lambda^i \cdot \varphi \tag{2.6}$$

where θ is of degree 2 with $\theta(\pi) = 2$, λ is the linear character of $\langle \pi \rangle$ such that $\lambda(\pi) = e^{2\pi i/5}$, φ is a linear character of V.

$$\chi \mid \langle \pi \rangle \times V = \varphi_1 + \varphi_2 + \sum_{i=0}^{4} \lambda^i \varphi \tag{2.7}$$

where λ, φ are as in (2.6) and φ_1, φ_2 are distinct linear characters of $c(\pi)$ conjugate in $N(\pi)$. Also $\varphi_i(\pi) = 1$, $i = 1, 2$. In the case (2.6)

V is nonabelian; in the case (2.7) V is abelian.

Suppose χ is exceptional. If $s = 4$ and $e = 1$ or 2 an examination of [2, II, Th. 1] shows χ cannot have degree 7. If $e = 4$ any of the characters in $B_1(5)$ can be chosen exceptional and so χ can be chosen nonexceptional. In any case X satisfies (2.6) or (2.7). If $s = 2$, there are at least two characters χ and χ' of degree 7 in $B_1(5)$. By [7, p. 579] $\chi\chi'$ has a constituent in $B_0(5)$. As $\chi \neq \chi'$ this constituent is not x_0 and so $B_0(5)$ has a nontrivial character whose degree is at most 49. By [2] there are three characters in $B_0(5)$ besides x_0. Their degrees must be congruent to ± 1 or $0 \bmod 7$. By examining the possibilities one sees the smallest such degree equation for $B_0(5)$ is $1 + 63 = 64$. This means χ cannot be exceptional and so χ must satisfy (2.6) or (2.7).

3. **The case $\chi\bar{\chi} = \chi_0 + \chi_1$, $\deg \chi_1 = 48$, $g = 7\cdot 5^a \cdot 3^b \cdot 2^c$.** In this section we consider the case IB where $\chi\bar{\chi} = \chi_0 + \chi_1$, $\deg \chi_1 = 48$. We still assume $s = 6$. This case is eliminated by first showing χ is rational when restricted to 5-Sylow group and so $a \leqq 1$. The case $a = 1$ is eliminated by finding $C(\pi)$ where π is an element of order 5. The case $a = 0$ is then eliminated using some results in [6].

Suppose χ_1 is not rational. In particular let σ be an element of the Galois group of a splitting field K for G over the rationals for which $(\chi_1)^\sigma \neq \chi_1$. Clearly $\chi^\sigma\bar{\chi}^\sigma = \chi_0 + \chi_1^\sigma$. This implies $\chi\chi^\sigma\bar{\chi}\bar{\chi}^\sigma$ has χ_0 as a constituent with multiplicity 1. This means $\chi\chi^\sigma$ of degree 49 is irreducible giving a contradiction. We see χ_1 is rational. Also $\chi\bar{\chi} = \chi_0 + \chi_1$ must be rational

Let π be an element of order 5. Suppose $\chi \mid \langle\pi\rangle = \sum_{i=1}^t b_i\lambda_i$ where the λ_i, $i = 1, 2, \cdots, t$ are distinct linear characters of $\langle\pi\rangle$, $b_i \neq 0$. Certainly $t \leqq 5$ and $\sum_{i=1}^t b_i = 7$. If $c = \sum_{i=1}^t (b_i)^2$ we have $\chi\bar{\chi}(\pi) = c - b$ where $c + 4b = 49$. This means $c \equiv 1 \pmod 4$. If the numbers $\{b_1, \cdots, b_t\}$ are arranged in decreasing order the following possibilities occur:

$$\{3, 1, 1, 1, 1\}; \{2, 2, 2, 1\}; \{4, 2, 1\}; \{3, 2, 2\}; \{6, 1\}; \{5, 2\}; \{4, 3\} .$$

Now let λ be the linear character of $\langle\pi\rangle$ with $\lambda(\pi) = e^{2\pi i/5}$. Suppose $\chi \mid \langle\pi\rangle = \sum_{i=0}^4 a_i\lambda^i$. This means

$$\chi\bar{\chi} \mid \langle\pi\rangle = \{\sum_{i=0}^4 a_i\lambda^i\}\{\sum_{i=0}^4 a_i\lambda^{-i}\} = \sum_{i=0}^4 \{\sum_{j=0}^4 a_j a_{j-i}\}\lambda^i .$$

As $\chi\bar{\chi}$ is rational we obtain

$$\sum_{j=0}^4 a_j a_{j-i} = b , \qquad i = 1, 2, 3, 4 . \tag{3.1}$$

The nonzero entries among $\{a_0, a_1, \cdots, a_4\}$ are the values $\{b_1, \cdots, b_t\}$ given in the above paragraph possibly rearranged. A routine check of these possibilities shows $\{3, 1, 1, 1, 1\}$ to be the only set for which (3.1) can be satisfied. The checking is facilitated by noting that the first two integers a_0, a_1 can be picked arbitrarily from $\{b_1, \cdots, b_t, 0, \cdots, 0\}$ without changing the form of (3.1). Here the bracketed integers are completed with zeroes to give 5 terms. The case $[4, 2, 1\}$ or $\{3, 2, 2\}$ can be chosen as $\{0, 0, \delta_1, \delta_2, \delta_3\}$ where $\delta_1, \delta_2, \delta_3$ is 4, 2, 1 or 322 in some order. Equation (3.1) gives $\delta_1\delta_2 + \delta_2\delta_3 = \delta_1\delta_3$. This is impossible for any of the choices of $\delta_1, \delta_2, \delta_3$. The case $\{2, 2, 2, 1\}$ can be taken as $\{0, 1, 2, 2, 2\}$. Equation (3.1) is not satisfied. In the cases $\{b_1, b_2\}$ we consider $\{b_1, b_2, 0, 0, 0\}$. Equation (3.1) cannot be satisfied. In the case $\{3, 1, 1, 1, 1\}$ the unimodularity of X gives $\chi \mid \langle\pi\rangle = 3\lambda^0 + \lambda + \lambda^2 + \lambda^3 + \lambda^4$. This means $\chi(\pi) = 2$.

Suppose π has order 5^2. As π^5 has order 5 the constituents of $\chi \mid \langle\pi^5\rangle$ are $\{3\lambda^0, \lambda, \lambda^2, \lambda^3, \lambda^4\}$ where $\lambda(\pi^5) = e^{2\pi i/5}$. Let the linear constituents of $\chi \mid \langle\pi\rangle$ be $\{\lambda_1, \lambda_2, \lambda_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ where $(\varepsilon_i)^5 \mid \langle\pi^5\rangle = \lambda^i$, $(\lambda_i)^5 \mid \langle\pi^5\rangle = \lambda^0$. Suppose $\lambda_i = \lambda_j$, $i \neq j$, $i, j = 1, 2$, or 3. This means $\varepsilon_1\bar{\lambda}_i$ and $\varepsilon_1\bar{\lambda}_j$ are equal and so all twenty conjugates appear with equal multiplicity at least 2 in $\chi\bar{\chi} \mid \langle\pi\rangle$. The following thirteen linear characters also appear in $\chi\bar{\chi} \mid \langle\pi\rangle$: $\varepsilon_i\bar{\varepsilon}_i$, $i = 1, 2, 3, 4$; $\lambda_j\lambda_k$, $j, k = 1, 2, 3$. None of these are conjugate to $\varepsilon_1\bar{\lambda}_i$ and so we have too many constituents. This means $\lambda_1, \lambda_2, \lambda_3$ are all distinct. Also $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are all distinct as their fifth powers are distinct. This means the trivial character of $\langle\pi\rangle$ occurs seven times. The number of conjugates of any nontrivial character is 4 or 20 and hence divisible by 4. However $4 \nmid 42$. This means there are no elements of order 5^2 in G.

We have shown χ is rational on a 5-Sylow group. In particular, by [21] $a \leqq 1$. We now consider this case, case IB(ii) of the flow chart. There are two possibilities for $\chi \mid C(\pi)$, (2.6) and (2.7). In case (2.7), $C(\pi)$ is abelian as $\chi \mid C(\pi)$ has seven linear constituents. In case (2.6), V is nonabelian as $\chi \mid V$ has a constituent of degree 2.

We may also consider the restriction $\chi_1 \mid V \times \pi$. Here $\chi_1(\pi) = 3$. The only possibility [2, II, Th. 1] is $\chi_1 \mid V \times \pi = \theta\cdot\lambda^0$, $\deg\theta = 3$. This means V is nonabelian as θ is irreducible. In particular, case (2.7) above does not occur. The character θ is rational as χ_1 is rational. As $\chi\bar{\chi} = \chi_0 + \chi_1$ we have $\theta_1\bar{\theta}_1 = \theta_0 + \theta$ where θ_0 is the trivial character of V, $\chi \mid V \times \langle\pi\rangle = \theta_1 + \sum_{i=0}^4 \lambda^i\cdot\varphi$.

We know $\chi \mid V = \theta_1 + 5\varphi$. If R is in the kernel of θ_1, $(\varphi(R))^5 = 1$ by the unimodularity of X. This means $\varphi(R) = 1$ and so θ_1 is faithful. Let V_3 be a 3-Sylow group of V. As θ_1 is faithful of degree 2, V_3 must be abelian. We see $X \mid V_3$ has at most 3 distinct linear characters and so $|V_3| \leqq 3^2$ [5, 3D]. Let V_2 be a 2-Sylow group of V. As

V_2 has a faithful representation of degree 2 there is an abelian subgroup A of index 2. As $X \mid A$ has at most 3-distinct linear characters, $|A| \leqq 2^2$ [5, 3D]. Here V_2 is nonabelian if and only if $|V_2| = 2^3$. In this case an involution in $Z(V_2)$ must satisfy $\theta_1(J) = -2$. Clearly $\varphi(J) = 1$. In particular $J \in Z(V)$.

We now consider θ. Let K be the kernel of θ. As $\theta_1\bar{\theta}_1 = \theta_0 + \theta$, if $R \in K$, $|\theta_1(R)| = 2$. In particular $R \in Z(V)$. Suppose there is an element R of order 3 in K. As θ is faithful and rational on V/K $|V| = 3^2 \cdot 2^\delta$. Also $\theta_1(R)$ is $2u$ or $2u^2$ where $u = e^{2\pi i/3}$. We can assume $\theta_1(R) = 2u$ by taking R^2 if necessary. By the unimodularity of X, $\varphi(R) = u^2$. If there is an element J in V for which $\theta_1(J) = -2$, $\varphi(J) = 1$ and Blichfeldt's theorem is violated for X and JR. This means $\delta \leqq 2$. As V/K has a representation of degree 3, $\delta = 2$. If V_2 is elementary abelian there is an element J for which $\theta_2(J) = -2$. We see V_2 is cyclic. This means there is a normal 2-complement. However, in this case there can be no character of degree 3 by Ito's theorem [12, 53.18]. This shows there are no elements of order 3 in K.

As V/K has a rational character of degree 3 $|V| = 3 \cdot 2^\delta$. Here V has characters of degree 3 and 2. As $3^2 + 2^2 > 12$ we see $|V| = 3 \cdot 2^3$. This means V_2 is nonabelian and so there is an involution in $Z(V)$ for which $\theta_1(J) = -2$. Let T be an element of order 3 in V. If $\theta_1(T) = u + \bar{u}$ where $u = e^{2\pi i/3}$ the element TJ would contradict Blichfeldt's theorem. This means $\theta_1(T) = 1 + u$ or $1 + u^2$. We see $\bar{\theta}_1 \neq \theta_1$. By the unimodularity of X, $\chi(T) = 1 + 6u$. We see T is not conjugate to T^{-1} in G. There must be a normal three complement to $\langle T \rangle$ in V and so the number of linear characters of V is a multiple of three.

The characters obtained so far have degrees 1, 1, 1, 3, 2, 2. There must be one further character of degree 2. As $\theta_1\bar{\theta}_1 = \theta_0 + \theta$ has two irreducible constituents, $\theta_1\theta_1 = P_2(\theta_1) + C_2(\theta_1)$ must have two irreducible constituents. These are the characters corresponding to the symmetric tensors of rank 2, $P_2(\theta_1)$, and the skew symmetric tensors of rank 2, $C_2(\theta_1)$. We see $P_2(\theta_1) = \theta$. Similarly $\chi\chi = P_2(\chi) + C_2(\chi)$. As $\chi\bar{\chi} = 1 + \chi_1$, $P_2(\chi)$ and $C_2(\chi)$ are irreducible. Clearly $\chi\chi(\pi v) = \theta_1\theta_1(v) = (\theta + C_2(\theta_1))(v)$, where $v \in V$. Let $\psi = P_2(\chi)$. Clearly $\psi(\pi v) = \theta(v)$. This means ψ is in the same 5-block as χ_1. Denote this 5-block by $B_1(5)$. Evaluating $\psi(T)$ we find $\psi(T) = \{(1 + 6u)^2 + (1 + 6u^2)\}/2 = -5 + 15u^2$. This means $\psi \neq \bar{\psi}$. However $\psi(\pi v) = \bar{\psi}(\pi v)$ for $v \in V$ and so $\bar{\psi} \in B_1(5)$. We show case I B(ii) is impossible by showing the block $B_1(5)$ cannot be completed without giving a contradiction.

There cannot be two exceptional characters in $B_1(5)$ or there would be too many characters. Here ψ, $\bar{\psi}$ cannot be the exceptional characters as $28 \equiv 48 \pmod 5$. This means there are two missing characters with degrees R and S. There are two cases for the degree equation $104 + R = S$ and $104 = R + S$. There must be one more character from $B_0(7)$ and one whose degree is not divisible by 3. If the degree of the character from $B_0(7)$ is not divisible by 3 it must be 8, 64, or 512. The only solution is $48 + 28 + 28 + 8 = 112$. If the character of degree 8 is denoted by χ_6, $\chi_6 \mid V = \theta + 5\zeta$ where ζ is linear. As $\theta(J) = 3$ we see $\zeta(J) = 1$. This means J is in the kernel of χ_6 contradicting the simplicity of G. This means R or S is of the form $7\cdot 2^B$ where $7\cdot 2^B \equiv \pm 2 \pmod 5$. The degree of the character in $B_0(7)$ is divisible by 3. We see $B = 2, 4, 6, 8, 10$, $7\cdot 2^B \equiv 1 \pmod 3$. We need only consider the cases $7\cdot 2^B \equiv 3 \pmod 5$. The values are 28, 448, 7168. There are no solutions. This case is therefore impossible and we can assume $g = 7\cdot 3^b\cdot 2^c$.

We now begin Case IB (iii). We assume $s = 6$, $g = 7\cdot 3^b\cdot 2^c$. We know $b \leqq 8$. By Sylow's theorem $b = 1, 3, 5, 7$. There must be a character of degree 27 or 729 and one of degree 8,64, or 512.

If a character of degree 729 occurs it must be in a 3-block containing 3 characters of degree 729, for if not there would be a character of degree at least $6\cdot 729$. The degree equation would then be $1 + 729 + 729 + 729 + \begin{Bmatrix} 8 \\ 64 \\ 512 \end{Bmatrix} = 48 + \chi_6$. There is no solution.

There must be a character χ_3 of degree 27. Let $g = 7\cdot 3^b\cdot 2^c$. Sylow's theorem gives $b = 3, 5, 7$. Suppose first $b = 3$. Then $c = 3, 6, 9$. If $c = 3$ or 6, $g < 20{,}000$. All simple groups of order at most 20,000 are listed in [19] and none have this order. If $c = 9$ the result [6, 1H] is contradicted as $2^9 \geqq 12\cdot 3^3 = 324$. When $b = 5$ or 7 [6, 1L] can be applied to show there is no character of degree 512. As there must be a character of degree 2^B it must be 8 or 64. Each of these cases can be eliminated with a routine elimination of degree equations using block separation and Schur's theorem [21]. We do not include the details.

4. The case $\chi\bar{\chi} = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 20$, $g = 7\cdot 5^a\cdot 3^b\cdot 2^c$. In this section we consider the case IC where $\chi\chi = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 20$. The case is eliminated by first showing $a = 1$. This case is eliminated by considering $\chi \mid C(\pi)$ where π is an element of order 5. The relations (2.1)–(2.7) are used.

We begin with a preliminary discussion regarding the tree for the prime 7 [2]. The character χ is a principal 7-indecomposable and so $\chi\bar{\chi}$ is a sum of principal 7-indecomposables [9]. There is exactly one principal indecomposable containing χ_0 as a constituent. This is $\chi_0 + \chi_j$ where χ_j is adjacent to χ_0 on the tree. This means $\chi_0 + \chi_j$ is a constituent of $\chi\bar{\chi}$. We have already eliminated the case

$x_j = 6$ or $x_j = 48$. This means x_j is 20 or 27. We are assuming $x_1 = 20$. If $x_j = 27$, $\chi\bar{\chi}$ would have two linear constituents contradicting the simplicity of G or the irreducibility of χ. This means $x_j = 20$.

Suppose $j \neq 1$. This would mean $\chi\bar{\chi} = \chi_0 + \chi_1 + \chi_j + \mu$. Here μ would be of degree 8. Clearly χ_j is real as it is adjacent to χ_0 on the stem. Also μ is real as it is the only constituent of $\chi\bar{\chi}$ of degree 8 and $\chi\bar{\chi}$ is real. Using (2.4) and (2.5) we see $\chi_0 + \mu + \chi_j \mid N$ has ψ_0 as constituent with multiplicity 3. However (2.2) and (2.3) imply $\chi\bar{\chi} \mid N$ has ψ_0 as constituent with multiplicity 2. This means $j = 1$ and so χ_1 is adjacent to χ_0 on the stem.

Let $\chi\bar{\chi} = \chi_0 + \chi_1 + \mu$. As above χ_1 is real. From (2.2)–(2.5) we see

$$\mu \mid N = 4\psi_6 + \psi_1 + \psi_5 + \psi_2 + \psi_4 \,. \tag{4.1}$$

In particular ψ_0 and ψ_3 are not constituents of $\mu \mid N$. This means by (2.1) that μ has no irreducible constituent of degree 7. By (2.5), μ has no real constituent of degree 20. As μ itself is real it can have no constituent of degree 20. By (2.4), μ has no real constituent of degree 8. If there is a nonreal constituent its conjugate also appears as μ is real. This leaves a remaining constituent of degree at most 12 which is impossible as no χ_i has degree 6. This means μ is irreducible or has two constituents of degree 14.

We can now show χ_1 is rational. Suppose there is some element σ of the Galois group of K such that $\chi_1^\sigma \neq \chi_1$. Then $(\chi\bar{\chi})^\sigma = (\chi^\sigma)(\bar{\chi}^\sigma) = \chi_0 + \chi_1^\sigma + \mu^\sigma$. As μ^σ has no constituent of degree 20, χ_1^σ must be adjacent to χ_0 on the stem. However this means $\chi_1 = \chi_1^\sigma$.

We now show that χ_1 is not in the principal 5-block $B_0(5)$. In fact we show that a character η of degree 20 in $B_0(5)$ must be irrational when restricted to a 5-Sylow group Q. Suppose not. If π has order 5, $\eta(\pi) = -5$. If there is an elementary abelian subgroup of Q of order 5^2 summing the character η over the subgroup gives $(-5)(24) + 20 < 0$ giving a contradiction. If Q has order 5^2, η has 5-defect 1 and so $\eta \notin B_0(5)$. This gives the result as there are no elements of order 5^3 in G. Let $B_2(5)$ be the 5-block containing χ_1.

We will now assume $a \geqq 2$. This is Case IC(i) in the flow chart. We have

$$\chi\bar{\chi} = \chi_0 + \chi_1 + \mu \,. \tag{4.2}$$

Again, let Q be a 5-Sylow group. As $a \geqq 2$, $\chi \mid Q$ cannot be rational by Schur [21]. Let σ be an element of the Galois group of a splitting field K which fixes all p-th roots of unity for primes p other than 5. Set

(4.3) $$\chi^{\sigma}\bar{\chi} = \sum a_i\zeta_i \; .$$

Here the ζ_i are irreducible characters of G. Let 4.2 be written in terms of ζ_i as

(4.4) $$\chi\bar{\chi} = \sum b_i\zeta_i \; .$$

We see $\chi\bar{\chi}$ and $\chi^{\sigma}\bar{\chi}$ are equal on 5-regular element. This means for B any 5-block

(4.5) $$\sum_{\zeta_i \in B} a_i\zeta_i(\rho) = \sum_{\zeta_i \in B} b_i\zeta_i(\rho)$$

for any 5-regular ρ.

We apply (4.5) with $B = B_2(5)$, the 5-block containing χ_1. The character χ_1 appears with multiplicity one on the right hand side. There is possibly one second character of degree 14 appearing with no zero coefficient on the right hand side. The degree is therefore 20 or 34. In particular it is congruent to $-1 \pmod 7$. This means the left hand side must contain a character χ_j of degree 20 or 27. As (4.3) is a sum of principal 7-indecomposables there is a character χ_k whose degree is congruent to $1 \pmod 7$. Its degree must be 8 or 15. Also χ_j and χ_k are adjacent on the tree.

Suppose $x_k = 8$. It follows from the discussion in the above paragraph that $\chi_k \notin B_2(5)$. As N is 5-regular we may use (4.5), (4.1), and (2.4) to see χ_k cannot be real. If $\chi_k \mid Q$ is not rational where Q is a 5-Sylow group of G, then Lemma 2.1 gives four nonreal conjugates of χ_j. The degree equation for $B_0(7)$ must now be

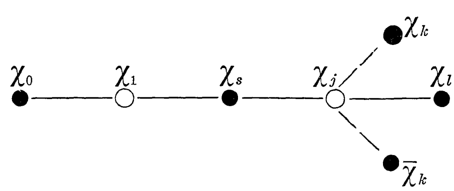
$$1 + 8 + 8 + 8 + 8 = 20 + 13 \; .$$

This is impossible and so $\chi_k \mid Q$ is rational. By Schur's theorem [21], $a \leqq 2$. As we are assuming $a \geqq 2$ we have $a = 2$. Also $B_2(5)$ is of defect 1 and so the right hand side of (4.5) is χ_1. This means χ_j has degree 20 and $\chi_1 = \chi_j$ for 5-regular elements.

Suppose $j \neq 1$ and so $\chi_j \neq \chi_1$. If χ_j is not real the degree equation becomes

$$1 + 8 + 8 + 43 = 20 + 20 + 20$$

which is impossible. If χ_j is real it is on the stem. The stem has 5 characters giving



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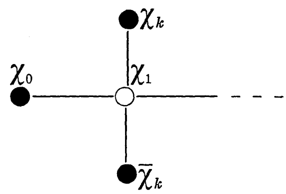
The character χ_l has degree 8 or 15. This is impossible as $8+8+8>20$ and so $2x_k + x_l > x_j$. This means $j = 1$ and so $\chi_j = \chi_1$.

As $\chi_j = \chi_1$ and there are no characters with degree smaller than 8, the tree must be

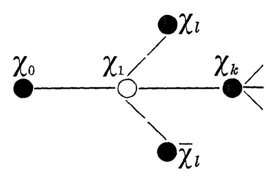


The 7-modular constituents of χ_1 are therefore of degree 1, 8, 8, and 3. By [23] the constituent of degree 3 is realizable in GF(7). However $5^2 \nmid |\mathrm{GL}(3,7)|$ giving a contradiction. This means $x_k = 15$.

Suppose now χ_k has degree 15. Let χ_k be in the 5-block $B_*(5)$. We assume first $B_*(5) \neq B_0(5)$. Apply equation (4.5) with $B = B_*(5)$. A character of degree 15 cannot be fitted into the sum (4.5) over $B_2(5)$ as that sum is of degree 20 or 34. This means $B_*(5) \neq B_2(5)$. The possible sums of degree over $B_*(5)$ on the right of (4.5) are 14 and 28. However $14 < 15$. If the sum is 28, (4.3) must have a linear constituent giving a contradiction. This means $\chi_k \in B_0(5)$.

Because $\chi_k \in B_0(5)$, $\chi_k(\pi)$ must be irrational for any element π of order 5. This means $\chi_k \mid Q$ is irrational where Q is a 5-Sylow group of G. If χ_i is not real Lemma 2.1 gives four nonreal conjugates of χ_k. The degree equation is impossible. Therefore χ_k is real. If χ_k^σ is a conjugate of χ_k a similar argument with χ^σ shows χ_k^σ is real. There is at least one such $\chi_k^\sigma \neq \chi_k$.

Assume first $\chi_j = \chi_1$. As χ_k and χ_k^σ are real they are on the stem. The stem contains at least five characters. Also χ_k is adjacent to χ_1 on the stem. This forces a branch at χ_1. The tree must be



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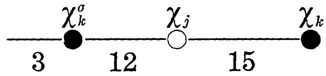
Clearly $x_l = 8$ and G has a 7-modular representation of degree 3. This contradicts $a \geqq 2$ and implies $\chi_j \neq \chi_1$.

We now eliminate this case using the tree and the degree equation. The tree has at least 5 real characters as $\chi_0, \chi_1, \chi_j, \chi_k, \chi_k^\sigma$ are all real. A branch at χ_1 implies a 7-modular character of degree 3 which is a contradiction. This means the character χ_l adjacent to χ_1 other than χ_0 must have degree at least 19 and so cannot be χ_k or χ_k^σ. In turn the stem must have 7 characters. If χ_j has degree 27 the configuration



implies χ_k^g has a 7-modular constituent of degree 3 again giving a contradiction as $a \geqq 2$ [23]. It follows that $x_j = 20$. The degree equation is

$$1 + 15 + 15 + x_l = 20 + 20 + x_m$$

where χ_m is the character adjacent to χ_k^g above. As χ_l is a constituent of $\chi_1\bar{\chi}_1 = \chi_1\chi_1$ its degree is at most $20\cdot 21/2$. The only such degrees are 36, 50, 64, 120, and 162. There is only one solution

$$1 + 15 + 15 + 36 = 20 + 20 + 27 .$$

This is eliminated by 5-block separation using $B_2(5)$. We have completed all cases where $a \geqq 2$ which is part IC(i) of the flow chart.

We now consider the case in which $a = 1$. This is IC(ii) of the flow chart. We use the results (2.6) and (2.7). The case is eliminated by a careful examination of the decompositions of $\chi\bar{\chi}$ and $\chi\chi$ and their restrictions to $C(\pi)$. The results of [2] provide contradictions for each of the possibilities for decompositions of $\chi\bar{\chi}$ and $\chi\chi$.

We know from (2.6) and (2.7) that χ is 5-rational, that is χ lies in the field of $g/5$th roots of unity. This can also be shown using (4.5). We know that $\chi\bar{\chi} = \chi_0 + \chi_1 + \mu$. From (4.1) we know that μ is either irreducible or has two constituents μ_1 and μ_2 of degree 14. Also (4.1) shows $\mu_1 \mid N \neq \mu_2 \mid N$. As χ is 5-rational so is μ. As N consists of 5-regular elements, μ_1 and μ_2 are 5-rational also when μ is reducible. This means that the constituents of $\chi\bar{\chi}$ of full 5-defect are all 5-rational and consequently nonexceptional for $p = 5$. [2].

Let S_0 be the character of $\langle\pi\rangle$ defined by $S_0(e) = 5$, $S_0(\pi) = 0$. In case (2.6) let $\gamma = \theta$, in case (2.7) let $\gamma = \varphi_1 + \varphi_2$. Then by (2.6) or (2.7), $\chi \mid C(\pi) = \gamma + \varphi S_0$. Let $\gamma\bar{\gamma} = \xi_0 + \xi_1$ where ξ_0 is the trivial character of $C(\pi)$. This means

$$\chi\bar{\chi} \mid C(\pi) = \xi_0 + \xi_1 + \gamma\bar{\varphi}S_0 + \bar{\gamma}\varphi S_0 + 5\varphi\bar{\varphi}S_0 . \tag{4.6}$$

Assume now μ is reducible. By [2, II, Th. 1] set $\mu_i \mid C(\pi) = \pm\bar{\psi}_i + S_i$ where ψ_i is a sum of τ_i irreducible characters θ_i^j, $j = 1, 2, \cdots, \tau_i$ containing π in their kernel. The S_i are $\gamma_i S_0$ where γ_i is a character of V. The θ_i^j, $j = 1, 2, \cdots, \tau_i$ are conjugate in $N(\pi)$ and so $\tau_i = 1, 2, 4$. Using 4.6 and 4.2 we see $\pm\psi_1 \pm \psi_2 = \xi_1$. If $\psi_1 \neq \psi_2$ both signs must be plus. Interchanging ψ_1 with ψ_2 if necessary we may assume ψ_1 has degree 1, $\tau_1 = 1$. But then degree $\mu_1 \equiv 1 \pmod 5$ giving a contradiction. If $\psi_1 = \psi_2$ then both signs must be equal.

As ξ_1 has odd degree this is impossible.

This contradiction implies that μ must be irreducible. We may set $\mu \mid C(\pi) = \pm \psi + S$ as before. This time $\psi = \xi_1$. As ψ is a sum of τ irreducible characters θ^j of $C(\pi)$ with the same degree and $\tau = 1, 2, 4$, we see $\tau = 1$ and θ^1 has degree 3. In particular V is nonabelian and so case (2.6) holds. This gives $\theta\bar{\theta} = \xi_0 + \theta^1$.

The same technique can be applied to $\chi\chi$. We have first that $\chi\chi = P_2(\chi) + C_2(\chi)$. Using (2.2), (2.3), (2.4) and (2.5) it can be seen that the constituents of $\chi\chi$ are all of zero 7-defect. As $\chi\bar{\chi}$ has three distinct irreducible constituents so does $\chi\chi$. Using (2.2) and (2.3) we see that either $P_2(\chi)$ has two distinct constituents of degree 14 and $C_2(\chi)$ is irreducible or $P_2(\chi)$ is irreducible and $C_2(\chi)$ has constituents of degrees 14 and 7. Let these be η_1, η_2, η_3. We may again check using (2.2), (2.3), (2.4) that they are 5-rational.

We also have $\theta^2 = P_2(\theta) + C_2(\theta)$. Also $P_2(\theta)$ and $C_2(\theta)$ are irreducible as $\theta\bar{\theta} = \xi_0 + \theta^1$. As before we may set $\eta_i \mid C(\pi) = \pm \psi_i + S_i$ where the ψ_i are sums of characters conjugate in $N(\pi)$ each with π in the kernel. We have

$$\pm\psi_1 \pm \psi_2 \pm \psi_3 = P_2(\theta) + C_2(\theta) .$$

If the ψ_i, $i = 1, 2, 3$ are all distinct there would be three constituents on the right. It is easy to see that we may assume $\psi_1 = \psi_2$ and the signs are opposite. Therefore $\psi_3 = P_2(\theta) + C_2(\theta)$ which is impossible as $P_2(\theta)$ is irreducible of degree 3 and ψ_3 is a sum of irreducible characters of the same degree. This contradiction finishes this section.

5. **The case $\chi\bar{\chi} = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 27$, $g = 7\cdot 5^a\cdot 3^b\cdot 2^c$.** In this section we consider the case ID where $\chi\bar{\chi} = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 27$. Here $s = 6$. There is exactly one group of this form $S_6(2)$ of order $7\cdot 5\cdot 3^4\cdot 2^9$. Cases are eliminated by first showing χ is rational when restricted to a 3-Sylow group. This is done much as in § 4 where it was shown that χ restricted to a 5-Sylow group was rational. Here the character χ_1 of degree 27 cannot be in the principal 3-block by [8]. We can therefore use relations like (4.5) for the 3-block containing χ_1. Once it is known that χ restricted to a 3-Sylow group is rational, the value of b is at most 4 by Schur [21]. As χ_1 is of degree 27, $b \geqq 3$. The two cases $b = 3$ and $b = 4$ are treated separately. For $b = 3$, the generalized decomposition numbers for χ on $C(\tau^2)$ are examined where τ^2 is of order 3 and normalizes a 7-Sylow group. These lead to a contradiction. For $b = 4$, the 3-Sylow group is determined explicitly. The character χ_1 is of 3-defect 1. The various possibilities for the tree are eliminated except of course the one

leading to $S_6(2)$. In this case it is shown there is an involution J for which $\chi(J) = -5$. Adjoining $-I$ to the matrices $X(G)$ gives a group generated by reflections. These groups are all known and we obtain $S_6(2)$.

The analysis here is much longer than in preceding sections and we do not give all the details. Where arguments are similar to earlier arguments they are not repeated. In eliminating cases, the various known techniques involving block separation, cyclic defect groups, etc., are used implicitly. Consequently, only the most troublesome cases are treated.

Let $\chi\bar{\chi} = \chi_0 + \chi_1 + \mu$. If μ contained a constituent of 7-defect 1 its degree would by 6, 8, or 20. We have ruled out a degree 6 in § 2. As μ has degree 21, μ has no constituents of 7-defect 1. As in § 4, $\chi_0 + \chi_1$ is a principal 7-indecomposable and so χ_1 adjoins χ_0 on the stem. In particular χ_1 is real. As in § 4, χ_1 is rational.

We now consider the case in which $\chi \mid S$ is not rational. Here S is a 3-Sylow group of G. This is part ID(i) of the flow chart. Let $B_1(3)$ be the 3-block containing χ_1. By [8] it is not of full defect. This means that $B_1(3) \neq B_0(3)$ where $B_0(3)$ is the principal 3-block. Also the degrees of the characters in $B_1(3)$ are all divisible by 3. Let σ be an element of the Galois group of K which fixes all roots of unity except 3rd roots of unity and for which $\chi^\sigma \mid S \neq \chi \mid S$. This will mean $\chi^\sigma\bar{\chi}$ and $\chi\bar{\chi}$ are equal on 3-regular elements. Let

$$\chi^\sigma\bar{\chi} = \sum b_i\zeta_i \; . \tag{5.2}$$

Let $\mu = \sum c_i\zeta_i$. We have for 3-regular elements ρ

$$\chi_1(\rho) + {\sum}^1 c_i\zeta_i(\rho) = {\sum}^1 b_i\zeta_i(\rho) \tag{5.3}$$

where the sum $\sum^1$ is taken only over the characters ζ_i in $B_1(3)$. If some $c_i \neq 0$ appearing in (5.3) the degree ζ_i must be 21 as such degrees must be divisible by 3. This means some constituent of (5.2) is linear giving a contradiction. Therefore the degree of $\sum^1 b_i\zeta_i$ is 27. There must be a $b_i \neq 0$ appearing in (5.3) for which the degree is congruent to $-1 \pmod 7$. It can only be 27. This gives for 3-regular elements ρ

$$\chi_1(\rho) = \zeta_1(\rho) \tag{5.3$'$}$$

where ζ_1 is an irreducible character of G in $B_1(3)$. As in § 4 there are two cases (i) $\zeta_1 = \chi_1$ and (ii) $\zeta_1 \neq \chi_1$.

In either case there must be exactly one further character ζ_2 in $B_0(7)$ appearing in (5.2). Its degree must be 8 or 15. As $\chi^\sigma\bar{\chi}$ must be a sum of principal indecomposables $\zeta_1 + \zeta_2$ must be a sum of principal indecomposables and so ζ_2 is adjacent to ζ_1 on the tree.

It is now possible to eliminate each of these cases by careful analysis of the tree. The method is routine using block separation, properties of the tree, and the degree equation. Lemma 2.1 and Schur's theorem [21] are used when ζ_2 is of degree 8 and nonreal. We do not give any further details of this enumeration.

We now treat the cases in which $\chi \mid S$ is rational. Again S is a 3-Sylow group. By Schur's theorem [21] the order of S, 3^b, is at most 3^4. As χ_1 has degree 27 we have either $b = 3$ or 4. We will first treat the case $b = 3$, ID (ii) of the flow chart. We then treat the case $b = 4$, ID (iii) of the flow chart.

Assume then that $|S| = 3^3$. We will show first that S is nonabelian. Suppose first S is abelian. Let $\chi \mid S = \sum_{i=1}^{7} \lambda_i$ where the λ_i are linear characters of S. Suppose there is an element T of order 9 in S. There must be an i for which $\lambda_i(T)$ is a primitive 9-th root of unity. As χ is rational all six conjugates must appear amongst the λ_i, $i = 1, 2, \cdots, 7$. For the remaining character λ_j we have $\lambda_j(T) = 1$ by the unimodularity of X. There can be no element of S independent from T. As there are no elements of order 27 in G [5, 3B] we see $|S| \leqq 9$. Therefore S must be elementary abelian. We can write $\chi \mid S = \lambda_1 + \bar{\lambda}_1 + \lambda_2 + \bar{\lambda}_2 + \lambda_3 + \bar{\lambda}_3 + \lambda_0$ where λ_0 is the trivial character. We know χ_1 of degree 27 appears as a constituent in $\chi\bar{\chi}$. As $\chi_1 \mid S$ is the character of the regular representation of S, we see all linear characters of S appear as constituents of $\chi\bar{\chi} \mid S$. Checking with the characters $\lambda_1, \lambda_2, \lambda_3$ shows that $\chi\bar{\chi} \mid S$ cannot contain 27 distinct linear characters as constituents. This means S is nonabelian.

As $\chi \mid S$ is faithful it must contain a nonlinear constituent. As S is a 3-group its degree must be 3. Let μ be this nonlinear character and let U be the representation corresponding to it. Here U must be faithful as any proper quotient group of a group of order 27 is abelian. There is an element R in $Z(S)$ for which $\mu(R) = 3u$ where $u = e^{2\pi i/3}$. Here $U(R) = uI_3$ where I_3 is the 3×3 identity matrix. This means μ is nonreal. As $\chi \mid S$ is rational $\bar{\mu}$ must also appear as a constituent. We see $\chi \mid S = \mu + \bar{\mu} + \lambda_1$ where λ_1 is linear.

The constituents of $\chi \mid S$ are all distinct and so $|C(S)|$ is divisible by the primes 3 and 7 only by [5, 3F]. As $7 \nmid |C(S)|$, we see $C(S) = Z(Q)$. This means that the principal 3-block $B_0(3)$ is the only 3-block of full defect [3, 16D]. We know $Z(S)$ has order 3. Let $\langle R \rangle = Z(S)$. Then $\chi(R) = 3u + 3\bar{u} + \lambda_1(R) = -3 + \lambda_1(R)$. Clearly $\lambda_1(R) = 1$ and so $\chi(R) = -2$. In particular, if S is a 3-element and $\chi(S) \neq -2$, then S is not in the center of any 3-Sylow group and so $3^3 \nmid |C(S)|$.

We will apply this to the element τ^2 of order 3 given in Table

I, § 2. Here $\tau^2 \in N(\xi)$ where ξ is an element of order 7. We may assume $\tau^2 \in S$. By (2.1), $\chi(\tau^2) = 1$ and so $3^3 \nmid |C(\tau^2)|$. Let $|C(\tau^2)| = 3^2 \cdot c_0$ where $3 \nmid c_0$. Here $|C(\tau^2)/\langle\tau^2\rangle| = 3c_0$. For the sake of simplicity we will replace τ^2 with T. This case will be eliminated by considering the generalized decomposition numbers for T.

Let $C = C(T)$, $\bar{C} = C/\langle T\rangle$. We know $|C| = 3^2 \cdot c_0$, $|\bar{C}| = 3c_0$. If b is a 3-block of full defect of C there is a corresponding block $\bar{b}$ of full defect for $\bar{C}$. The modular characters of b all have T in their kernel and can be considered as modular characters of the block $\bar{b}$ of $\bar{C}$. If $\bar{C}_1$ is the Cartan matrix for $\bar{b}$, the Cartan matrix, C_1, for b is $3\bar{C}_1$, [4, p. 154].

Any 3-block of full defect of $\bar{C}$ has a cyclic defect group of order 3. The theory of such defect blocks can readily be applied [2]. There are two cases, (a) and (b). In case (a) there is one modular character and three ordinary characters of the same degree. The Cartan matrix is (3). In case (b) there are two modular characters and three ordinary characters. If f_1 and f_2 are the degrees of the modular characters, the degrees of the ordinary characters are $f_1, f_1 + f_2, f_2$. Also $f_1 \equiv f_2 \pmod 3$. The Cartan matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

We apply these results to the principal 3-block $\overline{b_0(3)}$ of $\bar{C}$. We first show case (a) is impossible. Suppose, then, $\bar{C}$ had one modular character in $\bar{b}_0(3)$. Then the principal 3-block $b_0(3)$ of C has one modular character which is of course the trivial character φ_0. We know that $\chi \in B_0(3)$ as there is only one 3-block of full defect. Results on generalized decomposition numbers in [3, 4] show $\chi(TS) = d\varphi_0(S) = d$ where $S \in C(T)$ and S is 3-regular. We may pick $J = \tau^3$ of order 2. From (2.1) we see $\chi(T) = d = 1$. Therefore $d = 1$. However from (2.1) we see also $\chi(TJ) = -1$. This would mean $d = -1$ giving a contradiction. This shows case (a) does not occur.

This means case (b) occurs. There are two modular constituents of $C(T)$ in $b_0(3)$. One is φ_0. Let φ_1 be the second. Let $J = \tau^3$. If d_0 and d_1 are the decomposition numbers for χ we obtain

$$\begin{aligned} \chi(T) &= d_0 + d_1\varphi_1(e) = 1 \\ \chi(TJ) &= d_0 + d_1\varphi_1(J) = -1 \,. \end{aligned} \tag{5.4}$$

Subtracting we find $d_1(\varphi_1(e) - \varphi_1(J)) = 2$. As J is an involution $\varphi_1(e) - \varphi_1(J)$ is an even integer. Therefore $d_1 = 1$, $\varphi_1(e) - \varphi_1(J) = 2$. Equations (5.4) become

$$\begin{aligned} d_0 + \varphi_1(e) &= 1 \\ d_0 + \varphi_1(J) &= -1 \,. \end{aligned} \tag{5.4$'$}$$

This means d_0 is a rational integer. As $d_0\bar{d}_0 \leqq 6$ and $\varphi_1(e) \equiv 1 \pmod 3$

we see $\varphi_1(e) = 1$, $d_0 = 0$. This shows that for any character χ of degree 7 the decomposition numbers are $d_0 = 0$, $d_1 = 1$.

We are now in a position to analyze the decomposition matrix if there are at least four characters of degree 7. Suppose then that there are four characters of degree 7. Let D be the nonzero rows of the decomposition matrix of $B_0(3)$ with respect to T. We know ${}^t\overline{D}D = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$. The entries of D are in $Z[\rho]$ where $\rho = e^{2\pi i/3}$. We let the first column correspond to φ_0, the second to φ_1. By a small amount of trial and error, we find there is one possibility to within permutations of the rows and changes in sign. This is

$$D = \begin{bmatrix} 1 & 0 \\ \pm 1 & \pm 1 \\ \pm 2 & \pm 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

This shows there is exactly one character whose degree is not congruent to zero (mod 3) other than χ_0 and the four characters of degree 7. This is the character corresponding to the second row. As $B_0(3)$ is the only 3-block of full defect, these six characters are the only ones whose degrees are not divisible by 3. This new character must therefore be in $B_0(7)$ or the degree equation could not be satisfied.

We are now in a position to obtain a contradiction if $a \geqq 2$ where $g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. Certainly if $a \geqq 2$, $\chi \mid Q$ is irrational by Schur's theorem. Here Q is a 5-Sylow group of G. Suppose χ is not real. By Lemma 2.1 there are at least four conjugates of χ. The conclusions of the above paragraph apply. Let $P_2(\chi) = \sum a_i \eta_i$ where η_i are irreducible characters of G, $P_2(\chi)$ is the character corresponding to the symmetric tensors of rank two for X. As $\chi \neq \overline{\chi}$ none of the η_i are linear. By (2.3) no character of degree 7 can be a constituent of $P_2(\chi)$. As 28 is not divisible by 3, $P_2(\chi)$ is reducible by the above paragraph. In fact it is impossible to write $P_2(\chi)$ as a sum of characters satisfying the above paragraph. This means $\chi \mid Q$ is real.

Suppose $a \geqq 3$ where $g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. By [25, Th. 3.1], $\chi \mid Q$ is not real. The above paragraph applies and eliminates this case. Therefore $a \leqq 2$. Sylow's theorem gives $a = 2$ or 0.

We first treat the case $a = 2$. By the paragraphs above we know χ is real. Therefore $\chi\overline{\chi} = \chi\chi = P_2(\chi) + C_2(\chi)$ where $P_2(\chi)$ is the character corresponding to the symmetric tensors of rank two. As

χ_1 is a constituent of $\chi\chi$ we see by considering degrees that $P_2(\chi) = \chi_0 + \chi_1$. As χ_0 and χ_1 are rational this means $P_2(\chi)$ is rational. Using the formula for $P_2(\chi)$ we have that $\chi^2(R) + \chi(R^2)$ is rational for any $R \in G$. We prove a lemma regarding this situation. We assume $\chi \mid Q$ is real.

LEMMA 5.1. *If $P_2(\chi) \mid Q$ is rational, Q is cyclic. Here Q is the 5-Sylow group of G, $P_2(\chi)$ is the character associated with the symmetric tensors.*

Proof. Let π be an element of order 5. Let σ be an automorphism of $R[\lambda]$ mapping λ to λ^2 where $\lambda = e^{2\pi i/5}$, $R =$ rationals. We have $\chi^\sigma(\pi) = \chi(\pi^2)$, $\chi^{\sigma^2}(\pi) = \overline{\chi(\pi)} = \chi(\pi)$. Also

$$(\chi^2(\pi) + \chi(\pi^2))^\sigma = \chi^2(\pi) + \chi(\pi^2) ,$$
$$(\chi^\sigma)^2(\pi) + \chi(\pi) = \chi^2(\pi) + \chi^\sigma(\pi) ,$$
$$(\chi^\sigma - \chi)(\chi^\sigma + \chi - 1)(\pi) = 0 .$$

This implies $\chi^\sigma(\pi) = \chi(\pi)$ or $(\chi^\sigma + \chi - 1)(\pi) = 0$. Assume first $(\chi^\sigma + \chi)(\pi) = 1$. This will be true also for π, π^2, π^3, π^4. Therefore

$$(\chi^\sigma + \chi)(\pi + \pi^2 + \pi^3 + \pi^4 + e)$$

is 18 giving a contradiction. Therefore $\chi^\sigma(\pi) = \chi(\pi)$. In particular χ is rational when restricted to elements or order 5 in Q. As Q is abelian it must be cyclic by Schur's theorem [21]. This completes the proof of the lemma.

As there are no elements of order 125, $|Q| \leqq 25$. Let π_1 be a generator such that $(\pi_1)^5 = \pi$. We know $\chi(\pi) = 2$. Suppose

$$\chi \mid Q = \lambda_1 + \bar{\lambda}_1 + \lambda_2 + \bar{\lambda}_2 + \lambda_3 + \bar{\lambda}_3 + \lambda_0 ,$$

where the λ_i are linear characters of Q, λ_0 is the trivial character. Only two of $\lambda_1, \lambda_2, \lambda_3$ can represent π_1 by a primitive 25-th root of unity. This means there are at least five conjugates of χ contradicting the above paragraphs.

We have shown then that $g = 7 \cdot 3^3 \cdot 2^c$. Sylow's theorem gives $c = 3, 6, 9$. The cases 3,6 are well within the known range under 20,000 [19] and there are no simple groups with these orders. If $c = 9$ we again mention [6, 1H] which shows $2^c \leqq 12 \cdot 3^3 = 324$.

This eliminates all cases for which $b = 3$, Case ID(ii) of the flow chart. We begin now Case ID(iii) of the flow chart in which $b = 4$. We show there is exactly one group of this form $S_6(2)$.

We are considering groups whose orders are $7 \cdot 5^a \cdot 3^4 \cdot 2^c$. Sylow's theorem gives $a \equiv 1 \pmod 2$. As $a \leqq 6$ we have $a = 1, 3$, or 5. We will first show that $a \neq 5$.

Suppose then that $a = 5$. Let Q be a 5-Sylow group. We know from [25, Th. 2.1] that Q is abelian. By [25, Corollary 2.8] an elementary abelian subgroup of Q has order at most 5^4. In particular Q is not elementary abelian. Let π_1 be an element of order 25.

We know χ_1 is a rational character of degree 27. Let

$$\chi_1 \mid Q = \sum_{i=1}^{27} \lambda_i$$

where the λ_i, $i = 1, 2, \cdots, 27$ are linear characters of Q. As χ_1 is faithful there is an i such that $\lambda_i(\pi_1)$ is a primitive 25th root of 1. Each of the 20 conjugates of λ_i must appear as constituents in $\chi_1 \mid Q$ as χ_1 is rational. Let K be the kernel of λ_i. As there are no elements of order 125 in Q, K has order 125 and $Q \cong K \times \langle \pi_1 \rangle$. Clearly K is in the kernel of each of the conjugates of λ_i. If $\lambda_1, \cdots, \lambda_7$ are the characters λ_j not conjugate to λ_i we have $y = \sum_{i=1}^{7} \lambda_i$ a faithful rational representation of K. As K has order 5^3 this is impossible by Schur's theorem [21]. We have proved that $a \neq 5$ and so a must be 1 or 3.

The character χ_1 of degree 27 is in a 3-block of defect 1. Let $B_1(3)$ be the 3-block containing χ_1. As the defect group is cyclic of order 3 we may apply results in [13]. There are exactly three characters in $B_1(3)$. The three characters in $B_1(3)$ may all be of degree 27, in which case two are nonreal, or there will be two degrees y_1, y_2 besides 27 such that $27 + y_1 = y_2$. The degrees y_1 and y_2 will be divisible by 3^3 but not 3^4. By checking the various possibilities for the degrees of representations of the groups we are considering we find exactly one possibility. This is $y_1 = 189$, $y_2 = 216$. Block separation has been used in this elimination. This means that $B_0(7)$ contains characters with degrees 1, 27, 216 or characters with degrees 1, 27, 27, 27. In the latter case the tree has a branch.

It is now possible to eliminate all but eleven possible degree equations by straightforward techniques as described earlier. We do not include the details but list the degree equations not eliminated.

1. $1 + 15 + 120 + 512 = 27 + 216 + 405$
2. $1 + 64 + 36 + 162 = 27 + 216 + 20$
3. $1 + 120 + 120 + 162 = 27 + 216 + 160$
4. $1 + 960 + 120 + 162 = 27 + 216 + 1000$
5. $1 + 64 + 64 + 162 = 48 + 27 + 216$
6. $1 + 8 + 120 + 162 = 27 + 216 + 48$
7. $1 + 512 + 288 + 162 = 27 + 216 + 720$
8. $1 + 120 + 162 = 27 + 216 + 20 + 20$
9. $1 + 120 = 27 + 27 + 27 + 20 + 20$
10. $1 + 120 + 120 = 27 + 27 + 27 + 160$
11. $1 + 960 + 120 = 27 + 27 + 27 + 1000.$

We now separate the cases $g = 7\cdot 5\cdot 3^4\cdot 2^c$ and $g = 7\cdot 5^3\cdot 3^4\cdot 2^c$. We begin with $g = 7\cdot 5\cdot 3^4\cdot 2^c$ and show that all degree equations are impossible except (1) and that this leads to $S_6(2)$.

By Sylow's theorem $c = 3, 6, 9$. The only possible degree equations for $c = 3$ are (8) and (9) as the remaining equations contain degrees divisible by 16. Equations (8) and (9) can be eliminated using the two characters of degree 20 which must be in a 2-block of defect 1. However on the tree they are off the stem and so they must be complex conjugates. This is impossible. This leaves the two cases $g = 7\cdot 5\cdot 3^4\cdot 2^6$ and $g = 7\cdot 5\cdot 3^4\cdot 2^9$. We will do the latter case completely to obtain $S_6(2)$. The case $g = 7\cdot 5\cdot 3^4\cdot 2^6$ can be done similarly.

We assume then that $g = 7\cdot 5\cdot 3^4\cdot 2^9$ or $7\cdot 5\cdot 3^4\cdot 2^6$. The results and notation of § 2 for groups of order $7\cdot 5\cdot 3^b\cdot 2^c$ will be used, in particular equation (2.6) and (2.7). Let π be of order 5, $C(\pi) = \langle\pi\rangle \times V$. We know $|V| = 3^r\cdot 2^s$. Let V_2 be a 2-Sylow group of V and V_3 a 3-Sylow group of V. The restriction $\chi \mid V_3$ has linear constituents by (2.6), (2.7) and so V_3 is abelian. By (2.6), (2.7), $X \mid V_3$ has at most 3-distinct linear characters. Using [5, 3D] we see $|V_3| \leqq 3^2$. In particular $r \leqq 2$. If V_2 is abelian, the variety[1] of $\chi \mid V_2$ is at most 3 and so $s \leqq 2$. In case 2.7 this is always the case. In the case 2.7, V_2 at any rate has a subgroup of index two which is abelian with variety at most 3. In this case $s \leqq 3$. If V_2 is in fact nonabelian $|V_3| = 2^3$, and $\theta(J) = -2$ where $J \in Z(V_2)$.

We also know that $\chi \mid V_3$ is rational. Therefore there can be no element of order 9 in V_3 as the variety is at most 3. Furthermore if T is of order 3, the eigenvalues of $X(T)$ must be $\lambda, \bar{\lambda}, 1, 1, 1, 1, 1$ where $\lambda = e^{2\pi i/3}$. In case (2.6) then $\varphi(T) = 1$, $\theta(T) = -1$; in case (2.7), $\varphi(T) = 1$, $(\varphi_1 + \varphi_2)(T) = -1$. In particular $3^2 \nmid |V|$. If there is an involution J which commutes with T, then $X(J)$ will have either six or two eigenvalues -1. In particular, the eigenvalues of $X(TJ)$ will be $\{1, 1, 1, 1, 1, -\lambda, -\bar{\lambda}\}$ or $\{-1, -1, -1, -1, -1, \lambda, -\bar{\lambda}\}$ for T or T^2. In each case Blichfeldt's theorem [1, p. 96] is contradicted. This means that V cannot contain an element of order 6. In particular if $|V|$ is divisible by 24 this is the case. In case (2.7), V is abelian and so $|V|$ is not divisible by six.

We now consider the case $g = 7\cdot 5\cdot 3^4\cdot 2^9$. Let $w = |N(\pi)/C(\pi)|$. If $w = 2$ by Sylow's theorem $|V| \equiv 2 \pmod 5$. This means $|V| = 2$ or 12. If $|V| = 2$, V is abelian and so case (6.7) applies. However here φ_1 must equal φ_2 giving a contradiction. If $|V| = 12$, V must be nonabelian and so case (2.6) applies. Therefore V has an irreducible representation of degree 2 and so there is an element of order 6 in

[1] The variety is defined in [1] or in [24, Introduction].

V. This shows $w \neq 2$.

Suppose $g = 7 \cdot 5 \cdot 3^4 \cdot 2^9$ and $w = 4$. In this case Sylow's theorem gives $|V| \equiv 1 \pmod 5$. This shows $|V| = 1$ or 6. If $|V| = 1$ there would be one 5-block of full defect and so no character of degree 7. Therefore $|V| = 6$. As there can be no element of order 6 in V, V must be isomorphic to S_3 as there are only two nonisomorphic groups of order six. Case (2.6) applies.

The group V is generated by an element T of order 3 and an involution J such that $JTJ = T^{-1}$. There are three irreducible characters of V; η_0, η_1, θ. Here η_0 is the trivial character; $\eta_1(T) = 1$, $\eta_1(J) = -1$, $\eta_1(e) = 1$; $\theta(T) = -1$, $\theta(J) = 0$, $\theta(e) = 2$. There are therefore three 5-blocks of full defect: $B_0(5)$ corresponding to η_0, $B_1(5)$ corresponding to η_1, and $B_2(5)$ corresponding to θ. In each case $e = 4$ and so each block consists of 5 ordinary characters. The degrees of characters in $B_0(5)$ and $B_1(5)$ are congruent to $\pm 1 \pmod 5$. In $B_2(5)$ they are congruent to $\pm 2 \pmod 5$.

Suppose there is a character, say χ_2, of degree 162. Then $\chi_2 \in B_2(5)$ and so $\chi_2(\pi T) = \pm 1$. As χ_2 is of full 3-defect this is a contradiction [7, p. 579]. This shows that cases $2, 3, \cdots, 8$ are impossible.

The cases 9, 10, 11 can be eliminated by block separation as χ_0 is the only possible character in $B_0(7) \cap B_0(5)$.

This leaves (1) as the only possible degree equation remaining. The degrees in $B_2(5)$ are so far 7, 27, 512. If χ is not rational the degree equation would be $7 + 7 + 27 + 512 = 553 = 7.79$ which is absurd. Therefore χ is rational.

It is now possible to show $G \cong S_6(2)$. Let J be an involution in V. As $\theta(J) = 0$, $\varphi(J) = -1$ by the unimodularity of $X(J)$. Therefore $X(J)$ has six eigenvalues -1 and one eigenvalue 1. As G is simple the conjugates of J generate G.

We consider the group $\tilde{G} = G \times Z_2$ and a representation $\tilde{X}$ of $\tilde{G}$ given by $\tilde{X}(a, b) = X(a)\eta(b)$ where $a \in G$, $b \in Z_2$. Here η is the nontrivial character of Z_2, Z_2 is the group of order 2. As $\tilde{X}$ is rational and of odd degree it can be written in the real field [22]. We may assume the matrices are orthogonal. If $Z_2 = \langle J_1 \rangle$, $\tilde{X}(JJ_1)$ is a reflection in R^7. The same is true of any conjugate. The group generated by these conjugates is $\tilde{G}$ as can be quickly checked. This group is then a group generated by reflections in R^7. These groups have all been classified as Weyl groups of certain Dynkin diagrams containing seven elements [10, 11, 26]. These are A_7, B_7, D_2, E_7. The only group with the correct order is the Weyl group of E_7. It is known that the Weyl group of E_7 has a subgroup of index 2 isomorphic to $S_6(2)$ and that it has a complex irreducible representation of degree 7 [14].

The case $g = 7\cdot 5\cdot 3^4\cdot 2^6$ can be handled similarly. All degree equations can be eliminated.

We now proceed to the case $g = 7\cdot 5^3\cdot 3^4\cdot 2^c$. Sylow's theorem gives $c = 4, 7, 10$. The cases 1, 2, 6, 7, 8, 9 can be eliminated by a routine use of the established techniques. For example in cases 1 and 7 the character of degree 512 implies $c = 10$. However the character of degree 512 is then of 2-defect 1 and so must occur with multiplicity two. In cases 2, 8, and 9 the characters of degree 20 cannot be in $B_0(5)$ and the cases are eliminated by 5-block separation. In case 6 there is a rational representation of degree 8 contradicting [21].

The remaining cases 3, 4, 5, 10, 11 will be eliminated by examining a 3-Sylow group of G. It will be shown there is a self centralizing element π_9 of order 9. As $5^3 \mid g$ we know that $\chi \mid P_5$ is not real by [25, Th. 3.1]. By Lemma 2.1 there are at least four conjugates of χ. The cases can be eliminated by showing it is impossible to complete the π_9 column of the character table. The decompositions of $\chi\bar{\chi}$ and $\chi\chi$ together with equations (2.1)–(2.3) will be used. As no groups arise we sometimes only sketch the arguments.

We first show there is a self centralizing element of order 9. Let T generate the defect group of $B_1(3)$. Then, as $B_1(3)$ is not of defect 0, $\chi_1(T) \neq 0$ by [3]. As $g\cdot\chi_2(T)/|C(T)|\,27$ is an algebraic integer $3^4 \nmid |C(T)|$. Therefore Q is nonabelian where Q is a 3-Sylow group of G. As earlier using the fact that $\chi \mid Q$ is rational we obtain $\chi \mid Q = \mu + \bar{\mu} + \lambda_0$ where μ has degree three. The representation U corresponding to μ can be written in monomial form.

There must be an abelian subgroup of order 27. Let M be any abelian subgroup of order 27. Let $\chi \mid M = \sum_{i=1}^{7} \zeta_i$. Suppose there is an element of order 9 in M. If ζ_1 represents it faithfully all six conjugates must appear in $\chi \mid M$. Therefore ζ_1 is faithful and M is cyclic. This is a contradiction and shows M is elementary abelian.

The matrices $U(M)$ may be picked as all diagonal matrices of the form

$$\begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix}, \quad (\rho_i)^3 = 1\,.$$

If $R \notin M$ we may choose a basis so that

$$U(R) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ C & 0 & 0 \end{bmatrix}.$$

As $U(R^3) = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}$ we have $C^3 = 1$. We may choose R so that $C = 1$ or ρ where $\rho = e^{2\pi i/3}$. Picking R with $C = \rho$ gives an element π_9 of order 9. The eigenvalues of $\chi(\pi_9)$ are distinct and so $|C(\pi_9)|$ is a 3-group by [5, 3F] and the fact $7 \nmid |C(\pi_9)|$. If $3^3 \mid |C(\pi_9)|$ there would be an abelian subgroup of order 27 which was not elementary. This means π_9 is self centralizing. We have also shown $\chi(\pi_9) = 1$.

The value of χ_0 and the four characters of degree 7 on π_9 is 1. Let $\eta_1, \cdots, \eta_t$ be the remaining characters of G for which $\eta_i(\pi_9) \neq 0$. Let $\eta_i(\pi_9) = b_i$. The orthogonality relations give

$$\sum_{i=1}^{t} b_i\bar{b}_i = 4 \tag{5.5}$$
$$\sum_{i=1}^{t} b_i\eta_i(e) = -29 .$$

There must be at least one character from $B_0(7)$ amongst the η_i and in the 5 remaining cases such degrees can be given explicitly. In case 5 the characters of degree 64 must be amongst the η_i. In cases 3 and 10 the character of degree 160 and in cases 4 and 11 the character of degree 1000 must be one of the η_i. From the tree in each case it can be seen the value is real and except for case 5 is rational and so the value b_i can be obtained by its congruence mod 3.

We now show there are at least two η_i occurring as constituents of $\chi\chi$. As $\chi\bar{\chi}$ has at least three constituents the same is true of $\chi\chi$. This implies $P_2(\chi)$ or $C_2(\chi)$ must be reducible. From (2.2) and (2.3) we see that $P_2(\chi)$ has no constituents of degree 7 and $C_2(\chi)$ has at most one. If $P_2(\chi)$ is reducible there are two constituents of degree 14. As $P_2(\chi)(\pi_9)$ is 1 they are not both equal. If $C_2(\chi)$ is reducible there is one constituent of degree 7 and one of degree 14. In any case there are at least two new constituents. Their values on π_9 can be readily evaluated. In each case it is now impossible to satisfy (5.5). This completes the final case in Section ID of the flow chart.

6. **The case** $s = 2$. In this section we consider the case $s = 2$. There is one group PSL (2,8), of order 504, of this kind, case II in Theorem I. This is section II of the flow chart. There are five characters in $B_0(7)$, χ_0 the trivial character, χ_1 of degree congruent to ± 1 (mod 7) and three exceptional characters $\chi_0^1, \chi_0^2, \chi_0^3$ whose degrees are congruent to ± 2 (mod 7). There are two possible degree equations

$$\text{(a)} \quad 1 + x_0^1 = x_1 \quad \text{or} \quad \text{(b)} \quad 1 + x_1 = x_0^1$$

where $x_1 =$ degree χ_1, $x_0^1 =$ degree χ_0^1.

If χ is the character of degree 7 set

$$\chi\bar{\chi} = \chi_0 + a\chi_1 + b(\chi_0^1 + \chi_0^2 + \chi_0^3) + \eta \tag{6.1}$$

where η has constituents of zero 7-defect. Here a and b are nonnegative integers. As the degree of $\chi\bar{\chi}$ is 49 and the degree of η is divisible by 7 it is clear that either $a \neq 0$ and $x_1 \equiv -1 \pmod 7$ or $b \neq 0$ and $x_0^1 \equiv 2 \pmod 7$. This means that $x_1 = 6, 20, 27, 48$ or $x_0^1 = 2, 9, 16$. The only degree equations possible for G are then

(a) $1+5=6$, (b) $1+1=2$, (c) $1+8=9$, (d) $1+15=16$.

The first case (a) is impossible by [5] or [17]. In case (b) G' is of index 2. But then G' has a normal 7-complement and is not simple [5]. It can also be eliminated by [1] or [2]. In case (c) 2-block and 3-block separation imply $g = 7\cdot 5^a\cdot 3^2\cdot 2^3$. As there is a rational character of degree 8 $a \leqq 2$ by Schur [21]. Sylow's theorem gives $a = 0$. There is one simple group PSL (2,8) of order 504 [19]. It has a representation of degree 7 by [15]. One can also work out the character table quite easily.

In case (d) the character χ_1 of degree 15 is rational. A 5-Sylow group is abelian. Therefore χ_1 cannot be in $B_0(5)$ as $\chi_1(S)/15 \equiv 1 \pmod 5$ for any 5-element S. This would imply $\chi_1(S) = 15$ or $\chi_1(S) = -10$. Neither are possible in G. This argument is similar to one in § 4 where a character of degree 20 was involved. If $\chi_1 \notin B_0(5)$, 5-block separation implies $g = 7\cdot 5\cdot 3^b\cdot 2^c$. However the four characters $\chi_0, \chi_0^1, \chi_0^2, \chi_0^3$ are all in $B_0(5)$. By [2, Th. 11] there must be one further character in $B_0(5)$ of degree 49. This is a contradiction and completes this section.

7. The case $s = 3$. We now consider the case $s = 3$, $G = G'$. Thus G is simple by [5, 8A]. This is section III of the flow chart. In this case $B_0(7)$ contains χ_0, two characters χ_1, χ_2 whose degrees are congruent to $\pm 1 \pmod 7$, and two exceptional characters χ_0^1, χ_0^2 whose degrees are congruent to $\pm 3 \pmod 7$. If $x_i =$ degree χ_i, $x_0^i =$ degree χ_0^i for $i = 1, 2$ the degree equation becomes one of

$$(7.1)\quad \begin{array}{lll} \text{(a)} & 1 + x_2 = x_1 + x_0^1 & x_1 \equiv -1 \pmod 7,\ x_2 \equiv 1 \pmod 7\,, \\ \text{(b)} & 1 + x_0^1 = x_1 + x_2 & x_1, x_2 \equiv -1 \pmod 7\,, \\ \text{(c)} & 1 + x_1 + x_2 = x_0^1 & x_1 = x_2 \equiv 1 \pmod 7\,. \end{array}$$

As in §§ 2 and 6 we have

$$\chi\bar{\chi} = \chi_0 + a_1\chi_1 + a_2\chi_2 + b(\chi_0^1 + \chi_0^2) + \eta \tag{7.2}$$

where again a_1, a_2, b are nonnegative integers and the constituents of η are of zero 7-defect. We may assume that either (i) $a_1 \neq 0$, $x_1 \equiv -1 \pmod 7$ or (ii) $b \neq 0$ $x_0^1 \equiv 3 \pmod 7$. We may also assume that χ_1 or χ_0^1 adjoins χ_0 on the stem as $\chi\bar{\chi}$ is a sum of principal indecomposables [9]. The possibilities for x_1 in (i) are 6, 20, 27, 48. The possibilities for x_0^1 in (ii) are 3, 10, 24. It is clear that case c in (7.1) is impossible. This means the tree contains four real characters all on the stem.

If there is a character of degree 3, then $G \cong PSL\ (2,7)$ by [1, 2, or 17]. Also, $PSL\ (2,7)$ has a character of degree 7 by [15]. Alternatively, the character table for $PSL\ (2,7)$ can be quickly worked out. This is Case IV of Theorem I.

We will show that when $x_1 = 6$ $G \cong PSL\ (2,7)$; when $x_1 = 20$, $G \cong A_8$, when $x_1 = 27$, $G \cong U_3(3)$. The remaining cases can all be eliminated. The methods are similar to those of earlier chapters but in general much simpler because there are only two missing degrees in the degree equation. The details will not all be given.

Suppose that $x_1 = 6$ and χ_1 adjoins χ_0 on the stem. This means that $\chi_1\chi_1$ contains χ_2 as a constituent in case (a) or χ_0^1 as a constituent in case (b) by an argument involving the tree. In particular χ_2 or χ_0^1 has degree at most 21 as $(\chi_1)^2$ has constituents of degrees 15 and 21 corresponding to the symmetric and skew symmetric tensors. The possible degree equations are

$$\text{(i)}\quad 1 + 8 = 6 + 3 \quad\text{and}\quad \text{(ii)}\quad 1 + 15 = 6 + 10\,.$$

Case (i) is again by [1 or 2], $PSL\ (2,7)$.

In case (ii), χ_1 is in $B_0(5)$ by 5-block separation. [As χ_1 is rational, $5^2 \nmid g$ and $3^5 \nmid g$ by [21]. This means $g = 7\cdot5\cdot3^b\cdot2^c$; $b \leqq 4$. As in § 2 we apply the results in [2, II, Th. 1] for the prime 5 this time with χ replaced by χ_1. Let $C(\pi) = \langle\pi\rangle \times V$ where π is a 5-element. Then $\chi_1 \mid V = \varphi_0 + 5\varphi_1$ where φ_0 is the trivial character of V, φ_1 is a linear character of V. It is immediate from the unimodularity of the representation corresponding to χ_1 that $\varphi_1 = \varphi_0$. This means $|V| = 1$. In particular there is only one 5-block of full defect by [3] and so $\chi \in B_0(5)$. This means $B_0(5)$ contains χ_0, χ_1, χ and a fourth character χ^* conjugate to χ.

As there are no elements of order $5\cdot2$ or $5\cdot3$ block separation applies to $B_0(5)$. If $b = 1$, $B_0(5) \cap B_0(3)$ contains three characters χ_0, χ, χ^*. This contradicts the theory of cyclic 3-defect groups. Therefore $b \geqq 2$. By block separation $B_0(5) \cap B_0(3) = B_0(5)$. In particular $\chi_1 \in B_0(3)$ and so $b \geqq 3$ [12, 90.19]. Let π_3 be an element of order 3 in the center of a 3-Sylow group. As χ_1 is rational, $\chi_1(\pi_3) = 3, 0, -3$. As $\chi_1 \in B_0(3)$, $\chi_1(\pi_3) = -3$. This implies $\chi(\pi_3) = \chi^*(\pi_3) = -2$. Computing $a(\pi, \pi, \pi_3)$ [6–3.1, 3.2] now gives a negative value and so a contradiction. This case is therefore impossible.

Suppose $x_0^1 = 10$ and χ_0^1 adjoins χ_0 on the stem. Using arguments involving the tree we see $x_2 \leqq 100$. The only degree equations possible are

$$\text{(i)}\quad 1+15=6+10 \qquad \text{(ii)}\quad 1+36=10+27\,.$$

Case (i) is eliminated as in the above paragraph. In case ii, χ_0^1 is rational when restricted to a 5-Sylow group and so $5^3 \nmid g$. The case $g = 7 \cdot 5^2 \cdot 3^b \cdot 2^c$ is eliminated by 5-block separation on χ_0^1 and χ_0^2. The case $7 \cdot 5 \cdot 3^b \cdot 2^c$ is eliminated by 5-block separation with $B_0(5)$.

If $\chi_1 = 48$ or $x_0^1 = 24$ an argument similar to that of § 3 applies to give $g = 7 \cdot 3^b \cdot 2^c$. This case can be eliminated as there must be a degree congruent to ± 1 of the form 2^β or 3^γ. These few cases can all be easily eliminated.

There are two cases remaining, $x_1 = 20$ and $x_1 = 27$ with χ_1 adjacent to χ_0 on the stem. We can apply the same tree arguments to see that χ_0^1 or χ_2 has degree at most 210 if $x_1 = 20$ and at most 378 if $x_1 = 27$. There are only a few possibilities and all but the following three can be easily eliminated using techniques already discussed.

(i) $1 + 64 = 20 + 45$
(ii) $1 + 32 = 27 + 6$
(iii) $1 + 50 = 27 + 24$.

In case (ii) the order is $7 \cdot 5^a \cdot 3^3 \cdot 2^6$ by block separation. As χ_1 has degree 6 and is rational $a \leqq 1$ by [21]. By Sylow's theorem $a = 0$. There is exactly one simple group with this order, $U_3(3)$, [19]. It is known to have a representation of degree 7. For example, a character table is given in [16].

In case (i), χ_1 of degree 20 is rational. As in § 4, χ_1 cannot be in $B_0(5)$. Block separation now gives $g = 7 \cdot 5 \cdot 3^b \cdot 2^6$. This means $b = 2$ or 8. For $b = 2$ the order is 20160. There are two known simple groups of this order A_8 and PSL (3,4). Only A_8 has a character of degree 7. The character table can be completed in a routine way to the character table of A_8. By [20], G must be A_8. The case $b = 8$ can be eliminated by closely examining $C(\pi)$ and noting $|V| = 2$, 12, or 72.

The third case is, curiously enough, quite troublesome. We do not give full details but just sketch the argument. Block separation on the characters of degrees 50 and 27 gives $g = 7 \cdot 5^2 \cdot 3^3 \cdot 2^c$. Sylow's theorem gives $c = 3, 6, 9$.

If χ is real $P_2(\chi) = \chi_0 + \chi_1$ and so $P_2(\chi)$ is rational. By Lemma 5.1 a 5-Sylow group is cyclic. This case can now be eliminated using [13]. This shows χ is not real. By Lemma 2.1 there are at least four conjugates of χ.

As in § 5 it can be shown that the 3-Sylow group is nonabelian. Let T be an element of order 3 not in the center. The decomposition numbers for $C(T)$ can be analyzed. The analysis is not as easy as in § 5 as there is no involution inverting a 7-element. As in § 5 there are two possible Cartan matrices (9) and $\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$. Using χ_0, χ_2, and the four characters of degree 7 and carefully analyzing the possible decomposition matrices this case can be eliminated.

8. The case $G \neq G'$. We now discuss the case in which $G \neq G'$. This is case IV of the flow chart. It is shown in [5] that G' is a simple group and so $X(G')$ is one of the representations already obtained. The candidates for G' are PSL (2, 7), PSL (2,8), A_8 and $U_3(3)$. G' could not be $S_6(2)$ as $|N(P)/C(P)| = 6$ in this case and so any extension would have an element in $C(\xi)$ where ξ is an element of order 7. For the same reason $|G: G'| = 2$ if G' is PSL (2,7), A_8 or $U_3(3)$; $|G: G'| = 3$ if G' is PSL (2,8).

If α is an element of G not in G' the map $\theta_\alpha: \xi \rightarrow \xi^\alpha$, $\xi \in G'$ is an automorphism of G'. As α cannot commute with an element of order 7, θ_α cannot be the identity automorphism. It cannot be an inner automorphism as no element $\alpha\eta$, $\eta \in G'$, can commute with an element of order 7. Therefore θ_α is an outer automorphism. The automorphisms of our groups are all known. A very readable account without proofs using the fact they are all Chevalley groups can be found in [9].

In the case of PSL (2,8), $(\theta_\alpha)^3$ is an inner automorphism as $\alpha^3 \in G'$. By taking $\theta_\alpha\eta$ where η is an inner automorphism we can assume $(\theta_\alpha)^3 = \theta_e$ as in G a 7-element is self centralizing. We see then that G is the semidirect product of G by $\langle\theta_\alpha\rangle$. If A is the automorphism group of PSL (2,8), $I(A)$ the inner automorphism group, then $A/I(A)$ has an element of order 3 generated by a field automorphism. There are seven such extensions all isomorphic. We may assume then the extension is induced by a field automorphism. From the character table [15], PSL (2,8) has four characters of degree 7 and so one must lift.

In the remaining case $(\theta_\alpha)^2$ is an inner automorphism as $\alpha^2 \in G'$. In each of the remaining cases there is exactly one element of order 2 in $A/I(A)$. It must be θ_α. By taking an element $\alpha\eta$ instead of α we can assume $(\theta_\alpha)^2 = \theta_e$ the identity automorphism. In this case α^2 must commute with all elements of G' and so must be e. This shows that G is uniquely determined as the semidirect product of G' and $\langle\alpha\rangle$ with the automorphism θ_α.

In each of the groups PSL (2,7), A_8, and $U_3(3)$, there is exactly one rational character of degree 7. This means θ_α must leave it fixed and so the character of degree 7 can be lifted to G.

In the case of PSL (2,7) there is certainly only one representation of degree 7 as the sum of the squares of the degrees in $B_0(7)$ is $168 - 7^2$. The element α can be taken $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Big/ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in the usual matrix form of PSL (2,7).

In the case of A_8 one may check the character table [18] to see there is only one representation of degree 7. However, this is not necessary because S_8 is known to have an irreducible unimodular representation of degree 7 and so G is S_8.

For the final case $U_3(3)$ one must check a character table [16]. Here there are three representations of degree 7, two of them are conjugate and the third rational. The automorphism is a field automorphism. The group G must be the group $G_2(2)$ as $G_2(2)$ cannot be $Z_2 \times G'$ or it would have an automorphism of order 2 [9].

FLOW CHART. $|G| = g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$.

I. $s = 6$. $\chi\bar{\chi} = \chi_0 + \sum_{i=1}^{6} a_i\chi_i + \eta$. $G = G'$.
- A(§ 2). One of χ_i, $i = 1, 2, \cdots, 6$ has degree 6.
- B(§ 3). Some $a_i = 1$, $x_i = 48$. We assume $a_1 = 1$, $x_1 = 48$.
  - (i) χ irrational on a 5-Sylow group.
  - (ii) $g = 7 \cdot 5 \cdot 3^b \cdot 2^c$.
  - (iii) $g = 7 \cdot 3^b \cdot 2^c$.
- C(§ 4). Some $a_i = 1$, $x_i = 20$. We assume $a_1 = 1$, $x_1 = 20$.
  - (i) $a \geqq 2$.
  - (ii) $a = 1$.
- D(§ 5). Some $a_i = 1$, $x_i = 27$. We assume $a_1 = 1$, $x_1 = 27$.
  - (i) χ restricted to a 3-Sylow group is irrational.
  - (ii) $b = 3$.
  - (iii) $b = 4$. (This case gives $S_6(2)$).

II. (§ 6). $s = 2$ $G = G'$. (This case gives $PSL_2(8)$.)

III. (§ 7). $s = 3$ $G = G'$. (This case gives $PSL_2(7)$, A_8, and $U_3(3)$.)

IV. (§ 8). $G \neq G'$. (This is VII of Theorem I).

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AN ISOMORPHIC REFINEMENT THEOREM FOR ABELIAN GROUPS

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In this paper we find a class $\mathscr{C}$ of Abelian groups with the property that if a group A is a direct sum of groups in the class $\mathscr{C}$, then any two direct sum decompositions of A have isomorphic refinements. The class $\mathscr{C}$ includes those groups which are complete and Hausdorff in their natural topology and also the torsion-complete p-groups.

All groups in this paper are Abelian groups, additively written. The *natural topology* (or Z-topology) is defined on a group G by taking as neighborhoods of 0 the subgroups nG, for nonzero integers n. A group G is called *Hausdorff* if it is Hausdorff in this topology, or, equivalently, if $\cap nG = 0$ (where n ranges over all nonzero integers). G has *bounded order* if for some nonzero integer n, $nG = 0$. We will frequently use Prufer's theorem that a group of bounded order is a direct sum of cyclic groups [9, Th. 6]. The groups which are complete and Hausdorff in the natural topology are exactly the reduced algebraically compact groups in the terminology of [9]. A p-group is *torsion-complete* if it is Hausdorff and it is the maximal torsion subgroup of its completion in the natural topology (which in this case is the same as the p-adic topology).

We use the symbol $\sum$ for the direct sum of a family of groups, $A \oplus B$ for the direct sum of the groups A and B (either abstractly or as a subgroup of another group), and $A + B$ for the ordinary sum of two subgroups of a group (not necessarily a direct sum). If a group G has two direct sum decompositions, $G = \sum_{i \in I} A_i = \sum_{j \in J} B_j$, we say that these decompositions are isomorphic if there is a bijective mapping $\phi: I \to J$, such that $A_i \cong B_{\phi(i)}$ for all $i \in I$, and we say that the second decomposition is a *refinement* of the first if each B_j is contained in one of the A_i.

A group B has the *exchange property* if for any group A, if $A = B' \oplus C = \sum_{i \in I} D_i$, with $B \cong B'$, then there are subgroups $D'_i \subseteqq D_i$ such that $A = B' \oplus \sum_{i \in I} D'_i$. If this holds in every case where the index set I is finite, then B is said to have the *finite exchange property*. It is not known whether these two properties are equivalent. The exchange property has been exploited for the study of infinite direct sum decompositions by P. Crawley and B. Jónsson in [4].

DEFINITION. An Abelian group G is in the class $\mathscr{C}$ if it satisfies the following three conditions:

(i) G is Hausdorff;

(ii) G has the finite exchange property;

(iii) If $f: G \to M$ is a homomorphism of G into a Hausdorff group M and $M = \sum_{i \in I} M_i$ then there is a finite subset $J \subseteqq I$ and a decomposition of G, $G = G_1 \oplus G_2$, where G_1 is of bounded order and every nonzero element of G_2 has a nonzero multiple whose image under f is in $\sum_{i \in J} M_i$.

The main result of § 4 below is that complete Hausdorff groups are in $\mathscr{C}$. Torsion-complete p-groups are also in $\mathscr{C}$ since Crawley and Jónsson showed [4, Lemma 11.4] that they have the exchange property, and property (iii) is easy to check directly (using, for example, the completeness of the socle in the p-adic topology and applying the Baire category theorem as in § 4 below).

There are many other examples of groups in $\mathscr{C}$. Crawley proved [3, Lemma 3.5] that for p-groups properties (i) and (ii) above imply (iii) (his condition appears weaker than (iii) but is actually equivalent) so any Hausdorff p-group with the finite exchange property is in $\mathscr{C}$. He also constructs in [3] a class of "stiff" p-groups which are in $\mathscr{C}$, but which are not torsion-complete. For other examples, we remark that if G is a Hausdorff group whose maximal torsion subgroup T is a stiff p-group and if G/T is divisible of finite rank, then G is a mixed group in $\mathscr{C}$. Finite rank pure subgroups of the p-adic integers (for any prime p) are examples of torsionfree groups which are not complete but which are in $\mathscr{C}$ (see Proposition 1 and the proof of Proposition 4 in [14]).

We will need two important additional properties of $\mathscr{C}$.

LEMMA 1. *If G is in $\mathscr{C}$, so is any summand of G. If $G_i (i = 1, \cdots, n)$ are in $\mathscr{C}$, so is $G_1 \oplus \cdots \oplus G_n$.*

LEMMA 2. *If $G \in \mathscr{C}$, any two finite direct sum decompositions of G have isomorphic refinements.*

Lemma 1 is obvious except perhaps for property (ii) for which see [4, Lemma 3.10]. Lemma 2 is immediate from the finite exchange property. The groups in $\mathscr{C}$ actually have the exchange property (not just the finite exchange property). For a proof we refer to [3, Lemma 3.6], only remarking that one must use our Lemmas 6 and 7 below instead of Crawley's 3.2 and 3.3. We will not need this result.

We will state our main results for abstract classes of groups, since the class $\mathscr{C}$ is not the only class of groups for which these theorems can be proved.

THEOREM 1. *Let $\mathscr{D}$ be a class of groups such that*

(i) *Summands and finite direct sums of groups in $\mathscr{D}$ are in $\mathscr{D}$,*

(ii) *If $G \in \mathscr{D}$ then any two finite direct sum decompositions of G have isomorphic refinements, and*

(iii) *If $G \in \mathscr{D}$, and $f: G \to M$ is a homomorphism, where $M = \sum_{i \in I} M_i$, and the M_i are all in $\mathscr{D}$, then there is a finite subset $J \subseteqq I$ and a decomposition of G, $G = G_1 \oplus G_2$, where G_1 is of bounded order and every nonzero element of G_2 has a nonzero multiple whose image under f is in $\sum_{i \in J} M_i$.*

Then if A is any group which is a direct sum of groups in the class $\mathscr{D}$, any two decompositions of A into summands in the class $\mathscr{D}$ have isomorphic refinements.

THEOREM 2. *Let $\mathscr{D}$ be a class of Abelian groups satisfying the hypotheses of Theorem 1, and such that the elements of $\mathscr{D}$ have the finite exchange property. Then if a group A is a direct sum of groups in the class $\mathscr{D}$, any summand of A is also a direct sum of groups in the class $\mathscr{D}$.*

Sections 2 and 3 below are devoted to the proofs of these theorems.

COROLLARY. *If $\mathscr{D}$ is a class of groups satisfying the conditions of Theorem 2 and A is a direct sum of groups in the class $\mathscr{D}$ then any two direct sum decompositions of A have isomorphic refinements. In particular, this applies to the class $\mathscr{C}$, (by definition and Lemmas 1 and 2) so (specializing further) if A is a direct sum of complete Hausdorff groups or of torsion-complete p-groups, then any two direct sum decompositions of A have isomorphic refinements.*

The results of this paper for torsion-free and mixed groups are entirely new, but the corresponding questions for p-groups have a considerable history. Reinhold Baer completely solved the problem for countable p-groups in 1935 [1]. Kulikov proved in [11] that if an Abelian p-group is a direct sum of cyclic groups, then any two direct sum decompositions of the group have isomorphic refinements, thus generalizing one of the results obtained by Baer in the countable case. Kulikov also defined torsion-complete p-groups in [11] and showed that any two direct sum decompositions of a torsion-complete p-group have isomorphic refinements. E. Enochs, in work based partly on earlier work of Kolettis [10], proved in [5] the special case of Theorem 1 involving direct sums of torsion-complete p-groups. Our proof of Theorem 1 was motivated by his paper. Crawley generalized this result in [3], replacing torsion-complete p-groups by Hausdorff p-groups with the finite exchange property. In both of these cases one still needs to prove the corresponding special case of Theorem 2. This has previously been done only in the special case of direct sums of torsion-complete p-groups—for countable sums by Irwin, Richman and Walker [7], and in general by P. Hill [6] and the author [12], independently.

We close this introduction with some examples to illustrate the limitations of our results. Numerous examples of groups without the isomorphic refinement property are known, due to Baer [1] (for countable p-groups), Jónsson [8] (for torsion-free groups), and Corner and Crawley [2] (for Hausdorff p-groups). On the other hand, there are groups not in the class $\mathscr{C}$ for which a theorem such as ours should be provable. If G is a p-group such that the subgroup G^1 of elements of infinite height is torsion-complete and not zero, and such that G/G^1 is stiff (in the sense of [3]) then G has the exchange property and any two direct sum decompositions of G have isomorphic refinements, but G is not Hausdorff and therefore is not in $\mathscr{C}$. Possibly the class $\mathscr{C}$ could be enlarged by omitting condition (i) and suitably altering condition (iii).

1. Lemmas on pure subgroups and projections. We recall that if A is an Abelian group and B a subgroup, then B is *pure* in A if $nB = B \cap nA$ for all integers n. We define the *p-height* (denoted h_p) of an element x by setting $h_p(x) = n$ if $x = p^n y$ for some y but $x \neq p^{n+1}z$ for any z, and $h_p(x) = \infty$ if x is divisible by all powers of p. By the *height* of x we mean the function associating to each prime p the number $h_p(x)$. Clearly a subgroup B is pure in A if and only if the heights computed with respect to B and with respect to A are the same.

Most of the lemmas that follow are generalizations to mixed groups of well-known and widely used results for p-groups. The first two, for example, are generalizations of Lemmas 12 and 7 of [9].

LEMMA 3. *If M is a group and N a pure subgroup such that for every $x \in M$ $(x \neq 0)$ there is an integer n such that $nx \neq 0$ and $nx \in N$, then $N = M$.*

REMARK. This lemma is not true for modules over an arbitrary integral domain. For instance, if D is a divisible R-module which is not injective, and E is the injective envelope of D, then D is pure in E and every nonzero element of E has a nonzero multiple in D, but $E \neq D$.

Proof. By a finite reduction process, it suffices to show that if p is a prime and $px \in N$, then $x \in N$. If $px = 0$ this is trivial by hypothesis, since some nonzero multiple of x must be in N. Otherwise, $px = py$ for some $y \in N$ (by the purity of N) and $p(x - y) = 0$ so either $x = y$ (and we are done) or $x - y$ is of order p and hence in N, and $x = (x - y) + y$ is a sum of elements in N.

LEMMA 4. *Let M be a group and N a subgroup and say that for all $x \in N$ ($x \neq 0$) there is an integer n such that $nx \neq 0$, and nx has the same height in M and in N. Then N is pure.*

Proof. By a finite reduction we need only show that if px has the same height in N and M then so does x. Since x and px have the same q-height for all primes q, $q \neq p$, we need only consider p-height and we must show that if $x = p^n y$ for some $y \in M$ then there is a $z \in N$ with $x = p^n z$. If $px = 0$ the result is trivial since in this case, if $nx \neq 0$ then nx and x have the same height. We therefore assume $px \neq 0$.

Suppose $x = p^n y$, $y \in M$. Then $px = p^{n+1} y$ and by hypothesis, $px = p^{n+1} z_0$ for some $z_0 \in N$. Then $p(p^n z_0 - x) = 0$ so either $x = p^n z_0$ (and we are done) or $p^n z_0 - x$ is of order p and hence has the same height in M as in N. Since it is divisible by p^n in M (since $x = p^n y$) it is also divisible by p^n in N, so that there is a $z_1 \in N$ with $p^n z_1 = p^n z_0 - x$, that is, $x = p^n(z_0 - z_1)$, proving the result.

DEFINITION. If two pure subgroups A and B of a group M have the property that each nonzero element of A has a nonzero multiple in B and each nonzero element of B has a nonzero multiple in A, then A and B are said to be *essentially linked.*

LEMMA 5. *If M is a group with pure subgroups A and B which are essentially linked, such that A is a summand ($M = A \oplus A'$), then B is also a summand and $M = B \oplus A'$.*

Proof. The conclusion is equivalent to the statement that the projection $\theta: M \to A$ carries B isomorphically onto A. θ restricted to B is clearly injective since any nonzero element of B has a nonzero multiple which is left fixed by θ. We next note that $\theta(B)$ is a pure subgroup of A, by Lemma 4, since if $\theta(b)$ is in $\theta(B)$, then for some integer n, $n\theta(b) = nb \neq 0$, and $n\theta(b)$ has the same height in $\theta(B)$ as in B (since θ restricted to B is injective) and the same height in B as in A (since A and B are pure). Finally, any nonzero element in A has a nonzero multiple in $\theta(B)$, so $A = \theta(B)$ by Lemma 3.

LEMMA 6. *If M is a group with pure, essentially linked subgroups A and B, both summands (say $M = A \oplus A' = B \oplus B'$), then also $M = A \oplus B' = B \oplus A'$.*

Proof. Apply Lemma 5 twice.

LEMMA 7. *If A and B are summands of a group M and every nonzero element of B has a nonzero multiple in A, then the projection of B into A carries B isomorphically onto a summand of A.*

Proof. Let the projection into A be θ. $\theta(B)$ is pure by Lemma 4. By Lemma 5 therefore, $\theta(B)$ is a summand with the same complement as B.

LEMMA 8. *If M is a group and n a positive integer, we can decompose $M = A \oplus A'$ where $nA = 0$ and any nonzero element of A' has a nonzero multiple in nA'. Furthermore if $G = B \oplus B'$ is another such decomposition then $A \cong B$ and $A' \cong B'$.*

Proof. Choose A to be a subgroup of M maximal with respect to the properties that A is pure and $nA = 0$. A pure, bounded-order subgroup is a summand [9, Th. 7], so we can decompose $M = A \oplus A'$. We must show that any nonzero element of A' has a nonzero multiple in nA'. If the element has infinite order the result is trivial, and otherwise it has a nonzero multiple of prime order, so it will suffice to show that if $x \in A'$, $x \neq 0$, and $px = 0$, for some prime p, then x is divisible by n. This is equivalent to showing that $h_p(x) \geqq k$, where p^k is the highest power of p dividing n. If this were not the case, there would be an element $y \in A'$ with $p^m y = x$, where $h_p(x) = m$ and $m < k$. By Lemma 4, the subgroup $[y]$ generated by y would be pure, and hence a summand of A'. $A \oplus [y]$ would then be a pure subgroup satisfying $n(A \oplus [y]) = 0$, contradicting the maximality of A.

To prove the final statement, note that $nM = nA' = nB'$, so A' and B' are essentially linked, so by Lemma 6, $M = A \oplus B' = B \oplus A'$, which implies that $A \cong B$ and $A' \cong B'$.

LEMMA 9. *Let M be a group, and $M = A \oplus B \oplus C = A' \oplus D \oplus E$, and suppose that A and A' are essentially linked, and every nonzero element of D has a nonzero multiple in $A \oplus B$, and that π is the projection of M onto B from the first decomposition. Then $\pi(D)$ is a summand isomorphic to D, and the subgroups $A \oplus \pi(D)$ and $A' \oplus D$ are essentially linked.*

Proof. Let θ be the projection of M onto $B \oplus C$. By Lemma 6, $M = A \oplus D \oplus E$, so $\theta(D)$ is a summand of $B \oplus C$. By the condition on elements of D, every nonzero element of $\theta(B)$ has a nonzero multiple in B. By Lemma 7 therefore, $\pi(D) = \pi(\theta(D))$ is a summand of B, and it is clearly isomorphic to D. Note that $\pi(D)$ and $\theta(D)$ are essentially linked by construction. To prove the last statement of the lemma, it suffices to show that any nonzero element of D has a nonzero multiple in $A \oplus \pi(D)$ and that any nonzero element of $\pi(D)$ has a nonzero multiple in $A' \oplus D$. For the first, we note that $D \subseteqq A \oplus \theta(D)$, and any element of $A \oplus \theta(D)$ has a nonzero multiple in $A \oplus \pi(D)$. For the second, let x be a nonzero element of $\pi(D)$ and nx a nonzero multiple which is in $\theta(D)$. Then $nx = a + d$, where $a \in A$ and $d \in D$. If $a = 0$ we are done. Otherwise there is a nonzero multiple ra of a in A'. Certainly $rnx = ra + rd$ is in $A' \oplus D$, and, since $ra \neq 0$, $rnx \neq 0$ since $ra \in A'$, and $rd \in D$ and $A \cap D = 0$.

2. Proof of Theorem 1. We begin with three remarks which we will need to refer to.

(2.1). Hypothesis (iii) of Theorem 1 can be strengthened by adding the condition that none of the finite set of summands $M_i (i \in J)$ are of bounded order. To see this, let n be a positive integer such that $nM_i = 0$ for all of the $M_i (i \in J)$ which are of bounded order. This is possible since there are only a finite number of them. Let G, G_1 and G_2 be as in the statement of condition (iii) and use Lemma 8 to decompose G_2 so that $G_2 = G_2^* \oplus G_2'$, where $nG_2^* = 0$ and every nonzero element in G_2' has a nonzero multiple in nG_2'. We now let $G_1' = G_1 \oplus G_2^*$, so that $G = G_1' \oplus G_2'$, where G_1' is again of bounded order, and every nonzero element of G_2' has a nonzero multiple whose image under f is in the sum of those $M_i (i \in J)$ not of bounded order.

(2.2). If G is in the class $\mathscr{D}$ then any two direct sum decompositions of G have isomorphic refinements. For if $G = \sum_{i \in I} A_i = \sum_{j \in J} B_j$, then by condition (iii) of Theorem 1 there is a positive integer n such that $nA_i = nB_j = 0$ for all but a finite number of the i's and j's. We now apply Lemma 8 to each of the summands A_i and B_j, using this integer n, and obtain decompositions

$$A_i = A_i^* \oplus A_i', \ B_j = B_j^* \oplus B_j'$$

where $nA_i^* = nB_j^* = 0$ for all $i \in I$, $j \in J$, and any nonzero element of A_i' or B_j' has a nonzero multiple in nG. If $A = \sum_{i \in I} A_i^*$, $A' = \sum_{i \in I} A_i'$ and B and B' are defined similarly, then $G = A \oplus A' = B \oplus B'$. These decompositions satisfy the conditions of Lemma 8, so $A \cong B$ and

$A' \cong B'$. These four groups have decompositions inherited from the original decompositions of G. The decompositions of A and B are finite, so by hypothesis (ii) of Theorem 1 they have isomorphic refinements. A' and B' are of bounded order and hence are direct sums of cyclic groups, so their decompositions have isomorphic refinements by Kulikov's theorem ([11] or [9, Exercise 34]). Putting these results together, we have the required isomorphic refinements of the original decompositions.

(2.3). Applying (2.2) several times, it is easy to see that if G is a group which is decomposed in two ways as a direct sum of groups in the class $\mathscr{D}$, $G = \sum_{\gamma \in \Gamma} C_\gamma = \sum_{\lambda \in \Lambda} D_\lambda$, and if these decompositions have isomorphic refinements, then if $G = \sum_{i \in I} A_i$ is a refinement of the first decomposition, and $G = \sum_{j \in J} B_j$ is a refinement of the second, then these two decompositions also have isomorphic refinements.

We now outline the rest of the proof of Theorem 1. Suppose $G = \sum_{i \in I} A_i = \sum_{j \in J} B_j$ where the A_i and B_j are groups of the class $\mathscr{D}$. We regroup the summands A_i into finite sets, setting $C_\gamma = \sum_{i \in I_\gamma} A_i$, where γ is an ordinal in some initial segment of the ordinal numbers $(\gamma < \lambda)$, and the I_γ we construct will be disjoint and their union will be I. We similarly group the summands B_j defining $D_\gamma = \sum_{j \in J_\gamma} B_j$, where J_γ is a finite subset of J, and the J_γ are disjoint sets which together include at least all elements $j \in J$ for which B_j is not of bounded order.

We will then have

$$G = \sum_{\gamma < \lambda} C_\gamma = \Big(\sum_{\gamma < \lambda} D_\gamma\Big) \oplus \sum_{j \in J_*} B_j$$

where J_* is the set of all $j \in J$ not contained in any of the J_γ, and all of the $B_j (j \in J_*)$ are of bounded order. We will construct isomorphic refinements of these decompositions, which will prove Theorem 1 by (2.3). We will decompose the C's and D's as follows:

$$C_\gamma = C_\gamma^1 \oplus C_\gamma^2 \oplus C_\gamma^3 \quad D_\gamma = D_\gamma^1 \oplus D_\gamma^2 \oplus D_\gamma^3$$

and we will have by construction

(1) $D_\gamma^1 \cong C_\gamma^1$

(2) $D_\gamma^2 \cong C_{\gamma+1}^2$ where $C_\gamma^2 = 0$ if $\gamma = 0$ or γ is a limit ordinal.

(3) D_γ^3 and C_γ^3 are of bounded order.

(4) $\sum_{\gamma<\lambda} (D_\gamma^1 \oplus D_\gamma^2)$ and $\sum_{\gamma<\lambda} (C_\gamma^1 \oplus C_\gamma^2)$ are essentially linked.

We now note that in the above situation, the theorem is proved, since by (4) and Lemma 6,

$$G = \sum_{\gamma < \lambda} (C_\gamma^1 \oplus C_\gamma^2) \oplus \sum_{\gamma < \lambda} D_\gamma^3 \oplus \sum_{j \in J^*} B_j$$

so that $\sum_{\gamma<\lambda} D_\gamma^3 \oplus \sum_{j\in J_*} B_j \cong \sum_{\gamma<\lambda} C_\gamma^3$, and since these last groups are direct sums of finite cyclic groups, we can get isomorphic refinements by Kulikov's theorem. Hence our pairing in formulas (1) and (2) above and this remark together prove the theorem.

We now construct the subgroups C_γ, D_γ and their decompositions to satisfy (1), (2), (3), and (4). We say the process is completed up to k if

(a) for $n \leqq k$ (ordinal numbers) the finite sets I_n of indices are chosen, and for $n < k$ the sets J_n are chosen.

(b) for $n < k$ the D_n and C_n decompose as above and the summands D_n^i, C_n^i satisfy the statements (1), (2), (3) where they apply.

(c) C_k is chosen and C_k^2, a summand such that $C_k^2 \cong D_{k-1}^2$ if $k-1$ exists, and $C_k^2 = 0$ if k is a limit ordinal.

(d) $\sum_{n<k}(C_n^1 \oplus C_n^2) \oplus C_k^2$ and $\sum_{n<k}(D_n^1 \oplus D_n^2)$ are essentially linked.

Now let the induction hypothesis be that this has been done for all $k < \gamma$, and do it for γ. If γ is a limit ordinal the process is trivial. Take C_γ to be any summand A_i not previously included in $C_k (k < \gamma)$, and set $C_\gamma^2 = 0$. I_γ is the single chosen index i. (If no A_i remain then we are done, for no B_j can remain except possibly groups of bounded order, since by the previous argument, if we let K be the sum of the remaining summands B_j, we have $\sum_{k<\gamma} D_k^3 \oplus K \cong \sum_{k<\gamma} C_k^3$, a direct sum of finite cyclic groups, and by condition (iii), any element of $\mathscr{D}$ which is a direct sum of cyclic groups is necessarily of bounded order.)

If we are not at a limit ordinal, we change notation and assume that the process has been carried out for γ and do it for $\gamma + 1$. We are given C_γ and C_γ^2. Let C_γ^* be a complement to C_γ^2 in C_γ. Let $\sum^\gamma B_j$ be the sum of those summands B_j not in D_k for any $k < \gamma$.

We now apply condition (iii) of Theorem 1 to the subgroup C_γ^* and its natural inclusion mapping into G, using the decomposition

$$G = \sum_{k<\gamma}(D_k^1 \oplus D_k^2) \oplus \sum_{k<\gamma} D_k^3 \oplus \textstyle\sum^\gamma B_j \,.$$

We obtain a decomposition $C_\gamma^* = C_\gamma^1 \oplus C_\gamma^3$, where C_γ^3 is of bounded order, and also a finite subset J_γ of J disjoint from all of the J_k, $k < \gamma$, such that if $D_\gamma = \sum_{j\in J_\gamma} B_j$ then any nonzero element of C_γ^1 has a nonzero multiple in

$$\sum_{k<\gamma}(D_k^1 \oplus D_k^2) \oplus D_\gamma \,.$$

Hence we have used remark (2.1) to eliminate summands of the form D_k^3, $k < \gamma$. We now apply Lemma 9, where A and A' (in the terminology of that lemma) are $\sum_{k<\gamma}(D_k^1 \oplus D_k^2)$ and $\sum_{k<\gamma}(C_k^1 \oplus C_k^2) \oplus C_\gamma^2$, B is D_γ, and D is C_γ^1. We obtain a summand D_γ^1 of D_γ which is isomorphic to C_γ^1, and such that the subgroups

$$\sum_{k<\gamma} (C_k^1 \oplus C_k^2) \oplus C_\gamma^1 \oplus C_\gamma^2$$

and $\sum_{k<\gamma} (D_k^1 \oplus D_k^2) \oplus D_\gamma^1$ are essentially linked.

We now apply the same process in the other direction, choosing $I_{\gamma+1}$, $C_{\gamma+1}$, D_γ^2, D_γ^3, $C_{\gamma+1}^2$ exactly as we choose J_γ, D_γ, C_γ^1, C_γ^3, and D_γ^1, respectively. The proof is exactly the same, thus completing the induction and the proof of Theorem 1.

3. Proof of Theorem 2.

Suppose we have

$$M = \sum_{j \in J} M_j = A \oplus B\ ,$$

where the M_j are groups of the class $\mathscr{D}$. We group these, as in the proof of Theorem 1, defining summands N_i, where each N_i is the sum of a finite number of the M_j. The indices i of the N_i will form an initial segment ($i < \lambda$) of the ordinal numbers and the N_i will be constructed by transfinite induction, so that we will have $M = \sum_{i<\lambda} N_i$. For each i we will also construct summands A_i, B_i of A and B respectively, where A_i and B_i are in the class $\mathscr{D}$ and we will set $C_i = A_i \oplus B_i$. By construction the C_i will be independent, by which we mean that the subgroup generated by them is their direct sum. We will decompose the N_i as follows

$$N_i = N_i^1 \oplus N_i^2 \oplus N_i^3$$

where N_i^3 is of bounded order and $N_i^2 = 0$ if $\mathrm{i} = 1$ or i is a limit ordinal. We then regroup and decompose again, so that we will have

$$N_i^1 \oplus N_{i+1}^2 = P_i \oplus P_i^3$$

where P_i^3 is of bounded order (the superscript 3 will always mean this). Finally, the subgroups $\sum_{i<\lambda} P_i$ and $\sum_{i<\lambda} C_i$ will be essentially linked, so that by Lemma 5, $\sum_{i<\lambda} C_i$ is actually a summand of M.

Let us first show that when this construction has been carried out we will have proved the theorem. By Lemma 6 the summands $\sum_{i<\lambda} P_i$ and $\sum_{i<\lambda} C_i$ interchange and we have

$$M = \sum_{i<\lambda} C_i \oplus \sum_{i<\lambda} (P_i^3 \oplus N_i^3)$$

where the second term is a direct sum of finite cyclic groups. Now $\sum_{i<\lambda} C_i = \sum_{i<\lambda} A_i \oplus \sum_{i<\lambda} B_i$, and since $\sum_{i<\lambda} C_i$ is a summand of M, so are $\sum_{i<\lambda} A_i$ and $\sum_{i<\lambda} B_i$. Hence $\sum_{i<\lambda} A_i$ and $\sum_{i<\lambda} B_i$ are summands of A and B respectively, and we have $A = \sum_{i<\lambda} A_i \oplus A^*$, $B = \sum_{i<\lambda} B_i \oplus B^*$, and $A^* \oplus B^* \cong \sum_{i<\lambda} (P_i^3 \oplus N_i^3)$ (since both are complements to $\sum_{i<\lambda} C_i$) and since this is a direct sum of cyclic groups, so are A^* and B^* by

Kulikov's theorem. Any cyclic summand of M is in $\mathscr{D}$ (by hypothesis (i) of Theorem 1) since it is actually contained in (and therefore a summand of) the sum of a finite number of the original summands M_j. Hence $A = A^* \oplus \sum_{i<\lambda} A_i$ is a direct sum of groups in the class $\mathscr{D}$, which is what we wanted to prove.

To complete the proof, we must carry out the construction of the subgroups N_i, C_i and P_i in the way outlined above. We say the construction has been carried out for k if for $i \leqq k$, N_i is chosen and for $i < k$, C_i, A_i, and B_i are chosen (all belonging to the class $\mathscr{D}$), where A_i and B_i are summands of A and B respectively, $C_i = A_i \oplus B_i$, and all these are chosen so that

(a) $N_i = N_i^1 \oplus N_i^2 \oplus N_i^3 (i < k)$ with N_i^3 of bounded order and $N_i^2 = 0$ if $i = 1$ or a limit ordinal,

(b) N_k has a summand N_k^2 which is zero if k is 1 or a limit ordinal.

(c) For $i < k$, $N_k^1 \oplus N_{i+1}^2 = P_i \oplus P_i^3$ where P_i^3 is of bounded order.

(d) The C_i are independent and $\sum_{i<k} C_i$ is a pure subgroup of M.

(e) $\sum_{i<k} C_i$ and $\sum_{i<k} P_i$ and essentially linked, so that, in particular, $\sum_{i<k} C_i$ is a summand of M by Lemma 5.

We now suppose that this has been done for all $k < \gamma$ and do it for γ. Suppose first that γ is 1 or γ is a limit ordinal. We let N_γ be one of the remaining M_j (if any remain) and set $N_\gamma^2 = 0$ (as we must). Note that this choice guarantees that the process eventually terminates with the choice of all of the M_j. Conditions (a) and (c) are trivially verified, having already been assumed for $i < \gamma$, and (b) is immediate from our definition of N_γ^2. For (d), it is clear that the $C_i \, (i < \gamma)$ certainly are independent and their direct sum is a pure subgroup, since it is an ascending union of pure subgroups. For (e), we note that $\sum_{i<\gamma} C_i$ and $\sum_{i<\gamma} P_i$ are essentially linked, and since $\sum_{i<\gamma} P_i$ is a summand, we can apply Lemma 5 to show that $\sum_{i<\gamma} C_i$ is also a summand. This completes the induction in this case.

Suppose, then, we are not at a limit ordinal. For convenience we assume that the construction has been carried out for γ and do it for $\gamma + 1$. Say $N_\gamma = N_\gamma^2 \oplus N_\gamma^*$, and let the projections to A and B respectively be θ_A and θ_B. We can decompose M in three ways.

$$M = \sum_{i<\gamma} C_i \oplus A_\gamma^* \oplus B_\gamma^* \tag{1}$$

where A_γ^* is a complement in A of $\sum_{i<\gamma} A_i$, and B_γ^* is a complement in B of $\sum_{i<\gamma} B_i$.

$$M = \sum_{i<\gamma} P_i \oplus N_\gamma^* \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus \sum{}' M_j \tag{2}$$

where $\sum^{\gamma} M_j$ denotes the sum of those M_j not chosen to be in N_i for any i, $i \leqq \gamma$. Since $\sum_{i<\gamma} C_i$ and $\sum_{i<\gamma} P_i$ are essentially linked we also have

$$(3) \qquad M = \sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus \sum{}^{\gamma} M_j .$$

We now apply condition (iii) of Theorem 1 to the group N_γ^* and the two homomorphisms θ_A and θ_B (applying the condition twice), using the decomposition (3) above. We obtain a decomposition

$$N_\gamma^* = N_\gamma^1 \oplus N_\gamma^3$$

where N_γ^3 is of bounded order, and there are a finite number of the summands M_j not included in any N_i for $i \leqq \gamma$, such that if $L_{\gamma+1}$ is the sum of this finite number of subgroups, then any nonzero element of N_γ^1 has a nonzero multiple whose images under θ_A and θ_B are both in

$$\sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus L_{\gamma+1} .$$

Hence we have used remark (2.1) in order to eliminate summands of the form P_i^3 or N_i^3.

Now let π be the natural projection of M onto $A_\gamma^* \oplus B_\gamma^*$ from decomposition (1). We have immediately

$$(4) \qquad \sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus L_{\gamma+1} = \sum_{i<\gamma} C_i \oplus \pi(N_\gamma^* \oplus L_{\gamma+1}) .$$

We let $K = \sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus L_{\gamma+1}$. Note also that

$$(5) \qquad K \cap (A_\gamma^* \oplus B_\gamma^*) = \pi(N_\gamma^* \oplus L_{\gamma+1}) .$$

Now $\pi(N_\gamma^* \oplus L_{\gamma+1})$ is isomorphic to $N_\gamma^* \oplus L_{\gamma+1}$ and is therefore in $\mathscr{D}$ (since summands and finite direct sums of elements of $\mathscr{D}$ are in $\mathscr{D}$) and therefore has the finite exchange property, so that

$$A_\gamma^* \oplus B_\gamma^* = \pi(N_\gamma^* \oplus L_{\gamma+1}) \oplus A_\gamma^{**} \oplus B_\gamma^{**}$$

where $A_\gamma^{**} \subseteqq A_\gamma^*$, $B_\gamma^{**} \subseteqq B_\gamma^*$. We have natural decompositions

$$A_\gamma^* = A_\gamma^{**} \oplus D_\gamma^A, \; B_\gamma^* = B_\gamma^{**} \oplus D_\gamma^B$$

where the groups D_γ^A, D_γ^B can be identified as follows:

$$D_\gamma^A = A_\gamma^* \cap (\pi(N_\gamma^* \oplus L_{\gamma+1}) \oplus B_\gamma^{**})$$

$$D_\gamma^B = B_\gamma^* \cap (\pi(N_\gamma^* \oplus L_{\gamma+1}) \oplus A_\gamma^{**}) .$$

Note that the above formulas and statement (5) imply that

$$(6) \qquad K \cap A_\gamma^* \subseteqq D_\gamma^A, \text{ and } K \cap B_\gamma^* \subseteqq D_\gamma^B .$$

We let $D_\gamma = D_\gamma^A \oplus D_\gamma^B$, and we claim that any nonzero element of N_γ^1 has a nonzero multiple in $\sum_{i<\gamma} C_i \oplus D_\gamma$. By the original definition of N_γ^1, if $x \in N_\gamma^1$ and $x \neq 0$, then x has a nonzero multiple nx such that if $nx = y + z$, with $y \in A$ and $z \in B$, then y and z are in K. We will show that y is in $\sum_{i<\gamma} C_i \oplus D_\gamma$, and the proof for z will be the same. We have $y = a_1 + a_2$, where $a_1 \in \sum_{i<\gamma} A_i$ and $a_2 \in A_\gamma^*$. Since $a_1 \in \sum_{i<\gamma} C_i$, it will be enough to show that $a_2 \in D_\gamma$. Since y and a_1 are both in K, so is a_2, so $a_2 \in A_\gamma \cap K$, and thus is in D_γ by formula (6).

We have now shown that any nonzero element of N_γ^1 has a nonzero multiple in $\sum_{i<\gamma} C_i \oplus D_\gamma$. We apply Lemma 9 to obtain a summand D_γ^1 of D_γ such that the subgroups $\sum_{i<\gamma} P_i \oplus N_\gamma^1$ and $\sum_{i<\gamma} C_i \oplus D_\gamma^1$ are essentially linked.

Let D_γ^* be a complement to D_γ^1 in D_γ. As usual, we cannot handle all of D_γ^*, so we apply condition (iii) of Theorem 1 again, with respect to the decomposition

$$M = \textstyle\sum_{i<\gamma} P_i \oplus N_\gamma^* \oplus L_{\gamma+1} \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus \sum^{\gamma\prime} M_j .$$

where we use the notation $\sum^{\gamma\prime} M_j$ to denote the sum of those M_j not chosen to be in N_i for any $i \leqq \gamma$ or in $L_{\gamma+1}$. We obtain a decomposition $D_\gamma^* = D_\gamma^2 \oplus D_\gamma^3$, where D_γ^3 is of bounded order, and there are a finite number of summands M_j from the term $\sum^{\gamma\prime} M_j$ such that if we let $N_{\gamma+1}$ be the sum of $L_{\gamma+1}$ and this additional set of summands, then any nonzero element of D_γ^2 has a nonzero multiple in

$$\sum_{i<\gamma} P_i \oplus N_\gamma^1 \oplus N_{\gamma+1} .$$

Applying Lemma 9 again, (where this time the subgroups corresponding to the A and A' of that lemma are $\sum_{i<\gamma} P_i \oplus N_\gamma^1$ and $\sum_{i<\gamma} C_i \oplus D_\gamma^1$ respectively), we obtain a summand $N_{\gamma+1}^2$ of $N_{\gamma+1}$ such that the subgroups $\sum_{i<\gamma} C_i \oplus D_\gamma^1 \oplus D_\gamma^2$ and $\sum_{i<\gamma} P_i \oplus N_\gamma^1 \oplus N_{\gamma+1}^2$ are essentially linked.

Unfortunately, $D_\gamma^1 \oplus D_\gamma^2$ cannot be the C_γ we need for our induction since it is not necessarily the sum of its A and B components. We return then to D_γ, and compare decompositions. We have

$$D_\gamma = D_\gamma^A \oplus D_\gamma^B = D_\gamma^1 \oplus D_\gamma^2 \oplus D_\gamma^3 ,$$

where $nD_\gamma^3 = 0$ for some positive integer n. Applying Lemma 8 (using this integer n) we obtain decompositions

$$D_\gamma^A = A_\gamma \oplus A_\gamma^3,\ D_\gamma^B = B_\gamma \oplus B_\gamma^3$$

where A_γ^3 and B_γ^3 are of bounded order and every nonzero element of A_γ and B_γ has a nonzero multiple in $D_\gamma^1 \oplus D_\gamma^2$. Let

$$C_\gamma = A_\gamma \oplus B_\gamma \, .$$

Let σ be the projection onto $N_\gamma^1 \oplus N_{\gamma+1}^2$ from the decomposition

$$M = \sum_{i<\gamma} P_i \oplus (N_\gamma^1 \oplus N_{\gamma+1}^2) \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus N_\gamma^3 \oplus N_{\gamma+1}^* \oplus \sum\nolimits^{\gamma+1} M_j \, ,$$

where $N_{\gamma+1}^*$ is a complement to $N_{\gamma+1}^2$ in $N_{\gamma+1}$. Since $\sum_{i<\gamma} C_i$ and $\sum_{i<\gamma} P_i$ are essentially linked, and $\sum_{i<\gamma} C_i \oplus D_\gamma^1 \oplus D_\gamma^2$ and $\sum_{i<\gamma} P_i \oplus N_\gamma^1 \oplus N_{\gamma+1}^2$ are also essentially linked, we know that σ takes $D_\gamma^1 \oplus D_\gamma^2$ isomorphically onto $N_\gamma^1 \oplus N_{\gamma+1}^2$. Let $P_\gamma = \sigma(C_\gamma)$ and $P_\gamma^3 = \sigma(A_\gamma^3 \oplus B_\gamma^3)$. We then have

$$N_\gamma^1 \oplus N_{\gamma+1}^2 = P_\gamma \oplus P_\gamma^3 \, ,$$

where P_γ^3 is of bounded order. We now apply Lemma 9 once more, where the A, A', and D of that lemma correspond to $\sum_{i<\gamma} P_i$, $\sum_{i<\gamma} C_i$, and C_γ respectively, and we see that the subgroups $\sum_{i<\gamma} P_i \oplus P_\gamma$ and

$$\sum_{i<\gamma} C_i \oplus C_\gamma$$

are essentially linked. It is also clear that C_γ is in $\mathscr{D}$ since it is isomorphic to P_γ and P_γ is a summand of $N_\gamma \oplus N_{\gamma+1}$, which in turn is a direct sum of a finite number of groups in the class $\mathscr{D}$. We therefore have completed our induction and the proof of Theorem 2.

4. Complete Abelian groups.

For any Abelian group A there is a natural homomorphism

$$A \to \varprojlim A/nA$$

where the limit is taken over the nonzero integers n ordered by divisibility. The inverse limit is denoted $\hat{A}$ and it is the Hausdorff completion of A with respect to the uniform structure defined by taking as neighborhoods of zero the subgroups nA $(n \neq 0)$. The mapping $A \to \hat{A}$ is injective if and only if A is Hausdorff. We remark that the homomorphism $A \to \hat{A}$ induces an isomorphism $A/nA \to \hat{A}/n\hat{A}$, so that the image of A is a pure subgroup of $\hat{A}$ and the Z-topology on $\hat{A}$ agrees with the topology induced (by the completion process) from the Z-topology of A. The group A is complete and Hausdorff if and only if $A = \hat{A}$.

Note that a subgroup B of A is pure if and only if for all integers n, $n \neq 0$, the natural homomorphism $B/nB \to A/nA$ is injective. B is dense in A (with respect to the Z-topology) if and only if for all nonzero integers n, the natural homomorphism $B/nB \to A/nA$ is surjective.

LEMMA 10. *If B is a pure dense subgroup of a group A and f is a homomorphism from B into a complete Hausdorff group C then f extends in one and only one way to a homomorphism from A to C.*

This follows from standard inverse limit or topological arguments. (From the topological point of view, one needs to observe that any homomorphism between two groups is continuous in the Z-topology and that the Z-topology on a pure subgroup B agrees with the topology induced from the Z-topology on A.)

If A is any group, we let A^1 be the subgroup of A consisting of those elements divisible by all integers n. The proof of the following lemma is an elementary computation.

LEMMA 11. *If B is a subgroup of a group A, then the closure of B is the inverse image in A of $(A/B)^1$. In particular, B is closed if and only if A/B is Hausdorff, and B is dense in A if and only if A/B is divisible.*

For any prime p, we denote by Z_p the ring of rational numbers which can be written as fractions with denominators prime to p, and for any group A, we let $A_p = A \otimes Z_p$, regarded as a Z_p-module. A_p is the localization of A at the prime p. If A_p^1 is the submodule of A_p consisting of all elements divisible by all powers of p then we define the *Hausdorff localization*, A_p^* of A by $A_p^* = A_p/A_p^1$. We have natural homomorphisms $\phi_p: A \to A_p$, and hence a natural homomorphism

$$\phi: A \to \prod_p A_p^* .$$

If A is Hausdorff, this imbeds A as a pure, dense subgroup of $\prod_p A_p^*$. This proves the following lemma.

LEMMA 12. *If C is a complete Hausdorff group then the natural homomorphism $C \to \prod_p C_p^*$ is an isomorphism.*

To exploit this lemma, we need some results about modules over the rings Z_p. The results are actually valid for modules over any discrete valuation ring. A subset X of a Z_p-module is a *pure independent subset* if the elements are independent and the submodule $[X]$ generated by X is a pure submodule. A submodule B of M is a *basic submodule* if it is pure, dense, and a direct sum of cyclic modules. By [9, Lemma 21] any maximal pure independent subset generates a basic submodule, and it is trivial to verify that if X is a pure independent subset then X is maximal if and only if $[X]$ is dense (or equivalently, $M/[X]$ is divisible). The next lemma is a refinement of [9, Th. 23].

LEMMA 13. *If M is a Z_p-module and C a pure submodule which is complete and Hausdorff, X a maximal pure independent subset of C, and Y a set disjoint from X such that $X \cup Y$ is a maximal pure independent subset of M, then $M = C \oplus D$, where D is the closure of the submodule generated by Y.*

Proof. Define a function f on the set $X \cup Y$ by $f(x) = x$ if $x \in X$ and $f(y) = 0$ if $y \in Y$. This extends to a homomorphism of the basic submodule generated by $X \cup Y$, which can be regarded as a homomorphism of $[X \cup Y]$ into C. By Lemma 10, this extends to a homomorphism of M into C, which we also call f. Since f is the identity on $[X]$ and $[X]$ is dense in C, f is a projection onto C. If D is the kernel of the projection then D is closed since C is Hausdorff. To show that D is the closure of Y, we remark that $M/[X \cup Y]$ is divisible and $M/[X \cup Y] \cong C/[X] \oplus D/[Y]$, so $D/[Y]$ is divisible, which implies that Y is dense in D by Lemma 11.

LEMMA 14. *If M is a Z_p-module with torsion submodule T, and X is a subset of M, and X_0 and X_1 are the subsets of X consisting of the elements of finite and infinite order respectively, then X is a maximal pure independent subset if and only if X_0 is a maximal pure independent subset of T and X_1 is mapped bijectively onto a basis of the Z/pZ-vector space $M/(T + pM)$.*

Proof. Let X be a maximal pure independent subset of M and let $C = [X_1]$. Then the natural homomorphism $C/pC \to M/(T + pM)$ is an isomorphism by the proof of Lemma 21 of [9], and certainly X_0 is a maximal pure independent subset of T, which proves half of the lemma. Conversely, if the condition above is satisfied, and $\sigma\colon M \to M/T$ is the natural map, then σ takes X_1 bijectively onto a maximal pure independent subset of M/Y by [13, Lemma 3]. The submodule B generated by $\sigma(X_1)$ is therefore free, so

$$\sigma^{-1}(B) = T \oplus [X_1] .$$

It follows immediately that X is an independent set. Also, since B is pure in M/T, $\sigma^{-1}(B)$ is pure in M, and since $[X_0]$ is a pure submodule of the summand T, $[X]$ is a pure submodule of M. Finally, $M/[X]$ is clearly divisible, since $T/[X_0]$ and $M/(T + [X_1])$ are both divisible.

LEMMA 15. *Let M be a Z_p-module, Y a maximal pure independent subset of M, and X a pure independent subset of M. Then there* *is a subset Z of Y, disjoint from X, such that $X \cup Z$ is a maximal pure independent subset of M.*

Proof. This result was proved for p-groups in [4, Lemma 10.12]. We therefore know that if X_0 and X_1 are the sets of elements of finite order and infinite order respectively in X and Y_0 and Y_1 are the corresponding subsets of Y, then there is a subset Z_0 of Y_0, disjoint from X_0, such that $X_0 \cup Y_0$ is a maximal pure independent subset of the torsion submodule T of M. If ϕ is the natural map of M onto $M/(T + pM)$, (where T is the maximal torsion submodule of M), then ϕ takes Y_1 bijectively onto a Z/pZ-basis for $M/(T + pM)$, and X_1 bijectively onto an independent subset of $M/(T + pM)$. There is therefore a subset Z_1 of Y_1, disjoint from X_1, such that ϕ takes $X_1 \cup Z_1$ bijectively onto a basis for $M/(T + pM)$. This proves the lemma, setting $Z = Z_0 \cup Z_1$.

THEOREM 3. *A complete Hausdorff group has the exchange property.*

Proof. Let A be a group and C a complete Hausdorff summand of A, and say $A = \sum_{i \in I} D_i$. We will show that there are subgroups $D_i' \subseteq D_i$ with

$$A = C \oplus \textstyle\sum_{i \in I} D_i' .$$

We first prove the theorem in the local, Hausdorff case. Suppose that A, C, and the D_i are all Z_p-modules. Suppose in addition that A is Hausdorff. Let X be a maximal pure independent subset of C and Y_i a maximal pure independent subset of D_i. By Lemma 15, we can extend X to a maximal pure independent subset of A by adding elements from the sets Y_i. Let the added sets be $Y_i' \subseteq Y_i$, and let Z be the union of the sets Y_i' (so that $X \cup Z$ is a maximal pure independent subset of A). By Lemma 14, if E is the closure of the subgroup generated by Z, then $A = C \oplus E$. We let $E_i = E \cap D_i$, and we claim that $E = \sum_{i \in I} E_i$. Since A is Hausdorff, D_i is closed, so E_i is also closed. Since the E_i are in different summands, $\sum_{i \in I} E_i$ is closed, and it contains Z, so $E = \sum_{i \in I} E_i$ as desired. Hence, $A = C \oplus \sum_{i \in I} E_i$, proving the exchange theorem in this case.

We now prove the theorem in general. If $A = \sum_{i \in I} D_i$ then $A_p^* = \sum_{i \in I} (D_i)_p^*$ and C_p^* is a Hausdorff complete summand of A_p^*. By the previous case, there are submodules $E_i(p) \subseteq (D_i)_p^*$ such that

$$A_p^* = C_p^* \oplus \sum_{i \in I} E_i(p) .$$

This means that there is a projection $g_p \colon A_p^* \to C_p^*$ such that

$$\text{(1)} \qquad \operatorname{Ker}(g_p) = \sum_{i \in I} \ker(g_p) \cap (D_i)_p^* \ .$$

What we need to prove is that there is a projection $f: A \to C$ such that

$$\text{(2)} \qquad \operatorname{Ker}(f) = \sum_{i \in I} \operatorname{Ker}(f) \cap D_i \ .$$

The definition is now clear. Let

$$g: \prod_p A_p^* \to \prod_p C_p^*$$

be the homomorphism induced by the mappings $g_p: A_p^* \to C_p^*$, let ϕ be the natural homomorphism

$$\phi: A \to \prod_p A_p^*$$

with coordinate mappings ϕ_p, and let

$$\sigma: \prod_p C_p^* \to C$$

be the inverse of the isomorphism of Lemma 15. Let $f = \sigma g \phi$. To prove that (2) holds, we need only check that if $x \in A$ and $x = \sum x_i$ in the decomposition $\sum_{i \in I} D_i$ then if $f(x) = 0$, we also have $f(x_i) = 0$. If $f(x) = 0$, then $g_p(\sum \phi_p(x_i)) = 0$ for each prime p. By (1), this implies that $g_p(\phi_p(x_i)) = 0$ for each prime p, which shows that $f(x_i) = 0$ as desired. This proves that (2) holds, and if we define

$$E_i = \operatorname{Ker}(f) \cap D_i \ ,$$

then we have

$$A = C \oplus \sum_{i \in I} E_i \ .$$

This completes the proof of Theorem 3.

COROLLARY. *A complete Hausdorff group is in the class* $\mathscr{C}$.

Proof. Condition (1) is immediate and condition (ii) is contained in Theorem 3. For condition (iii), we suppose that C is a complete Hausdorff group and $f: C \to M$ a homomorphism of C into a Hausdorff group M which is a direct sum, $M = \sum_{i \in I} M_i$. We first remark that it will suffice to show that there is a finite subset $J \subseteqq I$, such that for some nonzero integer n, $f(nC) \subseteqq \sum_{i \in J} M_i$. For in this case we apply Lemma 8 to obtain a decomposition $C = C_1 \oplus C_2$, where $nC_1 = 0$ and any nonzero element of C_2 has a nonzero multiple in nC_2., whose image under f is therefore in $\sum_{i \in I} M_i$.

We assume first that the decomposition of M is countable, $M =$

$\sum_{i=1}^{\infty} M_i$. The subgroups $f^{-1}(\sum_{i=1}^{m} M_i)$ are closed subgroups of C whose union is all of C, so by the Baire category theorem, for some integer m, $f^{-1}(\sum_{i=1}^{m} M_i)$ contains a neighborhood of 0, namely nC, for some nonzero integer n.

If the result were false in the general case (with an arbitrarily large index set I) and if the mapping f and the group M in fact provided a counterexample, then we could find a sequence of integers $n_j (j = 1, 2, \cdots)$, a sequence of elements x_j of C, and a sequence of distinct summands of the original family, which we simply write N_j, such that x_j is divisible by n_j and $f(x_j)$ has a nonzero coordinate in N_j. If we let N_0 be the direct sum of all of the summands M_i not in our chosen list, then we have a decomposition $M = \sum_{j=0}^{\infty} N_j$ which provides another counterexample, this time with a countable number of summands. Since this has been shown to be impossible, the corollary is proved.

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THE AMBIENT HOMEOMORPHY OF AN INCOMPLETE SUBSPACE OF INFINITE-DIMENSIONAL HILBERT SPACES

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The pair (H, H_f) is studied from a topological point of view (where H is an infinite-dimensional Hilbert space and H_f is the linear span in H of an orthonormal basis), and a complete characterization is obtained of the images of H_f under homeomorphisms of H onto itself. As the characterization is topological and essentially local in nature, it is applicable in the context of Hilbert manifolds and provides a characterization of (H, H_f)-manifold pairs (M, N) (with M an H-manifold and N an H_f-manifold lying in M so that each coordinate chart f of M may be taken to be a homeomorphism of pairs $(U, U \cap N) \xrightarrow{f} (f(U), f(U) \cap H_f))$.

This implies that in the countably infinite Cartesian product of H with itself, the infinite (weak) direct sum of H_f with itself is homeomorphic to H_f (the two form such a pair), and that if K is a locally finite-dimensional simplicial complex equipped with the barycentric metric (inducing the Euclidean metric on each simplex) and if no vertex-star of K contains more than $\dim(H)$ vertices, then $(K \times H, K \times H_f)$ is an (H, H_f)-manifold pair.

These results are used in [10] to study H_f-manifolds much more intensively to obtain results previously available only for H-manifolds or in the case that H_f is separable, i.e., connected H_f-manifolds are homeomorphic to open subsets of H_f, homotopy-equivalent H_f-manifolds are homeomorphic, and there is an essentially unique completion of an H_f-manifold into an H-manifold, yielding an (H, H_f)-pair.

It should be remarked that this characterization has already been achieved for separable Hilbert spaces by R. D. Anderson [1] and by C. Bessaga and A. Pełczynski [5], and that the observations concerning (H, H_f)-manifold pairs have been made by T. A. Chapman [6, 7] in that case. (Chapman then proceeded to obtain most of the results of [10] in the separable case by methods which seem at the moment to be limited to separability.)

Throughout the discussion, X will denote some complete metric space, and $\mathscr{H}(X)$, the group of all homeomorphisms of X onto itself. The term "isotopy" ("isotopic") will be understood as an abbreviation for "invertible, ambient isotopy", that is, a map $F: X \times [0, 1] \rightarrow X$ such that the function $G: X \times [0, 1] \rightarrow X \times [0, 1]$ defined from F by setting $G(x, t) = (F(x, t), t)$ is a homeomorphism. (When an embedding

f of a subset of X into X is said to be isotopic to the identity, then, there will exist an extension g of f to an element of $\mathscr{H}(X)$ which is invertibly ambient isotopic to the identity.) If $\mathscr{U}$ is a collection of open sets of X, a map f of a subset Y of X into X will be said to be *limited by* $\mathscr{U}$ if for each point y of Y such that $y \neq f(y)$, there is a member of $\mathscr{U}$ containing both. A homotopy $F: Y \times [0, 1] \to X$ will be said to be *limited by* $\mathscr{U}$ if for each point y of Y such that $F(\{y\} \times [0, 1]) \neq \{y\}$, there is an element of $\mathscr{U}$ containing $F(\{y\} \times [0, 1])$. If $\mathscr{S}$ is a collection of subsets of X then $\mathscr{S}^*$ will denote their union, and $\mathscr{S}$ will be termed *normal* whenever there is an open cover $\mathscr{U}$ of $\mathscr{S}^*$ by mutually disjoint sets with the property that for each U in $\mathscr{U}$, $U \cap \mathscr{S}^* \in \mathscr{S}$. The letter N means the positive integers. Finally, if A is a subset of X and $\mathscr{S}$ is a collection of subsets of X, then $\mathrm{st}(A, \mathscr{S})$ denotes the star of A with respect to $\mathscr{S}$, that is, the union of all members of $\mathscr{S}$ meeting A, and $\mathrm{st}(\mathscr{S}) = \{\mathrm{st}(S, \mathscr{S}) \mid S \in \mathscr{S}\}$. Also,

$$\mathrm{st}^n(A, \mathscr{S}) = \mathrm{st}(\mathrm{st}^{n-1}(A, \mathscr{S}), \mathscr{S}),$$

and $\mathrm{st}^n(S) = \mathrm{st}(\mathrm{st}^{n-1}(\mathscr{S}))$. All refinements used will be understood to be composed of open sets, and $\mathscr{T}$ is a st^n-refinement of $\mathscr{S}$ provided that $\mathrm{st}^n(\mathscr{T})$ refines $\mathscr{S}$.

The first lemma is due to Anderson and Bing [2].

LEMMA 1. *Let $\{f_n\}_{n \in N}$ be a sequence of homeomorphisms of the complete metric space X onto itself, and let $\mathscr{U}$ be any open cover of X. If $\{U_n\}_{n=0}^{\infty}$ is a collection of open covers of X such that $\mathrm{st}^2(\mathscr{U}_0)$ refines $\mathscr{U}$ and for each n in N $\mathscr{U}_n$ is a star-refinement of $\mathscr{U}_{n-1}$ of mesh less than $1/2^n$, then $\{f_n \circ \cdots \circ f_1\}_{n \in N}$ converges (uniformly) to a member of $\mathscr{H}(X)$ which is limited by $\mathscr{U}$ provided that for each n in N f_{n+1} is limited by $\mathscr{U}_n$ and mesh*

$$(f_1^{-1} \circ \cdots \circ f_n^{-1}(\mathscr{U}_n)) < 1/2^n.$$

Proof. Anderson and Bing proved that $\{f_n \circ \cdots \circ f_1\}_{n \in N}$ converges uniformly to a member f of $\mathscr{H}(X)$. To verify that f is limited by $\mathscr{U}$, it is sufficient to observe that for each x in X and n in N, there is a $U(x, n)$ in $\mathscr{U}_n$ containing both $f_n \circ \cdots \circ f_1(x)$ and $f_{n+1} \circ \cdots \circ f_1(x)$, and there is also a $U(x, 0)$ in $\mathscr{U}_0$ containing both x and $f_1(x)$. If $V(x, n)$ is an element of $\mathscr{U}_{n-1}$ containing $\mathrm{st}(U(x, n), \mathscr{U}_n)$ for each x and n, then x and $f_{n+1} \circ \cdots \circ f_1(x)$ lie in

$$\begin{aligned}\bigcup_{m=0}^{n} U(x, m) &\subset \bigcup_{m=0}^{n-1} U(x, m) \cup V(x, n)\\ &\subset \bigcup_{m=0}^{n-2} U(x, m) \cup V(x, n-1) \subset \cdots \subset U(x, 0) \cup V(x, 1)\\ &\subset \mathrm{st}(U(x, 0), \mathscr{U}_0),\end{aligned}$$

so x and $f(x)$ must lie in the closure of $\mathrm{st}(U(x, 0), \mathscr{U}_0)$, which is contained in $\mathrm{st}^2(U(x, 0), \mathscr{U}_0)$, which lies in some member of $\mathscr{U}$.

LEMMA 2. *If $\mathscr{U}$ is a collection of pairwise disjoint open subsets of X, then there is an open cover $\mathscr{V}$ of $\mathscr{U}^*$, refining $\mathscr{U}$, with the property that if for each $U \in \mathscr{U}$ f_U is a homeomorphism of U onto itself which is limited by $\mathscr{V}$, then the function f defined by $f(x) = f_U(x)$, if $x \in U$, and $f(x) = x$, if $x \notin \mathscr{U}^*$, is a homeomorphism of X onto itself.*

Proof. Let $\mathscr{V} = \{V(x) = \{y \in X \mid d(y, x) < d(z, x)/2$ for each z in $X \backslash U\} \mid x \in U \in \mathscr{U}\}$, where $d(\cdot, \cdot)$ is the metric for X. Then for any points z of $X \backslash U^*$, and y of X, $d(z, f(y)) \leqq 3d(z, y)$, which establishes continuity. As f must be one-to-one and onto, and the same argument establishes the continuity of f^{-1}, f is a homeomorphism.

Let $\mathscr{K}$ be an hereditary collection of closed subsets of X which is invariant under the action of $\mathscr{H}(X)$, that is, each closed subset of a member of $\mathscr{K}$ is in $\mathscr{K}$ and $f(K) \in \mathscr{K}$ if $K \in \mathscr{K}$ and $f \in \mathscr{K}(X)$. A set A in X will be termed *$\mathscr{K}$-absorptive* if for each open cover $\mathscr{U}$ of a member K of $\mathscr{K}$ and each member K' of $\mathscr{K}$ contained in $K \cap A$, there is a homeomorphism f in $\mathscr{H}(X)$ which is limited by $\mathscr{U}$, is the identity on K', and carries K into A. If f may always be chosen so that there is an isotopy from it to the identity which is limited by $\mathscr{U}$, then A will be called *strongly $\mathscr{K}$-absorptive.*

LEMMA 3. *If A is $\mathscr{K}$-absorptive (strongly $\mathscr{K}$-absorptive), L is an open subset of a member of $\mathscr{K}$, and U is an open cover of L in X, then there is a member f of $\mathscr{H}(X)$ carrying L into A which is limited by $\mathscr{U}$ (is isotopic to the identity by an isotopy limited by $\mathscr{U}$).*

Proof. As $\mathscr{U}^*$ is an open subset of the complete metric space X, it may be given an equivalent metric under which it is itself complete, so Lemma 1 holds under the new metric. Let $\{V_n\}_{n \in N}$ be a sequence of open sets in X such that each contains its successor and $\bigcap_{n \in N} V_n = X \backslash U^*$, and let $\mathscr{W}$ be a refinement of $\mathscr{U}$ which covers $\mathscr{U}^*$ and has the property that any member of $\mathscr{H}(\mathscr{U}^*)$ which is limited by $\mathscr{W}$ extends to an element of $\mathscr{H}(X)$ which is also limited by $\mathscr{W}$. If $\mathscr{K}'$ is the collection of all members of $\mathscr{K}$ which lie in $\mathscr{U}^*$, then from the definition of (strong) $\mathscr{K}$-absorptivity it is immediate that as a subset of $\mathscr{U}^*$, $A \cap U^*$ is (strongly) $\mathscr{K}'$-absorptive. Using Lemma 1 and the fact that $L \backslash V_{n+1}$ contains $L \backslash V_n$ for all n in N and that both are in $\mathscr{K}'$, select a sequence $\{f_n\}_{n \in N}$ of members of $\mathscr{H}(\mathscr{U}^*)$ with $\{f_n \circ \cdots \circ f_1\}_{n \in N}$ converging to a member of $\mathscr{H}(\mathscr{U}^*)$ which is limited by $\mathscr{W}$ and such that for each n, f_n

carries $f_{n-1} \circ \cdots \circ f_1(L \backslash V_n)$ into $A \cap \mathscr{U}^*$ and is the identity on

$$f_{n-1} \circ \cdots \circ f_1(L \backslash V_{n-1}) .$$

This may be done because each of the functions $f_n \circ \cdots \circ f_1$ may be kept limited by $\mathscr{W}$, which ensures that they permute the elements of $\mathscr{K}'$. Extending the limit homeomorphism to all of X so that it is the identity off $\mathscr{U}^*$ produces the desired member of $\mathscr{H}(X)$. (In the case that an isotopy is desired, and that A is strongly $\mathscr{K}$-absorptive, consider the cover $\mathscr{W}' = \{W \times [0, 1] \mid W \in \mathscr{W}\}$ of $\mathscr{U}^* \times [0, 1]$ and construct a level-preserving homeomorphism of $\mathscr{U}^*$ which is limited by $\mathscr{W}'$, is the identity on $\mathscr{U}^* \times \{0\}$, and carried $L \times \{1\}$ into $A \times \{1\}$. The associated isotopy extends to X.)

A collection $\mathscr{A}$ of members of K will be called a *$\mathscr{K}$-complex* if it may be expressed as a countable union $\bigcup_{n=0}^{\infty} \mathscr{A}_n$ of subsets of itself such that $\mathscr{A}^n = \bigcup_{m=0}^{n} \mathscr{A}_m^*$ is closed for each n and $\mathscr{A}[n] = \{A \backslash \mathscr{A}^{n-1} \mid A \in \mathscr{A}_n\}$ is normal for all n. (Here, $\mathscr{A}^{-1} = \varnothing$.) The set $\mathscr{A}^*$ will be said to *admit the structure of a $\mathscr{K}$-complex.* If $\mathscr{A}^*$ is (strongly) $\mathscr{K}$-absorptive, then it will be referred to as a *(strong) $\mathscr{K}$-absorption base.*

THEOREM 1. *If $\mathscr{U}$ is an open cover of X and A^* and B^* are two (strong) $\mathscr{K}$-absorption bases in X, there is a homeomorphism f of X onto itself (an isotopy F of X), limited by $\mathscr{U}$, such that $f(A^*) = B^*(F(A^* \times \{1\}) = B^*)$.*

Proof. Let $\mathscr{A} = \bigcup_{n=0}^{\infty} \mathscr{A}_n$ and $\mathscr{B} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$ be $\mathscr{K}$-complex structures for A^* and B^* respectively. As the construction of an isotopy in the strong case may be handled from the construction of a homeomorphism in the other case as was done in the previous proof, only the latter construction will be made here. It is quite simple. Since $\mathscr{K}$ is invariant under the action of $\mathscr{H}(X)$, so is the collection of (strong) $\mathscr{K}$-absorption bases. A sequence $f_1, g_1, f_2, g_2, \cdots$ of members of $\mathscr{H}(X)$ is to be chosen with $\{g_n^{-1} \circ f_n \circ \cdots \circ g_1^{-1} \circ f_1\}_{n \in N}$ converging to an element f of $\mathscr{H}(X)$ which is limited by $\mathscr{U}$. Furthermore, $f_n(g_{n-1}^{-1} \circ \cdots \circ f(\mathscr{A}^n))$ is to be a subset of $\mathscr{B}^*$, $g_n(\mathscr{B}^n)$ is to be a subset of $f_n \circ g_{n-1}^{-1} \circ \cdots \circ g_1^{-1} \circ f_1(\mathscr{A}^*)$, f_n is to be the identity on $g_{n-1}^{-1} \circ f_{n-1} \circ \cdots \circ g_1^{-1} \circ f_1(\mathscr{A}^{n-1}) \cup \mathscr{B}^{n-1}$, and g_n is to be the identity on $f_n \circ g_{n-1}^{-1} \circ \cdots \circ g_1^{-1} \circ f_1(\mathscr{A}^n) \cup \mathscr{B}^{n-1}$. Then the limit homeomorphism f is limited by $\mathscr{U}$ and $f(\mathscr{A}^*) = \mathscr{B}^*$. The selection of these homeomorphisms may be made inductively so as to satisfy the convergence criterion of Lemma 1 because for each n, $\mathscr{A}[n]$ and $\mathscr{B}[n]$ are normal and $\mathscr{A}^{n-1}$ and $\mathscr{B}^{n-1}$ are closed, so Lemmas 2 and 3 may be applied and the homeomorphisms constructed piecemeal on collections of pair-wise disjoint open sets in X.

THEOREM 2. *If U is an open subset of X, A^* is a (strong) $\mathscr{K}$-absorption base for X, and $\mathscr{K}'$ is the set of all members of $\mathscr{K}$ contained in U, then $A^* \cap U$ is a (strong) $\mathscr{K}'$-absorption base for U.*

Proof. It has already been remarked that $A^* \cap U$ is (strongly) $\mathscr{K}'$-absorptive, so all that is necessary is to demonstrate that it admits the structure of a $\mathscr{K}'$-complex. If $A^* \cap U = \varnothing$, then $\mathscr{K}' = \{\varnothing\}$, and $A^* \cap U$ is a strong $\mathscr{K}'$-absorption base for U. Otherwise, let $\{V_n\}_{n \in N}$ be a collection of open sets with $X \backslash U \subset V_{n+1} \subset \bar{V}_{n+1} \subset V_n$ for each n, and with $\bigcap_{n \in N} V_n = X \backslash U$. Now, let

$$\mathscr{B}_{2n} = \bigcup_{m=0}^{n} \{A \backslash V_{2(n-m+1)} \mid A \in \mathscr{A}_m\}$$

and $\mathscr{B}_{2n+1} = \bigcup_{m=0}^{n+1} \{A \backslash V_{2(n-m+1)+1} \mid A \in \mathscr{A}_m\}$. If $\bigcup_{n=0}^{\infty} \mathscr{B}_n$ is denoted by $\mathscr{B}$, it is apparent that $\mathscr{B}^n$ is closed for each n. To see that $\mathscr{B}[n]$ is normal for each n, let $\{\mathscr{U}_n\}_{n \in N}$ be a collection of sets of mutually disjoint open sets of X with the property that $\mathscr{U}_n{}^*$ contains $\mathscr{A}[n]^*$ and that for each U in $\mathscr{U}_n$, $U \cap \mathscr{A}[n]^* \in \mathscr{A}[n]$. Then define $\mathscr{W}_{2n} = \bigcup_{m=0}^{n} \{U \cap V_{2(n-m)+1} \backslash \bar{V}_{2(n-m+1)+1} \mid U \in \mathscr{U}_m\}$ and

$$\mathscr{W}_{2n+1} = \bigcup_{m=0}^{n+1} \{U \cap V_{2(n-m+1)} \backslash \bar{V}_{2(n-m+2)} \mid U \in \mathscr{U}_m\} ,$$

for each $n = 0, 1, \cdots$ The collections $\mathscr{W}_n$ are composed of pairwise disjoint open sets separating members of $\mathscr{B}[n]$, so $\mathscr{B}$ is a $\mathscr{K}'$-complex. Since $\mathscr{B}^* = \mathscr{A}^* \cap U$, the proof is complete.

If $\{Y_n\}_{n \in N}$ is a collection of spaces, then $\prod_{n \in N} Y_n$ will denote their Cartesian product. If, for each n, $y_n \in Y_n$, then $\prod_{n \in N} (Y_n, y_n)$ will denote that subset of $\prod_{n \in N} Y_n$ composed of those points with n-th coordinate differing from y_n for at most finitely many n. Also, let $\mathscr{C}$ be a class of spaces which is closed under the operations of taking closed subsets and of taking finite products, and for each space Y, let $\mathscr{C}(Y)$ denote the collection of images of members of $\mathscr{C}$ under closed embeddings in Y.

THEOREM 3. *If $\{X_n\}_{n \in N}$ is a sequence of complete metric spaces and if, for each n, $\mathscr{A}(n)$ is a $\mathscr{C}(X_n)$-complex, and x_n is a point of $\mathscr{A}(n)^*$, then $\prod_{n \in N}(\mathscr{A}(n)^*, x_n)$ admits the structure of a $\mathscr{C}(\prod_{n \in N} X_n)$-complex.*

Proof. For each finite subset S of N, let f denote the natural injection of $\prod_{n \in S} X_n$ into $\prod_{n \in N} (X_n, x_n)$. Now, for each ordered n-tuple $(m_1, \cdots, m_n)$ of nonnegative integers, each of which is no greater than n, let $\mathscr{B}(n; m_1, \cdots, m_n) = \{f(\prod_{i=1}^{n} A_i) \mid A_i \in \mathscr{A}(i)_{m_i}\}$. Order the set of all these collections in such a manner that

$$\mathscr{B}(n; m_1, \cdots, m_n) \geqq \mathscr{B}(n'; m'_1, \cdots, m'_{n'})$$

if $n \geqq n'$ or if $n = n'$ and $m_j \geqq m'_j$ for all j. The order selected will be isomorphic to the nonnegative integers, so index the $\mathscr{B}$'s by them in a manner consistent with the above requirements. Let $\mathscr{B} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$. For each n, $\mathscr{B}_n^*$ is closed, so $\mathscr{B}^n$ is, also. Thus, in order to check that $\mathscr{B}$ is a $\mathscr{C}(\prod_{n \in N} X_n)$-complex, it is only necessary to verify that $\mathscr{B}[i]$ is normal for each i. However, for n and $(m_1, \cdots, m_n)$ such that $\mathscr{B}_i = \mathscr{B}(n; m_1, \cdots, m_n)$, and for B in $\mathscr{B}_i$, $B \backslash \mathscr{B}^{i-1} \subset B \backslash f(\prod_{j=1}^{n} \mathscr{A}(j)^{m_j - 1})$, so if for each n in N and each nonnegative integer m, $\mathscr{U}_m^n$ is an open cover of $\mathscr{A}(n)[m]^*$ in X_n by pairwise disjoint open sets U with the property that

$$U \cap \mathscr{A}(n)[m]^* \in \mathscr{A}(n)[m] ,$$

then $\mathscr{V}_i = \{\prod_{j=1}^{n} U_j \times \prod_{j=n+1}^{\infty} X_j \mid U_j \in \mathscr{U}_{m_j}^j \text{ for } j = 1, \cdots, n\}$ is a cover of $\mathscr{B}[i]$ by mutually disjoint open sets of $\prod_{n \in N} X_n$ with the property that the intersection of each with $\mathscr{B}[i]^*$ is a member of $\mathscr{B}[i]$. Thus, each $\mathscr{B}[i]$ is normal and $\mathscr{B}$ is a $\mathscr{C}(\prod_{n \in N} X_n)$-complex. As it is immediate that $\mathscr{B}^* = \prod_{n \in N}(\mathscr{A}(n)^*, x_n)$, the theorem has been proved.

REMARK. It was tacitly assumed above that there were infinitely many X_n's. Of course, the same proof works for a finite collection.

COROLLARY 1. *If, in the above,* $\prod_{n \in N} (\mathscr{A}(n)^*, x_n)$ *is (strongly)* $\mathscr{C}(\prod_{n \in N} X_n)$*-absorptive, then it is a (strong)* $\mathscr{C}(\prod_{n \in N} X_n)$*-absorption base.*

REMARK. It is clear from the definitions that if X and Y are homeomorphic, then any homeomorphism between them carries the $\mathscr{C}(X)$-complexes to the $\mathscr{C}(Y)$-complexes and the (strong) $\mathscr{C}(X)$-absorption bases to the (strong) $\mathscr{C}(Y)$-absorption bases.

From now on, $\mathscr{C}$ will denote the class of all finite-dimensional compact metric spaces. The next lemma is an extension of Proposition 4.5 of [5] to the nonseparable case and to isotopies. It consists of combining Theorem 4.2 of [3] with the Bartle-Graves Theorem.

LEMMA 4. *If X is an infinite-dimensional Fréchet space and K is a compact subset of X, then for each open cover $\mathscr{U}$ of K there is a second, $\mathscr{V}$, such that any embedding of K in X which is limited by $\mathscr{V}$ is (invertibly ambient) isotopic to the identity by an isotopy which is limited by $\mathscr{U}$.*

Proof. For a real number (positive) r and a point x in a metric space, $N(x, r)$ will denote the open ball centered at x with radius r.

Let λ be a Lebesgue number of $\mathscr{U}$ with respect to K, let $\mathscr{V}_1 = \{N(x, \lambda/3^6) \mid x \in K\}$, and, inductively, for $n > 1$, let

$$\mathscr{V}_n = \{N(x, \lambda/3^{n+5}) \mid x \in \mathscr{V}_{n-1}^*\} .$$

Now, let $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}_n$. If f embeds K in X and is limited by $\mathscr{V}$, let Y be the closed linear span in X of the image of $F: K \times [0, 1] \to X$ defined by $F(x, t) = (1 - t)x + tf(x)$. Let $p_Y: X \to X/Y$ be the canonical projection, and let $q_Y: X/Y \to X$ be a right inverse for p_Y sending 0 to 0. (This is by the Bartle-Graves Theorem. For a proof see [11].) Now, the function $h_f: X/Y \times Y \to X$ defined by $h_f = q_Y p_1 + p_2$ is a homeomorphism, where p_1 and p_2 denote the projections onto the first and second factors, respectively.

From the definition of $\mathscr{V}$, it follows that for each element V of $\mathrm{st}^4(\mathscr{V})$, $V + N(0, \lambda/3)$ is contained in some member of $\mathscr{U}$, where here "+" denotes the set of all sums of pairs of elements, one from the first set and one from the second. Letting W be a neighborhood of the origin in X/Y which q_Y carries into $N(0, \lambda/3)$, one sees that $h_f(W \times (\mathscr{V}^* \cap Y))$ lies in $\mathscr{U}^*$ and, indeed, that $\{h_f(W \times V) \mid V \in \mathrm{st}^4(\mathscr{V} \mid Y)\}$ refines $\mathscr{U}$. (Here, $\mathscr{V} \mid Y = \{V \cap Y \mid V \in \mathscr{V}\}$.)

Select a map $g: X/Y \to [0, 1]$ such that $g^{-1}(0) \supset (X/Y) \backslash W$ and $0 \in g^{-1}(1)$. Since Y is separable and $\mathscr{V}^* \cap Y$ is open in Y, [3] yields an isotopy $G: (\mathscr{V}^* \cap Y) \times [0, 1] \to \mathscr{V}^* \cap Y$ from the identity homeomorphism at $t = 0$ to an extension to $\mathscr{V}^* \cap Y$ of f at $t = 1$ which is limited by $\mathrm{st}^4(\mathscr{V} \mid Y)$. Then $H: X \times [0, 1] \to X$ given by

$$H(x, t) = \begin{cases} h_f(p_Y(x), G(p_2 \circ h_f^{-1}(x), t \cdot g \circ p_Y(x))), & \text{if } x \in h_f(W \times (\mathscr{V}^* \cap Y)) \\ x & , \text{if } x \notin h_f(W \times (\mathscr{V}^* \cap Y)) \end{cases}$$

is the desired isotopy.

Let H be an infinite-dimensional (real) Hilbert space, let E be a complete, orthonormal basis for H, and denote by H_f the collection of all (finite) linear combinations of members of E.

THEOREM 4. *H_f is a strong $\mathscr{C}(H)$-absorption base.*

Proof. Two things must be shown, namely, that H_f admits the structure of a $\mathscr{C}(H)$-complex and that it is strongly $\mathscr{C}(H)$-adsorptive. To see the first, let $\mathscr{K}_0$ be the set of all integral linear combinations of members of E. For $n > 0$, let

$$\mathscr{Q}_n = \left\{Q_n = \left\{\sum_{m=1}^{n} t_m e_m \mid t_m \in [0, 1],\ m = 1, \cdots, n\right\} \middle| e_1, \cdots, e_n \right.$$

$$\left. \text{are } n \text{ distinct elements of } E\right\},$$

and let $\mathscr{A}_n = \{A = Q_n + x \mid Q_n \in \mathscr{Q}_n, x \in \mathscr{A}_0\}$. It is readily seen that $\mathscr{A} = \bigcup_{n=0}^{\infty} \mathscr{A}_n$ is a $\mathscr{C}(H)$-complex with $\mathscr{A}^* = H_f$.

By Lemma 4, in order to demonstrate that H_f is strongly $\mathscr{C}(H)$-absorptive one must only show that for each member K of $\mathscr{C}(H)$, each open cover $\mathscr{U}$ of K, and for each closed subset K' of $K \cap H_f$, there is an embedding f of K in H_f, limited by $\mathscr{U}$, which is the identity on K'. Since K is compact, there exists a Lebesgue number λ for $\mathscr{U}$ with respect to K, so one must only find an embedding f of K in H_f which moves no point as much as λ and is the identity on K'. However, the total boundedness of K and the denseness in H of H_f lead to the existence of a sequence $\{e_i\}_{i \in N}$ in E and a sequence $\{n(i)\}_{i \in N}$ in N such that if p_i is the orthogonal projection of H onto the span of $\{e_j\}_{j=n(i-1)+1}^{n(i)}$, then $\| \sum_{i=1}^{m} p_i(x) - x \| < 2^{-m-2}\lambda$ for each $m \in N$ and $x \in K$. Also, since K is finite-dimensional, for each set S of $2\dim(K) + 2$ distinct elements of E, there is an embedding of K in the unit sphere (=elements of norm one) of the subspace spanned by S. Assume that for each i, $n(i) - n(i-1) \geqq 2\dim(K) + 2$, and let f_i be an embedding of K in the unit sphere of the span of $\{e_j\}_{j=n(i-1)+1}^{n(i)}$. Now, let g map K into $[0, 1]$ such that $K' = g^{-1}(0)$, and for each i let h_i map $[0, 1]$ into $[0, 1]$ such that $h_1^{-1}(0) = [0, 1/n(3)]$ and $h_1^{-1}(1) = [1/n(2), 1]$ and for $i > 1$,

$$h_i^{-1}(0) = [1/n(i-1), 1] \cup [0, 1/n(i+2)]$$

and $h_i^{-1}(1) = [1/n(i+1), 1/n(i)]$. Finally, set

$$f(x) = \sum_{i \in N} (\max_{j \geqq i} \{h_j \circ g(x)\}) p_i(x) + \sum_{i \in N} 2^{-i-1}\lambda \cdot h_i \circ g(x) f_{i+3}(x) .$$

This is the desired embedding.

COROLLARY 2. *If $\mathscr{U}$ is any collection of open sets of H and Y is any $\mathscr{C}(\mathscr{U}^*)$-absorption base in $\mathscr{U}^*$, then there is an ambient, invertible isotopy of H onto itself which is limited by $\mathscr{U}$, is the identity at $t = 0$, and at $t = 1$ is a homeomorphism h_1 such that $h_1(Y) = \mathscr{U}^* \cap H_f$.*

Proof. Lemma 4 shows the equivalence of the concepts of $\mathscr{C}(\mathscr{U}^*)$-absorption base and strong $\mathscr{C}(\mathscr{U}^*)$-absorption base, Theorem 4 combined with Theorem 2 gives that $\mathscr{U}^* \cap H_f$ is also a strong $\mathscr{C}(\mathscr{U}^*)$-absorption base, and Theorem 1 supplies the isotopy on $\mathscr{U}^*$ limited by an open cover given by Lemma 2 which refines $\mathscr{U}$ and has the property that any isotopy limited by it may be extended trivially to one on H.

COROLLARY 3. *Let $\{H_n\}_{n \in N}$ be an indexed, countably infinite collection of copies of H, and let Y be the subspace of $\prod_{n \in N} H_n$ consisting of all points with at most finitely many nonzero coordinates, each of which lies in the appropriate copy of H_f. Then Y is homeomorphic to H_f.*

Proof. It is easy to modify the proof of Theorem 4 to show that Y is $\mathscr{C}(\prod_{n \in N} H_n)$-absorptive. If the copy of H_f in H_n is denoted by $(H_f)_n$, then $Y = \prod_{n \in N}((H_f)_n, 0)$, so Corollary 1 applies to show that Y is a $\mathscr{C}(\prod_{n \in N} H_n)$-absorption base. However, $\prod_{n \in N} H_n$ is homeomorphic to H by a theorem of Bessaga and Pełczynski [4], so by the remark following Corollary 1, Y may be embedded in H as a $\mathscr{C}(H)$-absorption base. Corollary 2 now applies to finish the proof.

The above result is crucial to [10]. The next two results identify some simplicial complexes whose products with H_f are H_f-manifolds.

THEOREM 5. *If K is a metric simplicial complex and $K \times H$ is an H-manifold, then $K \times H_f$ is an H_f-manifold.*

Proof. By Theorem 3 (the remark after Theorem 3), $K \times H_f$ is a $\mathscr{C}(K \times H)$-complex, since K is by definition a $\mathscr{C}(K)$-complex. The strategy of the proof is to show that $K \times H_f$ is a $\mathscr{C}(K \times H)$-absorption base, to embed $K \times H$ component-wise in H as open subsets (using a theorem of Henderson [8]) and then to use Corollary 2 to find a homeomorphism of the open subsets in question onto themselves throwing the images of $K \times H_f$ onto $H_f \cap$ (the open subsets). Thus, all that is necessary is to establish the $\mathscr{C}(K \times H)$-absorptivity of $K \times H_f$. In fact, since for each vertex v of K, $\mathrm{st}^0(v, K)$— the open star of v in K— is a contractible open set, $\mathrm{st}^0(v, K) \times H$ will be homeomorphic to H by [9], so all that is needed is to show that $\mathrm{st}^0(v, K) \times H_f$ is $\mathscr{C}(\mathrm{st}^0(v, K) \times H)$-absorptive. Therefore, let X be a finite-dimensional compactum of $\mathrm{st}^0(v, K) \times H$, let $\mathscr{U}$ be an open cover of X in $\mathrm{st}^0(v, K) \times H$ and let X' be a closed subset of $X \cap (\mathrm{st}^0(v, K) \times H_f)$. Lemma 4 together with the fact that $\mathrm{st}^0(v, K) \times H$ is homeomorphic to H establishes that it is sufficient to find an embedding of X in $\mathrm{st}^0(v, K) \times H_f$ which is limited by $\mathscr{U}$, and is the identity on X'. Let λ be a Lebesgue number for $\mathscr{U}$ with respect to X, and let p_H denote the projection of $K \times H$ onto H. As noted in the proof of Theorem 4, there exists a sequence $\{e_i\}_{i \in N}$ in E and another sequence $\{n(i)\}_{i \in N}$ in N such that $n(i) - n(i-1) \geqq 2\dim(X) + 2$ for each i and $\| \sum_{i=1}^{m} p_i \circ p_H(x) - p_H(x) \| < 2^{-m-2}\lambda$ for each $m \in N$ and $x \in X$, the rest of the notation being as in the proof of Theorem 4. Constructing $f_0\colon X \rightarrow H_f$ by the same method as used in Theorem 4, except for the substitution of $p_i \circ p_H$ for p_i, and setting $f = (p_K, f_0)$ produces the desired embedding, if p_K denotes the projection of $K \times H$ onto K.

COROLLARY 4. *If K is a metric, locally finite-dimensional, simplicial complex such that no vertex-star contains more vertices than* $\dim(H)$, *then $K \times H_f$ is an H_f-manifold.*

Proof. By Theorem 4 of [12], $K \times H$ is an H-manifold, so Theorem 5 applies. (This metric is assumed that in the abstract.)

Actually, if a pair (X, Y) of spaces, $Y \subset X$, is called a (H, H_f)-*manifold pair* provided that X is a paracompact H-manifold and there is an open cover $\mathscr{U}$ of X by sets U for which there are open embeddings $f_U\colon U \to H$ such that $f_U(U \cup Y) = f_U(U) \cap H_f$, then the following have been established.

THEOREM 6. *The pair (X, Y) is a (H, H_f)-manifold pair if and only if Y is a $\mathscr{C}(X)$-complex, X is an H-manifold, and the following weak $\mathscr{C}(X)$-absorptivity condition is satisfied: For each finite-dimensional compactum C of X, each open cover $\mathscr{U}$ of C, and each compact subset C' of $C \cap Y$, there is an embedding of C in Y which is limited by $\mathscr{U}$ and extends the inclusion of C'. If (X, Z) is another (H, H_f)-manifold pair and $\mathscr{V}$ is an open cover of X, then there is an isotopy of X, limited by $\mathscr{V}$, from the identity to a pair homeomorphism of (X, Y) onto (X, Z).*

COROLLARY 5. *If (X, Y) and (X', Y') are (H, H_f)-manifold pairs, then $(X \times X', Y \times Y')$ is an (H, H_f)-manifold pair.*

COROLLARY 6. *If (X, Y) is an (H, H_f)-manifold pair and K is a metric, locally finite-dimensional, simplicial complex such that no vertex-star contains more than* $\dim(H)$ *vertices, then $(X \times K, Y \times K)$ is an (H, H_f)-manifold pair.*

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ADJOINT PRODUCT AND HOM FUNCTORS IN GENERAL TOPOLOGY

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The well known natural equivalence $[R \times S, T] \cong [R, T^S]$, valid in the category of sets and set mappings, can be derived in various ways in the category of topological spaces and continuous maps, provided suitable topologies are introduced on the product set $R \times S$ and on the set of all continuous maps from S to T. In this paper we will show how to construct topologies of this kind. The ordinary product topology on $R \times S$ and the compact-open topology on T^S will be given their proper setting in this context.

Given the category of topological spaces and continuous maps, we shall write Con (A, B) for the set of morphisms from space A to space B, $A \times B$ for the product of the carrier sets of A and B. If R, S, T are three topological spaces, suppose topologies have been fixed on $R \times S$ and on Con (S, T). $[R \times S, T]$ and $[R, \text{Con}\,(S, T)]$ will denote the sets of continuous maps from $R \times S$ to T and from R to Con (S, T), respectively. (No topologies introduced on these sets.)

As in the category of sets and set mappings, there is a naturally given function Φ from $[R \times S, T]$ to $[R, \text{Con}\,(S, T)]$ defined by

$$f \in [R \times S, T]; (\Phi f)(r/s) = f(r, s) \qquad (r \in R, s \in S) .$$

Its inverse is the function Ψ from $[R, \text{Con}(S, T)]$ to $[R \times S, T]$, given by

$$g \in [R, \text{Con}\,(S, T)]; (\Psi g)(r, s) = g(r/s) \qquad (r \in R, s \in S) .$$

The problem to be investigated in this paper is the following: what topologies on $R \times S$ and on Con (S, T) will make the couple (Φ, Ψ) a natural equivalence, i.e., will make the functors $- \times S$ and Con $(S, -)$ adjoint functors?

The best known example is probably the use of the product topology on $R \times S$ and of the compact-open topology on Con (S, T), restricting S to locally compact Hausdorff spaces. Starting from this standard situation, two lines of attack on the general problem have been opened in the literature. One can start with the product topology on $R \times S$ and look for conditions on Con (S, T), or else one starts at the other end by using the compact-open topology for Con (S, T), looking for suitable topologies on $R \times S$. (See [1], [2], [3]; the authors do not use categorical language and their aims are somewhat different from ours).

In this paper we shall use a different approach. For the space Con (S, T) a class K of topologies is chosen generalizing the class of set-open topologies of [1]. Requirements for Con (S, T) and for $R \times S$ to be functors and for (Φ, Ψ) to be a natural equivalence will on the one hand reduce K to a subclass of admissible topologies on Con (S, T) and will on the other hand force $R \times S$ to carry topologies uniquely determined by the topologies on Con (S, T). (The word "admissible" is used in its ordinary sense, not as in [1]).

By a suitable choice of topologies a natural equivalence $[R \times S, T] \cong [R, \mathrm{Con}(S, T)]$ can always be established in more than one way, irrespective of the nature of the spaces R, S, T. There is even a minimal and a maximal nontrivial solution to this problem (given the class K of topologies on Con (S, T)). This will be the content of §'s 2, 3 and 4.

The remaining part of the paper is concerned with the role of the compact-open topology on Con (S, T) and of the product topology on $R \times S$ in this context. One of the admissible topologies on Con (S, T) is determined by the compact sets of S; it turns out to be equal to the compact-open topology only in case S satisfies a condition which does not seem to be easily reducible to familiar properties of topological spaces, but holds for well-known classes of spaces, e.g., Hausdorff spaces. Given an admissible topology on Con (S, T) and the corresponding one on $R \times S$, if the latter is required to be the product topology the space S has to satisfy a local condition related to local compactness.

2. Topologies on Con (S, T) and on $R \times S$. Notation: R, S, T will always denote topological spaces. Given S, the letter $\mathfrak{S}$ will be used to describe the space of open sets of S as well as the corresponding lattice. $\mathfrak{F}$, $\mathfrak{G}$, $\cdots$ will be used for filters on $\mathfrak{S}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{M}$, $\mathscr{K}$, $\cdots$ for families of such filters. If K is any subset of S, $\mathfrak{F}(K)$ will denote the filter of all open sets of S containing K; $\mathfrak{S} = \mathfrak{F}(\varnothing)$ counts as a filter. Finally, Z will stand for a Sierpinski space, i.e., a space consisting of two points z_1, z_2 and with $\varnothing$, $\{z_1\}$, $\{z_1, z_2\}$ as its open sets.

Let $\mathscr{F}$ be a family of filters on $\mathfrak{S}$ containing the filter $\mathfrak{S}$ itself. For any $\mathfrak{F} \in \mathscr{F}$ and any open set U of a space T we define

$$\langle \mathfrak{F}, U \rangle = \{f \in \mathrm{Con}(S, T) : f^{-1}U \in \mathfrak{F}\} .$$

By requiring the family of all these sets to be an open subbasis one introduces a topology $\tau(\mathscr{F})$ on Con (S, T). If all filters of $\mathscr{F}$ are of the form $\mathfrak{F}(K)$, $\tau(\mathfrak{F})$ will be a set-open topology in the sense of [1].

Let $\mathfrak{F}, \mathfrak{G} \in \mathscr{F}$; obviously, $\langle \mathfrak{F}, U \rangle \cap \langle \mathfrak{G}, U \rangle = \langle \mathfrak{F} \cap \mathfrak{G}, U \rangle$, where $\mathfrak{F} \cap \mathfrak{G}$ is the intersection (meet) of the filters $\mathfrak{F}$ and $\mathfrak{G}$. Hence there is no loss of generality by assuming $\mathscr{F}$ to contain all finite intersections of its members; this will be tacitely understood in the sequel.

Next, look at each $\mathfrak{F} \in \mathscr{F}$ as a subset of the space $\mathfrak{S}$ and introduce a topology on $\mathfrak{S}$ by requiring $\mathscr{F}$ to be an open basis.

To determine a topology on $R \times S$ we introduce a special notation. Let $A \subset R \times S$; for each $r \in R$, let ϕr denote the set of all $s \in S$ such that $(r, s) \in A$. The function ϕ from R to the power set of S completely describes A and we shall write $A = [\phi]$.

Suppose topologies have been defined on $\mathfrak{S}$ and on Con (S, T) by means of a family $\mathscr{F}$ of filters on $\mathfrak{S}$, while $R \times S$ carries a topology yet to be specified. Consider the transformations Φ and Ψ introduced in §1. For Φ and Ψ to be natural transformations, the following conditions are obviously necessary: given $f \in [R \times S,\ T]$, $g \in [R, \text{Con}\,(S, T)]$, $r \in R$, the mappings $\Phi f\colon R \to \text{Con}\,(S,\ T)$, $(\Phi f)(r/-)\colon S \to T$ and $\Psi g\colon R \times S \to T$ must be continuous. We shall say that Φ and Ψ must *preserve continuity.*

THEOREM 1. *Φ and Ψ preserve continuity if and only if the following holds: a subset $[\phi]$ of $R \times S$ is open in the chosen topology if and only if ϕ is a continuous map from R to $\mathfrak{S}$.*

Proof. Assume the conditions on the open sets of $R \times S$. Let U be open in T, $f \in [R \times S,\ T]$, $r \in R$. Define ϕ by $[\phi] = f^{-1}U$. It is easy to see that for any $\mathfrak{F} \in \mathscr{F}$, $(\Phi f)^{-1}\langle \mathfrak{F},\ U\rangle = \phi^{-1}\mathfrak{F}$ and $((\Phi f)(r/-))^{-1}U = \phi r$. Φ preserves continuity if and only if $\phi^{-1}\mathfrak{F}$ is open in R and ϕr is open in S, which is just our assumption. Similarly, for $g \in [R, \text{Con}\,(S,\ T)]$ define ϕ' by $[\phi'] = (\Psi g)^{-1}U$. Ψ preserves continuity if and only if $[\phi']$ is open in $R \times S$. But $\phi' r = (g(r/-))^{-1}U$ and $\phi'^{-1}\mathfrak{F} = g^{-1}\langle \mathfrak{F},\ U\rangle$. Thus, ϕ' is a continuous map from R to $\mathfrak{S}$ and $[\phi']$ is open by assumption.

Suppose Φ, Ψ preserve continuity and let $T = Z$. If $[\phi]$ is any open set of $R \times S$, construct $f\colon R \times S \to Z$ by putting $f(r, s) = z_1$ for $(r, s) \in [\phi]$, $f(r, s) = z_2$ for $(r, s) \notin [\phi]$. Obviously, $f \in [R \times S,\ Z]$. Choosing $\{z_1\}$ as the set U and repeating the argument used above we see that ϕ is a continuous map from R to $\mathfrak{S}$. Conversely, let $\phi\colon R \to \mathfrak{S}$ be continuous. Construct $g\colon R \to \text{Con}\,(S,\ Z)$ by $g(r/s) = z_1$ for $s \in \phi r$, $g(r/s) = z_2$ for $s \notin \phi r$. We must show $g \in [R, \text{Con}\,(S,\ Z)]$. Since $(g(r/-))^{-1}\{z_1\} = \phi r$, we have $g(r/-) \in \text{Con}(S,\ T)$. For any $\mathfrak{F} \in \mathscr{F}$ and U open in Z, $g^{-1}\langle \mathfrak{F},\ U\rangle$ can only be equal to $\varnothing$, R or $\phi^{-1}\mathfrak{F}$. Hence g is continuous. Repeating the argument used to show that Ψ preserves continuity we see that $[\phi]$ is open in $R \times S$.

Theorem 1 restricts the system of filters $\mathscr{F}$ which may be used to define a topology on Con $(S,\ T)$, because the family of subsets $[\phi]$ of $R \times S$ described by Theorem 1 must satisfy the axioms of a topology. Actually, three of these axioms hold for any family $\mathscr{F}$. The sets $\varnothing$ and $R \times S$ are open in $R \times S$, because they correspond to the constant functions from R to $\mathfrak{S}$ with $\varnothing$ and S as values, respectively.

Let $\phi_1, \phi_2: R \to \mathfrak{S}$ be two continuous functions and define $\phi_1 \cap \phi_2$ by $(\phi_1 \cap \phi_2)r = \phi_1 r \cap \phi_2 r$, the intersection on the right to be taken in S. Obviously, $[\phi_1] \cap [\phi_2] = [\phi_1 \cap \phi_2]$, hence $\phi_1 \cap \phi_2$ must be continuous. But this is always the case, since for any filter $\mathfrak{F}$ of $\mathscr{F}$ we have $(\phi_1 \cap \phi_2)^{-1}\mathfrak{F} = \{r \in R: \phi_1 r \cap \phi_2 r \in \mathfrak{F}\} = \{r \in R: \phi_1 r \in \mathfrak{F}$ and $\phi_2 r \in \mathfrak{F}\} = \phi_1^{-1}\mathfrak{F} \cap \phi_2^{-1}\mathfrak{F}$, which is open in $R \times S$. There remains the union axiom. For any family $\phi_i: R \to \mathfrak{S}(i \in I)$ of continuous maps define $\cup \phi_i$ by $(\cup \phi_i)r = \cup \phi_i r(i \in I)$. Because $\cup [\phi_i] = [\cup \phi_i]$, $\cup \phi_i$ must be continuous. We investigate this requirement.

A filter $\mathfrak{F}$ of $\mathfrak{S}$ will be called *compact*, if for any system $A_i(i \in I)$ of open subsets of S, whenever $\cup A_i \in \mathfrak{F}(i \in I)$, there is a finite subset $K \subset I$ such that $\cup A_i \in \mathfrak{F}(i \in K)$. If $\mathfrak{F}$ and $\mathfrak{G}$ are compact, so is $\mathfrak{F} \cap \mathfrak{G}$. If $\mathfrak{F}$ is of the form $\mathfrak{F}(K)$, it is a compact filter if and only if K is a compact subset of S. Filters of this kind will be called *compactly generated*.

A family of filters $\mathscr{F}$ may satisfy the following condition:

(A) For any two open subsets A_1, A_2 of S and for any $\mathfrak{F} \in \mathscr{F}$, if $A_1 \cup A_2 \in \mathfrak{F}$, there are filters $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathscr{F}$ such that $A_1 \in \mathfrak{G}_1$, $A_2 \in \mathfrak{G}_2$, $\mathfrak{G}_1 \cap \mathfrak{G}_2 \subset \mathfrak{F}$.

THEOREM 2. *$\cup \phi_i(i \in I)$ is a continuous map from R to $\mathfrak{S}$, for any R, if and only if $\mathscr{F}$ is a system of compact filters satisfying* (A).

Proof. Let $\phi_i: R \to \mathfrak{S}(i \in I)$ be a family of continuous maps. For $\mathfrak{F} \in \mathscr{F}$, define the set $F = (\cup \phi_i)^{-1}\mathfrak{F} = \{r \in R: \cup (\phi_i r) \in \mathfrak{F}(i \in I)\}$. F must be shown to be open in R.

Assume $\mathfrak{F}$ compact; for each $r \in F$ there will be a finite subset $K_r \subset I$ such that $\cup (\phi_i r) \in \mathfrak{F}(i \in K_r)$. Write $F_r = \{s \in R: \cup (\phi_i s) \in \mathfrak{F}(i \in K_r)\}$. By definition, $F \subset \cup F_r(r \in F)$. On the other hand, for any $s \in F_r$ we have $\cup (\phi_i s) \subset \cup (\phi_j s)(i \in K_r, j \in I)$ and because $\mathfrak{F}$ is a filter, $\cup (\phi_j s) \in \mathfrak{F}$, i.e., $s \in F$. Hence $F = \cup F_r$ and to show F open it will be sufficient to consider only finite families of continuous maps ϕ_i. As the general finite case follows by induction, two functions ϕ_1, ϕ_2 will suffice.

$\phi_1 \cup \phi_2$ is continuous if and only if the set $G = (\phi_1 \cup \phi_2)^{-1}\mathfrak{F} = \{r \in R: \phi_1 r \cup \phi_2 r \in \mathfrak{F}\}$ is open in R for any $\mathfrak{F} \in \mathscr{F}$. Assume $\mathscr{F}$ to satisfy condition (A). To any $r \in G$ there correspond filters $\mathfrak{G}_r, \mathfrak{H}_r \in \mathscr{F}$ such that $\phi_1 r \in \mathfrak{G}_r$, $\phi_2 r \in \mathfrak{H}_r$, $\mathfrak{G}_r \cap \mathfrak{H}_r \subset \mathfrak{F}$. We claim $G = \cup (\phi_1^{-1}\mathfrak{G}_r \cap \phi_2^{-1}\mathfrak{H}_r)(r \in G)$. By definition, G is contained in the right-hand set. Let s be an element of $\phi_1^{-1}\mathfrak{G}_r \cap \phi_2^{-1}\mathfrak{H}_r$, for some $r \in R$. Then

$$\phi_1 s \in \mathfrak{G}_r,\ \phi_2 s \in \mathfrak{H}_r,\ \phi_1 s \cup \phi_2 s \in \mathfrak{H}_r \cap \mathfrak{G}_r \subset \mathfrak{F}\ ,$$

hence $s \in G$. This proves equality and shows G to be open.

For the converse, let $\mathfrak{F} \in \mathscr{F}$ and let $A_i(i \in I)$ be a family of open sets of S such that $\cup A_i \in \mathfrak{F}(i \in I)$. Consider the space $R = \mathfrak{S}^I$, equipped with the product topology and write $\pi_i \colon R \to \mathfrak{S}(i \in I)$ for the canonical projections. An open basis of R consists of finite intersections of sets $\pi_i^{-1}\mathfrak{G}$, where $\mathfrak{G} \in \mathscr{F}$. The functions π_i are continuous. By hypothesis, the same is true for $\cup \pi_i(i \in I)$. Thus the set $H = \{u \in R \colon \cup(\pi_i u) \in \mathfrak{F} \ (i \in I)\}$ is open in R and each point of H belongs to a basic open set contained in H.

Define $a \in R$ by $\pi_i a = A_i$ for all $i \in I$. Then $a \in H$ and there are a finite set $J \subset I$ as well as filters $\mathfrak{H}_j \in \mathscr{F}$ such that $a \in \cap \pi_j^{-1}\mathfrak{H}_j \subset H$ $(j \in J)$. This implies $\pi_j a \in \mathfrak{H}_j$ for all $j \in J$. Define $b \in R$ by $\pi_j b = A_j$ $(j \in J)$, $\pi_i b = \varnothing(i \in I - J)$. For $j \in J$, $\pi_j b = \pi_j a$, thus $b \in \cap \pi_j^{-1}\mathfrak{H}_j$. Since $\cap \pi_j^{-1}\mathfrak{H}_j$ is a subset of H, $b \in H$ and $\cup \pi_i b = \cup A_j \in \mathfrak{F}(i \in I, j \in J)$. This shows $\mathfrak{F}$ to be compact.

Finally, let I be the set $\{1, 2\}$, otherwise using the same notation as above. The element $a \in R = \mathfrak{S} \times \mathfrak{S}$, defined by $\pi_1 a = A_1$, $\pi_2 a = A_2$ will again satisfy $a \in \pi_1^{-1}\mathfrak{H}_1 \cap \pi_2^{-1}\mathfrak{H}_2 \subset H$ for two appropriately chosen filters $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathscr{F}$; thus $A_1 \in \mathfrak{H}_1$, $A_2 \in \mathfrak{H}_2$. Let the set C be a member of $\mathfrak{H}_1 \cap \mathfrak{H}_2$ and consider the element $c \in R$, defined by $\pi_1 c = \pi_2 c = C$. Because $c \in \pi_1^{-1}\mathfrak{H}_1 \cap \pi_2^{-1}\mathfrak{H}_2 \subset H$, we have $\pi_1 c \cup \pi_2 c = C \in \mathfrak{F}$. Since this holds for any such C, $\mathfrak{H}_1 \cap \mathfrak{H}_2 \subset \mathfrak{F}$ and $\mathscr{F}$ satisfies condition (A).

A system of compact filters satisfying (A) will be called an *adjoining system*. Suppose all filters of a system $\mathscr{F}$ are compactly generated. Our next theorem describes condition (A) in this case.

For any subset X of S define X^* to be the intersection of all open sets containing X. Obviously, $(X_1 \cup X_2)^* = X_1^* \cup X_2^*$; if K is compact in S, the same is true for K^*, because any open cover of K^* is also an open cover of K, hence contains a finite subcover which must be above K^*.

We shall call a compact set K for which $K = K^*$, *fully compact*. All compact sets of a topological space are fully compact if and only if the space is T_1.

Let $\mathfrak{F}(K)$ be a compactly generated filter; it is an immediate consequence of our definition that $\mathfrak{F}(K) = \mathfrak{F}(K^*)$. Moreover, $\mathfrak{F}(K_1^*) \cap \mathfrak{F}(K_2^*) = \mathfrak{F}((K_1 \cup K_2)^*)$ and $\mathfrak{S} = \mathfrak{F}(\varnothing) = \mathfrak{F}(\varnothing^*)$. One may therefore assume that the compact sets generating the filters of the given family $\mathscr{F}$ are all fully compact.

We need one more fact, easily seen to hold: if K and L are compact sets in S, then $\mathfrak{F}(K) \subset \mathfrak{F}(L)$ if and only if $L^* \subset K^*$.

THEOREM 3. *Let $\mathscr{F}$ be a family of compactly generated filters on $\mathfrak{S}$, and let $\mathfrak{K}$ denote the family of fully compact sets of S which generate the filters of $\mathscr{F}$. Then $\mathscr{F}$ satisfies condition (A) if and only if the following holds: if A_1, A_2 are open sets of S and if $K \in \mathfrak{K}$ is such that $K \subset A_1 \cup A_2$, there are $K_1, K_2 \in \mathfrak{K}$ with $K_1 \subset A_1$, $K_2 \subset A_2$,*

$K \subset K_1 \cup K_2$.

Theorem 3 is an immediate consequence of the facts established above.

3. Functorial requirements. Given an adjoining system $\mathscr{F}$ on $\mathfrak{S}$, Theorems 1 and 2 show that $\mathscr{F}$ uniquely determines topologies on Con (S, T) as well as on $R \times S$ such that Φ and Ψ preserve continuity. (Φ, Ψ) determine a natural equivalence between $[R \times S, T]$ and $[R, \mathrm{Con}\,(S, T)]$ if, given the space S, Con (S, T) and $R \times S$ are (object mappings of) functors in T and R, respectively. This will now be shown.

Let T, T' be topological spaces, $p\colon T \to T'$ a continuous map. As usual, the morphism mapping $C(p)\colon \mathrm{Con}\,(S, T) \to \mathrm{Con}\,(S, T')$ of the functor $\mathrm{Con}\,(S, -)$ will be defined by $C(p)f = pf$, for any $f \in \mathrm{Con}\,(S, T)$. It must be shown that $C(p)$ is continuous with respect to the topologies defined on Con (S, T) and on Con (S, T'). For U' open in T' and for $\mathfrak{F} \in \mathscr{F}$ we have $C(p)^{-1}\langle \mathfrak{F}, U' \rangle = < \mathfrak{F}, p^{-1}U' >$, which implies continuity of $C(p)$.

Similarly, let R, R', $q\colon R \to R'$ be given, q continuous. The morphism mapping $P(q)\colon R \times S \to R' \times S$ of the functor $- \times S$ is defined by $P(q)(r, s) = (qr, s)$, for any $(r, s) \in R \times S$. We show that $P(q)$ is continuous with respect to the topologies chosen for $R \times S$ and $R' \times S$. Let $[\phi']$ be open in $R' \times S$ and define $\phi\colon R \to S$ by $\phi = \phi' q$. It is easy to see that $P(q)^{-1}[\phi'] = [\phi]$, which is open in $R \times S$ by Theorem 2.

The variety of adjoining systems for a given space S depends of course on the nature of this space. To obtain a categorically significant simultaneous choice of adjoining systems $\mathscr{F}(S)$ for each space S we now require Con (S, T) and $R \times S$ to be functors in S as well. To investigate this requirement we have to define two operations on systems $\mathscr{F}$ and list their properties.

Let $\mathscr{F}$ be a family of filters on $\mathfrak{S}$ and let $\mathfrak{F}_i \in \mathscr{F}$, $i \in I$. In general, $\cup \mathfrak{F}_i (i \in I)$ will not be a filter. We want to adjoin to $\mathscr{F}$ all unions of its members which are themselves filters. Hence we define $\overline{\mathscr{F}}$ to be the family of all filters on $\mathfrak{S}$ which can be written as unions of filters belonging to $\mathscr{F}$. If all filters of $\mathscr{F}$ are compact, the same holds for $\overline{\mathscr{F}}$, and if $\mathscr{F}$ satisfies condition (A) of §2, so does $\overline{\mathscr{F}}$. Consequently, if $\mathscr{F}$ is adjoining, then $\overline{\mathscr{F}}$ is also adjoining. $\overline{\mathscr{F}}$ can actually be larger than $\mathscr{F}$. The following example will incidentally prove the existence of compact, not compactly generated filters.

Let the space S consist of the set of natural numbers N together with an element $w \notin N$. Write $S_0 = N$, $S_n = N - \{0, 1, 2, \cdots, n-1\}$ $(n \geqq 1)$. Open sets of S shall be the sets S_n, $\{w\}$, $S_n \cup \{w\}$ and the empty set. Write $\mathfrak{F}(n)$ for the filter generated by the compact set $\{n\}$ on $\mathfrak{S}(n \in N)$. Then $\mathfrak{F} = \cup \mathfrak{F}(n)(n \in N)$ is a filter, as is easily seen.

If it were compactly generated by a fully compact set $K \subset S$, the intersection of all its members would be K. But this intersection is the empty set $\varnothing$, while $\mathfrak{F} \neq \mathfrak{F}(\varnothing) = \mathfrak{S}$, since $\{w\} \notin \mathfrak{F}$. Hence for $\mathscr{F} = \{\mathfrak{F}(n): n \in N\}$, $\mathscr{F}$ is properly contained in $\overline{\mathscr{F}}$.

THEOREM 4. *Let $\mathscr{F}, \mathscr{G}$ be two (not necessarily adjoining) families of filters on a space $\mathfrak{S}$, and let $\tau(\mathscr{F}, T)$, $\tau(\mathscr{G}, T)$ be the corresponding topologies on the space* Con (S, T). *Writing $\sigma \leqq \tau$, if the topology τ is finer than the topology σ, we have $\tau(\mathscr{G}, T) \leqq \tau(\mathscr{F}, T)$ for all T if and only if $\mathscr{G} \subset \overline{\mathscr{F}}$.*

COROLLARY. *$\tau(\mathscr{F}, T) = \tau(\overline{\mathscr{F}}, T)$ for all T.*

Proof. Let $T = Z$, a Sierpinski space, and let $\mathfrak{G} \in \mathscr{G}$. The set $\langle \mathfrak{G}, \{z_1\} \rangle$ is not empty, since the constant function $f: S \rightarrow Z$ with value z_1 belongs to it. On the other hand $\langle \mathfrak{G}, \{z_1\} \rangle = \mathrm{Con}\,(S, Z)$ if and only if $\mathfrak{G} = \mathfrak{S}$. Suppose $\mathfrak{G} \neq \mathfrak{S}$ and assume $\tau(\mathscr{G}, Z) \leqq \tau(\mathscr{F}, Z)$. Then $\langle \mathfrak{G}, \{z_1\} \rangle = \cup \langle \mathfrak{F}_i, \{z_1\} \rangle (i \in I)$, where $\mathfrak{F}_i \in \overline{\mathscr{F}}$. We want to show $\mathfrak{G} = \cup \mathfrak{F}_i$. For $A \in \mathfrak{G}$, define $f \in \mathrm{Con}\,(S, Z)$ by $fa = z_1 (a \in A)$, $fa = z_2 (a \in S - A)$. Because $f \in \langle \mathfrak{G}, \{z_1\} \rangle$, there is an $i \in I$ such that $f \in \langle \mathfrak{F}_i, \{z_1\} \rangle$, which implies $A \in \mathfrak{F}_i$ and $\mathfrak{G} \subset \cup \mathfrak{F}_i$. By a similar argument, $\cup \mathfrak{F}_i \subset \mathfrak{G}$. This proves $\mathscr{G} \subset \overline{\mathscr{F}}$.

The converse is an easy consequence of the definitions. The corollary is implied by the fact that $\mathscr{F}$ is a basis for the topology $\tau(\mathscr{F}, T)$.

Let S, S' be two topological spaces and let $q: S \rightarrow S'$ be continuous. For any filter $\mathfrak{F}$ on $\mathfrak{S}$ define the subset $q'\mathfrak{F}$ of $\mathfrak{S}'$ by $q'\mathfrak{F} = \{A' \in \mathfrak{S}': q^{-1}A' \in \mathfrak{F}\}$. $q'\mathfrak{F}$ is a filter on $\mathfrak{S}'$, and if $\mathfrak{F}$ is compact, so is $q'\mathfrak{F}$; this follows easily from the algebraic properties of q^{-1}. If $\mathfrak{F} = \mathfrak{F}(K)$, then $q'\mathfrak{F}(K) = \mathfrak{F}(qK)$; if, therefore, $\mathfrak{F}$ is compactly generated, so is $q'\mathfrak{F}$.

Given a family $\mathscr{F}$ of filters on $\mathfrak{S}$, let us write $q'\mathscr{F}$ for the family of filters $q'\mathfrak{F}$, $\mathfrak{F} \in \mathscr{F}$. If $\mathscr{F}$ satisfies condition (A) of §2, the same is true for $q'\mathscr{F}$. This again is an immediate consequence of the definitions. Together with the facts noted above we have: if $\mathscr{F}$ is an adjoining system, then $q'\mathscr{F}$ is also adjoining.

Returning to the investigation of the functorial requirements, let T be a fixed topological space and consider Con (S, T) as the object mapping of the (contravariant) functor Con $(-, T)$. For a continuous $q: S \rightarrow S'$, define the morphism mapping $D(q): \mathrm{Con}\,(S', T) \rightarrow \mathrm{Con}\,(S, T)$ by $D(q)f' = f'q$ $(f' \in \mathrm{Con}\,(S', T))$. The topologies on the two spaces Con (S, T) and Con (S', T) are assumed to be given by adjoining systems $\mathscr{F}$ and $\mathscr{F}'$, respectively. $D(q)$ must be continuous, i.e., for each $\mathfrak{F} \in \mathscr{F}$ and for each open subset $U \subset T$, the set $D(q)^{-1}\langle \mathfrak{F}, U \rangle$ has to be open in Con (S', T). It is easy to see that $D(q)^{-1}\langle \mathfrak{F}, U \rangle = \langle q'\mathfrak{F}, U \rangle$, which is a basic open set of the topology defined by $q'\mathscr{F}$ on Con (S', T). Thus, this topology must be coarser than the given one. Conversely, if this is true, $D(q)$ will be continuous. Theorem 4 now implies

THEOREM 5. *If for each space* Con (S, T), T *fixed, a topology is defined by the choice of an adjoining system* $\mathscr{F}(S)$ *on the corresponding space* S, *then* Con $(-, T)$ *together with the mapping* D *is a functor if and only if for each continuous map* $q: S \to S'$ *the relation* $q'\mathscr{F}(S) \subset \overline{\mathscr{F}(S')}$ *holds.*

We shall describe this relation briefly by "$\mathscr{F}(S)$ is functorial".

If adjoining systems have been chosen according to Theorem 5, then $R \times S$ with the topology induced by $\mathscr{F}(S)$ is also, for fixed R, (the object mapping of) a functor. For any continuous $q: S \to S'$ define the morphism mapping $Q(q): R \times S \to R \times S'$ by $Q(q)(r, s) = (r, qs)$ $((r, s) \in R \times S)$. $Q(q)$ is continuous. To see this, let $[\phi']$ be open in $R \times S'$, where $\phi': R \to \mathfrak{S}'$. The inverse q^{-1} of q induces a mapping $q^{-1}: \mathfrak{S}' \to \mathfrak{S}$; define $\phi: R \to \mathfrak{S}$ by $\phi = q^{-1}\phi'$. Then $Q(q)^{-1}[\phi'] = [\phi]$, as is easy to see. For any $\mathfrak{F} \in \mathscr{F}(S)$, we have $\phi^{-1}\mathfrak{F} = \phi'^{-1}(q'\mathfrak{F})$. By assumption, $q'\mathfrak{F}$ is open in the space $\mathfrak{S}'$ with respect to the topology given by $\mathscr{F}(S')$. This shows $\phi^{-1}\mathfrak{F}$ to be open in $\mathfrak{S}$, ϕ to be continuous, $[\phi]$ to be open in $R \times S$ and finally $Q(q)$ to be continuous.

4. Minimal and maximal adjoining systems. Let the variable E range over the finite subsets of a topological space S. By $\mathscr{E}$ we shall denote the family of all filters $\mathfrak{F}(E)$. $\mathscr{E}$ is adjoining: every $\mathfrak{F}(E)$ is compactly generated, and if A, B are open sets in S with $A \cup B \in \mathfrak{F}(E)$ for some E, then $A \in \mathfrak{F}(A \cap E)$, $B \in \mathfrak{F}(B \cap E)$, $\mathfrak{F}(A \cap E) \cap \mathfrak{F}(B \cap E) = \mathfrak{F}((A \cup B) \cap E) = \mathfrak{F}(E)$. It is also easy to see that $\mathscr{E}(S)$ is functorial: for any continuous $q: S \to S'$, $q'\mathfrak{F}(E) = \mathfrak{F}(qE)$, hence $q'\mathscr{E}(S) \subset \mathscr{E}(S') \subset \overline{\mathscr{E}(S')}$.

Before stating the next theorem we need a preliminary discussion. Let S, S' be two topological spaces. Given a point $t \in S'$, write q_t for the constant map $q_t: S \to S'$ with value t. If $\mathfrak{F}$ is a filter on $\mathfrak{S}$, then $q_t'\mathfrak{F} = \mathfrak{F}(\{t\})$, provided $\mathfrak{F} \neq \mathfrak{S}$; otherwise, $q_t'\mathfrak{S} = \mathfrak{S}'$. This follows from $q_t^{-1}A' = S(t \in A' \subset S')$, $q_t^{-1}A' = \varnothing (t \notin A')$.

Consider the system of filters $\mathscr{T}$ on $\mathfrak{S}$, consisting of the filter $\mathfrak{S} = \mathfrak{F}(\varnothing)$ alone. It is a trivial fact that $\mathscr{T}$ is adjoining and $\mathscr{T}(S)$ is functorial. One can prove a slightly stronger result: if $\mathscr{F}(S)$ is functorial and $\mathscr{F}(S_0) = \mathscr{T}(S_0)$ for some space S_0, then $\mathscr{F}(S) = \mathscr{T}(S)$ for all spaces S. Let $t \in S_0$ (we exclude the empty space); for any space S, consider the constant function $q_t: S \to S_0$. Since $\mathscr{F}(S)$ is functorial, we must have $q_t'\mathscr{F}(S) \subset \overline{\mathscr{T}(S_0)} = \mathscr{T}(S_0)$, hence $q_t'\mathfrak{F} = \mathfrak{S}_0$ for all $\mathfrak{F} \in \mathscr{F}(S)$. By our discussion above, this forces $\mathfrak{F} = \mathfrak{S}$ and

$\mathscr{F}(S) = \mathscr{T}(S)$. Of course, the choice of $\mathscr{T}(S)$ leads to a trivial solution of our original problem: Con (S, T) carries the indiscrete, $R \times S$ the discrete topology.

THEOREM 6. *$\mathscr{E}(S)$ is a minimal adjoining system for any space S, in the following sense: if $\mathscr{F}(S)$ is a functorial choice of adjoining systems different from $\mathscr{T}(S)$, then $\mathscr{E}(S) \subset \overline{\mathscr{F}(S)}$.*

Proof. Given S, let $s \in S$ and consider the constant map $q_s: S \to S$. By assumption, there is a filter $\mathfrak{F} \in \mathscr{F}(S)$ with $\mathfrak{F} \neq \mathfrak{S}$. Since $q'_s\mathfrak{F} = \mathfrak{F}(\{s\})$, Theorem 5 implies $\mathfrak{F}(\{s\}) \in \overline{\mathfrak{F}}$ and $\mathscr{E} \subset \overline{\mathscr{F}}$.

The topology induced by $\mathscr{E}$ on Con (S, T) is equal to the subspace topology with respect to the space T^S of all functions from S to T, carrying the ordinary product topology. In keeping with the term "set-open" a name sometimes used for this topology is "*point-open*"; it is called p-topology in [3], usually also the topology of pointwise convergence.

The topology on $R \times S$ corresponding to $\mathscr{E}$ can be described in different ways. Suppose $r: R \to \mathfrak{P}S$ is a mapping from R to the power set of S, $s: S \to \mathfrak{P}R$ a mapping from S to the power set of R. We shall call r, s *reciprocal*, if for any $v \in S$, $s(v) = \{u \in R: v \in r(u)\}$, or equivalently, if for any $u \in R$, $r(u) = \{v \in S: u \in s(v)\}$. Obviously, to each r there corresponds exactly one reciprocal s, and conversely. If r maps R to points of S, s is simply the inverse mapping r^{-1}.

Suppose r is a map from R to $\mathfrak{S}$, the space of open subsets of S. We shall call r and its reciprocal s, *topologically reciprocal*, if the same is true for s, i.e., if s maps S into $\mathfrak{R}$, the space of open sets of R.

The topology on $R \times S$, induced by the adjoining system $\mathscr{E}(S)$, can now be described as follows: a subset $[\phi]$ of $R \times S$ is open in this topology if and only if ϕ and its reciprocal ψ are topologically reciprocal.

The filters $\mathfrak{F}(\{s\})(s \in S)$ constitute an open subbasis for the topology given by $\mathscr{E}$ on $\mathfrak{S}$. Because $[\phi]$ is open if and only if $\phi: R \to \mathfrak{S}$ is continuous, $[\phi]$ will be open if and only if $\phi^{-1}\mathfrak{F}(\{s\}) = \{r \in R: s \in \phi r\} = \psi s$ is open in R. This is exactly what we have claimed.

Another description of the topology on $R \times S$ has been given by R. Brown (see [2]), who also briefly comments on its connection with the point-open topology on Con (S, T). ([2], Remark 1.15).

To obtain maximal adjoining systems, we first show how to construct new systems out of a family of given ones. Let $\mathscr{F}_i (i \in I)$ be a family of adjoining systems on a space $\mathfrak{S}$. Define $\cup^* \mathscr{F}_i (i \in I)$ as the set of all filters which can be written as finite intersections of filters belonging to $\cup \mathscr{F}_i$. By a straightforward application of the definitions involved one proves easily: $\bigcup^* \mathscr{F}_i (i \in I)$ is an adjoining system; if all filters belonging to $\bigcup \mathscr{F}_i$ are compactly generated, the same holds for the filters of $\bigcup^* \mathscr{F}_i$.

This construction allows to define two distinguished adjoining systems on any space $\mathfrak{S}$. The first one is the system $\mathscr{M}$, obtained by applying our construction to the family of all adjoining systems on $\mathfrak{S}$. $\mathscr{M}$ is of course the maximal adjoining system on $\mathfrak{S}$; furthermore, $\mathscr{M}(S)$ is functorial.

The second distinguished system will be denoted by $\mathscr{K}$; it is constructed from all adjoining systems consisting of compactly generated filters only. Since $\mathscr{E}$ is such a system, $\mathscr{K}$ is not empty. $\mathscr{K}(S)$ is functorial: for any continuous $q: S \to S'$ and any compactly generated filter $\mathfrak{F}(K)$ we have $q'\mathfrak{F}(K) = \mathfrak{F}(qK)$; qK is compact.

Note that other adjoining systems satisfying the functorial requirement can be obtained by cardinality arguments. Consider for instance, on a space $\mathfrak{S}$, the family of all adjoining systems of compactly generated filters, where for each such filter a generator may be found with cardinality $\leqq m$, m a fixed infinite cardinal. Application of our construction to this family yields an adjoining and functorial system $\mathscr{K}_m(S)$ for each space S.

5. The compact-open topology.

The compact-open topology on a space Con (S, T) is defined by the system $\mathscr{K}_0$ of all compactly generated filters on $\mathfrak{S}$. $\mathscr{K}_0$ need not be adjoining, as will be shown presently; hence the compact-open topology does not always provide a solution to our problem. However, due to the importance of this topology for the theory of function spaces an investigation of spaces S for which the compact-open topology on Con (S, T) is induced, for any T, by some adjoining system is indicated.

Let $\mathscr{F}$ be such a system. According to Theorem 4, we have $\overline{\mathscr{F}} = \overline{\mathscr{K}_0}$; conversely, a system $\mathscr{F}$ satisfying this equality defines, for any T, the compact-open topology on Con (S, T).

THEOREM 7. *If $\overline{\mathscr{F}} = \overline{\mathscr{K}_0}$ for some adjoining system $\mathscr{F}$ on $\mathfrak{S}$, the space S satisfies*:

(D) *If K is a compact subset, and if A_1, A_2 are open subsets of S such that $K \subset A_1 \cup A_2$, there are compact sets K_1, K_2 in S with $K_1 \subset A_1$, $K_2 \subset A_2$, $K \subset K_1 \cup K_2$.*

Conversely, if (D) *holds, the adjoining system $\mathscr{K}$ defined in* § 4 *equals $\mathscr{K}_0$.*

Proof. Suppose $\overline{\mathscr{F}} = \overline{\mathscr{K}_0}$ and let K, A_1, A_2 satisfy the hypothesis of condition (D). Then $A_1 \cup A_2 \in \mathfrak{F}(K)$ and $\mathfrak{F}(K) \in \overline{\mathscr{F}}$. Since $\overline{\mathscr{F}}$ is adjoining, there exist filters $\mathfrak{F}_1, \mathfrak{F}_2 \in \overline{\mathscr{F}}$ with $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$, $\mathfrak{F}_1 \cap \mathfrak{F}_2 \subset$

$\mathfrak{F}(K)$. $\mathfrak{F}_1$, $\mathfrak{F}_2$ being members of $\overline{\mathscr{K}_0}$, there are compact sets L_{1i}, L_{2j} $(i \in I, j \in J)$ such that $\mathfrak{F}_1 = \cup \mathfrak{F}(L_{1i})$, $\mathfrak{F}_2 = \cup \mathfrak{F}(L_{2j})(i \in I, j \in J)$. $A_1 \in \mathfrak{F}_1$ implies $L_1 \subset A_1$, where L_1 is one of the sets L_{1i}; similarly, $L_2 \subset A_2$. Furthermore, $\mathfrak{F}_1 \cap \mathfrak{F}_2 = \cup (\mathfrak{F}(L_{1i}) \cap \mathfrak{F}(L_{2j})) = \cup \mathfrak{F}(L_{1i} \cup L_{2j})$, thus $\mathfrak{F}(L_1 \cup L_2) \subset \mathfrak{F}(K)$. As was shown in §2, the latter relation is equivalent to $K \subset K^* \subset L_1^* \cup L_2^*$. By letting $K_1 = L_1^*$, $K_2 = L_2^*$ we have established the first part of the theorem.

Suppose (D) holds in a space S; then it also holds if "compact" is replaced by "fully compact". Theorem 3 now shows $\mathscr{K}_0$ to be adjoining. Obviously, $\overline{\mathscr{K}_0} = \mathscr{K}$.

Examples of spaces satisfying (D) are easily provided: discrete, indiscrete and pseudo-finite spaces (all compact sets finite; see [4] or [5]) are instances in case. We give a sufficient condition for (D) to hold, which covers some important classes of spaces.

We shall use the following notation: $T_i(i = 1, 2, 3, 4, 5)$ denotes the usual separation axioms, where for $i \geqq 2$ we do not assume T_1. A space is called a KC space (according to [5]), if every compact set is closed; it will be called a K^*C space if every fully compact set is closed. This is equivalent to: if K is compact, A open, and if $K \subset A$, then $\bar{K} \subset A$ (where $\bar{K}$ is the closure of K). Finally, a space will be said to be KT_4, if every compact subspace is a T_4 space.

THEOREM 8. *Any KT_4 space satisfies condition* (D). *If the space is a K^*C space, KT_4 is equivalent to* (D).

Proof. Let K be a compact subset, A_1, A_2 open subsets of a KT_4 space S, such that $K \subset A_1 \cup A_2$. The sets $K - A_1$, $K - A_2$ are closed subsets of the subspace K and $(K - A_1) \cap (K - A_2) = \varnothing$. By assumption, there are open sets B_1, B_2 in S such that $K - A_1 \subset B_1$, $K - A_2 \subset B_2$, $K \cap B_1 \cap B_2 = \varnothing$. Writing $K_1 = K - B_1$, $K_2 = K - B_2$, we have $K_1 \subset A_1$, $K_2 \subset A_2$ and $K_1 \cup K_2 = K$. The sets K_1, K_2 being closed in the compact subspace K, they are compact in S.

Let S be a K^*C space satisfying (D), and let K be a compact subspace of S. If C_1, C_2 are closed disjoint subsets of K, write D_1, D_2 for their complements in S. Because $K \subset D_1 \cup D_2$ and D_1, D_2 are open, there are compact sets L_1, L_2 in S with $L_1 \subset D_1$, $L_2 \subset D_2$, $K \subset L_1 \cup L_2$. By assumption, $\bar{L}_1 \subset D_1$, $\bar{L}_2 \subset D_2$. Writing M_1, M_2 for the complements of $\bar{L}_1, \bar{L}_2$ in S, we have $C_1 \subset M_1$, $C_2 \subset M_2$, $K \cap M_1 \cap M_2 = \varnothing$. Hence K is a T_4 space.

It is easy to see that T_2, T_3 and T_5 spaces are KT_4 spaces and thus satisfy (D). This is trivial for T_5. Any subspace of a $T_2(T_3)$ space is itself $T_2(T_3)$; as is well known, a compact $T_2(T_3)$ space is also a T_4 space.

We shall see in the next section that (D) holds also in locally compact spaces.

On the other hand, neither T_1 (or the stronger KC) nor compactness imply (D). Consider the space Q^+, the one-point-compactification of the space Q of rationals. Q^+ is a compact KC space (see e.g., [5]), hence also T_1 and K^*C. By Theorem 8, Q^+ will satisfy (D) only if it is a T_4 space. Because T_1 and T_4 imply T_2, Q^+ would have to be a Hausdorff space, which it is not.

6. **The product topology.** The topology induced on the space $R \times S$ by an adjoining system on $\mathfrak{S}$ need not be the product topology. In this section we give necessary and sufficient condition in order that a given adjoining system induces the product topology on $R \times S$, for any R.

Let $\mathfrak{F}$ be a filter on $\mathfrak{S}$ and let $F_i (i \in I)$ be the family of its members. We write $\mathfrak{F}^0$ for the set $(\cap F_i)^0 (i \in I)$. (X^0 denotes the interior of X in the space S). If $\mathfrak{F} = \mathfrak{F}(K)$ is compactly generated, then $\mathfrak{F}^0 = K^{*0}$.

Let R, S be two topological spaces, A an open subset of R, B an open subset of S. We define a mapping $\psi(A, B)\colon R \to \mathfrak{S}$ by $\psi(A, B)r = B$ $(r \in A)$, $\psi(A, B)r = \varnothing (r \in R - A)$. Obviously, with the notation of §2, $[\psi(A, B)] = A \times B$. Let $\mathfrak{F}$ be any filter on $\mathfrak{S}$. It is easy to see that $\psi(A, B)^{-1}\mathfrak{F}$ is equal either to A or to $\varnothing$ or to R. Hence $\psi(A, B)$ is continuous for all A, B and $A \times B$ is open no matter what adjoining system is chosen on $\mathfrak{S}$ to determine a topology on $R \times S$. This shows that any such topology is finer than the product topology.

Since the family of sets $A \times B$ is an open basis for the product topology on $R \times S$, an adjoining system on $\mathfrak{S}$ determines this topology if and only if to any continuous map $\phi\colon R \to \mathfrak{S}$ there correspond open sets $A_i \subset R$, $B_i \subset S (i \in I)$ such that $[\phi] = \cup [\psi(A_i, B_i)] (i \in I)$.

THEOREM 9. *An adjoining system $\mathscr{F}$ on $\mathfrak{S}$ induces the product topology on $R \times S$, for any R, if and only if any open set $A \subset S$ satisfies $A = \cup \mathfrak{F}^0$, where $\mathfrak{F}$ runs through all filters of $\mathscr{F}$ containing A as an element.*

Proof. The condition is necessary. Let R be the space $\mathfrak{S}$ with the topology given by $\mathscr{F}$. We claim: the family of sets $[\psi(\mathfrak{F}, B)]$ $(\mathfrak{F} \in \mathscr{F}, B$ open in $S)$ is an open basis for the product topology on $\mathfrak{S} \times S$. Indeed, an open set in $\mathfrak{S}$ is given by $\mathfrak{A} = \cup \mathfrak{F}_j (\mathfrak{F}_j \in \mathscr{F}, j \in J)$ and $[\psi(\mathfrak{A}, B)] = \cup [\psi(\mathfrak{F}_j, B)] (j \in J)$. Consider the identical mapping $\varepsilon\colon \mathfrak{S} \to \mathfrak{S}$. Since ε is continuous, $[\varepsilon]$ is open in $\mathfrak{S} \times S$ and there exist filters $\mathfrak{F}_i \in \mathscr{F}$ and open sets $B_i \subset S (i \in I)$ such that $[\varepsilon] = \cup [\psi(\mathfrak{F}_i, B_i)]$ $(i \in I)$, or equivalently, $\varepsilon = \cup \psi(\mathfrak{F}_i, B_i)$ (according to the notation introduced in §2). For any open $A \subset S$ we have $\varepsilon A = A = \cup \psi(\mathfrak{F}_i, B_i)A = \cup B_j (i \in I, j \in J(A))$, where the subset $J(A) \subset I$ is determined by $i \in J(A)$ if and only if $A \in \mathfrak{F}_i$. Hence for any $i \in I$ and any $A \in \mathfrak{F}_i$ we have

$B_i \subset A$ and consequently $B_i \subset \mathfrak{F}_i^0$. Fix an open $A \subset S$ and consider the index set $J(A)$. Then $\mathfrak{F}_j^0 \subset A$ for all $j \in J(A)$, hence $\cup \mathfrak{F}_j^0 \subset A$, but also $A = \cup B_j \subset \cup \mathfrak{F}_j^0$. This proves $A = \cup \mathfrak{F}_j^0$ and also $A = \cup \mathfrak{F}^0$, as stated in the theorem.

The condition is sufficient. Let R be any topological space, $\phi: R \to \mathfrak{S}$ a continuous map. For any $\mathfrak{F} \in \mathscr{F}$, define $\phi(\mathfrak{F}) = \psi(\phi^{-1}\mathfrak{F}, \mathfrak{F}^0)$. By definition, for $r \in R$, $\phi(\mathfrak{F})r = \mathfrak{F}^0$ if $\phi r \in \mathfrak{F}$, otherwise $\phi(\mathfrak{F})r = \varnothing$. Taking the union of all the sets $\phi(\mathfrak{F})r$ for all $\mathfrak{F} \in \mathscr{F}$, r fixed, one obtains $\cup \phi(\mathfrak{F})r = \cup \mathfrak{G}^0$, where $\mathfrak{G}$ runs through all filters of $\mathscr{F}$ containing ϕr. By assumption, this is equal to ϕr and we have $\phi = \cup \phi(\mathfrak{F}) = \cup \psi(\phi^{-1}\mathfrak{F}, \mathfrak{F}^0)(\mathfrak{F} \in \mathscr{F})$. Hence $[\phi]$ is open in the product topology on $R \times S$.

We shall call a space S satisfying the condition of Theorem 9, *locally* $\mathscr{F}$. The reason for this terminology is the following: let A be open in S, p a point of A. The condition is satisfied if and only if there is a filter $\mathfrak{F} \in \mathscr{F}$ containing A and such that $p \in \mathfrak{F}^0 \subset A$.

If $\mathscr{F}$ consists of compactly generated filters and if $\mathfrak{K}(\mathscr{F})$ is the set of its fully compact generators, S is locally $\mathscr{F}$ if and only if for any open $A \subset S$ and any point $p \in A$ there is $K \in \mathfrak{K}(\mathscr{F})$ such that $p \in K^0 \subset K \subset A$, i.e., if and only if $\mathfrak{K}(\mathscr{F})$ is a local neighborhood base for each point of S. If $\mathfrak{K}(\mathscr{F})$ is the family of all fully compact sets of S, this is just (a version of) local compactness, as is easily seen.

As was shown above, the product topology on $R \times S$ is always coarser than any topology induced by an adjoining system. This remark leads to.

THEOREM 10. *Let $\mathscr{F}$ be a family of compact filters on $\mathfrak{S}$ and let the space S be locally $\mathscr{F}$. Then $\overline{\mathscr{F}}$ consists of all compact filters on $\mathfrak{S}$ and is adjoining; in fact, $\overline{\mathscr{F}}$ equals the maximal adjoining system $\mathscr{M}$ on $\mathfrak{S}$.*

Proof. Let $\mathfrak{G}$ be any compact filter on $\mathfrak{S}$ and let $A \in \mathfrak{G}$. By assumption, $A = \cup \mathfrak{F}^0$, where $\mathfrak{F}$ runs through all filters of $\mathscr{F}$ containing A. Since $\mathfrak{G}$ is compact, there is a finite number of such filters, say $\mathfrak{F}_1, \cdots, \mathfrak{F}_n$, such that $\mathfrak{F}_1^0 \cup \cdots \cup \mathfrak{F}_n^0 \in \mathfrak{G}$. Consider $\mathfrak{H} = \mathfrak{F}_1 \cap \cdots \cap \mathfrak{F}_n$; it is easy to check that $\mathfrak{H}^0 \supset \mathfrak{F}_1^0 \cup \cdots \cup \mathfrak{F}_n^0$. Consequently, $\mathfrak{H}^0 \in \mathfrak{G}$ and $\mathfrak{H} \subset \mathfrak{G}$. For any $A \in \mathfrak{G}$ we have found a filter $\mathfrak{H}(A) \in \mathscr{F}$ with $A \in \mathfrak{H}(A) \subset \mathfrak{G}$; hence $\mathfrak{G} = \cup \mathfrak{H}(A)(A \in \mathfrak{G})$ and $\mathfrak{G} \in \overline{\mathscr{F}}$.

To prove $\overline{\mathscr{F}} = \mathscr{M}$, we show that $\mathscr{F}$ is adjoining, i.e., satisfies condition (A) of §2. Let $\mathfrak{F} \in \mathscr{F}$, A, B open in S, $A \cup B \in \mathfrak{F}$. Writing, as above, $A = \cup \mathfrak{G}^0$, $B = \cup \mathfrak{H}^0(A \in \mathfrak{G} \in \mathscr{F}, B \in \mathfrak{H} \in \mathscr{F})$ we can find filters $\mathfrak{G}(A)$, $\mathfrak{H}(B) \in \mathscr{F}$ with $A \in \mathfrak{G}(A)$, $B \in \mathfrak{H}(B)$, $\mathfrak{G}(A)^0 \cup \mathfrak{H}(B)^0 \in \mathfrak{F}$. The latter relation implies $\mathfrak{G}(A) \cap \mathfrak{H}(B) \subset \mathfrak{F}$, as is easily seen by using the formula $\mathfrak{G}^0 \cup \mathfrak{H}^0 \subset (\mathfrak{G} \cap \mathfrak{H})^0$.

COROLLARY. *If S is locally compact (in the sense introduced above, i.e., locally $\mathscr{K}_0$), then $\overline{\mathscr{K}_0} = \mathscr{M}$ and the compact-open topology on* Con (S, T) *corresponds to the product topology on $R \times S$.*

This corollary together with Theorem 7 implies that a locally compact space satisfies condition (D) of Theorem 7. This fact can of course be shown directly.

We shall now give an example of a space S which is not locally $\mathscr{F}$ for any system of compact filters. By our results the product topology on $R \times S$ will not, in general, be induced by any adjoining system on $\mathfrak{S}$.

Let S be an uncountable set with the countable topology (a set is closed if and only if it is at most countable or the whole space). As is well known, the resulting space is pseudo-finite (compact = finite). We want to show that every compact filter $\mathfrak{F}$ on $\mathfrak{S}$ is compactly generated.

Let K be the intersection of all members of $\mathfrak{F}$ and suppose $K \subset A$, where A is open in S. We shall prove $A \in \mathfrak{F}$. A is either cofinite or cocountable. In the first case, let $A = S - \{v_1, \cdots, v_n\}$. For each i, $1 \leqq i \leqq n$, there is $F_i \in \mathfrak{F}$ with $v_i \notin F_i$. Then $F_1 \cap \cdots \cap F_n \subset S - \{v_1, \cdots, v_n\} = A$ and $A \in \mathfrak{F}$. In the second case, let $A = S - V$, where $V = \{v_1, v_2, \cdots\}$. For each natural i, there is again $F_i \in \mathfrak{F}$ with $v_i \notin F_i$. We use, for any i, the notation $V_i = \{v_1, \cdots, v_i\}$, $A_i = A \cup V_i$. Obviously, $F_1 \cap F_2 \cap \cdots \cap F_i \subset S - V_i$, which implies $S - V_i \in \mathfrak{F}$. The union of all A_i is equal to S and therefore belongs to the filter $\mathfrak{F}$. Since $\mathfrak{F}$ is compact, there is a finite number of sets A_i whose union is a member of $\mathfrak{F}$. As is easily seen, this union is equal to some A_j. Then $A_j \cap (S - V_j) \in \mathfrak{F}$; but $A_j \cap (S - V_j) = A$.

Our result implies $\mathfrak{F} = \mathfrak{F}(K)$, thus $\mathfrak{F}$ is compactly generated. It is now evident that S cannot be locally $\mathscr{F}$ for any system of compact filters $\mathscr{F}$, for this would imply the existence of finite open sets on S.

The author wishes to dedicate this paper to Professor Hugo Hadwiger of the University of Bern on the occasion of his sixtieth birthday.

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A NOTE ON THE CHARACTERIZATION OF CONDITIONAL EXPECTATION OPERATORS

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Let $(X, \sum, \mu)$ be an arbitrary measure space. A complete characterization is presented for the norm one positive projections P of $L^1(X, \sum, \mu)$ into itself such that $||f||_\infty \geqq ||Pf||_\infty$ for each essentially bounded, summable function f.

If $(X, \sum, \mu)$ is a probability measure space it is known [1], [2], and [4] that such operators coincide precisely with the conditional expectation operators defined on L^1 (see definitions below). In this note we show that this characterization extends to an arbitrary measure space. The proof presented here is a direct, constructive proof requiring only basic measure theory. Although this extension is not unexpected, it does not seem to be a consequence of the methods used in the case for the finite measure spaces. An independent proof for this result, using the ergodic theory of Markov processes, was found by S. Foguel. Also extensions to arbitrary measure spaces of related theorems in [1], have recently been done by L. Tzafriri [5].

DEFINITION. Let $(X, \sum, \mu)$ be a measure space. Let $\sum_0$ be a σ-subring of $\sum$. We call a projection P on $L^1(X, \sum, \mu)$ a *conditional expectation operator* with respect to $\sum_0$ if Pf is $\sum_0$ measurable for all f, and if for each U in $\sum_0$ we have

$$\int_U Pfd\mu = \int_U fd\mu .$$

Clearly a conditional expectation operator is positive projection of norm one.

Notation. Let P be a norm one positive projection from $L^1(X, \sum, \mu)$ onto E. Let

$$\textstyle\sum_0 = \{K \subseteqq X : K = \operatorname{supp} f, f \geqq 0, f \text{ in } E\} .$$

We use $\operatorname{supp} f$ to denote the support of a function f. The characteristic function of a set A is written 1_A.

LEMMA 1. *E is a lattice.*

Proof. Let f and g be in E. Since $f \vee g$ dominates f and g, $P(f \vee g)$ dominates f, g and hence $f \vee g$. Since $|| f \vee g || \geqq || P(f \vee g)||$, we have that $f \vee g = P(f \vee g)$.

LEMMA 2. *$\sum_0$ is a σ-ring.*

Proof. We need to show that countable unions of members of $\sum_0$ are in $\sum_0$, and differences of members in $\sum_0$ are in $\sum_0$. Suppose U_i is in $\sum_0$ for $i = 1, 2, \cdots$. Let f_i be norm one positive functions in E whose support in U_i. Then $f = \sum_{i=1}^{\infty} (1/2)^i f_i$ is a positive function in E whose support is $\bigcup_{i=1}^{\infty} U_i$.

Suppose U and V are in $\sum_0$, and are the supports of positive fuction f and g in E.
Let

$$f_n = (f - ng) \vee 0 .$$

Let

$$f'(x) = \begin{cases} f(x) & \text{for } x \text{ in } U - V \\ 0 & \text{otherwise} . \end{cases}$$

From the dominated convergence theorem f_n converges to f'. Thus $U - V$ is in $\sum_0$

LEMMA 3. *Suppose f vanishes off some member of $\sum_0$. The following is true.*

(i) *$P(|f|) = 0$ implies $f = 0$.*

(ii) *if K is in $\sum_0$ then $P(1_K f) = 1_K P(f)$.*

(iii) *$\int_K Pfd\mu = \int_K fd\mu$ for all K in $\sum_0$.*

Proof. We will assume that f is nonnegative. Suppose g is a nonnegative number of E such that f vanishes off $\operatorname{supp}(g)$. Since $f \wedge ng$ increases monotonically to f, it suffices to assume that f is bounded by a member of E. We will assume therefore that $0 \leqq f \leqq g$.

(i) $P(g - f) = g$ so $\|g\| \leqq \|g - f\|$, but $0 \leqq g - f \leqq g$. Therefore $f = 0$.

(ii) Suppose that h is a nonnegative member of E such that $\operatorname{supp} h = K$. Since $g \wedge nh$ converges monotonically to $1_K g$, it follows that $1_K g$ is in K.

Now $0 \leqq P(1_K f) \leqq P(1_K g) = 1_K g$. Hence $P(1_K f)$ vanishes off K, and $P(1_K f) \leqq 1_K P(f)$. We also have $1_K g - P(1_K f) = P(1_K(\dot{g} - f)) \leqq 1_K P(g - f) = 1_K g - 1_K P(f)$. Thus $1_K Pf = P(1_K f)$.

(iii) $\int_K fd\mu = \|1_K f\| \geqq \|P(1_K f)\| = \int_K Pfd\mu$. Similarly

$$\begin{aligned}\int g d\mu - \int_K f d\mu &= \int_K (g - f) d\mu = \| 1_K(g - f) \| \\ &\geqq \| P(1_K(g - f)) \| = \int_K (Pg - Pf) d\mu .\end{aligned}$$

Hence $\int_K Pfd\mu = \int_K fd\mu$.

For the remainder of the paper we will also assume that for each essentially bounded f in L^1, $\| f \|_\infty \geqq \| Pf \|_\infty$.

LEMMA 4. *Each member of E is $\sum_0$-measurable.*

Proof. We first show that $g \wedge c$ is in E for each constant function c and each g in E. It suffices to prove this assertion for positive functions g and for $c > 0$. However this is almost obvious for since P is positive $g \geqq P(g \wedge c) \geqq 0$, and from the hypothesis $c \geqq P(g \wedge c)$. Thus $g \wedge c \geqq P(g \wedge c)$. Now with $K = \operatorname{supp} g$ Lemma 3 (iii) implies that $g \wedge c = P(g \wedge c)$.

It follows that for any c, $g - g \wedge c$ is in E. Hence if g is a positive function in E, the set $\{x \text{ in } X : g(x) > c\}$ is also the support of $g - g \wedge c$, and thus is in $\sum_0$. Hence Pf is $\sum_0$ measurable for each f in L^1.

PROPOSITION. *P is the conditional expectation operator with respect to $\sum_0$.*

Proof. Let f be in L^1. Let $r = \sup\left\{\int_K | f | d\mu : K \text{ in } \sum_0\right\}$. Let K be a member of $\sum_0$ such that $\int_K | f | d\mu = r$. Writing $f = 1_K f + (f - 1_K f)$ we see that f is the sum of a function which vanishes off a member of $\sum_0$ and a function which vanishes on each member of $\sum_0$. Thus in view of all the previous lemmas it remains only to show that $Pf = 0$ if f vanishes on each member of $\sum_0$. We may assume that f is bounded and nonnegative. Since the support of nPf is in $\sum_0$, and since f vanishes on all members of $\sum_0$, we have

$$\| f + nPf \|_\infty = \max (\| f \|_\infty, n \| Pf \|_\infty) ,$$

but

$$\| f + nPf \|_\infty \geqq \|P(f + nPf) \|_\infty = (n + 1) \| Pf \|_\infty .$$

This implies that $\| Pf \|_\infty = 0$.

REMARKS. The hypothesis that $\| f \|_\infty \geqq \| Pf \|_\infty$ is equivalent to the assumption that $P(1_A) \leqq 1$ for all sets A of finite measure.

The referee has pointed out that the main result in this note is valied for norm one projections defined on L_p spaces. Besides the proofs presented here, one would also use the fact that there do not exist two distinct norm one projections of a smooth space onto a subspace. (For L_p spaces this result is in [1]. For smooth spaces a proof is in [6, Lemma 1]). The organization of this note was also suggested by the referee, and adapts to L_p operators more readily than the original.

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