ON SUBGROUPS OF A PSEUDO LATTICE ORDERED GROUP

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The purpose of this note is to investigate some problems raised in a recent paper of Conrad and Teller concerning o-ideals and p-subgroups in an abelian pseudo lattice ordered group.

The concept of a pseudo lattice ordered group ("p-group") has been introduced by Conrad [1]. In recent papers by Teller [5] and Conrad and Teller [2] there is developed a systematic theory of p-groups. Let \( G \) be an abelian p-group. In § 3 it is proved that if \( M \) is a subgroup of \( G \) such that \( \{a, b\} \cap M \neq \emptyset \) for any pair of p-disjoint elements \( a, b \in G \), then \( M \) contains a prime o-ideal; this generalizes a result from [2]. In § 4 we prove that the intersection of two p-subgroups of a p-group \( G \) need not be a p-subgroup of \( G \). Moreover, if \( \mathcal{A} \) is a partially ordered set and for each \( \delta \in \mathcal{A} \) \( H_\delta \neq \{0\} \) is a linearly ordered group, then for the mixed product \( G = \bigvee (\mathcal{A}, H_\delta) \) the following conditions are equivalent: (i) for any two p-subgroups \( A, B \) of \( G \) their intersection \( A \cap B \) is a p-subgroup of \( G \) as well; (ii) \( G \) is an l-group. If \( A \) is an o-ideal of a p-group \( G \) and \( B \) is a p-subgroup of \( G \), then \( A + B \) is a p-subgroup of \( G \).

2. Preliminaries. Let \( G \) be a partially ordered group. \( G \) is a Riesz group (cf. Fuchs [3], [4]) if it is directed and if from \( a_i, b_j \in G, a_i \leq b_i \ (i, j = 1, 2) \) it follows that there exists \( c \in G \) satisfying \( a_i \leq c \leq b_i \ (i, j = 1, 2) \). \( G \) is a p-group (cf. [1] and [5]) if it is Riesz and if each \( g \in G \) has a representation \( g = a - b \) such that \( a, b \in G, a \geq 0, b \geq 0 \) and

\[
(\ast) \quad x \in G, x \leq a, x \leq b \implies nx \leq a, nx \leq b
\]

for any positive integer \( n \).

Throughout the paper \( G \) denotes an abelian p-group. Elements \( a, b \in G, a \geq 0, b \geq 0 \) satisfying (\( \ast \)) are called p-disjoint. A subgroup \( M \) of \( G \) is a p-subgroup, if for each \( m \in M \) there are elements \( a, b \in M \) such that \( a, b \) are p-disjoint in \( G \) and \( m = a - b \). A subgroup \( C \) of \( G \) is an o-ideal, if it is directed and if \( 0 \leq g \leq c \in C, g \in G \) implies \( g \in C \). Let \( O(G) \) be the system of all o-ideals of \( G \) (partially ordered by the set inclusion). An o-ideal \( C \) of \( G \) is called prime, if \( G/C \) is a linearly ordered group. For any pair \( a, b \) of p-disjoint elements \( H(a, b) \) denotes the subgroup of \( G \) generated by the set.
Then $H(a, b) \in O(G)$ (cf. [2]).

Let $\mathcal{A}$ be a partially ordered set and let $H_\delta \neq \{0\}$ be a linearly ordered group for each $\delta \in \mathcal{A}$. Let $V = V(\mathcal{A}, H_\delta)$ be the set of all $\mathcal{A}$-vectors $v = (\cdots, v_\gamma, \cdots)$ where $v_\gamma \in H_\delta$, for which the support $S(v) = \{\delta \in \mathcal{A} \mid v_\delta \neq 0\}$ contains no infinite ascending chain. An element $v \in V$, $v \neq 0$ is defined to be positive if $v_\delta > 0$ for each maximal element $\delta \in S(v)$. Then ([2], Th. 5.1) $V$ is a $p$-group; $V$ is an 1-group if and only if $\mathcal{A}$ is a root system (i.e., $\{\delta \in \mathcal{A} \mid \delta \geq \gamma\}$ is a chain for each $\gamma \in \mathcal{A}$).

3. Subgroups containing a prime o-ideal. The following assertion has been proved in [2] (Proposition 4.3):

(A) For $M \in O(G)$, the following are equivalent: (1) $M$ is prime; (2) the o-ideals of $G$ that contain $M$ form a chain; (3) if $a$ and $b$ are $p$-disjoint in $G$, then $a \in M$ or $b \in M$.

Further it is remarked in [2] that each subgroup $M$ of $G$ fulfilling (3) is a $p$-subgroup and any subgroup containing a prime o-ideal satisfies (3) if and only if it contains a prime o-ideal (a similar assertion is known to be valid for lattice ordered groups).

We shall prove that the answer is positive.

We need the following propositions (cf. [2] and [5]):

(B) Let $g = a - b \in G$ where $a$ and $b$ be $p$-disjoint. Then $g = x - y$, where $x$ and $y$ are $p$-disjoint, if and only if $x = a + m$ and $y = b + m$ for some $m \in H(a, b)$.

(C) If $a$ and $b$ are $p$-disjoint, then $na$ and $nb$ are $p$-disjoint for any positive integer $n$ and $H(a, b) = H(na, nb)$ ([2], Proposition 3.1).

**Lemma 1.** Let $M$ be a subgroup of $G$ fulfilling (3) and let $a$, $b$ be $p$-disjoint elements in $G$. Then $H(a, b) \subseteq M$.

**Proof.** Let $h \in H(a, b)$. According to (3) we may assume without loss of generality that $a \in M$. Suppose (by way of contradiction) that $h \notin M$. Then $a + h \notin M$, hence by (B) $b + h \in M$, and analogously $b - h \in M$, thus $2b \in M$. Further $2a + h \in M$ and therefore according to (C) and (B) $2b + h \in M$, which implies $h \in M$.

**Lemma 2.** Let $M$ be a subgroup of $G$ satisfying (3) and let $X = \{X_i\}$ be the system of all o-ideals of $G$ such that $X_i \subseteq M$. Then the system $X$ has a largest element.

**Proof.** Let $Y$ be the subgroup of $G$ generated by the set $\bigcup X_i$. 

\[ \{0 \leq m \in G \mid m \leq a, m \leq b\}. \]
Then $Y \subseteq M$ and $Y$ is the supremum of the system $\{X_i\}$ in the lattice $\mathcal{O}$ of all subgroups of $G$. Since $O(G)$ is a complete sublattice of $\mathcal{O}$ ([2], Th. 2.1), $Y \in O(G)$ and thus $Y \in X$.

Let $H$ be the subgroup of $G$ generated by the set $\bigcup H(a, b)$ where $a, b$ runs over the system of all $p$-disjoint pairs of elements in $G$. Since each set $H(a, b)$ is an $o$-ideal ([2]), $H = \bigvee H(a, b)$ ($a$ and $b$ $p$-disjoint in $G$) where $\bigvee$ denotes the supremum in the lattice $O(G)$. According to Lemma 1 $H \subset M$ whenever the subgroup $M$ of $G$ satisfies (3).

For any $u, v \in G$, $u \leq v$, the interval $[u, v]$ is the set

$$\{x \in G \mid u \leq x \leq v\}.$$

**Lemma 3.** Let $M$ be a subgroup of $G$ satisfying (3) and let $N$ be the largest $o$-ideal of $G$ that is contained in $M$. Let $g \in G$, $g > 0$. Then

$$[0, g] \subset M \implies g \in N.$$

**Proof.** According to Lemma 2 the largest $o$-ideal $N$ in $M$ exists. Assume that $g \in G$, $g > 0$, $[0, g] \subset M$. The set

$$Z = \bigcup_{n=1}^{\infty} [-ng, ng]$$

is clearly an $o$-ideal in $G$. Let $z \in Z$, hence $z \in [-ng, ng]$ for a positive integer $n$. This implies $0 \leq y \leq 2ng$ where $y = z + ng$. Since $G$ is a Riesz group, according to [3, p. 158, Th. 27] there are elements $g_1, \ldots, g_{2n} \in G$, $0 \leq g_i \leq g$ such that $y = g_1 + \cdots + g_{2n}$. Thus $g_i \in M$, therefore $y \in M$ and $Z \subset M$. Now we have $Z \subset N$ and so $g \in N$.

**Lemma 4.** Let $M$ be a subgroup of $G$ fulfilling (3) and let $N$ be the largest $o$-ideal of $G$ contained in $M$. Then $G/N$ is a linearly ordered group.

**Proof.** Assume (by way of contradiction) than $G/N$ is not linearly ordered. According to Lemma 1 $H \subset N$, hence by [2], Theorem 4.1 $G/N$ is a lattice ordered group. Thus there exist elements $X, Y \in G/N$ such that $X \wedge Y = 0$, $X > 0$, $Y > 0$ ($0$ being the neutral element of $G/N$). From [2] (Proposition 2.2, (ii)) it follows that there are elements $x \in X$, $y \in Y$ such that $x$ and $y$ are $p$-disjoint in $G$ and hence $x \in M$ or $y \in M$. Clearly $x \notin N$, $y \notin N$ and thus according to Lemma 3 there exist elements $x, y \in G$ such that

$$0 < x, \ 0 < y, \ x \in M, \ y \in M.$$
Then in $G/N$ we have $0 < x_i + N \leq x + N = X$, $0 < y_i + N \leq y + N = Y$, whence

$$(x_i + N) \wedge (y_i + N) = \overline{0}.$$  

Thus by using repeatedly [2], Proposition 2.2, we can choose elements $x_2 \in x_1 + N$, $y_2 \in y_1 + N$ such that $x_2$ and $y_2$ are $p$-disjoint in $G$. Therefore (without loss of generality) we may assume $x_2 \in M$ and this implies $x_1 \in x_1 + N = x_2 + N \subset M$, a contradiction. The proof is complete.

**Theorem 1.** Let $M$ be a subgroup of a $p$-group $G$. Then (3) $\Rightarrow$ (2) and the condition (3) is equivalent to (1') $M$ contains a prime $o$-ideal.

**Proof.** According to Lemma 4 (3) $\Rightarrow$ (1'). By [2] (1') $\Rightarrow$ (3). Assume that $M$ is a subgroup of $G$ fulfilling (3). Let $K_1$, $K_2$ be $o$-ideals of $G$ such that $M \subset K_1 \cap K_2$. Let $N$ have the same meaning as in Lemma 4. Since $N \subset M$,

$$K_1 \subset K_2 \iff K_1/N \subset K_2/N.$$  

$K_1/N$ and $K_2/N$ are $o$-ideals of $G/N$ and $G/N$ is linearly ordered, hence $K_1/N \subset K_2/N$ or $K_2/N \subset K_1/N$; therefore (2) holds.

If $M$ is an $o$-ideal of $G$ satisfying (3), then by Theorem 1 $M$ contains a prime $o$-ideal $N$; according to [2] (Corollary 1 to the Induced Homomorphism Theorem) $G/M$ is isomorphic to $(G/N)/(M/N)$ and hence $(G/N$ being linearly ordered) $G/M$ is a linearly ordered group and $M$ is prime. Thus it follows from Theorem 1 that (3) $\Rightarrow$ (1) for $M \in O(G)$ (cf. (A)).

Let us remark that if $M$ is a subgroup of $G$ fulfilling (3) then $M$ need not contain any nonzero $o$-ideal that is a lattice; further (3) is not implied by (2).

**Example 1.** Let $B$ be an infinite Boolean algebra that has no atoms and put $\Delta = \{b \in B \mid b \neq 0\}$. For each $\delta \in \Delta$ let $H_\delta = E$ where $E$ is the additive group of all integers with the natural order, $G = V(\Delta, H_\delta)$. Let $M = \{v \in G \mid v_1 = v_2 = 0\}$ (by 1 we denote the greatest element of $B$). Then $M$ is a prime $o$-ideal of $G$, hence $M$ satisfies (3) and $M$ contains no lattice ordered $o$-ideal different from $\{0\}$.

**Example 2.** Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 < \delta_2 < \delta_3$ and $\delta_1, \delta_2$ are incomparable. Put $H_{\delta_i} = E(i = 1, 2, 3)$, $G = V(\Delta, H_\delta)$, $M = \{v \in G \mid v_{\delta_1} = v_{\delta_2} = 0\}$. Then the only $o$-ideal that contains $M$ is $G$, thus (2) holds. Let $a, b \in G$ such that $a_{\delta_1} = 1$, $a_{\delta_2} = a_{\delta_3} = 0$, $b_{\delta_1} = 1$, $b_{\delta_2} = b_{\delta_3} = 0$. The elements $a$ and $b$ are $p$-disjoint in $G$ and $a \in M$, $b \in M$, hence $M$ does not fulfil (3).
4. Intersections and sums of two $p$-subgroups. Another problem formulated in [2] is whether the intersection of two $p$-subgroups of a $p$-group $G$ must be a $p$-subgroup of $G$; there is remarked in [2] that this conjecture seems rather dubious. The answer to this problem is negative.

EXAMPLE 3. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 > \delta_2, \delta_2 > \delta_3$ and $\delta_1, \delta_2$ are incomparable. Let $H_i = E(i = 1, 2, 3), G = V(\Delta, H_i)$. We write $v(\delta_i)$ instead of $v_{\delta_i}$. Let $c_i \neq 0 (i = 1, 2)$ be positive integers, $c_1 \neq c_2$. Denote

$$A_i = \{v \in G \mid v(\delta_i) = c_i[v(\delta_1) + v(\delta_2)]\}$$

$i = 1, 2$. Let $i \in \{1, 2\}$ be fixed. For proving that $A_i$ is a $p$-subgroup of $G$ we have to verify that to each $v \in A_i$ we can choose $a, b \in A_i, a \geq 0, b \geq 0$ such that (*) holds and $v = a - b$. It is easy to verify that it suffices to consider the case when $0$ and $v$ are incomparable, hence we may assume $v(\delta_i) > 0, v(\delta_2) < 0$ (the case $v(\delta_i) < 0, v(\delta_2) > 0$ being analogous). Let $a, b \in G,$

$$a(\delta_1) = v(\delta_1), a(\delta_2) = 0, a(\delta_3) = c_i a(\delta_1),$$

$$b(\delta_1) = 0, b(\delta_2) = -v(\delta_2), b(\delta_3) = -c_i v(\delta_1).$$

Then $a$ and $b$ have the desired properties, hence $A_i$ is a $p$-subgroup of $G$. Denote $C = A_1 \cap A_2$. If $v \in C$, we have

$$c_i[v(\delta_1) + v(\delta_2)] = v(\delta_3) = c_i[v(\delta_1) + v(\delta_2)]$$

and thus (since $c_i \neq c_2) v(\delta_3) = 0, v(\delta_2) = -v(\delta_1).$. Therefore any element $v \in C, v \neq 0$ is incomparable with $0$ and $C$ is not a $p$-subgroup of $G$.

The method used in this example can be employed for proving the following theorem:

THEOREM 2. Let $\Delta$ be a partially ordered set and for each $\delta \in \Delta$ let $H_\delta \neq \{0\}$ be a linearly ordered group, $V = V(\Delta, H_\delta)$. If $V$ is not lattice ordered, then $V$ contains infinitely many pairs of $p$-subgroups $A_1, A_2$ such that $A_1 \cap A_2$ is not a $p$-subgroup of $V$.

Proof. Assume that $V$ is not lattice ordered. Then $\Delta$ is no root system, hence there exist elements $\delta_1, \delta_2, \delta_3$, such that $\delta_1 > \delta_2$, $\delta_2 > \delta_3$ and $\delta_1, \delta_2$ are incomparable. Choose $e_i \in H_{\delta_i}, e_i > 0$ and let $c_1, c_2$ be positive integers, $c_1 \neq c_2$. Let $V_1 = \{v \in V \mid v_\delta = 0$ for each $\delta \in \{\delta_1, \delta_2, \delta_3\}\}$,

$$A_i = \{v \in V_1 \mid v(\delta_1) = n_i e_i, v(\delta_2) = n_i e_i e_2, v(\delta_3) = c_i (n_i + n_2) e_3\}$$

where $n_i$ and $n_2$ run over the set of all integers ($i = 1, 2$). Analo-
gously as in Example 3 we can verify that $A_1$ and $A_2$ are $p$-subgroups of $V$. Let $v \in C = A_1 \cap A_2$. Then $c_i(n_i + n_2) = c_i(n_i + n_2)$, thus $n_2 = -n_1$ and $v(\delta_2) = 0$. Therefore no element of $C$ is strictly positive and $C$ is no $p$-subgroup of $G$. Since the positive integers $c_1 \neq c_2$ are arbitrary there exist infinitely many such pairs $A_1, A_2$.

As a corollary, we obtain:

**Proposition 1.** Let $V = V(\Delta, H_\delta)$, where each $H_\delta$ is linearly ordered. Then the following conditions are equivalent: (i) $V$ is lattice ordered; (ii) if $A$ and $B$ are $p$-subgroups of $V$, then $A \cap B$ is a $p$-subgroup of $V$ as well.

**Proof.** By Theorem 2 (ii) implies (i). Let $V$ be lattice ordered. Then a subgroup $A$ of $V$ is a $p$-subgroup of $V$ if and only if it is an 1-subgroup of $V$; since the intersection of two 1-subgroups is an 1-subgroup, (ii) is valid.

**Proposition 2.** Let $\Delta$ be a partially ordered set and for any $\delta \in \Delta$ let $H_\delta \neq \{0\}$ be a linearly ordered group. Assume that there exist $\delta_1, \delta_2, \delta_3 \in \Delta$ such that $\delta_1 < \delta_2, \delta_2 < \delta_3$ and $\delta_1, \delta_2$ are incomparable, $V = V(\Delta, H_\delta)$. Then there are infinitely many $p$-subgroups $A, B$ of $V$ such that $A + B$ is not a $p$-subgroup of $V$.

**Proof.** Denote $V_i = \{v \in V \mid v(\delta) = 0 \text{ for each } \delta \in \{\delta_1, \delta_2, \delta_3\}\}$ and let $c$ be a fixed positive integer, $e_1 \in H_{\delta_1}, e_i > 0$ ($i = 1, 2, 3$). Put

$$A = \{v \in V_i \mid v(\delta_1) = ne_1, v(\delta_2) = -cne_2, v(\delta_3) = ne_3\},$$

$$B = \{v \in V_i \mid v(\delta_1) = v(\delta_2) = 0, v(\delta_3) = ne_3\}$$

where $n$ runs over the set of all integers. $A$ and $B$ are linearly ordered subgroups of $V$, hence they are $p$-subgroups of $V$. The set $C = A + B$ is the system of all elements $v \in V_i$ such that $v(\delta) = n_ie_i$, $v(\delta_2) = -cne_2$, $v(\delta_3) = ne_3$ where $n_1, n_2$ are arbitrary integers. Hence there is $g \in C$ satisfying

$$g(\delta_1) = e_1, \quad g(\delta_2) = -ce_2, \quad g(\delta_3) = 0.$$ 

If $g = a - b$, $a \in C$, $b \in C$, $a \geq 0$, $b \geq 0$, then $a \neq 0 \neq b$ (since $g \geq 0$, $g < 0$), thus $a(\delta_3) = b(\delta_3) \geq e_3$. There exists $v \in V_i$ such that $v(\delta_i) = a(\delta_i)$, $v(\delta_2) < a(\delta_3)$ and $b(\delta_2)$, $v(\delta_3) < a(\delta_3)$ and $b(\delta_3)$. Thus $v < a$, $v < b$, but $2v < a$, $2v < b$. Therefore $a$ and $b$ are not $p$-disjoint in $G$ and $C$ is no $p$-subgroup of $G$.

One of the problems raised in [2] is affirmatively solved by
THEOREM 3. Let $A$ be an o-ideal of $G$ and let $B$ be a p-subgroup of $G$. Then $A + B$ is a p-subgroup of $G$.

Proof. Let us denote $G/A = \bar{G}$ and for any $t \in G$ write $t + A = \bar{t}$. Let $A + B = X$, $x \in X$. There are elements $a \in A$, $b \in B$ such that $x = a + b$ and since $B$ is a p-subgroup there exist $b_1, b_2 \in B$ such that $b = b_1 - b_2$ and $b_1, b_2$ are p-disjoint in $G$. Further $x = u - v$, $u, v \in G$, where $u$ and $v$ are p-disjoint in $G$. According to [2] $\bar{G}$ is a p-group and by [2], Proposition 2.2, $\bar{b}_1$ and $\bar{b}_2$ ($\bar{u}$ and $\bar{v}$) are p-disjoint in $G$. Further we have

$$\bar{x} = \bar{b}_1 - \bar{b}_2 = \bar{u} - \bar{v},$$

hence if we apply (B) (§ 3) to the p-group $\bar{G}$ it follows that there exists $\bar{m} \in H(\bar{u}, \bar{v})$ fulfilling

$$\bar{b}_1 = \bar{u} + \bar{m}, \quad \bar{b}_2 = \bar{v} + \bar{m}.$$ 

Again, by Proposition 2.2 of [2], there is $m_1 \in \bar{m}$ such that $m_1 \in H(u, v)$. Thus according to (B) the elements $u_i = u + m_1$ and $v_i = v + m_1$ are p-disjoint in $G$ and $x = u_i - v_i$. Since

$$u_i \in \bar{u}_i = \bar{u} + \bar{m}_1 = \bar{u} + \bar{m} = \bar{b}_1 = b_1 + A \subset A + B = X$$

and analogously $v_i \in X$, the set $X$ is a p-subgroup of $G$.

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Received July 1, 1969.

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