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In this paper, we shall consider an important class of probabilistic pseudometric spaces, the so-called pseudometrically generated spaces, i.e., spaces with a collection of pseudometrics on which a probability measure has been defined. Specifically, we shall examine the relationship between the uniformity introduced on the space probabilistically by means of the so-called ε, λ uniform neighborhoods and the uniformity obtained by considering all the uniform neighborhoods generated by each of the pseudometrics as a subbase.

A *probabilistic metric* (*PM*) *space* is a pair $(S, \mathscr{F})$ where S is a set, $\mathscr{F}$ is a mapping from $S \times S$ into Δ, the set of all one-dimensional left continuous distribution functions, whose value $\mathscr{F}(p, q)$ at any $(p, q) \in S \times S$ is usually denoted by F_{pq}, satisfying

(I) $F_{pp} = H$

(II) $F_{pq} = H$ implies $p = q$

(III) $F_{pq}(0) = 0$

(IV) $F_{pq} = F_{qp}$

(V) $F_{pq}(x) = F_{qr}(y) = 1$ implies $F_{pr}(x + y) = 1$,

where H is the distribution function defined by

$$H(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0 . \end{cases}$$

A *Menger space* is a triple $(S, \mathscr{F}, T)$ where $(S, \mathscr{F})$ is a *PM* space, T is a mapping (called a t-norm) from the unit square $[0, 1] \times [0, 1]$ into $[0, 1]$ which is nondecreasing in each place, symmetric, associative, satisfies boundary condition

$$T(a, 1) = a ,$$

and with the additional property

$$(\mathrm{Vm}) F_{pr}(x + y) \geqq T(F_{pq}(x), F_{qr}(y)) .$$

A *probabilistic pseudometric* (*pseudo-PM*) *space* is a pair $(S, \mathscr{F})$ satisfying (I), (III), (IV), and (V). Similarly, a *pseudo-Menger space* is a triple $(S, \mathscr{F}, T)$ satisfying (I), (III), (IV), and (Vm).

For further information on the basic properties of *PM* spaces, the reader is referred to Schweizer and Sklar [3].

DEFINITION 1. A *metrically generated* (*MG*) *space* is a *PM* space $(S, \mathscr{F})$ together with a probability space $(\mathscr{D}, \mathscr{B}, \mu)$ such that $\mathscr{D}$ is a set of metrics on S and such that for any $(p, q) \in S \times S$ and any $x > 0$

$$\{d \in \mathscr{D} : d(p, q) < x\} \in \mathscr{B} \tag{1}$$

and

$$F_{pq}(x) = \mu\{d \in \mathscr{D} : d(p, q) < x\} \,. \tag{2}$$

A *pseudometrically generated* (*pseudo-MG*) *space* is a pseudo-*PM* space $(S, \mathscr{F})$ together with a probability space $(\mathscr{D}, \mathscr{B}, \mu)$ of pseudometrics on S such that conditions (1) and (2) hold.

In the sequel, we will use the notation $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ to denote *MG* and pseudo-*MG* spaces.

In his paper [5], R. Stevens showed that any *MG* space is a Menger space under the t-norm T_m where

$$T_m(a, b) = \max\{a + b - 1, 0\} \,.$$

His proof may be easily generalized to show that any pseudo-*MG* space is a pseudo-Menger space under T_m.

DEFINITION 2. Let S be a set and let $\mathscr{D}$ be a collection of pseudometrics on S. Then the *gage uniformity* of $\mathscr{D}$ on S (denoted by $\mathscr{U}_{\mathscr{D}}$) is the uniformity generated by the following subbase

$$[\{(p, q) \in S \times S : d(p, q) < x\}]_{d \in \mathscr{D}, x > 0} \,.$$

It is shown in Kelley [1] that any uniformity on a set may be regarded as the gage uniformity of some collection of pseudometrics on that set.

THEOREM 1. *Let $(S, \mathscr{F}, T)$ be a pseudo-Menger space with the property that* $\sup_{x<1} T(x, x) = 1$. *Then the sets*

$$U(\varepsilon, \lambda) = \{(p, q) \in S \times S : F_{pq}(\varepsilon) > 1 - \lambda\}$$

form a base for a pseudometrizable uniformity on S.

The above theorem was proven by Schweizer, Sklar, and Thorp [4]. Since pseudo-*MG* spaces are pseudo-Menger spaces under T_m, a continuous t-norm, it follows that the sets

$$\begin{aligned} U(\varepsilon, \lambda) &= \{(p, q) \in S \times S : F_{pq}(\varepsilon) > 1 - \lambda\} \\ &= \{(p, q) : \mu\{d \in \mathscr{D} : d(p, q) < \varepsilon\} > 1 - \lambda\} \end{aligned}$$

form a base for a uniformity on the pseudo-*MG* space $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$. This uniformity will be referred to as the $\mathscr{F}$ uniformity and will be denoted by $\mathscr{U}_{\mathscr{F}}$.

Given a pseudo-*MG* space $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$, it follows from the above that we can put two uniformities on S, namely the gage uniformity $\mathscr{U}_{\mathscr{D}}$ and the $\mathscr{F}$ uniformity $\mathscr{U}_{\mathscr{F}}$. A natural question that arises is whether there is any relationship between the two uniformities. We shall first examine this question for pseudo-*MG* spaces generated by a countable family of pseudometrics.

THEOREM 2. *If $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ is a pseudo-MG space and $\mathscr{D}$ is countable, then $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.*

Proof. We shall first show that $\mathscr{B} = 2^{\mathscr{D}}$.

First of all, since for any $(p, q) \in S \times S$ and any $\varepsilon > 0$

$$\{d \in \mathscr{D} : d(p, q) < \varepsilon\} \in \mathscr{B} ,$$

it follows that its complement $\{d : d(p, q) \geqq \varepsilon\}$ is also μ-measurable. Similarly

$$\{d : d(p, q) \leqq \varepsilon\} = \bigcap_{n=1}^{\infty} \left\{d : d(p, q) < \varepsilon + \frac{1}{n}\right\} \in \mathscr{B} .$$

Hence, we have for any $(p, q) \in S \times S$ and any $\varepsilon > 0$

$$\{d : d(p, q) = \varepsilon\} = \{d : d(p, q) \leqq \varepsilon\} \cap \{d : d(p, q) \geqq \varepsilon\} \in \mathscr{B} .$$

Now pick any $d_0 \in \mathscr{D}$ and well order $\mathscr{D} - \{d_0\}$ as

$$\{d'_1, d'_2, \cdots\} .$$

Now since $d_0 \neq d'_k$, there is a pair $(p_k, q_k) \in S \times S$ for which

$$d_0(p_k, q_k) \neq d'_k(p_k, q_k) .$$

Hence it follows that

$$\{d_0\} = \bigcap_{k=1}^{\infty} \{d : d(p_k, q_k) = d_0(p_k, q_k)\} \in \mathscr{B} .$$

Since any subset of $\mathscr{D}$ is a countable union of unit sets $\{d\}$, it follows that $\mathscr{B} = 2^{\mathscr{D}}$.

To show that $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$, it suffices to show that any base element $U(\varepsilon, \lambda)$ of $\mathscr{U}_{\mathscr{F}}$, contains a base element

$$V = \bigcap_{d \in A} \{(p, q): d(p, q) < \varepsilon_d\} ,$$

of $\mathscr{U}_{\mathscr{D}}$, where A is a finite subset of $\mathscr{D}$ and each $\varepsilon_d > 0$. Well order $\mathscr{D}$ as

$$\{d_1, d_2, \cdots\} .$$

Clearly

$$\mu(\mathscr{D}) = \sum_{k=1}^{\infty} \mu\{d_k\} = 1 .$$

Pick n large enough so that

$$\mu\Bigl(\bigcup_{k=1}^{n} \{d_k\}\Bigr) = \sum_{k=1}^{n} \mu\{d_k\} > 1 - \lambda .$$

Let V be defined by

$$V = \bigcap_{k=1}^{n} \{(p, q): d_k(p, q) < \varepsilon\} .$$

Clearly, if $(p_0, q_0) \in V$, then

$$\bigcup_{k=1}^{n} \{d_k\} \subseteqq \{d: d(p_0, q_0) < \varepsilon\}$$

and

$$1 - \lambda < \mu\Bigl(\bigcup_{k=1}^{n} \{d_k\}\Bigr) \leqq \mu\{d: d(p_0, q_0) < \varepsilon\} = F_{p_0 q_0}(\varepsilon) ,$$

so that $(p_0, q_0) \in U(\varepsilon, \lambda)$. In other words,

$$V \subseteqq U(\varepsilon, \lambda) ,$$

which is what we wished to prove.

THEOREM 3. *Let $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ be a pseudo-MG space with the property that $\mathscr{D}$ is countable, and μ is nonzero on all nonempty measurable subsets of $\mathscr{D}$. Then $\mathscr{U}_{\mathscr{D}} = \mathscr{U}_{\mathscr{F}}$.*

Proof. In view of the preceding theorem, it is sufficient to show $\mathscr{U}_{\mathscr{D}} \subseteqq \mathscr{U}_{\mathscr{F}}$. In the proof of the preceding theorem we have already shown that all subsets of $\mathscr{D}$ are μ-measurable. It follows that $\mu(\{d_0\}) > 0$ for any $d_0 \in \mathscr{D}$. Now, for any $\varepsilon > 0$,

$$\mu\{d: d(p, q) < \varepsilon\} > 1 - \mu\{d_0\} \text{ implies } d_0(p, q) < \varepsilon .$$

It follows that

$$U(\varepsilon, \mu\{d_0\}) \subseteqq \{(p, q) : d_0(p, q) < \varepsilon\} .$$

Taking finite intersections, we have that any base element of $\mathscr{U}_{\mathscr{D}}$ contains a base element of $\mathscr{U}_{\mathscr{F}}$ and the desired result is an immediate consequence of this.

THEOREM 4. *Let* $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ *be a pseudo-MG space such that* $\mathscr{D}$ *is countable. Let* $\mathscr{D}' \subseteqq \mathscr{D}$ *be defined by*

$$\mathscr{D}' = \{d \in \mathscr{D} : \mu\{d\} > 0\} .$$

Then $\mathscr{U}_{\mathscr{F}} = \mathscr{U}_{\mathscr{D}'}$.

Proof. Let $(\mathscr{D}', \mathscr{B}', \mu')$ be the probability space naturally induced by $(\mathscr{D}, \mathscr{B}, \mu)$. By the previous theorem, the $\mathscr{F}'$ uniformity of $(S, \mathscr{F}'; \mathscr{D}', \mathscr{B}', \mu')$, $\mathscr{U}_{\mathscr{F}'}$, is equivalent to $\mathscr{U}_{\mathscr{D}'}$. Since $\mathscr{D} - \mathscr{D}'$ is a countable union of sets of μ-measure 0,

$$F'_{pq}(x) = \mu'\{d \in \mathscr{D}' : d(p, q) < x\} = \mu\{d \in \mathscr{D} : d(p, q) < x\} = F_{pq}(x) ,$$

so that $\mathscr{U}_{\mathscr{F}} = \mathscr{U}_{\mathscr{F}'} = \mathscr{U}_{\mathscr{D}'}$.

Thus, we have essentially solved our problem for spaces generated by a countable family of pseudometrics. It is reasonable to ask whether any of these results can be extended to arbitrary pseudo-*MG* spaces. The following example shows that this is not the case.

EXAMPLE 1. Let S be the set of all real-valued measurable functions on the unit interval [0, 1]. For any $t \in [0, 1]$ define a pseudometric d_t on S by

$$d_t(f, f^*) = |f(t) - f^*(t)|$$

for any f, f^* in S. Let $\mathscr{D} = \{d_t : t \in [0, 1]\}$, and let μ be the probability measure on $\mathscr{D}$ induced by the Lebesgue measure on [0, 1]. Let $\mathscr{F} : S \times S \to \Delta$ be defined by

$$\mathscr{F}(f, f^*)(x) = \mu\{d_t : d_t(f, f^*) < x\} .$$

Hence $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ is a pseudo-*MG* space. (The pseudometrics d_t may be interpreted as giving the distance between two particles at time t)

It is easy to show that $\mathscr{U}_{\mathscr{F}}$ and $\mathscr{U}_{\mathscr{D}}$ are not even comparable. For two particles may be close to one another at any finite number of instants but still be far away from each other the rest of the time. Conversely, given our finite number of instants we can find two particles which are far apart at these instants but arbitrarily close to each other at all other times.

However, the question still remains, whether any of our results on countably generated pseudo-*MG* spaces can be generalized to the uncountable case when sufficiently strong restrictions are placed upon the generating family of pseudometrics. A natural restriction that comes to mind is the requirement that all the pseudometrics be comparable.

DEFINITION 3. Two pseudometrics d_1 and d_2 on a set S are said to be *comparable* if one of the following relations holds

(i) $d_1(p, q) \geqq d_2(p, q)$ for all $(p, q) \in S \times S$; or

(ii) $d_2(p, q) \geqq d_1(p, q)$ for all $p, q \in S \times S$.

DEFINITION 4. A linearly ordered set $(S, \leqq)$ is said to be *countably bounded* if there exists a countable subset $A \subseteqq S$ such that for every element $s \in S$, there exists an element $\alpha \in A$ such that $s \leqq \alpha$.

The real numbers with the usual ordering are countably bounded where as the collection of ordinals less than the first uncountable is not countably bounded.

THEOREM 5. *Let* $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ *be a pseudo-MG space, such that any two pseudometrics of* $\mathscr{D}$ *are comparable. If* $\mathscr{D}$ *is countably bounded under the induced linear ordering, then* $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.

Proof. If $\mathscr{D}$ has an upper bound, this result may be proven very easily. If $\mathscr{D}$ does not have an upper bound, then neither does A, the countable bounding set, and we can construct from A a strictly increasing sequence $\{d_k\}_{k=1}^{\infty}$ such that for every $d \in \mathscr{D}$, there exists a k such that $d < d_k$.

Let (p_k, q_k) be a point of $S \times S$ such that $d_k(p_k, q_k) < d_{k+1}(p_k, q_k)$. Let A_k be defined by

$$A_k = \{d \in \mathscr{D}: d(p_k, q_k) < d_{k+1}(p_k, q_k)\} .$$

It is obvious that $\{A_k\}_{k=1}^{\infty}$ forms an increasing sequence of μ-measurable sets. It is also obvious that $\lim_{k\to\infty} A_k = \mathscr{D}$, whence $\lim_{k\to\infty} \mu(A_k) = \mu(\mathscr{D}) = 1$. Thus for any $\lambda > 0$ there exists a N such that $\mu(A_N) > 1 - \lambda$. Hence for any $d \in A_N$

$$d_{N+1}(p, q) < \varepsilon \text{ implies } d(p, q) < \varepsilon$$

and

$$\mu\{d: d(p, q) < \varepsilon\} \geqq \mu(A_N) > 1 - \lambda ,$$

so that

$$\{(p, q): d_{N+1}(p, q) < \varepsilon\} \subseteqq U(\varepsilon, \lambda)$$

which proves that $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.

Theorem 5 might seem to indicate that perhaps $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$ holds for all pseudo-*MG* spaces with comparable pseudometrics. But even this is false as the following example shows.

EXAMPLE 2. Let Ω denote the set of all ordinal numbers less than the first ordinal having the power of the continuum. Let φ denote a one-to-one correspondence from the closed unit interval $I = [0, 1]$ onto Ω. Now define a function $f_y: I \rightarrow I$ for every $y \in I$ as follows:

$$f_y(x) = \begin{cases} 1, \text{ if } \varphi(x) \leqq \varphi(y) \\ x/4, \text{ if } \varphi(x) > \varphi(y) . \end{cases}$$

Also define for every $y \in I$ a function $d_y: I \times I \rightarrow R$ by

$$d_y(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = x_2 \\ f_y(x_1) + f_y(x_2), & \text{if } x_1 \neq x_2 . \end{cases}$$

Define a measure μ on the Boolean σ-algebra $\mathscr{B}$ of $2^{\mathscr{D}}$ (where $\mathscr{D} = \{d_y: y \in I\}$) consisting of all subsets of $\mathscr{D}$ which have a cardinal numbers less than that of the continuum and of the complements of these sets by

$$\mu(A) = \begin{cases} 0, \text{ if card } (A) < \mathfrak{C} \\ 1, \text{ if card } (\mathscr{D} - A) < \mathfrak{C} . \end{cases}$$

One may easily verify that μ satisfies all the conditions for a probability measure.

It may also be easily verified that d_y is a metric on I for every $y \in I$ and that $\varphi(y_1) < \varphi(y_2)$ implies that $d_{y_1}(x_1, x_2) \leqq d_{y_2}(x_1, x_2)$ for every $(x_1, x_2) \in I \times I$.

To show I, $\mathscr{D}$, and μ determine an *MG*-space, it suffices to show that for $(x_0, z_0) \in I \times I$ and any $\varepsilon_0 > 0$, the set

$$\{d_y \in \mathscr{D}: d_y(x_0, z_0) < \varepsilon_0\}$$

is μ-measurable. If $x_0 = z_0$ or $\varepsilon_0 > 2$, this is obviously true. If $\varepsilon_0 \leqq 2$, let $x' = \varphi^{-1}\{\max\{\varphi(x_0), \varphi(z_0)\}\}$. Then

$$A = \{d_y \in \mathscr{D}: d_y(x_0, z_0) < \varepsilon_0 \leqq 2\} \subseteqq \{d_y \in \mathscr{D}: \varphi(y) < \varphi(x')\} = B ,$$

since if $\varphi(x') \leqq \varphi(y)$ held, then $\varphi(x_0) \leqq \varphi(y)$ and $\varphi(z_0) \leqq \varphi(y)$ and

$$d_y(x_0, z_0) = f_y(x_0) + f_y(z_0) = 1 + 1 = 2 .$$

We have card $(B) < \mathfrak{C}$ and so card $(A) < \mathfrak{C}$. Hence A is μ-measurable and $\mu(A) = 0$.

We shall now show that the proper inclusion $\mathscr{U}_{\mathscr{D}} \subsetneqq \mathscr{U}_{\mathscr{F}}$ holds (instead of $\mathscr{U}_{\mathscr{F}} \subsetneqq \mathscr{U}_{\mathscr{D}}$). We have

$$U(1, \tfrac{1}{2}) = \{x_1, x_2) \in I \times I: \mu\{d_y: d_y(x_1, x_2) < 1\} > \tfrac{1}{2}\} = D_I$$

where D_I is the diagonal set on $I \times I$, since, as shown in the preceding paragraph, $x_1 \neq x_2$ implies

$$\mu\{d_y: d_y(x_1, x_2) < 1\} = 0 .$$

To show that proper inclusion holds, assume the contrary. Then we would have to have

$$\bigcap_{i=1}^{n} \{(x_1, x_2): d_{y_i}(x_1, x_2) < \varepsilon_i\} = D_I$$

for some $\{d_{y_i}\}_{i=1}^{n}$ and $\{\varepsilon_i\}_{i=1}^{n}$, $\varepsilon_i > 0$. Let

$$\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\} \text{ and } d_{y_0} = \max\{d_{y_1}, \cdots, d_{y_n}\} .$$

Then

$$B = \{(x_1, x_2): d_{y_0}(x_1, x_2) < \varepsilon_0\} = D_I .$$

Since card $\{x: \varphi(x) \leqq \varphi(y_0)\} < \mathfrak{C}$, there exist two points $x_0 \neq z_0$ in the open interval $(0, \varepsilon_0)$ such that $\varphi(x_0) > \varphi(y_0)$ and $\varphi(z_0) > \varphi(y_0)$. Thus

$$d_{y_0}(x_0, z_0) = f_y(x_0) + f_y(z_0) = (x_0/4) + (z_0/4) \leqq (\varepsilon_0/4) + (\varepsilon_0/4) < \varepsilon_0 ,$$

so that $(x_0, z_0) \in B$, but $(x_0, z_0) \notin D_I$, which contradicts our assumption.

Hence Theorem 5 cannot be extended to arbitrary pseudo-*MG* spaces with comparable pseudometrics. However, Theorem 5 does admit generalization in another direction. For it may be easily seen that a pseudo-*MG* space $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ with comparable pseudometrics such that $\mathscr{D}$ is countably bounded, also has the property that the gage uniformity of $\mathscr{D}$ is also generated by some countable subfamily $\mathscr{A} \subseteqq \mathscr{D}$, for instance the countable bounding set; i.e., $\mathscr{U}_{\mathscr{D}} = \mathscr{U}_{\mathscr{A}}$. We shall now show that in any pseudo-*MG* space with this property, $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$ holds. We shall derive this result by first proving an even more general result.

THEOREM 6. *Let $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ be a pseudo-MG space. Let $\mathscr{R}$ be an arbitrary countable collection of pseudometrics upon S with the property that $\mathscr{U}_{\mathscr{D}} \subseteqq \mathscr{U}_{\mathscr{R}}$. Then $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{R}}$.*

Proof. Consider the countable collection of uniform neighborhoods

$$\mathscr{C} = \left\{\left\{(p, q): d(p, q) < \frac{1}{n}\right\}: d \in \mathscr{R}, n = 1, 2, \cdots\right\}.$$

Well-order $\mathscr{C}$ as

$$\{V_1, V_2, V_3, \cdots\}.$$

We shall now show that for every uniform neighborhood $U(\varepsilon, \lambda) \in \mathscr{U}_{\mathscr{F}}$, there exists an M such that

$$\bigcap_{i=1}^{M} V_i \subseteqq U(\varepsilon, \lambda).$$

For, consider the sets $A_m \subseteqq \mathscr{D}$ defined as follows

$$A_m = \left\{d \in \mathscr{D}: \bigcap_{i=1}^{m} V_i \subseteqq \{(p, q): d(p, q) < \varepsilon\}\right\}.$$

Obviously $\{A_m\}_{m=1}^{\infty}$ is an increasing sequence of sets. It is also very easy to show that $\lim_{m\to\infty} A_m = \mathscr{D}$. Now extend μ to an outer measure μ_0^* on $\mathscr{D}$ by defining

$$\mu_0^*(A) = \inf\{\mu(B): A \subseteqq B \text{ and } B \in \mathscr{B}\}.$$

It may be shown (see, for instance, Munroe [2], p. 99) that μ_0^* is a regular outer measure on $\mathscr{D}$. We then have

$$\lim_{m\to\infty} \mu_0^*(A_m) = \mu_0^*\left(\lim_{m\to\infty} A_m\right) = \mu_0^*(\mathscr{D}) = 1.$$

Therefore, there exists an M such that

$$\mu_0^*(A_M) > 1 - \lambda.$$

Now if $(p_0, q_0) \in \bigcap_{i=1}^{M} V_i$,

$$A_M \subseteqq \{d \in \mathscr{D}: d(p_0, q_0) < \varepsilon\}.$$

Hence

$$\mu\{d \in \mathscr{D}: d(p_0, q_0) < \varepsilon\} \geqq \mu_0^*(A_M) > 1 - \lambda,$$

so that $(p_0, q_0) \in U(\varepsilon, \lambda)$ and $\bigcap_{i=1}^{M} V_i \subseteqq U(\varepsilon, \lambda)$. This completes the proof that $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{R}}$.

COROLLARY. *Let* $(S, \mathscr{F}; \mathscr{D}, \mathscr{B}, \mu)$ *be a pseudo-MG space. If there exists a countable collection* $\mathscr{R}$ *of pseudometrics on S such that* $\mathscr{U}_{\mathscr{R}} = \mathscr{U}_{\mathscr{D}}$, *then* $\mathscr{U}_{\mathscr{F}} \subseteqq \mathscr{U}_{\mathscr{D}}$.

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