INTEGRAL DOMAINS THAT ARE NOT EMBEDDABLE IN DIVISION RINGS

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A class of totally ordered rings $V$ is constructed having the property $1 < \alpha \in V \Rightarrow 1/\alpha \in V$, but such that $V$ cannot be embedded in any division ring.

1. Inverses in semigroup power series rings. This note has only one objective—to construct the above class of counterexamples (see [6]).

Notation 1.1. Throughout $\Gamma$ will be a totally ordered cancellative semigroup with identity $e$; $R$ will denote any totally ordered division ring. If $\alpha: \Gamma \to R$ is any function, then the support of $\alpha$ is the set $\text{supp } \alpha = \{s \in \Gamma | \alpha(s) \neq 0\}$. The set $V = V(\Gamma, R)$ of all functions $\alpha$ such that $\text{supp } \alpha$ satisfies the a.c.c. (ascending chain condition) form a totally ordered abelian group. If $\Gamma$ is cancellative, then under the usual power series multiplication (see [3]), $V$ is a totally ordered ring.

1.2. Any $1 < \alpha \in V$ with $\alpha(s) = 0$ for $s > e$ may be written as $\alpha = \alpha(e)(1 - \lambda)$, where $1 \leq \alpha(e)$ and $\lambda = \sum(\lambda(a) a | a < e)$. It will be shown that

$$(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots = 1 + \sum_s \sum' \lambda(a(1))\lambda(a(2)) \cdots \lambda(a(n)),$$

where the finite sum $\sum'$ is over all integers and over all distinct $n$-tuples of $\Gamma^n$ satisfying $s = a(1)a(2) \cdots a(n)$ with each $a(i) < e$; the sum $\sum$ is over all $s < e$. To prove that $1/\alpha \in V$ it suffices to establish conditions (a) and (b) below.

(a) For each $s \in \Gamma$, there are only a finite number of $n$ with $\lambda^n(s) \neq 0$;
(b) $\text{supp } (1 - \lambda)^{-1}$ satisfies the a.c.c.

Assuming (a) and (b), the main theorem follows at once. By adjoining an identity as in [8; p. 158] to the semigroup in [2] a semigroup that actually satisfies the hypothesis in (ii) below can be constructed.

Main Theorem 1.3. If $\Gamma$ is a totally ordered cancellative semigroup with identity $e$ and $R$ any totally ordered division ring, then the power series ring $V = V(\Gamma, R)$ has the following properties:

(i) $1 < \alpha \in V$ and $\alpha(s) = 0$ for $s > e \Rightarrow 1/\alpha \in V$.
(ii) If in addition $\Gamma$ cannot be embedded in a group, then $V$
cannot be embedded in a division ring.

An already known result ([8; p. 135]) follows immediately from 1.3 (i).

**COROLLARY 1.4.** *If in addition* $\Gamma$ *is a group, then* $V(\Gamma, R)$ *is a division ring.*

2. Proof of the main theorem. Assume 1.2 (a) or (b) fails. Then a lengthy but elementary argument shows there exists a doubly indexed matrix $\{a(i, j) \in \text{supp}\lambda | 1 \leq i < \infty; 1 \leq j \leq n(i)\}$ such that the products $u(i) = a(i, 1)a(i, 2) \cdots a(i, n(i))$ of the rows form an infinite properly ascending chain. Eventually a contradiction will be derived from this. Without loss of generality assume $\Gamma \leq e$.

**DEFINITION 2.1.** For any totally ordered semigroup $\Gamma$ with identity $e$ and any element $a \in \Gamma$ with $a \leq e$, define a semigroup by

$$\Gamma(a) = \{q \in \Gamma | \exists \text{ an integer } m > 0, q^m \leq a\} .$$

**LEMMA 2.2.** With $\Gamma$ as above, for any $a(1), \cdots, a(m) \in \Gamma$ with each $a(j) \leq e$, set $u = a(1)a(2) \cdots a(m)$ and define

$$a^* = \min \{a(1), \cdots, a(m)\} .$$

Then $\Gamma(u) = \Gamma(a^*)$.

2.3. Consider a fixed subset $L\subseteq \Gamma$ all of whose elements satisfy $L \leq e$ and where $L$ satisfies the a.c.c., e.g., $L = \text{supp}\lambda < e$. Consider an array of elements $A = \|a(i, j)\|$ with $\{a(i, j) | 1 \leq i < \infty, 1 \leq j \leq n(i)\} \subseteq L$, where repetitions in the $a(i, j)$ are allowed. Assume all $n(i) \geq 2$. Define $u(i) = u(i, A)$ by

$$u(i) = u(i, A) = a(i, 1)a(i, 2) \cdots a(i, n(i)) .$$

Let $\mathcal{H}$ be the set of all such $A = \|a(i, j)\|$ for which $u(1) < u(2) < \cdots < u(i) < \cdots$ is strictly ascending at each $i$. With each member $A = \|a(i, j)\| \in \mathcal{H}$, we next associate three objects

$$\{a(i)^* | 1 \leq i < \infty\}, m = m(A), \text{ and } G = G(A) .$$

Define $a(i)^* \equiv \min \{a(i, j) | 1 \leq j \leq n(i)\}$. Note that $u(1) < u(2) < \cdots$ implies that $\Gamma(a(1)^*) \subseteq \Gamma(a(2)^*) \subseteq \Gamma(a(i)^*) \subseteq \cdots$. Thus since $L$ satisfies the a.c.c., there is a unique smallest integer $m = m(A)$ such that the semigroups $G \equiv \Gamma(a(m)^*) = \Gamma(a(m + 1)^*) = \cdots$ are all equal. The following schematic diagram of all these quantities may be helpful.
\[ \Gamma(a(1)^*) = \Gamma(u(1)) \quad u(1) = a(1, 1)a(1, 2) \cdots a(1)^* \cdots a(1, n(1)) \]
\[ \Gamma(a(2)^*) = \Gamma(u(2)) \quad u(2) = a(2, 1)a(2, 2) \cdots a(2)^* \cdots a(2, n(2)) \]
\[ \Gamma(a(m)^*) = \Gamma(u(m)) \quad u(m) = a(m, 1)a(m, 2) \cdots a(m)^* \cdots a(m, n(m)) \]
\[
G = \Gamma(u(m + 1)) .
\]

2.4. Among the elements of \( \mathcal{K} \), let \( \mathcal{N} \subset \mathcal{K} \) be all those \( A = \|a(i, j)\| \) such that this associated \( G = G(A) \) is as big as possible and call this particular \( G = M \). If \( \mathcal{K} \neq \emptyset \), also \( \mathcal{N} \neq \emptyset \). Define \( \bar{a} = \max \{a(m)^* \mid A \in \mathcal{K}, m = m(A)\} \). Pick and element \( B = \|b(i, j)\| \in \mathcal{N} \). Then by our choice of \( M, \Gamma(\bar{a}) = M \). Thus \( M = G(B) = \Gamma(b(i)^*) = \Gamma(b(i, j)) = \Gamma(u(i)) = \Gamma(\bar{a}) \) for \( i \geq m(B) \equiv m \). Finally, with each element \( B \) of \( \mathcal{N} \), we associate an integer \( r = r(B) \). Since \( \bar{a} \in \Gamma(u(m)) \), there is a unique smallest integer \( r \equiv r(B) \geq 1 \) such that \( \bar{a}^r \leq u(m) < \bar{a}^{r-1} \).

2.5. By omitting some of the rows of \( B \) and renumbering the remaining ones, it may be assumed as a consequence of the a.c.c. without loss of generality that \( m = 1 \), and also that \( b(1)^* \geq b(2)^* \geq \cdots \) is not ascending. Each \( u(i) \) is of one of the following three forms:

1. \[ u(i) = q(i)b(i)^* , \]
2. \[ u(i) = b(i)^*w(i) , \]
3. \[ u(i) = q(i)b(i)^*w(i) , \]

where the \( q(i), w(i) \) are certain products of the \( b(i, j) \). If there are an infinite number of \( u(i) \) of the forms (1) or (2), then since

\[
\begin{align*}
  u(i + 1) = q(i + 1)b(i + 1)^* &> u(i) = q(i)b(i)^*, \quad b(i + 1)^* \leq b(i)^* \\
  \implies q(i + 1) &> q(i) ;
\end{align*}
\]

it follows (after omitting some rows and renumbering) that there is a properly infinite ascending chain:

**Case 1.** \( q(1) < q(2) < \cdots ; \)

**Case 2.** \( w(1) < w(2) < \cdots . \)

If neither Case 1 nor Case 2 applies, then

\[
\begin{align*}
  u(i + 1) = q(i + 1)b(i + 1)^*w(i + 1) &> q(i)b(i)^*w(i) \\
  \text{and } b(i + 1)^* &\leq b(i)^*
\end{align*}
\]

implies that one of the inequalities \( q(i + 1) > q(i) \) or \( w(i + 1) > w(i) \)
must necessarily hold. It is asserted that there is a subsequence \( \{i(k) | k = 1, 2, \cdots \} \) such that

\[
\text{Case 3. either (a): } q(i(1)) < q(i(2)) < \cdots \\
\text{or (b): } w(i(1)) < w(i(2)) < \cdots.
\]

For if not, then the a.c.c. must hold in both the sets \( \{q(i)\} \) and \( \{w(i)\} \). Then by omitting some rows and renumbering the remaining ones it may be assumed that we have an element \( B \) in \( \mathcal{N} \) with \( q(1) \geq q(2) \geq \cdots \) and \( w(1) \geq w(2) \geq \cdots \). However, then

\[
q(1)b(1)w(1) \geq q(2)b(2)w(2) \geq \cdots
\]
gives a contradiction.

2.6. We may assume \( q(1) < q(2) < \cdots \) or \( w(1) < w(2) \cdots \) are properly ascending, depending on which of the Cases 1, 2, 3(a) or 3(b) is applicable. Set \( t = r(B) \), so that \( a^t \leq u(m) = u(1) \leq u(i) \).

2.7. It is next shown that either \( q(i) \geq a^{t-1} \) or \( w(i) \geq a^{t-1} \) holds for all \( i \). Suppose that the following holds.

\[
\text{Case 1. } q(1)b(1) < q(2)b(2) < \cdots; \\
q(1) < q(2) < \cdots; \\
b(1) \geq b(2) \geq \cdots.
\]

Then \( a^t \leq u(1) \leq u(i) = q(i)b(i) \), and \( a \geq b(i) \) implies that

\[
a^{t-1} \leq q(1) \leq q(i).
\]

(For if \( a^{t-1} > q(i) \), then \( a \geq b(i) \) implies that \( a^t > q(i)b(i) \).) (If \( t = 1 \), then \( a^0 = e_0 \).) Similarly, in Case 2 also \( a^{t-1} \leq w(1) \leq w(i) \).

Only Case 3(b) will be proved, since 3(a) is entirely parallel.

\[
\text{Case 3(b). } q(1)b(1)w(1) < q(2)b(2)w(2) < \cdots; \\
w(1) < w(2) < \cdots; \\
b(1)^* \geq b(2)^* \geq \cdots.
\]

Then again \( a^t \leq u(1) \leq u(i) = q(i)b(i)w(i) \) and \( a \geq b(i)^* \geq q(i)b(i)^* \) imply that \( a^{t-1} \leq w(1) \leq w(i) \). (Otherwise, if \( a^{t-1} > w(i) \), then \( a^t > q(i)b(i)^*w(i) \).)

The basic idea motivating the proof is that for \( B \in \mathcal{N} \), a new \( C \in \mathcal{N} \) can be constructed with \( r(C) \leq r(B) - 1 \).
2.8. Thus either \( g(1) < g(2) < \cdots \) and all \( q(i) \geq \bar{a}^{-1} \); or \( w(1) < w(2) < \cdots \) and all \( w(i) \geq \bar{a}^{-1} \). Assume the latter. Let

\[
C = \{ (c(i,j)) \mid c(i,j) \in \mathcal{H} \}
\]

be defined by taking as its \( i \)-th row all the \( b(i,j) \) appearing in \( w(i) \). (In view of \( w(1) < w(2) < \cdots \), there does not exist an infinite number of rows of \( C \) containing only one element. By omitting a finite number of rows it may be assumed that all rows of \( C \) contain two or more elements of \( L \).) Define \( c(i)^* = \inf \{ c(i,j) \mid j \geq 1 \} \). Since \( b(i)^* \leq c(i)^* \leq \bar{a} \), it follows that

\[
M = \Gamma(b(i)^*) \subseteq \Gamma(c(i)^*) \subseteq \Gamma(\bar{a}) = M.
\]

Consequently, \( G(C) = M \) and \( C \in \mathcal{H} \). Since \( w(1) \geq \bar{a}^{-1} \), \( r(C) \leq t - 1 \). By repetition of this process, we may reduce the \( r \) to one so that finally \( \bar{a}^r = \bar{a} \leq w(1) < w(2) \cdots \). Since all \( c(i,j) \in L \) satisfy \( c(i,j) \leq e \) and since \( w(i) \) is a product of these, it follows that \( \bar{a} \geq c(i)^* \geq w(i) \). Thus \( \bar{a} = w(1) = w(2) = \cdots \) gives a contradiction. Thus \( \mathcal{H} = \emptyset \) and the main theorem has been proved.

**References**


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