LOCALLY COMPACT SPACES AND TWO CLASSES OF C*-ALGEBRAS

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Let $X$ be a topological space which is second countable, locally compact, and $T_0$. Fell has defined a compact Hausdorff topology on the collection $\mathcal{G}(X)$ of closed subsets of $X$. $X$ may be identified with a subset of $\mathcal{G}(X)$, and in the first part of this paper, the original topology on $X$ is related to that induced from $\mathcal{G}(X)$. The main result is a necessary and sufficient condition for $X$ to be almost strongly separated. In the second part, these results are applied to the primitive ideal space $\text{Prim}(A)$ of a separable C*-algebra $A$, giving in particular a necessary and sufficient condition for $\text{Prim}(A)$ to be almost separated. Further information concerning ideals in $A$ which are central as C*-algebras is obtained.

Most of the theorems in the paper were suggested by the results for simplex spaces recently obtained by Effros [10], Effros and Gleit [11], Gleit [14], and Taylor [17]. The notion of a simplex space was introduced by Effros in [9]. If $\mathfrak{A}$ is a simplex space, then max $\mathfrak{A}$, $P_1(\mathfrak{A})$, and $EP_1(\mathfrak{A})$ denote the closed maximal ideals in $\mathfrak{A}$, the bounded positive linear functionals on $\mathfrak{A}$ of norm at most one, and its set of extreme points, resp., the first set provided with the hull-kernel topology and the latter two sets with the weak* topology. The sets max $\mathfrak{A}$ and $EP_1(\mathfrak{A})$-$\{0\}$ are in a natural one-to-one correspondence, but the topologies do not agree in general. Information about the simplex space $\mathfrak{A}$ can be obtained by comparing these two topologies (see [11], [14], [17]).

In trying to develop an analogous theory for a C*-algebra $A$, the first problem is to decide on replacements for max $\mathfrak{A}$, $P_1(\mathfrak{A})$, and $EP_1(\mathfrak{A})$. For simplicity, assume that $A$ is separable and has a $T_1$ structure space. An obvious substitute for max $\mathfrak{A}$ is the structure space of $A$, $\text{Prim}(A)$ (the primitive ideals in $A$, or in this case the maximal proper closed two-sided ideals in $A$, with the hull-kernel topology). To replace $P_1(\mathfrak{A})$ and $EP_1(\mathfrak{A})$ by the corresponding sets of linear functionals on $A$ does not seem to lead to a fruitful theory. Instead, $P_1(\mathfrak{A})$ and $EP_1(\mathfrak{A})$-$\{0\}$ are replaced by $N(A)$ and $EN(A)$-$\{0\}$, resp., where $N(A)$ is the compact Hausdorff space of C*-semi-norms on $A$, and $EN(A)$ is the set of “extreme” points of $N(A)$ (see [4; § 1. 9. 13], [8], [12]). Then $\text{Prim}(A)$ and $EN(A)$-$\{0\}$ are in a natural one-to-one correspondence which is in general not a homeomorphism. By identifying these sets, the primitive ideals in $A$ are endowed with
two topologies. Regarding Prim(A) as a subset of \( \mathcal{C}(\text{Prim}(A)) \), the identification of Prim(A) and EN(A)\{-0\} extends naturally to a homeomorphism of \( \mathcal{C}(\text{Prim}(A)) \) and \( N(A) \). Thus the second topology on Prim(A) is just its relative topology in \( \mathcal{C}(\text{Prim}(A)) \). It is therefore natural to attempt to formulate those theorems about a simplex space \( A \) which involve only the two topologies on max \( A \) in terms of a locally compact space \( X \) and the associated space \( \mathcal{C}(X) \).

The paper is organized as follows. §2 contains theorems which relate the topology of \( X \) to that of \( \mathcal{C}(X) \). The applications to C*-algebras are in §3. Two classes of C*-algebras, called GM- and GC-algebras, are investigated; they correspond to the GM- and GC-simplex spaces of [11]. A C*-algebra is a GM-algebra if its structure space is almost strongly separated, and a GC-algebra if it has a composition series \((I_{\alpha})\) of closed two-sided ideals such that the \( I_{\alpha+1}/I_{\alpha} \) are all central C*-algebras. These algebras were studied by Delaroche [2], who in particular showed that the GC-algebras are just the GM-algebras with only modular primitive ideals. A new proof of this fact (Theorem 3.7) is included. Finally, §4 points out how the GM- and GC-algebras are related to some of the classes of C*-algebras in the literature.

2. Locally compact spaces. Throughout this section \( X \) is assumed to be a locally compact topological space satisfying the \( T_1 \) separation axiom. Recall that \( X \) is \( T_1 \) means that if \( x, y \in X \) are such that \( \{x\}^- = \{y\}^- \) (bar indicates closure), then \( x = y \), and that \( X \) is locally compact means that if \( x \in X \), then each neighborhood of \( x \) contains a compact neighborhood of \( x \). It is important to remember that although a closed subset of a compact set must be compact, the converse need not be true in a non-Hausdorff space. Let \( X_i \) denote the closed points in \( X \), i.e., those \( x \) for which \( \{x\}^- = \{x\} \). If \( X = X_i \), then \( X \) is said to be \( T_1 \).

The following construction is due to J. M. G. Fell [13]. Let \( \mathcal{C}(X) \) denote the collection of all closed subsets of \( X \). The function \( \lambda : X \to \mathcal{C}(X) : x \mapsto \{x\}^- \) is one-to-one. If \( C \) is a compact subset of \( X \) and if \( \mathcal{F} \) is a (possibly empty) finite collection of open subsets of \( X \), then \( \mathcal{U}(C; \mathcal{F}) \) will denote the collection of all those \( F \in \mathcal{C}(X) \) such that \( F \cap C = \emptyset \) and \( F \cap G \neq \emptyset \) for each \( G \in \mathcal{F} \). The sets \( \mathcal{U}(C; \mathcal{F}) \) form a basis for a compact Hausdorff topology on \( \mathcal{C}(X) \) [13]. It is readily verified that a net \((F_n)\) in \( \mathcal{C}(X) \) will converge to an element \( F \) in \( \mathcal{C}(X) \) if and only if (1) for each \( x \) in \( F \) and neighborhood \( N \) of \( x \), eventually \( F_n \cap N \neq \emptyset \), and (2) if \( P \) is the complement of a compact set with \( F \subset P \), then eventually \( F_n \subset P \). This topology is metrizable whenever \( X \) is second countable [6; Lemma 2] (see Corollary 2.7 for a partial converse). A simple argu-
LEMMA 2.1. (1) \( \lambda \) is open onto its image, and (2) \( X \) is Hausdorff if and only if \( \lambda : X \to \lambda(X) \) is a homeomorphism.

The first object is to find sets on which \( \lambda \) restricts to a homeomorphism. A set \( \mathcal{I} \subset \mathcal{C}(X) \) will be called dilated if \( x \in F \) for some \( F \in \mathcal{I} \) implies that \( \lambda(x) \in \mathcal{I} \). In particular, if \( F \in \mathcal{C}(X) \), the set \( F^\perp = \{ E \in \mathcal{C}(X) : E \subseteq F \} \) is compact and dilated.

**LEMMA 2.2.** If \( \mathcal{I} \) is a compact and dilated subset of \( \mathcal{C}(X) \), then \( \lambda^{-1}(\mathcal{I}) \) is closed.

**Proof.** Suppose that \( x_0 \in X \) and \( x_0 \notin \lambda^{-1}(\mathcal{I}) \). Say \( F \in \mathcal{I} \). As \( \mathcal{I} \) is dilated, \( x_0 \notin F \), and so there is a compact neighborhood \( C(F) \) of \( x_0 \) which is disjoint from \( F \). The sets \( \mathcal{U}(C(F); \varnothing) \), \( F \in \mathcal{I} \), form an open covering for \( \mathcal{I} \); hence there are sets \( F_1, \ldots, F_n \in \mathcal{I} \) such that

\[
\mathcal{I} \subset \bigcup_{i=1}^n \mathcal{U}(C(F_i); \varnothing).
\]

Suppose \( x \in C = \bigcap_{i=1}^n C(F_i) \) and \( \lambda(x) \in \mathcal{I} \). Then \( \lambda(x) \cap C(F_i) = \varnothing \) for some \( i \), hence \( x \notin C(F_i) \), a contradiction. This shows that \( C \) is a neighborhood of \( x_0 \) which is disjoint from \( \lambda^{-1}(\mathcal{I}) \).

If \( T \) is a subset of \( X \), then \( \lambda(T) \) is dilated; hence

**COROLLARY 2.3.** If \( T \) is a subset of \( X \) for which \( \lambda(T) \) is compact, then \( \lambda \) restricts to a homeomorphism of \( T \) onto \( \lambda(T) \).

The following shows that convergence in \( X \) is closely related to that in \( \mathcal{C}(X) \). The trick employed in the proof of (ii) was used by both Gleit [14] and Taylor [17].

**THEOREM 2.4.** (i) Let \( (x_a) \) be a net in \( X \) such that \( \lambda(x_a) \to F \) for some \( F \in \mathcal{C}(X) \). Then \( x_a \to x \) for any \( x \in F \).

(ii) Let \( (x_n) \) be a sequence in \( X \) such that \( \lambda(x_n) \to F \) for some \( F \in \mathcal{C}(X) \). Then the limit points of the set \( \{ x_n : n \geq 1 \} \) lie in \( F \).

**Proof.** (i) Say \( x \in F \), and let \( G \) be an open set containing \( x \). Then since \( F \cap G \neq \varnothing \), eventually \( \lambda(x_a) \cap G \neq \varnothing \), hence \( x_a \in G \).

(ii) For each \( m \) the set \( \{ \lambda(x_n) : n \geq m \} \cup F^\perp \) is both closed and dilated, hence its inverse image \( F_m = \{ x_n : n \geq m \} \cup F \) is closed. If \( x \)
is a limit point of \( \{x_n : n \geq 1\} \), it must lie in each of the sets \( F_m \), and thus is an element of \( F \).

**Corollary 2.5.** Suppose that \( X \) is second countable. If \( \emptyset \in \lambda(X)^{-} \), then neither \( X \) nor \( X \) can be compact.

*Proof.* \( \mathcal{E}(X) \) is metrizable, hence there is a sequence \((x_n)\) in \( X \) with \( \lambda(x_n) \to \emptyset \). It follows from Theorem 2.4 (ii) that no subsequence of \((x_n)\) can converge to a point in \( X \).

**Corollary 2.6.** Suppose that \( \lambda(X)^{-} \) is first countable (this is the case if \( X \) is second countable), and that \( T \) is a compact subset of \( X \). If \( F \in \mathcal{E}(X) \) and \( T \cap F = \emptyset \), then \( \lambda(T)^{-} \cap F^\perp = \emptyset \).

*Proof.* If \( E \in \lambda(T)^{-} \cap F^\perp \), there is a sequence \((x_n)\) in \( T \) with \( \lambda(x_n) \to E \). Since \( T \) is compact, the set \( \{x_n : n \geq 1\} \) has a limit point \( x \) in \( T \). Then \( x \in E \) from Theorem 2.4 (ii), and since \( E \in F^\perp \), \( x \in F \). But this is a contradiction.

**Corollary 2.7.** Suppose that \( X \) is locally compact and \( T \). If \( \lambda(X)^{-} \) is second countable, then so is \( X \).

*Proof.* Let \( \mathcal{T}, \mathcal{T}, \ldots \) be a basis of open sets for the topology of \( \lambda(X)^{-} \); with no loss in generality, the sets \( \mathcal{T}_n \) may be assumed to be closed under finite unions. Suppose that an \( x \in X \) and an \( F \in \mathcal{E}(X) \) with \( x \in F \) are given. It is sufficient to show that for some \( n, \lambda^{-1}(\mathcal{T}_n) \) contains \( x \) in its interior and is disjoint from \( F \). Using the local compactness of \( X \), choose a compact neighborhood \( C \) of \( x \) disjoint from \( F \). Corollary 2.6 and the fact that \( F^\perp \) is closed give

\[
\lambda(C)^- \subset \lambda(X)^- \cap F^\perp = \bigcup_k \mathcal{T}_{n_k}
\]

for suitable integers \( n_k \). As \( \lambda(C)^- \) is compact and as the \( \mathcal{T}_n \) are closed under finite unions, there is an \( n \) for which \( \mathcal{T}_n \cap F^\perp = \emptyset \) and \( \lambda(C) \subset \mathcal{T}_n \). This completes the proof.

The following will be useful in § 3.

**Corollary 2.8.** Suppose that \( X \) is second countable and that \( f: \mathcal{E}(X) \to [0, \infty) \) is continuous and monotone in the sense that \( E, F \in \mathcal{E}(X) \) and \( E \subseteq F \) imply \( f(E) \leq f(F) \). Suppose further that \( f(\lambda(x)) > 0 \) for all \( x \) in some compact subset \( T \) of \( X \). Then there is an \( \alpha > 0 \) such that \( f(\lambda(x)) \geq \alpha \) for all \( x \in T \).
Proof. If there is no such \( a \), choose a sequence \( (x_n) \) in \( T \) such that \( f(\lambda(x_n)) \to 0 \). Using first the compactness of \( S(X) \) and then that of \( T \), it may be assumed that \( \lambda(x_n) \to F \) for some \( F \in S(X) \) and that \( x_n \to x \) for some \( x \in T \). From Lemma 2.4 (ii), it follows that \( x \in F \). Consequently, \( 0 < f(\lambda(x)) \leq f(F) \) and \( f(F) = 0 \), a contradiction.

For simplex spaces, the following result is due to P. D. Taylor.

Corollary 2.9. Suppose that \( X \) is second countable and that \( f \) is a continuous complex-valued function on \( \lambda(X)^{-} \). For each \( x \in X \), let \( c(x) \) denote the set of all those \( F \in \lambda(X)^{-} \) which contain \( x \). Then \( f \circ \lambda \) is continuous on \( X \), if and only if \( f \) is constant on the sets \( c(x), x \in X \).

Proof. Notice that \( \lambda(x) \in c(x) \) for each \( x \in X \). Suppose that \( f \circ \lambda \) is continuous on \( X \). Say \( x \in X \) and \( F \in c(x) \). Then there is a sequence \( (x_n) \) in \( X \) such that \( \lambda(x_n) \to F \). From Theorem 2.4 (i), \( x_n \to x \), and

\[
 f(F) = \lim_{n \to \infty} f(\lambda(x_n)) = f(\lambda(x)) .
\]

Conversely, suppose that \( f \) is constant on the \( c(x), x \in X \). Let \( (x_n) \) be a sequence in \( X \) converging to an \( x \in X \). To show that

\[
 f(\lambda(x_n)) \to f(\lambda(x)) ,
\]

it is sufficient (since \( f(\lambda(x)) \) lies in the compact set \( f(\lambda(X)) \)) to show that every convergent subsequence of \( f(\lambda(x_n)) \) converges to \( f(\lambda(x)) \). Passing to a subsequence, suppose that \( f(\lambda(x_n)) \to \alpha \) for some complex number \( \alpha \). Using the fact that \( S(X) \) is a compact metric space and passing to a further subsequence, it may even be assumed that \( \lambda(x_n) \to F \) for some \( F \in \lambda(X)^{-} \). Then from Theorem 2.4, (ii), \( x \in F \), i.e., \( F \in c(x) \), and therefore

\[
 f(\lambda(x)) = f(F) = \lim_{n \to \infty} f(\lambda(x_n)) = \alpha .
\]

If \( G \) is a nonempty open subset of \( X \), then \( G \) is locally compact and \( T_0 \) in its relative topology. Let \( \rho_G \) be the map \( F \to F \cap G \) of \( S(X) \) onto \( S(G) \), and let \( \sigma_G \) be its restriction to \( \lambda_x(G) \). Then \( \sigma_G \circ \lambda_x = \lambda_G \) and \( \sigma_G \) is a bijection of \( \lambda_x(G) \) onto \( \lambda_G(G) \). Using the fact that \( G \) is open in \( X \), it is easily checked that \( \rho_G \) is continuous; however, \( \sigma_G \) is in general not a homeomorphism.

Lemma 2.10. Let \( G \) be a nonempty open subset of \( X \), and suppose that \( \lambda(X)^{\perp} \subset \lambda(X) \cup (X - G)^\perp \). If \( \mathcal{T} \) is a subset of \( \lambda_x(G) \) and if \( \sigma_G(\mathcal{T}) \) is compact, then so is \( \mathcal{T} \).
Proof. As $\rho_\sigma$ is continuous,
\[ \rho_\sigma(\mathcal{T}^-) \subset [\rho_\sigma(\mathcal{T})]^- = [\sigma_\sigma(\mathcal{T})]^- = \sigma_\sigma(\mathcal{T}) \subset \lambda_\sigma(G), \]
and since $\emptyset \notin \lambda_\sigma(G)$, $\mathcal{T}^- \cap (X - G) = \emptyset$. But
\[ \mathcal{T}^- \subset \lambda(X)^- \subset \lambda(X) \cup (X - G)^- \subset \lambda_x(G) \cup (X - G)^-, \]
so that $\mathcal{T}^-$ is contained in $\lambda_x(G)$, the domain of $\sigma_\sigma$. Since
\[ \sigma_\sigma(\mathcal{T}^-) = \rho_\sigma(\mathcal{T}^-) \subset \sigma_\sigma(\mathcal{T}) \]
and $\sigma_\sigma$ is one-to-one, $\mathcal{T}$ must be closed in $\mathcal{E}(X)$.

A point $x$ in $X$ will be said to be strongly separated in $X$ if for each $y \neq x$, there are disjoint neighborhoods of $x$ and $y$ (i.e., $x$ is closed, and separated in the sense of [3; §1]). A nonempty subset $Y$ of $X$ will be called strongly separated in $X$ provided each of its points is strongly separated in $X$. Finally, $X$ will be called almost strongly separated if each nonempty closed subset $F$ of $X$ contains a nonempty relatively open subset $G$ which is strongly separated in $F$ (equivalently, every open subset $U$ of $X$ distinct from $X$ is properly contained in an open subset $V$ such that $V - U$ is strongly separated in $X - U$).

Proposition 2.11. A nonempty open subset $G$ of $X$ is strongly separated in $X$ if and only if $\lambda(X)^- \subset \lambda(X) \cup (X - G)^-$. 

Proof. Assume first that $G$ is strongly separated in $X$. Suppose that there is a net $(x_a)$ in $X$ and an $F \in \lambda(X) \cup (X - G)^-$ such that $\lambda(x_a)$ converges to $F$. Then $F$ must contain two distinct points, at least one of which is in $G$, which is impossible by Theorem 2.4 (i). Conversely, suppose that $\lambda(X)^- \subset \lambda(X) \cup (X - G)^-$. From this inclusion it is immediate that $G \subset X_1$. As $\rho_\sigma(\lambda(X)^-)$ is compact and contains $\lambda_\sigma(G)$,
\[ \lambda_\sigma(G)^- \subset \rho_\sigma(\lambda(X)^-) \subset \lambda_\sigma(G) \cup \{\emptyset\}, \]
and therefore $\lambda_\sigma(G) \cup \{\emptyset\}$ is compact. For any relatively closed subset $\mathcal{T}$ of $\lambda_\sigma(G)$, $\mathcal{T} \cup \{\emptyset\}$ is compact and dilated, hence $\lambda_\sigma^{-1}(\mathcal{T})$ is a closed subset of $G$ in the relative topology (Lemma 2.2). This shows that $\lambda_\sigma$ is continuous; since it is always open onto its image, $\lambda_\sigma$ is a homeomorphism and $G$ is Hausdorff. To show that $G$ is strongly separated, suppose $x \in G$ and $y \in G$ are given. Let $U \subset G$ be a compact neighborhood of $x$; it will suffice to show that $U$ is closed in $X$. As $\lambda_\sigma(U)$ is compact and as $\lambda_\sigma(U) = \sigma_\sigma(\lambda_x(U))$, $\lambda_x(U)$ is compact (Lemma 2.10). $\lambda_x(U)$ is dilated since $U \subset X_1$, and so $U =$
\(\lambda_x^{-1}(\lambda_x(U))\) is closed, by Lemma 2.2.

A topological space which is a countable union of compact sets will be called a \(K_\sigma\).

**Lemma 2.12.** If \(X\) is second countable and if \(G\) is an open nonempty strongly separated subset of \(X\), then \(\lambda_x(G)\) is \(K_\sigma\).

*Proof.* Since \(G\) is Hausdorff, \(\lambda_\sigma(G) \subset \lambda_\sigma(G) \cup \{\emptyset\}\) by Proposition 2.11, and \(\lambda_\sigma(G)\) is locally compact. Now \(\mathcal{E}(G)\) is second countable, for as \(G\) is second countable, \(\mathcal{E}(G)\) is a compact metric space [6; Lemma 2]. Therefore \(\lambda_\sigma(G)\) is \(K_\sigma\). The equality \(\lambda_\sigma(G) = \sigma_\sigma(\lambda_x(G))\), Lemma 2.10 and Proposition 2.11 now imply that \(\lambda_x(G)\) is \(K_\sigma\).

**Lemma 2.13.** Let \(E\) be a nonempty closed subset of \(X\). Then the map \(\theta: E^\perp \rightarrow \mathcal{E}(E)\) defined by \(\theta(F) = F\) for all \(F \in E^\perp\) is a homeomorphism onto, where \(E^\perp\) has the relative topology from \(\mathcal{E}(X)\).

*Proof.* That \(\theta\) is a bijection is clear. Since \(E^\perp\) is compact Hausdorff, it is enough to show that \(\theta\) is continuous. But this follows from the definition of the topologies and the fact that \(E\) is closed.

**Lemma 2.14.** If \(X\) is almost strongly separated, so is any nonempty subset of \(X\) which is either open or closed.

*Proof.* See [11; §3].

**Theorem 2.15.** Suppose that \(X\) is second countable, locally compact, and \(T_\sigma\). Then \(X\) is almost strongly separated if and only if

1. \(X\) is \(T_1\),
2. \(\lambda(X)\) is \(K_\sigma\), and
3. every nonempty closed subset of \(X\) is second category in itself.

*Proof.* Say that (1)-(3) hold. Let \(F\) be a nonempty closed subset of \(X\). Then \(F\) is \(T_1\) and second category, and \(\lambda_F(F)\) is \(K_\sigma\) by Lemma 2.13. Replacing \(F\) by \(X\), it is therefore sufficient to show that if \(X\) satisfies (1) and (2) and is second category, then \(X\) contains a nonempty open strongly separated set. Write \(\lambda(X) = \bigcup_{n=1}^\infty \mathcal{T}_n\), where each \(\mathcal{T}_n\) is compact. Since the \(\mathcal{T}_n\) are dilated, the \(\lambda^{-1}(\mathcal{T}_n)\) are closed by Lemma 2.2. \(X\) is second category, hence for some \(n\), \(\lambda^{-1}(\mathcal{T}_n)\) contains a nonempty set \(G\) which is open in \(X\). As \(\lambda^{-1}(\mathcal{T}_n)\) is closed in \(X\) and is Hausdorff in the relative topology (Corollary
2.3), $G$ is strongly separated in $X$.

Conversely, suppose that $X$ is almost strongly separated. By a transfinite induction (see [11; Proposition 3.1]), there is an ordinal $\alpha_0$ and a family $(G_\alpha)$ of open subsets of $X$, indexed by those ordinals $\alpha$ with $0 \leq \alpha \leq \alpha_0$, such that: (i) $G_0 = \emptyset$, $G_{\alpha_0} = X$; (ii) if $\alpha \leq \alpha_0$ is a limit ordinal, then $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$; and (iii) if $\alpha < \alpha_0$, then $G_\alpha \subset G_{\alpha+1}$ and $G_{\alpha+1} - G_\alpha$ is a nonempty strongly separated subset of $X - G_\alpha$.

To see that (1) holds, say $x \in X$. Let $\beta$ be the least ordinal such that $x \in G_\beta$. By (ii), $\beta$ cannot be a limit ordinal; let $\alpha + 1 = \beta$. Then $x \in G_{\alpha+1} - G_\alpha$, so that $\{x\}$ is closed in $X - G_\alpha$, and therefore in $X$.

The natural map $\theta_\alpha$ of $(X - G_\alpha)$ onto $\mathcal{C}(X - G_\alpha)$ is a homeomorphism, where $(X - G_\alpha)$ has the relative topology from $\mathcal{C}(X)$ (Lemma 2.13). Since $\theta_\alpha$ carries $\lambda_x(G_{\alpha+1} - G_\alpha)$ onto $\lambda_{x-G_\alpha}(G_{\alpha+1} - G_\alpha)$ and since the latter is $K_\sigma$ by (iii) and Lemma 2.12, $\lambda_x(G_{\alpha+1} - G_\alpha)$ must be $K_\sigma$. Now

$$X = \bigcup_{\alpha < \alpha_0}(G_{\alpha+1} - G_\alpha)$$

by the above and $\alpha_0$ is countable (see [16; §19, II]), so (2) holds. If $F_1, F_2, \ldots$ are closed and nowhere dense subsets of $X$, then $F_1 \cap G_1, F_2 \cap G_1, \ldots$ are closed and nowhere dense in the relative topology of $G_1$. Being locally compact and Hausdorff, $G_1$ is Baire, so the $F_\alpha \cap G_1$ do not cover $G_1$. Thus $X$ is second category. By Lemma 2.14, this is enough to show that (3) holds.

**Corollary 2.16.** If $X$ is second countable and almost strongly separated, then all nonempty closed and all nonempty open subsets of $X$ are Baire.

**Proof.** This follows from Lemma 2.14 and Theorem 2.15.

Suppose that $X$ is second countable. If all nonempty closed subsets of $X$ are Baire, then $\lambda(X)$ is $G_\delta$ [6; Th. 7]; in view of [16; §30, VI], this fact may be useful in deciding whether $X$ satisfies (2) of Theorem 2.15. As examples in §4 will show, (1) and (2) are independent of one another even if all nonempty closed subsets of $X$ are Baire. The set of integers with the Zariski (or cofinite) topology is second countable, locally compact, $T_\sigma$, and satisfies conditions (1) and (2), but not (3), of Theorem 2.15.

3. **C*-Algebras.** Let $A$ be a $C^*$-algebra. Throughout this section and the next, an ideal in $A$ will always mean a closed two-sided ideal. Let $Z(A)$ be the center of $A$, and let $\text{Id}(A)$ [resp.,
Prim \( (A) \), Max \( (A) \), and Mod \( (A) \) denote the set of all ideals [primitive ideals, maximal ideals, and modular ideals] in \( A \). For \( a \in A \) and \( I \in \text{Id} \( (A) \) \), define \( a(I) \) as the canonical image of \( a \) in \( A/I \) and \( I^\perp \) as the set of all those ideals \( J \) in \( A \) which contain \( I \). Prim \( (A) \) with the hull-kernel topology (sometimes called the structure, or Jacobson, topology) is the structure space of \( A \). The following facts about the structure space (see [4]) will be used frequently without explicit mention: its closed points are the elements of Max \( (A) \); it is locally compact and \( T_0 \); it is second countable whenever \( A \) is separable; and \( I \to \text{Prim} \( (A) \cap I^\perp \) \) is a one-to-one correspondence between Id \( (A) \) and the closed subsets of Prim \( (A) \). The weakest topology on Id \( (A) \) making each of the maps \( I \to ||a(I)|| \), \( a \in A \), continuous will be called the weak* topology on Id \( (A) \). It is not hard to show that \( I \to \text{Prim} \( (A) \cap I^\perp \) \) is a homeomorphism of Id \( (A) \) onto \( \scr{C}(\text{Prim} \( (A) \)) \) which restricts to \( \lambda \) on Prim \( (A) \) and carries \( I^\perp \) onto (Prim \( (A) \cap I^\perp \))^\perp \) (where the second \( \perp \) is taken in the sense of §2) [12, Th. 2.2]. In what follows, Id \( (A) \) and \( \scr{C}(\text{Prim} \( (A) \)) \) will be identified. Recall that if \( A \) is separable, Id \( (A) \) and Prim \( (A) \) with the weak* topology may be identified with the spaces \( N(A) \) and \( EN(A)-\{0\} \) of §1.

In view of the above, the results of §2 may be applied to \( C^* \)-algebras. Save for one, these will not be explicitly mentioned. For any \( a \in A \), \( I \to ||a(I)|| \) is a function of the type described in Corollary 2.8. This has the following amusing consequence: If \( A \) is separable and if \( T \) is a structurally compact subset of Max \( (A) \), then \( \bigcup \{P; P \in T\} \) is a norm-closed subset of \( A \).

A nonzero ideal \( I \) in \( A \) will be called an \( M \)-ideal in \( A \) if \( \text{Prim} \( (A) \cap I^\perp \) \) is a strongly separated subset of the structure space of \( A \), and \( A \) will be called an \( M \)-algebra [resp., a GM-algebra] if the structure space of \( A \) is Hausdorff [almost strongly separated]. Clearly \( A \) is an \( M \)-algebra if and only if \( A \) is an \( M \)-ideal in itself. Using [4; §3.2], it is easily verified that \( A \) is a GM-algebra if and only if every nonzero quotient of \( A \) contains a nonzero \( M \)-ideal.

**Proposition 3.1.** The following are equivalent for a nonzero ideal \( I \) in a \( C^* \)-algebra \( A \):

1. \( I \) is an \( M \)-ideal
2. \( \text{Prim} \( (A) \cap I^\perp \subseteq \text{Max} \( (A) \cup I^\perp \), where \( \text{Prim} \( (A) \cap I^\perp \) is the weak* closure of \( \text{Prim} \( (A) \) in \( \text{Id} \( (A) \) \n
3. for each \( a \in I \), \( P \to ||a(P)|| \) is continuous on \( \text{Prim} \( (A) \) in the structure topology.

**Proof.** (1) \( \Rightarrow \) (2): This is Proposition 2.11.

(1), (2) \( \Rightarrow \) (3): Suppose that an \( a \in I \) and an \( \alpha > 0 \) are given. The map \( p \to ||a(P)|| \) is lower semi-continuous on Prim \( (A) \) with the
structure topology, so it is enough to show that $T = \{ P \in \text{Prim}(A) : \| a(P) \| \geq \alpha \}$ is structurally closed. Now $T$ is a structurally compact subset of $\text{Prim}(A) - I^\perp$, and as $I$ is an $M$-ideal in $A$, $\text{Prim}(A) - I^\perp$ is Hausdorff in the relative structure topology. The map $\sigma$ which sends $P$ into $P \cap I$ is a homeomorphism of $\text{Prim}(A) - I^\perp$ onto $\text{Prim}(I)$ for the structure topologies, hence the structure space of $I$ is Hausdorff. From Lemma 2.1, this means that the structure and weak* topologies coincide on $\text{Prim}(I)$. Then $\sigma(T)$ is a weak* compact subset of $\text{Prim}(I)$, and $T$ is a weak* compact subset of $\text{Prim}(A)$ (Lemma 2.10). Since $T$ is contained in $\text{Max}(A)$, it is dilated and therefore structurally closed by Lemma 2.2.

(3) $\Rightarrow$ (1): Say $P \in \text{Prim}(A) - I^\perp$ and $Q \in \text{Prim}(A)$ are distinct. If $Q \in I^\perp$, choose an $a \in I$ with $\| a(P) \| = 2$. Then $\{ R \in \text{Prim}(A) : \| a(R) \| > 1 \}$ and $\{ R \in \text{Prim}(A) : \| a(R) \| < 1 \}$ are disjoint structurally open sets containing $P$ and $Q$, resp. Now suppose that $Q \not\in I^\perp$. For $R \in \text{Prim}(A) - I^\perp$ and $a \in I$, $R \cap I \in \text{Prim}(I)$ and

$$\| a(R \cap I) \| = \max \{ \| a(R) \| , \| a(I) \| \} = \| a(R) \| .$$

This equality together with the homeomorphism $\sigma$ of the previous paragraph implies that the structure and weak* topologies on $\text{Prim}(I)$ coincide, and therefore that $\text{Prim}(A) - I^\perp$ is Hausdorff in the relative structure topology. As $\text{Prim}(A) - I^\perp$ is a structurally open subset of $\text{Prim}(A)$, there are disjoint structure neighborhoods of $P$ and $Q$.

**Theorem 3.2.** If $A$ is a separable C*-algebra, then $\text{Prim}(A)$ is a $G_\delta$ in the weak* topology, and $A$ is a GM-algebra if and only if

1. $\text{Max}(A) = \text{Prim}(A)$, i.e., the structure space of $A$ is $T_1$, and
2. $\text{Prim}(A)$ is $K_\sigma$ in the weak* topology.

**Proof.** This is an immediate consequence of Theorem 2.15, [6; Th. 7], and the fact that all nonempty closed subsets of the structure space are Baire [4; Corollaire 3.4.13].

Section 4 contains examples which show that neither (1) nor (2) is a consequence of the other, even for separable C*-algebras. This completes the analogy between GM-simplex spaces and GM-C*-algebras. In studying the second class of C*-algebras, the following two lemmas will be useful.

**Lemma 3.3.** For any ideal $I$ in a C*-algebra $A$, $Z(I) = I \cap Z(A)$.

**Proof.** See [1; Lemma 6].

**Lemma 3.4.** The following are equivalent for a C*-algebra $A$:
(i) \( Z(A) \not\subseteq P \) for each \( P \in \text{Prim}(A) \) and the structure space of \( A \) is Hausdorff, and
(ii) \( P \to P \cap Z(A) \) is a one-to-one map from \( \text{Prim}(A) \) into \( \text{Prim}(Z(A)) \).

If these conditions are satisfied, then the map in (ii) is a homeomorphism of \( \text{Prim}(A) \) onto \( \text{Prim}(Z(A)) \) for the structure topologies.

Proof. For the equivalence of (i) and (ii), see [1; Proposition 3] or [18; Corollary 3.1.2]. The last statement is contained in [15; Th. 9.1].

A \( C^* \)-algebra satisfying one of the equivalent conditions of the last lemma is called central; for other equivalent definitions, see [1; Proposition 3].

Several results from [7; §4] will now be recalled. Consider an \( a \in Z(A) \) and a primitive ideal \( P \) in \( A \). Choose an irreducible representation \( \pi \) of \( A \) with kernel \( P \). As \( \pi(a) \) is in the center of \( \pi(A) \), it must be a multiple \( \alpha \) of the identity operator on the space of \( \pi \). Then \( \pi(a)\pi(b) = \alpha \pi(b) \), i.e., \( ab - \alpha b \in P \), for all \( b \in A \). This last condition determines \( \alpha \) uniquely, and shows that it depends only on \( P \) (and not on \( \pi \)). Set \( f_\alpha(P) = \alpha \). The function \( f_\alpha \) is clearly bounded on \( \text{Prim}(A) \). It is easy to show that \( \varphi(a) = f_\alpha(P) \) for any \( \varphi \in \theta^{-1}(P) \), where \( \theta \) is the natural mapping of \( P(A) \), the pure states on \( A \), onto \( \text{Prim}(A) \).

Because \( \theta \) is an open map,

\[
\theta^{-1}(U) = \{P \in \text{Prim}(A): f_\alpha(P) \in U\}
= \theta(\{\varphi \in P(A): f_\alpha(\varphi) \in U\})
= \theta(\{\varphi \in P(A): \varphi(a) \in U\})
\]

is structurally open for any open set \( U \) of complex numbers. This shows that \( f_\alpha \) is structurally continuous. If \( A \) is central, then \( P \in \text{Prim}(A) \) implies \( P \cap Z(A) \in \text{Max}(Z(A)) = \text{Prim}(Z(A)) \), and regarding \( a \in Z(A) \) as a function on \( \text{Max}(Z(A)) \), \( f_\alpha(P) = a(P \cap Z(A)) \). Since \( Z(A) \cong C_0(\text{Max} Z(A)) \), we may identify the functions \( f_\alpha \) with \( C_0(\text{Prim}(A)) \).

A \( C^* \)-algebra \( A \) will be said to have local identities if given \( P_0 \in \text{Prim}(A) \), there is an \( a \in A \) such that \( a(P) \) is an identity in \( A/P \) for all \( P \) in some structure neighbourhood of \( P_0 \). A nonzero ideal \( I \) in \( A \) will be called a \( C \)-ideal in \( A \) if \( I \) is a central \( C^* \)-algebra. \( A \) will be called a \( C \)-algebra if it is a \( C \)-ideal in itself (i.e., is central), and a \( GC \)-algebra if every nonzero quotient of \( A \) contains a nonzero \( C \)-ideal.

PROPOSITION 3.5. A nonzero ideal \( I \) in \( A \) is a \( C \)-ideal if and
only if it is an M-ideal with local identities.

Proof. Suppose that \( I \) is a C-ideal. Let \( P \) and \( Q \) be distinct primitive ideals in \( A \) with \( P \notin I \). If \( Q \notin I \), then since \( I \) is central, \( P \cap Z(I) \) and \( Q \cap Z(I) \) are distinct maximal ideals in \( Z(I) \) hence there is an \( a \in Z(I) \subset Z(A) \) with \( f_a(P) \neq 0 \) and \( f_a(Q) = 0 \). If \( Q \in I \), let \( a \) be any element of \( Z(I) \) with \( a(P) \neq 0 \). Then \( f_a \) will provide disjoint neighborhoods for \( P \) and \( Q \), and \( A \) is an M-ideal.

Thus it suffices to show that a C*-algebra \( A \) is a C-algebra if and only if it is an M-algebra with local identities. If \( A \) is a C-algebra, \( Z(A) \) may be identified with \( C_0(\text{Prim}(A)) \), hence it is trivial that \( A \) has local identities. Conversely, suppose that \( A \) is an M-algebra with local identities. Say \( P_0 \in \text{Prim}(A) \), and choose an \( a \in A \) such that \( a(P) \) is an identity in \( A/P \) for all \( P \) in some neighborhood \( T \) of \( P_0 \). Consider a continuous bounded complex-valued function \( f \) on \( \text{Prim}(A) \) with \( f(P_0) = 1 \) and whose support is contained in \( T \). From the Dauns-Hofmann theorem (see [7; §7]), there is a \( b \in A \) such that \( b(P) = f(P)a(P) \) for all \( P \in \text{Prim}(A) \). Then \( (bc - cb)(P) = 0 \) if \( c \in A \) and \( P \in \text{Prim}(A) \), so that \( b \in Z(A) \). Since \( b \in P_0 \), \( A \) must be a C-algebra.

**Lemma 3.6.** For a nonzero C-ideal \( I \) in \( A \),
1. \( P \mapsto ||a(P)|| \) is structurally continuous on \( \text{Prim}(A) - I \) for each \( a \in A \), and
2. \( \text{Prim}(A)^- \subset [\text{Max}(A) \cap \text{Mod}(A)] \cup I \).

**Proof.** To prove (1), fix \( a \in A \), and suppose \( P_0 \in \text{Prim}(A) - I \) is given. It is sufficient to show that \( P \mapsto ||a(P)|| \) is structurally continuous on some structure neighborhood of \( P_0 \). From the structure homeomorphism of \( \text{Prim}(A) - I \) onto \( \text{Prim}(I) \) and the fact that \( I \) has local identities, there is a structure neighborhood \( T \) of \( P_0 \) contained in \( \text{Prim}(A) - I \) and a \( b \in I \) such that \( b(P \cap I) \) is an identity in \( I/(P \cap I) \) for each \( P \in T \). As \( I \) is an M-ideal in \( A \), each \( P \in T \) is a structurally closed point in \( \text{Prim}(A) \), and so is a maximal ideal. Therefore \( P + I = A \) and there is an *-isomorphism of \( A/P \) onto \( I/(I \cap P) \) which carries \( c(P) \) into \( c(I \cap P) \), \( c \in I \) [4; Corollaire 1.8.4]. Hence \( b(P) \) is an identity in \( A/P \) for each \( P \in T \), and since \( ab \in I \), Proposition 3.1 implies that \( P \mapsto ||(ab)(P)|| = ||a(P)|| \) is structurally continuous on \( T \). Turning to (2), suppose \( P \in \text{Prim}(A)^- \), \( P \notin I \).

Since \( I \) is an M-ideal in \( A \), Proposition 3.1 gives \( P \in \text{Max}(A) \). As \( I \) is central, there is an \( a \in Z(I) \subset Z(A) \) with \( a \notin P \). Since \( a(P) \) is a nonzero central element of \( A/P \), \( P \) must be modular.

In the case of simplex spaces, the analogues of (1) and (2) of the previous lemma are each equivalent to \( I \) being a C-ideal. This is not
the case for $C^*$-algebras. In fact, there is an example of a noncentral $C^*$-algebra $A$ which satisfies (1) and (2) with $I$ replaced by $A$, viz, the algebra of all functions $a$ from $\{1, 2, \cdots\}$ into the two-by-two matrices with complex entries such that $\lim_{n \to \infty} a_{ij}(n)$ exists and is equal to zero unless $i = j = 1$ (this example was also used by Delaroche in [2; § 6]).

The following result is due to Delaroche [2, Proposition, 14].

**Theorem 3.7.** A separable $C^*$-algebra $A$ is a GC-algebra if and only if

1. $A$ is a GM-algebra, and
2. every primitive ideal in $A$ is modular.

**Proof.** Suppose that $A$ is a GC-algebra. Then by Proposition 3.5, $A$ is a GM-algebra. If $P \in \text{Prim}(A)$, then since $P$ is a maximal ideal in $A$ (Theorem 3.2), $A/P$ must be central. But then $A/P$ is primitive and has a nontrivial center, implying that $P$ is modular.

Conversely, suppose that (1) and (2) hold, and let $I \neq A$ be an ideal in $A$. From Lemma 2.14, $A/I$ is a GM-algebra. Since any primitive ideal in $A/I$ is of the form $P/I$ for some $P \in \text{Prim}(A) \cap I$ [4; Proposition 2.11.5 (i)], and since $(A/I)/(P/I) \cong A/P$ for such $P$, every primitive ideal in $A/I$ is modular. So to show that $A$ is a GC-algebra, it is only necessary to show that $A$ possesses a nonzero C-ideal. Let $I$ be a nonzero $M$-ideal in $A$. The structure space of $I$, being homeomorphic to $\text{Prim}(A) - I^\circ$ with the relative structure topology [4; Proposition 3.2.1], is Hausdorff. Since any $P \in \text{Prim}(A) - I^\circ$ is a maximal ideal in $A$, $P + I = A$ and $I/(P \cap I) \cong (P + I)/P = A/P$ [4; Corollaire 1.8.4]. So any primitive ideal in $I$, being of the form $P \cap I$ for some $P \in \text{Prim}(A) - I^\circ$, must be modular. This and [4; Proposition 1.8.5] show that it is sufficient to establish the following: If $A$ is a separable $C^*$-algebra all of whose primitive ideals are modular and whose structure space is Hausdorff, then $A$ has a nonzero C-ideal.

For such a $C^*$-algebra $A$, the structure and weak* topologies coincide on $\text{Prim}(A)$ (Lemma 2.1). Let $1_P$ be the identity in $A/P$, $P \in \text{Prim}(A)$. Let $(u_n)$ be an approximate identity in $A$ indexed on the positive integers, and set

$$T_n = \{P \in \text{Prim}(A) : \|u_n(P) - 1_P\| \leq 1/2\},$$

$n = 1, 2, \cdots$. Since $u_n(P) \to 1_P$ as $n \to \infty$ for each $P$, $\text{Prim}(A) = \bigcup_{n=1}^{\infty} T_n$. Let $A'$ be the $C^*$-algebra obtained by adjoining an identity 1 to $A$. Then $\text{Prim}(A') \cong \text{Prim}(A) \cup \{A\}$ and $A^1 = \{A\}$. Fix a $P' \in \text{Prim}(A') - A^1$, and set $P = P' \cap A$. Then $a(P) \to a(P')$, $a \in A$, is an isomorphism of $A/P$ onto $(A + P')/P'$. Choose a $b \in A$ such that
\( b(P) = 1_p \). Then \( b(P') \) must be an identity in \( (A + P')/P' \). The latter is an ideal in \( A'/P' \), and from Lemma 3.3, \( b(P') \) is a central idempotent in \( A'/P' \). Since \( A'/P' \) is primitive, \( b(P') = 1(P') \). Consequently,
\[
\| (u_n - 1)(P') \| = \| (u_n - b)(P') \| = \| (u_n - b)(P) \| = \| u_n(P) - 1_P \|. 
\]
Therefore
\[
T_n = \{ P' \cap A : P' \in \text{Prim}(A') \text{ and } \| (u_n - 1)(P') \| \leq 1/2 \},
\]
and \( T_n \) is a closed subset of \( \text{Prim}(A) \). Since the structure space of \( A \) is Baire [4; Corollaire 3.4.13], some \( T_n \) contains a nonempty open set \( T \). Because \( u_n \geq 0 \) and \( \| u_n \| \leq 1 \), \( \text{Sp } u_n(P) \subset [1/2, 1] \) for each \( P \in T \). Choosing a continuous real-valued function \( f \) on \([0, 1] \) with \( f(0) = 0 \) and \( f = 1 \) on \([1/2, 1] \) and setting \( a = f(u_n), a(P) = 1_p \) for each \( P \in T \) [4; Proposition 1.5.3]. Let \( I \) be the ideal in \( A \) with \( \text{Prim}(A) - I = T \). Say \( P \in T \). Since \( \text{Prim}(A) \) is locally compact and Hausdorff, there is a continuous bounded function \( g \) on \( \text{Prim}(A) \) such that \( g(P) = 1 \) and \( g \) vanishes off \( T \). From the Dauns-Hofmann theorem (see [7; §7]), there is a \( b \in A \) with \( b(Q) = g(Q)a(Q) \) for all \( Q \in \text{Prim}(A) \). Then \( b(Q) = 0 \) if \( I \subseteq Q \subseteq \text{Prim}(A) \) and \( (bc - cb)(Q) = 0 \) if \( c \in A \) and \( Q \in \text{Prim}(A) \), which imply (by [4; Th. 2.9.7 (ii)]) that \( b \in Z(I) \). Therefore \( I \) satisfies condition (i) of Lemma 3.4, and so is a \( C \)-ideal in \( A \). This completes the proof of Theorem 3.7.

It is not known whether the conclusion of Theorem 3.7 is true for nonseparable \( C^* \)-algebras.

4. Concluding remarks. Let \( A \) be a \( C^* \)-algebra. Recall that \( A \) is a \( CCR \)-algebra ("liminaire") if the image of \( A \) by any irreducible representation is contained in the algebra of compact operators on the representing Hilbert space. A nonzero ideal \( I \) in \( A \) is a \( CCR \)-ideal in \( A \) if it is a \( CCR \)-algebra, and \( A \) is a \( GCR \)-algebra ("post-liminaire") if every nonzero quotient of \( A \) contains a nonzero \( CCR \)-ideal.

The spectrum of \( A \) is the set \( \hat{A} \) of all equivalence classes of irreducible representations of \( A \) provided with the inverse image topology by the natural map \( \pi \to \text{Ker } \pi \) of \( \hat{A} \) onto the structure space of \( A \). Dixmier [4; § 4.5] has shown that the closure \( J(A) \) of the finite linear combinations of those \( a \in A^+ \) for which \( \pi \to \text{Tr } \pi(a) \) is finite and continuous on \( \hat{A} \) is an ideal in \( A \). A nonzero ideal \( I \) in \( A \) will be called a \( CTC \)-ideal in \( A \) if \( I \subseteq J(A) \), and \( A \) will be called a \( CTC \)-algebra [resp., \( GTC \)-algebra] if \( A \) is a \( CTC \)-ideal in itself [every
nonzero quotient of $A$ contains a nonzero $CTC$-ideal]. These algebras have been studied in the literature, where they are sometimes called "$C^*$-algèbre à trace continue" ["$C^*$-algèbre à trace continue généralisée"]). Recall that a $CTC$-algebra has Hausdorff structure space and that a $GTC$-algebra is $CCR$ ([4; §4]).

A $CCR$-algebra $A$ with a Hausdorff structure space will be said to satisfy the Fell condition if the canonical field of $C^*$-algebras defined by $A$ satisfies the Fell condition of Dixmier [4; §10.5]. This amounts to saying that given $P_0 \in \text{Prim}(A)$, there is an $a \in A$ such that $\alpha(P)$ is a one-dimensional projection in $A/P$ for all $P$ in some structure neighborhood of $P_0$. The following are some of the relations between the various classes of $C^*$-algebras:

1. if $A$ is separable, then it is both $GM$ and $GCR$ if and only if it is $GTC$ ([5; Proposition 4.2]),
2. if $A$ is separable, then it is both $GC$ and $GCR$ if and only if it is $GTC$ and all its irreducible representations are finite-dimensional ((1) and Theorem 3.7),
3. $A$ is $GCR$ and $M$ and satisfies the Fell condition if and only if it is $CTC$ ([4; Propositions 4.5.3 and 10.5.8]; recall that $A$ is $CCR$ if it is $GCR$ and $M$),
4. $A$ is a central $GCR$-algebra and satisfies the Fell condition if and only if it is a $CTC$-algebra with local identities ((3) and Proposition 3.7), and
5. if $A$ is separable, then it is $GM$ if either it is a $CCR$-algebra with compact structure space or its irreducible representations are all finite-dimensional ([3; §1]).

Let $H$ be a separable infinite-dimensional Hilbert space. Let $B$ denote the $C^*$-algebra obtained by adjoining an identity to $CC(H)$, the compact operators on $H$. The structure space of $B$ (see [4; Exercise 4.7.14 (a)]) fails to be $T_1$, and therefore is not almost strongly separated. Yet $\text{Prim}(B)$ is $K_\sigma$ in the weak* topology.

In [3; §2], Dixmier has constructed a separable $CCR$-algebra $D$ whose structure space contains no nonempty strongly separated subset. In particular, $D$ is not $GM$. Nevertheless, there is an open subset of the structure space of $D$ which is homeomorphic to $[0, 1]$, and $D$ contains an ideal $C$ isomorphic to the $C^*$-algebra of continuous maps of $[0, 1]$ into $CC(H)$. So $C$ is an $M$-algebra, yet no nonzero ideal in $C$ is an $M$-ideal in $D$. Since $D$ is a $CCR$-algebra, $\text{Prim}(D)$ is $T_1$ in the structure topology, so that $\text{Prim}(D)$ cannot be $K_\sigma$ in the weak* topology (Theorem 3.2). These two examples are the ones promised after Theorems 2.15 and 3.2.

Finally, one further point of contact between $C^*$-algebras and simplex spaces will be mentioned. Fell has shown that a $C^*$-algebra
A can be described (to within isomorphism) as the set of all functions on $\text{Prim}(A)^{-}$ satisfying certain conditions, the value of such a function at an $I \in \text{Prim}(A)^{-}$ being an element of $A/I$ [12]. Moreover, the Dauns-Hofmann theorem (see [7; §7]) may be deduced from this representation theorem [Fell, unpublished]. There is an analogous representation theorem for simplex spaces, due to Effros [10; Corollary 2.5]. The analogue of the Dauns-Hofmann theorem for simplex spaces can be deduced from this representation theorem (however, this is not the manner in which it is proven in the literature; cf. [10; Th. 2.1]).

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