ON GENERALIZED FORMS OF APOSYNDESIS

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If a point set is both connected and closed it is called a continuum. The structure of a nonlocally connected continuum can be described in terms of its aposyndetic properties. In this paper various forms of continuum aposyndesis, that is, aposyndesis with respect to subcontinua, are considered. It is shown that the presence of any of these forms of aposyndesis in a compact metric continuum which is totally nonconnected im kleinen (not connected im kleinen at any point) insures nonsemi-local-connectedness on a dense open subset of the continuum and the set of weak cut points in each open subset of the continuum has cardinality at least c.[1] A weak cut point theorem for compact plane continua is established. An example is given which indicates that this result does not hold in Euclidean 3-space. Near aposyndesis, a generalization of aposyndesis, is introduced. It is shown that the presence of this property in a totally nonaposyndetic, separable, metric continuum implies the existence of uncountably many weak cut points.

DEFINITION. Let x, y, and z be distinct points of a continuum M. If every subcontinuum of M which contains x and y also contains z, then z is said to *cut M weakly* between x and y. A point z of M is said to be a *weak cut point* of M if there exist two points x and y in M such that z cuts M weakly between x and y.

DEFINITION. Let S be a subset of a continum M and let x be a point of $M - S$. If M contains a continuum H and an open set U such that $x \in U \subset H \subset M - S$, then M is said to be *aposyndetic* at x with respect to S. Note that if M is a regular Hausdorff continuum, M being aposyndetic at a point p with respect to every closed set in $M - \{p\}$ is equivalent to M being connected im kleinen at p. Let x be a point of a continuum M; if for each point y of $M - \{x\}$, M is aposyndetic at x with respect to y, then M is said to be *aposyndetic* at x.

Let S is a subset of a continuum M. If x is a point of $M - S$ and M is not aposyndetic at x with respect to S, then M is said to be *nonaposyndetic* at x with respect to S.

2. **Continuum aposyndesis.** In the introduction it is pointed

[1] For a related result see [5, Th. 15]. For definitions of unfamiliar terms and phrases see [7] and [9].

out that connectedness im kleinen at a point of a regular Hausdorff continuum can be thought of in terms of aposyndesis at that point with respect to closed sets which do not contain the point. This concept can be generalized by considering aposyndesis at a point with respect to closed connected sets (i.e., continua) which do not contain the point.

DEFINITION. Let M be a continuum and p and q be two distinct points in M. If, for each continuum K in $M - \{p\}$ which contains q, M is aposyndetic at p with respect to K, then M is said to have *property A at p with respect to q*. If, for every point x in $M - \{p\}$, M has property A at p with respect to x, then M is said to have *property A at p*.

Obviously, if a regular Hausdorff continuum M is connected im kleinen at a point p in M, then M has property A at p. An example due to F. B. Jones indicates that the converse of this statement is false [6, Example 3]. The compact plane continuum described by Jones has property A at a point y and is not connected im kleinen at y. The point y in this continuum is a weak cut point. The following theorem indicates that a compact plane continuum has these properties (property A and nonconnectedness im kleinen) at a point only if the point is a weak cut point of the continuum.

LEMMA. *If a compact plane continuum M is not connected im kleinen at a point x, then for each open set U in the plane which contains x, there exists a pair of points $\{y, z\}$ in $U \cap M$ such that M is nonaposyndetic at x which respect to $\{y, z\}$.*[2]

Proof. Assume that there is an open set U containing x such that M is aposyndetic at x with respect to every pair of points in $U \cap M$. Since M is not connected im kleinen at x, there exists a circular region G such that Cl G (the closure of G) is contained in U and a sequence $K_1, K_2, K_3, \cdots$ of distinct components of $M \cap$ Cl G such that (1) for each positive integer i, K_i contains the point y_i of a sequence $y_1, y_2, y_3, \cdots$ of points of J (the boundary of G) converging to the point y and (2) x is in the limit inferior of $K_1, K_2, K_3, \cdots$.

Since M is aposyndetic at x with respect to any pair of points of $U \cap M$, M is aposyndetic at x with respect to y. Therefore, there exists a continuum H in $M - \{y\}$ such that x is contained in Int H (the interior of H). Each component of $H \cap G$ has a limit point in J. Hence there exists a subsequence $K'_1, K'_2, K'_3, \cdots$ of $K_1, K_2, K_3, \cdots$

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[2] This lemma is stated in [6]. The proof does not appear in the literature.

such that for each i, K_i' contains the point z_i of a sequence $z_1, z_2, z_3, \cdots$ of points of J converging to the point z of $J - \{y\}$.

By assumption M is aposyndetic at x with respect to $\{y, z\}$. Hence there is a continuum L such that $x \in \operatorname{Int} L \subset L \subset M - \{y, z\}$. Let A and B denote disjoint subarcs of J containing y and z respectively (as nonendpoints) such that $(A \cup B) \cap L = \varnothing$. Since every component of $L \cap G$ has a limit point in J, there exist three positive integers, i, j, and k, such that each of K_i', K_j', and K_k' intersects each of A, B, and $J - (A \cup B)$. Since $J - (A \cup B)$ has exactly two components, some two of these three continua must intersect the same component of $J - (A \cup B)$. This leads to a contradiction of Theorem 28 of [7, p. 156].

THEOREM 1. *If a compact plane continuum M has property A at a point x and is not connected im kleinen at x, then x is a weak cut point of M.*

Proof. By the preceding lemma, there exists a pair of points $\{y, z\}$ in $M - \{x\}$ such that M is nonaposyndetic at x with respect to $\{y, z\}$. M must be aposyndetic at x with respect to each continuum in $M - \{x\}$ since M has property A at x. Therefore no subcontinuum of M in $M - \{x\}$ contains both y and z.

EXAMPLE 1. A compact continuum M in Euclidean 3-space which has property A at a point p, and is not connected im kleinen at p, may fail to be cut weakly by p. To see this define $A_i = \{(0, 0, 1/n) \mid n = i, i + 1, i + 2, \cdots\}$. Let C_1 be the join of $\operatorname{Cl} A_1$ with the point $(1, 0, 0)$. For $i = 2, 3, \cdots$, define C_i to be the join of $\operatorname{Cl} A_i$ with the point $(1, 1/i, 0)$. Let $M = \bigcup_{i=1}^{\infty} C_i$. See Figure 1. Let p be the point $(0, 0, 0)$. Any subcontinuum in the complement of p intersect only finitely many of the C_i's. It follows that M has property A at p. Clearly M is not connected im kleinen at p and p does not cut M weakly between any two points in $M - \{p\}$. Note that M is not semi-locally-connected at p.

THEOREM 2. *If a regular Hausdorff continuum M is semi-locally-connected at a point p and has property A at p, then M is connected im kleinen at p.*[3]

Proof. Assume that M is not connected im kleinen at p. First it will be shown that under this assumption p must be a weak cut

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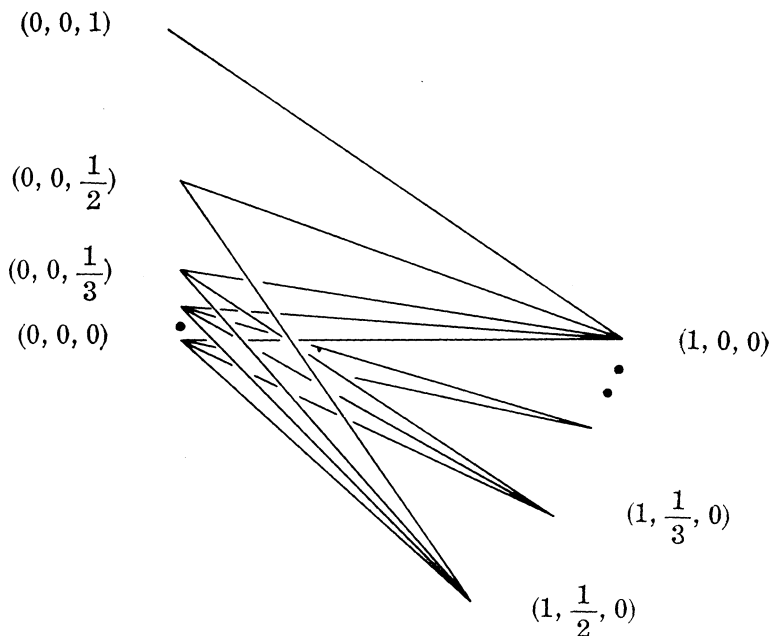[3] This has been previously observed for compact Hausdorff continua [1, Th. 4].

FIGURE 1.

point of M.

Since M is not connected im kleinen at p and is semi-locally-connected at p, there exists an open set U in M containing p such that M is nonaposyndetic at p with respect to $M - U$, and $M - U$ has only a finite number of components. If p is not a weak cut point then the components of $M - U$ can be joined together by a finite number of continua in $M - \{p\}$. Let L be the union of these continua. Every continuum containing p in its interior must meet the continuum $(M - U) \cup L$. But this contradicts the fact that M has property A at p. Therefore p must be a weak cut point.

Since M is semi-locally-connected at p and p is a weak cut point, p must separate M [8, Th. 6.2]. Suppose that for each component C of $M - \{p\}$ which meets $M - U$, the set $C \cup \{p\}$ is a connected subspace of M which is aposyndetic at p with respect to $(C \cup \{p\}) - U$. Then in each of these subspaces there is a continuum which contains p in its interior, relative to the subspace, which does not meet $M - U$. The sum of these continua and the components of $M - \{p\}$ which do not meet $M - U$ form a continuum in M which contains p in its interior and misses $M - U$. This contradicts the choice of U. Therefore there is a component C in $M - \{p\}$ such that the subspace $C \cup \{p\}$ is nonaposyndetic at p with respect to $(C \cup \{p\}) - U$. It follows that the subspace $C \cup \{p\}$ is not connected im kleinen at p. Let H be the subspace $C \cup \{p\}$. Note that H is semi-locally-connected at p, since

M is semi-locally-connected at p.

The subspace H has property A at p. To see this let Q denote a subcontinuum of $H - \{p\}$. Because M has property A at p, and Q is a subcontinuum of M, there exists a subcontinuum K of $M - Q$ which contains p in its interior. Since p is a separating point, the set $H \cap K$ is a subcontinuum of H and p is in the interior of $H \cap K$ relative to H. Therefore H has property A at p.

By applying to the subspace H the argument presented in the second paragraph of this proof, one can conclude that p must cut H weakly and therefore separate H. But this is impossible since H consists of the point p and a component of $M - \{p\}$. Hence M is connected im kleinen at p.

In bicompact T_1 continua, property A and local connectedness are equivalent as global properties, since if a bicompact T_1 continuum has property A everywhere, then it is aposyndetic at each of its points and therefore semi-locally-connected [5, Th. 0], and from Theorem 2 it follows that the continuum is everywhere connected im kleinen and therefore locally connected.

It is clear that a T_1 continuum M has property A at a point p with respect to a point q if and only if for each open set G in $M - \{q\}$ which contains p, M is aposyndetic at p with respect to the q-component of $M - G$. From this point of view one can generalize this property as follows.

DEFINITION. Let p and q be distinct points of a continuum M. If for each open set G in $M - \{q\}$ which contains p, there exists a point r in G such that M is aposyndetic at r with respect to the q-component of $M - G$, then M is said to have *property B* at p with respect to q. If for each point x in $M - \{p\}$, M has property B at p with respect to x, then M is said to have *property B* at p. If M has property B at each point of M, then M is said to have *property B*.

Obviously, if a continuum M contains a dense subset D such that M has property A at each point of D, then M has property B. The following example indicates that property B is considerably weaker than property A.

EXAMPLE 2. There exists a compact plane continuum M which has property B and is totally nonaposyndetic (not aposyndetic at any of its points) hence does not have property A at any point.

Let $M_1, M_2, \cdots$ be a sequence of closed plane point sets defined by induction as follows. Let S be the unit disk and let M_1 be the closure of the union of the (topological) disk sequence $D_1, D_2, \cdots$
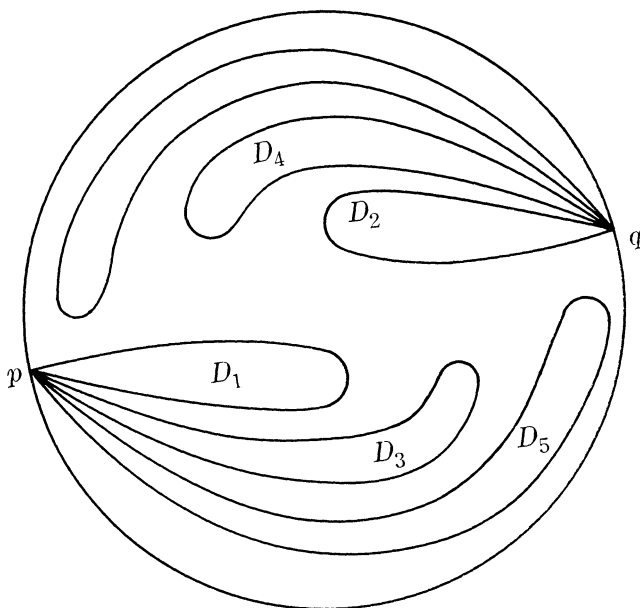
indicated in Figure 2. The boundary of S is the limiting set of D_1, D_2, $\cdots$. The diameter of each of D_1, D_2, $\cdots$ is greater than $3/4$. Let p and q be the two separating points in M_1. Note that M_1 is non-aposyndetic at each point of the boundary of S with respect to one of p and q.

Assume M_n to be defined and let D_1', D_2', $\cdots$ be a counting of the disks in M_n. Let f_i be a homeomorphism of S onto D_i' such that the distance from $f_i(p)$ to $f_i(q)$ is greater than $(1/4) + (1/n + 3)$. Let $M_{n+1} = \mathrm{Cl}\,[\bigcup_{i=1}^{\infty} f_i(M_1)]$. The homeomorphisms of S onto the disks of M_n are chosen in such a way that the disks in M_{n+1} will be of diameter greater than $(1/2) + (1/n + 3)$ and the set of separating points in M_{n+1} is $1/n + 1$ dense in M_n. Define $M = \bigcap_{n=1}^{\infty} M_n$.

M has property B. To see this let x and y be two points in M and let G be an open set in $M - \{y\}$ containing x. Let C be the y-component of $M - G$. Since the point set consisting of homeomorphic images of $\{p, q\}$ is dense in M, there is a point s in $M \cap G$ which is a separating point in M_n (for some n). There exists a disk D in M_n such that y is not in D and s separates $D - \{s\}$ from $M_n - D$ in M_n. It follows that $M \cap D$ is a continuum in $M - C$ which has an interior point in G.

It is clear that M is nonaposyndetic at each point which is on the boundary of some defining continuum M_n in S. Let x be a point of M which is in the interior of M_n for every positive integer n. There exist a nest of disks d_1, d_2, $\cdots$ and two sequences of points, s_1, s_2, $\cdots$

and t_1, t_2, $\cdots$, such that (1) for each positive integer n, the point x is contained in d_n; (2) d_n is a maximal disk in M_n; (3) the point s_n separates $d_{n+1} - \{s_n\}$ from $(d_n - d_{n+1}) \cap M_{n+1}$ in M_{n+1}; and (4) t_n is the other point which separates $d_n \cap M_{n+1}$. Note that for each n, M is nonaposyndetic at s_n with respect to t_n. If x is a limit point of s_1, s_2, $\cdots$, then there exists a point t in M, distinct from x, which is a limit point of the sequence t_1, t_2, $\cdots$, such that M is nonaposyndetic at x with respect to t [5, Th. 1]. If x is not a limit point of s_1, s_2, $\cdots$, then there exists a point s in M, distinct from x, which is a limit point of this sequence. It follows that M is nonaposyndetic at x with respect to s. Therefore M is totally nonaposyndetic.

DEFINITION. Let p be a point of a continuum M. If there exists a point q in $M - \{p\}$ such that M has property B at p with respect to q, then M is said to have *property C* at p.

Obviously, if a continuum has property B at a point p, then it has property C at p. One can see that property C is weaker than property B by considering the Cantor Cone. This continuum has property C at each point which is in the interior of an arc, but it has property B only at the vertex.

THEOREM 3. *If a compact metric continuum M has property C at each point of a dense G_δ subset of M and is totally nonconnected im kleinen on a dense G_δ subset of M, then M is totally nonsemi-locally-connected on a dense open subset of M.*

Proof. Let U be an open set in M. If one can show that there exists an open subset G of U such that M is not semi-locally-connected at any point of G, then the existence of the dense open subset of M with the desired condition with follow immediately.

The open set U contains an open set V, no component of which contains an open set [3, Th. 2]. It follows that for each point x in V, M is nonaposyndetic at x with respect to $M - V$. Define $D_i = \{x \in V \mid$ for some y in $M - S(x, 1/i)$ $(S(x, 1/i)$ is the circular open set in M with the point x as center and with radius $1/i)$, M has property B at x with respect to $y\}$. Since $\bigcup_{i=1}^{\infty} D_i$ is a second category subset of V, for some positive integer n, the set $\operatorname{Cl} D_n$ contains an open set in V. It follows that there exists an open set W in V such that W contains a dense subset D with the condition that for each point x in D, there exists a point y in $M - W$ such that M has property B at x with respect to y.

If M is totally nonsemi-locally-connected on W, then W has the

required conditions. Assume that this is not the case. That is, there exists a point p in W such that M is semi-locally-connected at p. There exists an open set Q containing p in W such that $M - Q$ has only a finite number of components. Suppose that p is not a weak cut point. Join together the components of $M - Q$ with a finite number of continua in $M - \{p\}$. Let L be the union of these continua and let K be the continuum $(M - Q) \cup L$. The set $M - K$ is open and contained in Q. Since D is dense in W and $M - W$ is contained in K there exist a point x in $D \cap (M - K)$ and a point y in K such that M has property B at x with respect to y. But this is impossible since M is nonaposyndetic at each point of $M - K$ with respect to K. Therefore p is a weak cut point in M.

Since M is semi-locally-connected at p, p is a separating point in M and each component of $M - \{p\}$ is both open and closed relative to $M - \{p\}$. Let X be a component of $M - \{p\}$ which meets $M - Q$. The set $S = X \cup \{p\}$ is a connected subspace of M. S is semi-locally-connected at p. The point p is not a weak cut point in S for if it were it would also separate S which is clearly impossible. $S - Q$ has only a finite number of components. Join there components together with a finite number of continua in X. Let F denote the union of these continua with $S - Q$. The set F is a continuum in $S - \{p\}$ which contains $S - Q$. The set $G = X - F$ is open in M. Let x be a point in G and assume that M is semi-locally-connected at x. Each open subset of G which contains x cuts M weakly between p and F (i.e., each continuum in M which meets both F and p must also meet each open subset of G which contains x). To see this suppose that there exist an open set R containing x in G and there exists a continuum H in $M - R$ which meets both p and F. It follows that $M - Q$ is contained in one component of $M - R$. But for some point s in R there exists a point t in $M - W$ such that M has property B at s with respect to t. Therefore there exists a point r in R such that M is aposyndetic at r with respect to the t-component of $M - R$. This is a contradiction since $M - Q$ is contained in the t-component of $M - R$ and M is nonaposyndetic at r with respect to $M - Q$. It follows that x cuts M weakly between p and F. Since M is semi-locally-connected at x, M is separated by x between p and F. $S - \{x\}$ can be written as the union of two mutually separated sets P and E such that p is contained in P and F is contained in E. $P \cup \{x\}$ is a subcontinuum of M which has a nonvoid interior and is contained in Q. This contradicts the fact that no component of Q contains an open set. Therefore M is totally nonsemi-locally-connected on G.

THEOREM 4. *If a compact metric continuum M is nonsemi-locally-connected at each point of a G_δ subset which is dense in M, then the*

set of weak cut points in each open subset of M has cardinality at least c.

Proof. Let U be an open subset of M. Define the set $D = \{x \in M \mid$ for each open set V containing x, there exists an open subset W of V containing x such that M is aposyndetic at x with respect to each point of the boundary of $W\}$. If D is dense in U, then there exists a dense G_δ subset J of U such that each point of J is a weak cut point in M [4, Th. 4].

Suppose that D is not dense in U. There exists an open set G in $U - D$. Since G is second category, there exist a point x in G and a point r in M such that if y is a point of M and M is non-aposyndetic at x with respect to y, then y cuts M weakly between x and r [2, Th. 4]. There exists an open set Q containing x in G such that for each open subset R of Q there is a point y in the boundary of R such that M is nonaposyndetic at x with respect to y. There are c open sets in Q which contain x and have mutually disjoint boundaries. It follows that there are c points in Q which cut M weakly between x and r.

THEOREM 5. *If a compact metric continuum M has property C at each point of a dense G_δ subset of M and is totally nonconnected im kleinen on a dense G_δ subset of M, then each open subset of M contains a set of weak cut points of M which has cardinality c.*

Proof. M is totally nonsemi-locally-connected on a dense open subset of M (Theorem 3). The conclusion follows from Theorem 4.

COROLLARY. *If a compact metric continuum M has property B and is totally nonconnected im kleinen on a dense G_δ subset of M, then each open subset of M contains a set of weak cut points of M which has cardinality c.*

Note that if a compact metric continuum M is totally nonconnected im kleinen on a dense G_δ subset of M and contains a dense subset D such that M has property A at each point of D, then M contains a dense G_δ set of weak cut points of M [4, Th. 4]. However, the existence of such continua is still an open question.

3. Near aposyndesis. In § 2 property B is introduced as a weaker form of continuum aposyndesis. In this section aposyndesis (aposyndesis with respect to points) is generalized in a similar fashion.

DEFINITION. A continuum M is said to be *nearly aposyndetic* at

a point p in M with respect to a point q in M if each open set in M containing p contains a point r such that M is aposyndetic at r with respect to q. Let p be a point of M; if for each point q in $M - \{p\}$, M is nearly aposyndetic at p with respect to q, then M is said to be *nearly aposyndetic* at p.

It is easily seen that if a continuum M has property C at a point p and is not nearly aposyndetic at p, then p must be a weak cut point in M. Note that the Cantor Cone has these two properties at each point of a dense open set.

THEOREM 6. *A compact metric continuum M is not nearly aposyndetic at a point p with respect to a point s if and only if there exists an open set G in M containing p, such that if U is a nonvoid open subset of G, then s cuts M weakly between some two points in U.*

Proof. If M is not nearly aposyndetic at p with respect to s, then there exists an open set G containing p such that M is nonaposyndetic at each point of G with respect to s. This open set G has the desired property [2, Th. 2].

To see that the condition is sufficient, assume that M is nearly aposyndetic at p with respect to s. Let G be an open set in M containing p. There exists a point x in G such that M is aposyndetic at x with respect to s. Therefore there is a continuum K and an open set U such that $x \in U \subset K \subset M - \{s\}$. It follows that s does not cut M weakly between any two points of the open set $G \cap U$ in G.

If a continuum M has property B, then M is nearly aposyndetic (that is, M is nearly aposyndetic at each of its points). It follows that the totally nonaposyndetic continuum M in Example 2 is nearly aposyndetic. One can see from this example that near aposyndesis is considerably weaker than aposyndesis. M in Example 2 is totally nonsemi-locally-connected. The following example indicates that this is not necessarily the case for totally nonaposyndetic continua which are nearly aposyndetic.

EXAMPLE 3. There exists a compact nearly aposyndetic, totally nonaposyndetic continuum M in E^3 (Euclidean 3-space) which is semi-locally-connected on a dense open subset of M.

Let C be the Cantor set and its image on the interval $[-1, 0]$. For each point z of C define the set

$$S_z = \{(x, y, z) \in E^3 \mid x = 0 \text{ or } x = 1 \text{ and } 0 \leq y \leq 1, \text{ or }$$
$$y = 0 \text{ or } y = 1 \text{ and } 0 \leq x \leq 1\}.$$

Let $S = \bigcup_{z \in C} S_z$. Define the continuum M to be the decomposition of S obtained as follows. For each positive real number z in C, identify the point $(z, 0, z)$ with the point $(z, 0, 0)$. For each negative real number z in C, identify the point $(-z, 1, z)$ with the point $(-z, 1, 0)$. See Figure 3. M is semi-locally-connected at each point which is not in the XY-plane. Note that M contains a Cantor set of weak cut points.
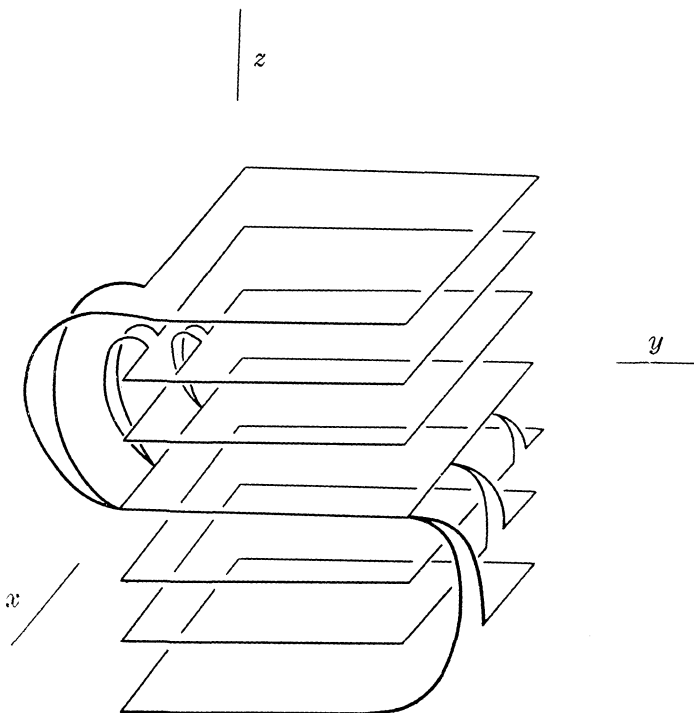

FIGURE 3.

It is possible for a compact totally nonaposyndetic metric continuum to have only one weak cut point [5, Example 1]. However, if the continuum is also nearly aposyndetic then one is assured of the existence of more than countably many weak cut points.

THEOREM 7. *If a compact metric continuum M is nearly aposyndetic and totally nonaposyndetic, then M has uncountably many weak cut points.*

Proof. Assume that M has only countably many weak cut points. Let $s_1, s_2, \cdots$ be a counting of these points. Let Q denote a countable dense subset of M. Since M is totally nonaposyndetic, M contains a dense G_δ subset I such that if x is a point in I and M is nonaposyn-

detic at x with respect to a point y, then y cuts x weakly from each point of $Q - \{y\}$ in M [2, Corollary 1]. For each positive integer i, define

$$D_i = \{x \in I \mid s_i \text{ cuts } x \text{ from each point of } Q - \{s_i\}\}.$$

$\bigcup_{i=1}^{\infty} D_i = I$. Since I is second category, there is a positive integer n such that D_n is somewhere dense. Let G be an open set in $\operatorname{Cl} D_n$ which does not contain s_n. Note that G has the property described in Theorem 6. It follows that M is not nearly aposyndetic at any point of G with respect to s_n. But this is a contradiction. Therefore M must contain uncountably many weak cut points.

<h2 style="text-align:center">BIBLIOGRAPHY</h2>

1. H. S. Davis, *A note on connectedness im kleinen*, Proc. Amer. Math. Soc. **19** (1968), 1237-1241.

2. E. E. Grace, *Cut sets in totally nonaposyndetic continua*, Proc. Amer. Math. Soc. **9**, (1958), 98-104.

3. ———, *Totally nonconnected im kleinen continua*, Proc. Amer. Math. Soc. **9**, (1958), 818-821.

4. ———, *Cut points in totally non-semi-locally-connected continua*, Pacific J. Math. **14** (1964), 1241-1244.

5. F. B. Jones, *Concerning non-aposyndetic continua*, Amer. J. Math. **70** (1948), 403-413.

6. ———, *Concerning aposyndetic and non-aposyndetic continua*, Bull. Amer. Math. Soc. **58** (1952), 137-151.

7. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Coll. Pub. **13** Rhode Island, 1962.

8. G. T. Whyburn, *Semi-locally-connected sets*, Amer. J. Math. **61** (1939), 733-749.

9. ———, *Analytic topology*, Amer. Math. Soc. Coll. Pub. **18**, Rhode Island, 1963.

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