

# Pacific Journal of Mathematics

**INTEGRAL DOMAINS THAT ARE NOT EMBEDDABLE IN  
DIVISION RINGS**

JOHN DAUNS

## INTEGRAL DOMAINS THAT ARE NOT EMBEDDABLE IN DIVISION RINGS

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**A class of totally ordered rings  $V$  is constructed having the property  $1 < \alpha \in V \Rightarrow 1/\alpha \in V$ , but such that  $V$  cannot be embedded in any division ring.**

1. **Inverses in semigroup power series rings.** This note has only one objective—to construct the above class of counterexamples (see [6]).

NOTATION 1.1. Throughout  $\Gamma$  will be a totally ordered cancellative semigroup with identity  $e$ ;  $R$  will denote any totally ordered division ring. If  $\alpha: \Gamma \rightarrow R$  is any function, then the *support* of  $\alpha$  is the set  $\text{supp } \alpha = \{s \in \Gamma \mid \alpha(s) \neq 0\}$ . The set  $V = V(\Gamma, R)$  of all functions  $\alpha$  such that  $\text{supp } \alpha$  satisfies the a.c.c. (ascending chain condition) form a totally ordered abelian group. If  $\Gamma$  is cancellative, then under the usual power series multiplication (see [3]),  $V$  is a totally ordered ring.

1.2. Any  $1 < \alpha \in V$  with  $\alpha(s) = 0$  for  $s > e$  may be written as  $\alpha = \alpha(e)(1 - \lambda)$ , where  $1 \leq \alpha(e)$  and  $\lambda = \sum \{\lambda(a)a \mid a < e\}$ . It will be shown that

$$(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \dots = 1 + \sum_s \sum' \lambda(a(1))\lambda(a(2)) \dots \lambda(a(n)),$$

where the finite sum  $\Sigma'$  is over all integers and over all distinct  $n$ -tuples of  $\Gamma^n$  satisfying  $s = a(1)a(2) \dots a(n)$  with each  $a(i) < e$ ; the sum  $\Sigma$  is over all  $s < e$ . To prove that  $1/\alpha \in V$  it suffices to establish conditions (a) and (b) below.

- (a) For each  $s \in \Gamma$ , there are only a finite number of  $n$  with  $\lambda^n(s) \neq 0$ ;
- (b)  $\text{supp}(1 - \lambda)^{-1}$  satisfies the a.c.c.

Assuming (a) and (b), the main theorem follows at once. By adjoining an identity as in [8; p. 158] to the semigroup in [2] a semigroup that actually satisfies the hypothesis in (ii) below can be constructed.

**MAIN THEOREM 1.3.** *If  $\Gamma$  is a totally ordered cancellative semigroup with identity  $e$  and  $R$  any totally ordered division ring, then the power series ring  $V = V(\Gamma, R)$  has the following properties:*

- (i)  $1 < \alpha \in V$  and  $\alpha(s) = 0$  for  $s > e \implies 1/\alpha \in V$ .
- (ii) *If in addition  $\Gamma$  cannot be embedded in a group, then  $V$*

cannot be embedded in a division ring.

An already known result ([8; p. 135]) follows immediately from 1.3 (i).

**COROLLARY 1.4.** *If in addition  $\Gamma$  is a group, then  $V(\Gamma, R)$  is a division ring.*

**2. Proof of the main theorem.** Assume 1.2 (a) or (b) fails. Then a lengthy but elementary argument shows there exists a doubly indexed matrix  $\{a(i, j) \in \text{supp } \lambda \mid 1 \leq i < \infty; 1 \leq j \leq n(i)\}$  such that the products  $u(i) = a(i, 1)a(i, 2) \cdots a(i, n(i))$  of the rows form an infinite properly ascending chain. Eventually a contradiction will be derived from this. Without loss of generality assume  $\Gamma \leq e$ .

**DEFINITION 2.1.** For any totally ordered semigroup  $\Gamma$  with identity  $e$  and any element  $a \in \Gamma$  with  $a \leq e$ , define a semigroup by

$$\Gamma(a) = \{q \in \Gamma \mid \exists \text{ an integer } m > 0, q^m \leq a\}.$$

**LEMMA 2.2.** *With  $\Gamma$  as above, for any  $a(1), \dots, a(m) \in \Gamma$  with each  $a(j) \leq e$ , set  $u = a(1)a(2) \cdots a(m)$  and define*

$$a^* = \min \{a(1), \dots, a(m)\}.$$

*Then  $\Gamma(u) = \Gamma(a^*)$ .*

**2.3.** Consider a fixed subset  $L \subseteq \Gamma$  all of whose elements satisfy  $L \leq e$  and where  $L$  satisfies the a.c.c., e.g.,  $L = \text{supp } \lambda < e$ . Consider an array of elements  $A = \|a(i, j)\|$  with  $\{a(i, j) \mid 1 \leq i < \infty, 1 \leq j \leq n(i)\} \subseteq L$ , where repetitions in the  $a(i, j)$  are allowed. Assume all  $n(i) \geq 2$ . Define  $u(i) = u(i, A)$  by

$$u(i) = u(i, A) = a(i, 1)a(i, 2) \cdots a(i, n(i)).$$

Let  $\mathcal{A}$  be the set of all such  $A = \|a(i, j)\|$  for which  $u(1) < u(2) < \cdots < u(i) < \cdots$  is strictly ascending at each  $i$ . With each member  $A = \|a(i, j)\| \in \mathcal{A}$ , we next associate three objects

$$\{a(i)^* \mid 1 \leq i < \infty\}, m = m(A), \text{ and } G = G(A).$$

Define  $a(i)^* \equiv \min \{a(i, j) \mid 1 \leq j \leq n(i)\}$ . Note that  $u(1) < u(2) < \cdots$  implies that  $\Gamma(a(1)^*) \subseteq \Gamma(a(2)^*) \subseteq \Gamma(a(i)^*) \subseteq \cdots$ . Thus since  $L$  satisfies the a.c.c., there is a unique smallest integer  $m \equiv m(A)$  such that the semigroups  $G \equiv \Gamma(a(m)^*) = \Gamma(a(m+1)^*) = \cdots$  are all equal. The following schematic diagram of all these quantities may be helpful.

$$\begin{aligned}
 \Gamma(a(1)^*) &= \Gamma(u(1)) & u(1) &= a(1, 1)a(1, 2) \cdots a(1)^* \cdots a(1, n(1)) \\
 &\quad \cap \parallel \\
 \Gamma(a(2)^*) &= \Gamma(u(2)) & u(2) &= a(2, 1)a(2, 2) \cdots a(2)^* \cdots a(2, n(2)) \\
 &\quad \cap \parallel \\
 \Gamma(a(m)^*) &= \Gamma(u(m)) & u(m) &= a(m, 1)a(m, 2) \cdots a(m)^* \cdots a(m, n(m)) \\
 &\quad \parallel \\
 \mathbf{G} &= \Gamma(u(m+1)) .
 \end{aligned}$$

2.4. Among the elements of  $\mathcal{K}$ , let  $\mathcal{N} \subset \mathcal{K}$  be all those  $A = \parallel a(i, j) \parallel$  such that this associated  $\mathbf{G} = \mathbf{G}(A)$  is as big as possible and call this particular  $\mathbf{G} \equiv \mathbf{M}$ . If  $\mathcal{K} \neq \emptyset$ , also  $\mathcal{N} \neq \emptyset$ . Define  $\bar{a} = \max \{a(m)^* \mid A \in \mathcal{K}, m = m(A)\}$ . Pick an element  $B = \parallel b(i, j) \parallel \in \mathcal{N}$ . Then by our choice of  $\mathbf{M}$ ,  $\Gamma(\bar{a}) = \mathbf{M}$ . Thus  $\mathbf{M} = \mathbf{G}(B) = \Gamma(b(i)^*) = \Gamma(b(i, j)) = \Gamma(u(i)) = \Gamma(\bar{a})$  for  $i \geq m(B) \equiv m$ . Finally, with each element  $B$  of  $\mathcal{N}$ , we associate an integer  $r = r(B)$ . Since  $\bar{a} \in \Gamma(u(m))$ , there is a unique smallest integer  $r \equiv r(B) \geq 1$  such that  $\bar{a}^r \leq u(m) < \bar{a}^{r-1}$ .

2.5. By omitting some of the rows of  $B$  and renumbering the remaining ones, it may be assumed as a consequence of the a.c.c. without loss of generality that  $m = 1$ , and also that  $b(1)^* \geq b(2)^* \geq \cdots$  is not ascending. Each  $u(i)$  is of one of the following three forms:

$$\begin{aligned}
 (1) \quad & u(i) = q(i)b(i)^* , \\
 (2) \quad & u(i) = b(i)^*w(i) , \\
 (3) \quad & u(i) = q(i)b(i)^*w(i) ,
 \end{aligned}$$

where the  $q(i)$ ,  $w(i)$  are certain products of the  $b(i, j)$ . If there are an infinite number of  $u(i)$  of the forms (1) or (2), then since

$$\begin{aligned}
 u(i+1) &= q(i+1)b(i+1)^* > u(i) = q(i)b(i)^*, \quad b(i+1)^* \leq b(i)^* \\
 &\implies q(i+1) > q(i) ;
 \end{aligned}$$

it follows (after omitting some rows and renumbering) that there is a properly infinite ascending chain:

$$\text{Case 1. } q(1) < q(2) < \cdots ;$$

$$\text{Case 2. } w(1) < w(2) < \cdots .$$

If neither Case 1 nor Case 2 applies, then

$$\begin{aligned}
 u(i+1) &= q(i+1)b(i+1)^*w(i+1) > q(i)b(i)^*w(i) \\
 &\quad \text{and } b(i+1)^* \leq b(i)^*
 \end{aligned}$$

implies that one of the inequalities  $q(i+1) > q(i)$  or  $w(i+1) > w(i)$

must necessarily hold. It is asserted that there is a subsequence  $\{i(k) \mid k = 1, 2, \dots\}$  such that

$$\begin{aligned} \text{Case 3. either (a): } & q(i(1)) < q(i(2)) < \dots \\ & \text{or (b): } w(i(1)) < w(i(2)) < \dots. \end{aligned}$$

For if not, then the a.c.c. must hold in both the sets  $\{q(i)\}$  and  $\{w(i)\}$ . Then by omitting some rows and renumbering the remaining ones it may be assumed that we have an element  $B$  in  $\mathcal{N}$  with  $q(1) \geq q(2) \geq \dots$  and  $w(1) \geq w(2) \geq \dots$ . However, then

$$q(1)b(1)^*w(1) \geq q(2)b(2)^*w(2) \geq \dots$$

gives a contradiction.

2.6. We may assume  $q(1) < q(2) < \dots$  or  $w(1) < w(2) < \dots$  are properly ascending, depending on which of the Cases 1, 2, 3(a) or 3(b) is applicable. Set  $t = r(B)$ , so that  $\bar{a}^t \leq u(m) = u(1) \leq u(i)$ .

2.7. It is next shown that either  $q(i) \geq \bar{a}^{t-1}$  or  $w(i) \geq \bar{a}^{t-1}$  holds for all  $i$ . Suppose that the following holds.

$$\begin{aligned} \text{Case 1. } & q(1)b(1)^* < q(2)b(2)^* < \dots; \\ & q(1) < q(2) < \dots; \\ & b(1)^* \geq b(2)^* \geq \dots. \end{aligned}$$

Then  $\bar{a}^t \leq u(1) \leq u(i) = q(i)b(i)^*$ , and  $\bar{a} \geq b(i)^*$  implies that

$$\bar{a}^{t-1} \leq q(1) \leq q(i).$$

(For if  $\bar{a}^{t-1} > q(i)$ , then  $\bar{a} \geq b(i)^*$  implies that  $\bar{a}^t > q(i)b(i)^*$ .) (If  $t = 1$ , then  $\bar{a}^0 = e$ .) Similarly, in Case 2 also  $\bar{a}^{t-1} \leq w(1) \leq w(i)$ .

Only Case 3(b) will be proved, since 3(a) is entirely parallel.

$$\begin{aligned} \text{Case 3(b). } & q(1)b(1)^*w(1) < q(2)b(2)^*w(2) < \dots; \\ & w(1) < w(2) < \dots; \\ & b(1)^* \geq b(2)^* \geq \dots. \end{aligned}$$

Then again  $\bar{a}^t \leq u(1) \leq u(i) = q(i)b(i)^*w(i)$  and  $\bar{a} \geq b(i)^* \geq q(i)b(i)^*$  imply that  $\bar{a}^{t-1} \leq w(1) \leq w(i)$ . (Otherwise, if  $\bar{a}^{t-1} > w(i)$ , then  $\bar{a}^t > q(i)b(i)^*w(i)$ .)

The basic idea motivating the proof is that for  $B \in \mathcal{N}$ , a new  $C \in \mathcal{N}$  can be constructed with  $r(C) \leq r(B) - 1$ .

2.8. Thus either  $q(1) < q(2) < \dots$  and all  $q(i) \geq \bar{a}^{t-1}$ ; or  $w(1) < w(2) < \dots$  and all  $w(i) \geq \bar{a}^{t-1}$ . Assume the latter. Let

$$C = \|c(i, j)\| \in \mathcal{K}$$

be defined by taking as its  $i$ -th row all the  $b(i, j)$  appearing in  $w(i)$ . (In view of  $w(1) < w(2) < \dots$ , there does not exist an infinite number of rows of  $C$  containing only one element. By omitting a finite number of rows it may be assumed that all rows of  $C$  contain two or more elements of  $L$ .) Define  $c(i)^* \equiv \inf\{c(i, j) \mid j \geq 1\}$ . Since  $b(i)^* \leq c(i)^* \leq \bar{a}$ , it follows that

$$M = \Gamma(b(i)^*) \subseteq \Gamma(c(i)^*) \subseteq \Gamma(\bar{a}) = M.$$

Consequently,  $G(C) = M$  and  $C \in \mathcal{N}$ . Since  $w(1) \geq \bar{a}^{t-1}$ ,  $r(C) \leq t - 1$ . By repetition of this process, we may reduce the  $r$  to one so that finally  $\bar{a}^r = \bar{a} \leq w(1) < w(2) \dots$ . Since all  $c(i, j) \in L$  satisfy  $c(i, j) \leq e$  and since  $w(i)$  is a product of these, it follows that  $\bar{a} \geq c(i)^* \geq w(i)$ . Thus  $\bar{a} = w(1) = w(2) = \dots$  gives a contradiction. Thus  $\mathcal{K} = \emptyset$  and the main theorem has been proved.

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