

# Pacific Journal of Mathematics

**CHARACTERIZATION OF SEPARABLE IDEALS**

BRYCE L. ELKINS

# CHARACTERIZATION OF SEPARABLE IDEALS

B. L. ELKINS

A  $k$ -algebra  $A$  is called separable if the exact sequence of left  $A^e = A \otimes_k A^0$ -modules:  $0 \rightarrow J \rightarrow A^e \xrightarrow{\phi} A \rightarrow 0$  splits, where  $\phi(a \otimes b^0) = a \cdot b$ ; a two-sided ideal  $\mathfrak{A}$  of  $A$  is separable in case the  $k$ -algebra  $A/\mathfrak{A}$  is separable.

In this note, we present two characterizations of separable ideals. In particular, one finds that a monic polynomial  $f \in k[x]$  generates a separable ideal if, and only if,  $f = g_1 \cdots g_s$ , where the  $g_i$  are monic polynomials which generate pairwise comaximal indecomposable ideals in  $k[x]$ , and  $f'(a)$  is a unit in  $k[a] = k[x]/f \cdot k[x]$  ( $a = x + f \cdot k[x]$ ).

Throughout this paper, we assume that all rings have units and all ring morphisms preserve units, further, all modules will be assumed unitary. We will denote the center of the ring  $A$  by  $Z(A)$ . Each  $k$ -algebra  $A$  induces an exact sequence of left  $A^e = A \otimes_k A^0$ -modules:

$$(1) \quad 0 \longrightarrow J \longrightarrow A^e \xrightarrow{\phi} A \longrightarrow 0$$

where  $\phi(a \otimes b^0) = a \cdot b$ .

DEFINITION 2 [1].  $A$  will be called a *separable  $k$ -algebra* if the sequence (1) splits. More generally, a two-sided ideal  $\mathfrak{A}$  in the  $k$ -algebra  $A$  will be called a *separable ideal* if the quotient algebra  $A/\mathfrak{A}$  ( $k \rightarrow A \rightarrow A/\mathfrak{A}$ ) is separable. Denote by  $\text{Sep}_k(A)$  the set of all such ideals in  $A$ ; of particular interest is the subset  $\text{Sep}_k^*(A)$  of all separable ideals  $\mathfrak{A}$  for which  $A/\mathfrak{A}$  is a projective  $k$ -module.

PROPOSITION 3 [6]. *Let  $A$  be a  $k$ -algebra.*

(a)  $\mathfrak{A} \in \text{Sep}_k(A) \wedge \mathfrak{A} \leq \mathfrak{A}' \Rightarrow \mathfrak{A}' \in \text{Sep}_k(A)$  ( $\mathfrak{A}'$  is any two-sided ideal of  $A$ ).

(b) If  $\{\mathfrak{A}_i\}_{i=1}^n \subset \text{Sep}_k(A)$  is a family of pairwise comaximal ideals, then  $\bigcap_{i=1}^n \mathfrak{A}_i \in \text{Sep}_k(A)$ .

The following result found in [1] provides a criterion for answering the question, is  $\text{Sep}_k(A) = \emptyset$  or  $\text{Sep}_k(A) \neq \emptyset$ .

PROPOSITION 4. *Let  $A$  be a  $k$ -algebra, and let  $K$  be a commutative  $k$ -algebra. If  $\phi(0:J) \otimes_k K$  generates  $Z(A) \otimes_k K$  as an ideal, then  $A \otimes_k K$  is a separable  $K$ -algebra.*

COROLLARY 5. (a) *If  $\alpha < k$  is an ideal such that*

$$\alpha \cdot Z(A) + \phi(0: J) = Z(A),$$

then  $\alpha A \in \text{Sep}_k(A)$ .

(b) If  $Z(A) = k$ , and either  $\phi(0: J)$  is not nil or  $\phi(0: J) \not\subseteq \text{Rad}(k)$ , then  $\text{Sep}_k(A) \neq \emptyset$ , where  $\text{Rad}(k)$  is the Jacobson radical of  $k$ .

### 1. Representation of separable ideals.

**THEOREM 1.1.** *Let  $A$  be a  $k$ -algebra and  $\mathfrak{A} \in \text{Sep}_k(A)$ . If the  $k$ -module  $A/\mathfrak{A}$  is of finite type, then for each maximal ideal  $m < k$ , there is a family  $(M_i)_{i=1}^s \subset \text{Sep}_k(A)$  of maximal two-sided ideals such that*

$$(1.2) \quad \mathfrak{A} + (m \cdot A) = M_1 \cap \cdots \cap M_s.$$

*Proof.* For each maximal ideal  $m < k$ , the  $k/m$ -algebra  $k/m \otimes A/\mathfrak{A}$  is separable and of finite type as a  $k/m$ -module, it follows from [2] Proposition 3.2 that  $k/m \otimes A/\mathfrak{A} \cong (A/\mathfrak{A})/m(A/\mathfrak{A}) \cong A/(m \cdot A + \mathfrak{A}) \cong B_1 \oplus \cdots \oplus B_s$ , where each  $B_i$  is a simple  $k/m$ -algebra with  $Z(B_i)$  being a separable field extension of  $k/m$ ; in particular, each  $B_i$  is a separable  $k$ -algebra. Denoting by  $M_i$  the kernel of the mapping  $A \rightarrow A/(m \cdot A + \mathfrak{A}) \rightarrow B_i$ , we find that the family  $(M_i)_{i=1}^s$  has the desired properties.

**REMARK 1.3.** If, in (1.1), we assume  $\mathfrak{A} \in \text{Sep}_k^*(A)$ , it follows from (1.1) of [9], that we can drop the assumption that  $A/\mathfrak{A}$  is a  $k$ -module of finite type.

We obtain immediately from the local criteria for separability ([2], p. 100) the following theorem.

**THEOREM 1.4.** *Let  $A$  be a  $k$ -algebra with two-sided ideal  $\mathfrak{A}$  such that the  $k$ -module  $A/\mathfrak{A}$  is of finite type. Suppose either that  $k$  is Noetherian or that  $A/\mathfrak{A}$  is a projective  $k$ -module.*

*If, for each maximal ideal  $m < k$ ,  $\mathfrak{A} + m \cdot A$  has a representation (1.2) with separable maximal ideals, then  $\mathfrak{A} \in \text{Sep}_k(A)$ .*

**COROLLARY 1.5.** *Let  $k$  be a field.*

(a)  $\mathfrak{A}$  is a separable maximal ideal of  $A$  if, and only if,  $A/\mathfrak{A}$  is a simple  $k$ -algebra whose center is a separable field extension of  $k$ .

(b)  $\mathfrak{A} \in \text{Sep}_k(A)$  if, and only if,  $\mathfrak{A}$  is the intersection of a finite family of separable maximal ideals of  $A$ .

**REMARKS 1.6.** (1.5) generalizes a result of [6] where a different

definition of  $\mathfrak{A} \in \text{Sep}_k(A)$  is given. (1.5) also leads to the following fact. For a field  $k$ ,  $f \in k[x]$  generates a separable ideal if, and only if,  $f'(a)$  is a unit in  $k[a] = k[x]/f \cdot k[x]$ ,  $a = x + f k[x]$ , and  $f$  is the product of distinct polynomials of  $k[x]$ .

DEFINITION 1.7. [5]. A monic polynomial  $f \in k[x]$  is separable if the ideal  $f k[x]$  is separable.

PROPOSITION 1.8. *If  $f \in k[x]$  is separable, then  $f'(a)$  is a unit in  $k[a] := k[x]/f \cdot k[x]$ .*

*Proof.* Assume, first, that  $k$  is local with maximal ideal  $m$ . Denote by  $\bar{f}$  the reduction of  $f$  modulo  $m$ , then

$$k[x]/(m, f) \simeq k/m \otimes_k k[a] \simeq k/m[x]/\bar{f}k/m[x] = k/m[\bar{a}]$$

is a separable  $k/m$ -algebra, hence  $\bar{f}$  is a separable polynomial. Whence, by (1.6),  $\bar{f} = \bar{g}_1 \cdots \bar{g}_s$  in  $k/m[x]$ , where each  $\bar{g}_i$  is irreducible and  $\bar{f}'(\bar{a})$  is a unit in  $k/m[\bar{a}]$ .

Now suppose  $f'(a)$  is a nonunit in  $k[a]$ ; by [7], p. 29, Lemma 4, each maximal ideal of  $k[a]$  has the form  $(g_i(a), m)$ , where  $g_i \in k[a]$  has reduction  $\bar{g}_i$  modulo  $m$ . Thus,  $f'(a) \in (g_i(a), m)$  for some  $i \in [1, s]$ , and this implies  $f'(x) \in (g_i(x), m)$ . But then

$$f'(x) \in \ker(k[x] \rightarrow k[x]/f k[x] \rightarrow k[x]/(m, f k[x]) \rightarrow k[x]/(g_i k[x], m)) ,$$

so that  $\bar{f}'(\bar{a})$  could not be a unit in  $k/m[\bar{a}]$ . This contradiction establishes our claim that  $f'(a)$  is a unit when  $k$  is local.

In general, observe that  $f'(a)$  is a unit in  $k[a]$  if, and only if  $f'_m(a_m)$  is a unit in  $k_m[a_m] = k_m \otimes_k k[a]$  for each maximal ideal  $m < k$ , and then apply the foregoing result.

PROPOSITION 1.9. *Let  $f \in k[x]$  be a monic polynomial satisfying the conditions.*

- (i)  $f'(a)$  is a unit in  $k[a] = k[x]/f k[x]$ ;
- (ii)  $f(x) = f_1 \cdots f_s$  in  $k[x]$ , where the monic polynomials  $f_i$  generate indecomposable ideals which are pairwise comaximal.

*Then  $f$  is separable.*

*Proof.* Let  $m < k$  be a maximal ideal of  $k$ , and denote by  $\bar{f}$  the reduction of  $f$  modulo  $m$ , then  $\bar{f}'(\bar{a})$  is a unit in  $k/m[\bar{a}] = k/m \otimes_k k[a]$ . Since  $\bar{f} = \bar{f}_1 \cdots \bar{f}_s$  in  $k/m[x]$ , we see that  $\bar{f}'_i = 0$  in  $k/m[x]$  entails  $\bar{f}' = \bar{f}'_i g + \bar{f}_i g' \in \bar{f}_i k/m[x]$ . But this implies  $\bar{f}'(\bar{a})$  is a nonunit in  $k/m[\bar{a}] := k/m[x]/\bar{f} k/m[x]$ , since  $\bar{f}_i k/m[x] < k/m[x]$ . Thus, each of the  $\bar{f}_i$  separable polynomials in  $k/m[x]$  which generate pairwise comaximal

by (ii). An application of [5] (2.3) shows that  $f$  is a separable polynomial in  $k[x]$ .

**COROLLARY 1.10.** *Suppose  $k$  has no proper idempotents and  $f \in k[x]$  is monic. A necessary and sufficient condition that  $f$  be separable is that conditions (i) and (ii) of (1.9) holds.*

*Proof.* We need only verify that when  $f$  is separable,  $f = f_1 \cdots f_s$  where each of the ideals  $f_i k[x]$  is indecomposable and they are pairwise comaximal. But  $k[x]/f \cdot k[x]$  has only a finite number of idempotents, since it is a free  $k$ -module of rank equal  $\deg(f)$ ; hence  $k[x]/f \cdot k[x] = B_1 \pi \cdots \pi B_s$ , where each  $B_i$  is connected and separable as well as projective as a  $k$ -module. Then, by [5] (2.9),  $B_i = k[x]/f_i k[x]$  and we see that  $f = f_1 \cdots f_s$  as usual.

## 2. Another representation of separable ideals.

**DEFINITION 2.1.** Let  $A$  be a  $k$ -algebra. The two-sided ideal  $\mathfrak{A} < A$  will be called *decomposable* if  $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$ , where  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are proper two-sided comaximal ideals of  $A$ ; otherwise  $\mathfrak{A}$  will be called *indecomposable*.  $A$  will be called decomposable or indecomposable according to whether or not  $0$  is.

**THEOREM 2.2.** *Let  $k$  be a commutative ring without proper idempotents. Assume  $\mathfrak{A} \in \text{Sep}_k^*(A)$ . Then there is a unique family  $(M_i)_{i=1}^s$  of pairwise comaximal indecomposable separable ideals of  $A$  such that*

$$(2.3) \quad \mathfrak{A} = M_1 \cap \cdots \cap M_s .$$

*Proof.* Since the projective  $k$ -module  $A/\mathfrak{A}$  has finite rank, we can write  $A/\mathfrak{A} \simeq B_1 \pi \cdots \pi B_s$ , where the  $B_i$  are indecomposable separable  $k$ -algebras. Putting  $M_i = \ker [A \rightarrow A/\mathfrak{A} \rightarrow B_i]$  we obtain the desired family.

If  $\mathfrak{A} = N_1 \cap \cdots \cap N_t$ , where the  $N_j$  are as the  $M_i$ , then  $A/\mathfrak{A} = A/M_1 \pi \cdots \pi A/M_s = A/N_1 \pi \cdots \pi A/N_t$  implies that

$$\mathbf{1} = e_1 + \cdots + e_s = f_1 + \cdots + f_t, \quad e_i, f_j$$

being orthogonal central idempotents. Since all the factors are indecomposable, for each  $i$  there is a unique  $j$  such that  $f_i = f_i e_j$ ; hence  $t \leq s$ , and by symmetry,  $s \leq t$ , so  $s = t$ . The indecomposability also implies (after reordering) that  $e_i = f_i$ , so that

$$M_i = \ker [A \rightarrow (A/\mathfrak{A})e_i] = \ker [A \rightarrow (A/\mathfrak{A})f_i] = N_i ,$$

completing the proof.

REMARK 2.4. (2.2) generalizes a result obtained in [5], see p. 471, (2.10).

#### BIBLIOGRAPHY

1. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367-409.
2. H. Bass, *Lectures on topics in algebraic K-theory*, Tata Institute of Fundamental Research, Bombay 1967.
3. M. Bourbaki, *Algebre*, Chap. 8, Hermann, 1958.
4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956.
5. G. J. Janusz, *Separable algebras over commutative rings*, Trans. Amer. Math. Soc. **122** (1966), 461-79.
6. R. E. MacRae, *On the notion of separable ideals*, Archiv Der Mathematik **28** (1967), 591-594.
7. J. P. Serre, *Corps locaux*, Hermann, Paris, 1962.
8. J. Sonn and H. Zassenhaus, *On the theorem on the primitive element*, Amer. Math. Monthly **74** (1967).
9. O. E. Villamayor and D. Zelinsky, *Galois theory for rings with finitely many idempotents*, Nagoya Math. J. **27** (1966), 721-731.

Received November 11, 1969.

THE OHIO STATE UNIVERSITY  
COLUMBUS, OHIO



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON  
Stanford University  
Stanford, California 94305

J. DUGUNDI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

RICHARD PIERCE  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

Johan Aarnes, Edward George Effros and Ole A. Nielsen, <i>Locally compact spaces and two classes of <math>C^*</math>-algebras</i> .....	1
Allan C. Cochran, R. Keown and C. R. Williams, <i>On a class of topological algebras</i> .....	17
John Dauns, <i>Integral domains that are not embeddable in division rings</i> ....	27
Robert Jay Daverman, <i>On the number of nonpiercing points in certain crumpled cubes</i> .....	33
Bryce L. Elkins, <i>Characterization of separable ideals</i> .....	45
Zbigniew Fiedorowicz, <i>A comparison of two naturally arising uniformities on a class of pseudo-PM spaces</i> .....	51
Henry Charles Finlayson, <i>Approximation of Wiener integrals of functionals continuous in the uniform topology</i> .....	61
Theodore William Gamelin, <i>Localization of the corona problem</i> .....	73
Alfred Gray and Paul Stephen Green, <i>Sphere transitive structures and the triality automorphism</i> .....	83
Charles Lemuel Hagopian, <i>On generalized forms of aposynthesis</i> .....	97
J. Jakubík, <i>On subgroups of a pseudo lattice ordered group</i> .....	109
Cornelius W. Onneweer, <i>On uniform convergence for Walsh-Fourier series</i> .....	117
Stanley Joel Osher, <i>On certain Toeplitz operators in two variables</i> .....	123
Washek (Vaclav) Frantisek Pfeffer and John Benson Wilbur, <i>On the measurability of Perron integrable functions</i> .....	131
Frank J. Polansky, <i>On the conformal mapping of variable regions</i> .....	145
Kouei Sekigawa and Shûkichi Tanno, <i>Sufficient conditions for a Riemannian manifold to be locally symmetric</i> .....	157
James Wilson Stepp, <i>Locally compact Clifford semigroups</i> .....	163
Ernest Lester Stitzinger, <i>Frattini subalgebras of a class of solvable Lie algebras</i> .....	177
George Szeto, <i>The group character and split group algebras</i> .....	183
Mark Lawrence Teply, <i>Homological dimension and splitting torsion theories</i> .....	193
David Bertram Wales, <i>Finite linear groups of degree seven. II</i> .....	207
Robert Breckenridge Warfield, Jr., <i>An isomorphic refinement theorem for Abelian groups</i> .....	237
James Edward West, <i>The ambient homeomorphy of an incomplete subspace of infinite-dimensional Hilbert spaces</i> .....	257
Peter Wilker, <i>Adjoint product and hom functors in general topology</i> .....	269
Daniel Eliot Wulbert, <i>A note on the characterization of conditional expectation operators</i> .....	285