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# ON SUBGROUPS OF A PSEUDO LATTICE ORDERED GROUP

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**The purpose of this note is to investigate some problems raised in a recent paper of Conrad and Teller concerning $o$-ideals and $p$-subgroups in an abelian pseudo lattice ordered group.**

The concept of a pseudo lattice ordered group ("$p$-group") has been introduced by Conrad [1]. In recent papers by Teller [5] and Conrad and Teller [2] there is developped a systematic theory of $p$-groups. Let $G$ be an abelian $p$-group. In § 3 it is proved that if $M$ is a subgroup of $G$ such that $\{a, b\} \cap M \neq \emptyset$ for any pair of $p$-disjoint elements $a, b \in G$, then $M$ contains a prime $o$-ideal; this generalizes a result from [2]. In § 4 we prove that the intersection of two $p$-subgroups of a $p$-group $G$ need not be a $p$-subgroup of $G$. Moreover, if $\Delta$ is a partially ordered set and for each $\delta \in \Delta$ $H_\delta \neq \{0\}$ is a linearly ordered group, then for the mixed product $G = V(\Delta, H_\delta)$ the following conditions are equivalent: (i) for any two $p$-subgroups $A, B$ of $G$ their intersection $A \cap B$ is a $p$-subgroup of $G$ as well; (ii) $G$ is an $l$-group. If $A$ is an $o$-ideal of a $p$-group $G$ and $B$ is a $p$-subgroup of $G$, then $A + B$ is a $p$-subgroup of $G$.

## 2. Preliminaries.

Let $G$ be a partially ordered group. $G$ is a Riesz group (cf. Fuchs [3], [4]) if it is directed and if from $a_i$, $b_j \in G$, $a_i \leqq b_j$ $(i, j = 1, 2)$ it follows that there exists $c \in G$ satisfying $a_i \leqq c \leqq b_j$ $(i, j = 1, 2)$. $G$ is a $p$-group (cf. [1] and [5]) if it is Riesz and if each $g \in G$ has a representation $g = a - b$ such that $a, b \in G$, $a \geqq 0$, $b \geqq 0$ and

$$(*) \qquad x \in G,\ x \leqq a,\ x \leqq b \Longrightarrow nx \leqq a,\ nx \leqq b$$

for any positive integer $n$.

Throughout the paper $G$ denotes an abelian $p$-group. Elements $a, b \in G$, $a \geqq 0$, $b \geqq 0$ satisfying $(*)$ are called $p$-disjoint. A subgroup $M$ of $G$ is a $p$-subgroup, if for each $m \in M$ there are elements $a, b \in M$ such that $a, b$ are $p$-disjoint in $G$ and $m = a - b$. A subgroup $C$ of $G$ is an $o$-ideal, if it is directed and if $0 \leqq g \leqq c \in C$, $g \in G$ implies $g \in C$. Let $O(G)$ be the system of all $o$-ideals of $G$ (partially ordered by the set inclusion). An $o$-ideal $C$ of $G$ is called prime, if $G/C$ is a linearly ordered group. For any pair $a, b$ of $p$-disjoint elements $H(a, b)$ denotes the subgroup of $G$ generated by the set

$$\{0 \leqq m \in G \mid m \leqq a,\ m \leqq b\} .$$

Then $H(a, b) \in O(G)$ (cf. [2]).

Let $\Delta$ be a partially ordered set and let $H_\delta \neq \{0\}$ be a linearly ordered group for each $\delta \in \Delta$. Let $V = V(\Delta, H_\delta)$ be the set of all $\Delta$-vectors $v = (\cdots, v_\delta, \cdots)$ where $v_\delta \in H_\delta$, for which the support $S(v) = \{\delta \in \Delta \mid v_\delta \neq 0\}$ contains no infinite ascending chain. An element $v \in V$, $v \neq 0$ is defined to be positive if $v_\delta > 0$ for each maximal element $\delta \in S(v)$. Then ([2], Th. 5.1) $V$ is a $p$-group; $V$ is an 1-group if and only if $\Delta$ is a root system (i.e., $\{\delta \in \Delta \mid \delta \geqq \gamma\}$ is a chain for each $\gamma \in \Delta$).

**3. Subgroups containing a prime *o*-ideal.** The following assertion has been proved in [2] (Proposition 4.3):

(A) For $M \in O(G)$, the following are equivalent: (1) $M$ is prime; (2) the $o$-ideals of $G$ that contain $M$ form a chain; (3) if $a$ and $b$ are $p$-disjoint in $G$, then $a \in M$ or $b \in M$.

Further it is remarked in [2] that each subgroup $M$ of $G$ fulfilling (3) is a $p$-subgroup and any subgroup containing a prime $o$-ideal satisfies (3); then it is asked whether a subgroup $M$ of a $p$-group $G$ satisfies (3) if and only if it contains a prime $o$-ideal (a similar assertion is known to be valid for lattice ordered groups). We shall prove that the answer is positive.

We need the following propositions (cf. [2] and [5]):

(B) Let $g = a - b \in G$ where $a$ and $b$ be $p$-disjoint. Then $g = x - y$, where $x$ and $y$ are $p$-disjoint, if and only if $x = a + m$ and $y = b + m$ for some $m \in H(a, b)$.

(C) If $a$ and $b$ are $p$-disjoint, then $na$ and $nb$ are $p$-disjoint for any positive integer $n$ and $H(a, b) = H(na, nb)$ ([2], Proposition 3.1).

LEMMA 1. *Let $M$ be a subgroup of $G$ fulfilling* (3) *and let $a$, $b$ be $p$-disjoint elements in $G$. Then $H(a, b) \subset M$.*

*Proof.* Let $h \in H(a, b)$. According to (3) we may assume without loss of generality that $a \in M$. Suppose (by way of contradiction) that $h \notin M$. Then $a + h \notin M$, hence by (B) $b + h \in M$, and analogously $b - h \in M$, thus $2b \in M$. Further $2a + h \notin M$ and therefore according to (C) and (B) $2b + h \in M$, which implies $h \in M$.

LEMMA 2. *Let $M$ be a subgroup of $G$ satisfying* (3) *and let $X = \{X_i\}$ be the system of all o-ideals of $G$ such that $X_i \subset M$. Then the system $X$ has a largest element.*

*Proof.* Let $Y$ be the subgroup of $G$ generated by the set $\bigcup X_i$.

Then $Y \subset M$ and $Y$ is the supremum of the system $\{X_i\}$ in the lattice $\mathscr{G}$ of all subgroups of $G$. Since $O(G)$ is a complete sublattice of $\mathscr{G}$ ([2], Th. 2.1), $Y \in O(G)$ and thus $Y \in X$.

Let $H$ be the subgroup of $G$ generated by the set $\bigcup H(a, b)$ where $a$, $b$ runs over the system of all $p$-disjoint pairs of elements in $G$. Since each set $H(a, b)$ is an $o$-ideal ([2]), $H = \bigvee H(a, b)$ ($a$ and $b$ $p$-disjoint in $G$) where $\bigvee$ denotes the supremum in the lattice $O(G)$. According to Lemma 1 $H \subset M$ whenever the subgroup $M$ of $G$ satisfies (3).

For any $u, v \in G$, $u \leqq v$, the interval $[u, v]$ is the set

$$\{x \in G \ u \leqq x \leqq v\} .$$

LEMMA 3. *Let $M$ be a subgroup of $G$ satisfying* (3) *and let $N$ be the largest o-ideal of $G$ that is contained in $M$. Let $g \in G$, $g > 0$. Then*

$$[0, g] \subset M \Longrightarrow g \in N .$$

*Proof.* According to Lemma 2 the largest $o$-ideal $N$ in $M$ exists. Assume that $g \in G$, $g > 0$, $[0, g] \subset M$. The set

$$Z = \bigcup_{n=1}^{\infty} [-ng, ng]$$

is clearly an $o$-ideal in $G$. Let $z \in Z$, hence $z \in [-ng, ng]$ for a positive integer $n$. This implies $0 \leqq y \leqq 2ng$ where $y = z + ng$. Since $G$ is a Riesz group, according to [3, p. 158, Th. 27] there are elements $g_1, \cdots, g_{2n} \in G$, $0 \leqq g_i \leqq g$ such that $y = g_1 + \cdots + g_{2n}$. Thus $g_i \in M$, therefore $y \in M$ and $Z \subset M$. Now we have $Z \subset N$ and so $g \in N$.

LEMMA 4. *Let $M$ be a subgroup of $G$ fulfilling* (3) *and let $N$ be the largest o-ideal of $G$ contained in $M$. Then $G/N$ is a linearly ordered group.*

*Proof.* Assume (by way of contradiction) than $G/N$ is not linearly ordered. According to Lemma 1 $H \subset N$, hence by [2], Theorem 4.1 $G/N$ is a lattice ordered group. Thus there exist elements $X, Y \in G/N$ such that $X \wedge Y = \bar{0}$, $X > \bar{0}$, $Y > \bar{0}$ ($\bar{0}$ being the neutral element of $G/N$). From [2] (Proposition 2.2, (ii)) it follows that there are elements $x \in X$, $y \in Y$ such that $x$ and $y$ are $p$-disjoint in $G$ and hence $x \in M$ or $y \in M$. Clearly $x \notin N$, $y \notin N$ and thus according to Lemma 3 there exist elements $x_1, y_1 \in G$ such that

$$0 < x_1 \leqq x,\ 0 < y_1 \leqq y,\ x_1 \notin M,\ y_1 \notin M .$$

Then in $G/N$ we have $\overline{0} < x_1 + N \leqq x + N = X$, $\overline{0} < y_1 + N \leqq y + N = Y$, whence

$$(x_1 + N) \wedge (y_1 + N) = \overline{0} .$$

Thus by using repeateadly [2], Proposition 2.2, we can choose elements $x_2 \in x_1 + N$, $y_2 \in y_1 + N$ such that $x_2$ and $y_2$ are $p$-disjoint in $G$. Therefore (without loss of generality) we may assume $x_2 \in M$ and this implies $x_1 \in x_1 + N = x_2 + N \subset M$, a contradiction. The proof is complete.

THEOREM 1. *Let $M$ be a subgroup of a $p$-group $G$. Then* $(3) \Rightarrow (2)$ *and the condition* (3) *is equivalent to* (1′) *$M$ contains a prime o-ideal.*

*Proof.* According to Lemma 4 $(3) \Rightarrow (1')$. By [2] $(1') \Rightarrow (3)$. Assume that $M$ is a subgroup of $G$ fulfilling (3). Let $K_1$, $K_2$ be $o$-ideals of $G$ such that $M \subset K_1 \cap K_2$. Let $N$ have the same meaning as in Lemma 4. Since $N \subset M$,

$$K_1 \subset K_2 \Longleftrightarrow K_1/N \subset K_2/N .$$

$K_1/N$ and $K_2/N$ are $o$-ideals of $G/N$ and $G/N$ is linearly ordered, hence $K_1/N \subset K_2/N$ or $K_2/N \subset K_1/N$; therefore (2) holds.

If $M$ is an $o$-ideal of $G$ satisfying (3), then by Theorem 1 $M$ contains a prime $o$-ideal $N$; according to [2] (Corollary 1 to the Induced Homomorphism Theorem) $G/M$ is isomorphic to $(G/N)/(M/N)$ and hence ($G/N$ being linearly ordered) $G/M$ is a linearly ordered group and $M$ is prime. Thus it follows from Theorem 1 that $(3) \Rightarrow (1)$ for $M \in O(G)$ (cf. ($A$)).

Let us remark that if $M$ is a subgroup of $G$ fulfilling (3) then $M$ need not contain any nonzero $o$-ideal that is a lattice; further (3) is not implied by (2).

EXAMPLE 1. Let $B$ be an infinite Boolean algebra that has no atoms and put $\Delta = \{b \in B \mid b \neq 0\}$. For each $\delta \in \Delta$ let $H_\delta = E$ where $E$ is the additive group of all integers with the natural order, $G = V(\Delta, H_\delta)$. Let $M = \{v \in G \mid v_1 = 0\}$ (by 1 we denote the greatest element of $B$). Then $M$ is a prime $o$-ideal of $G$, hence $M$ satisfies (3) and $M$ contains no lattice ordered $o$-ideal different from $\{0\}$.

EXAMPLE 2. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 < \delta_3$, $\delta_2 < \delta_3$ and $\delta_1$, $\delta_2$ are incomparable. Put $H_{\delta_i} = E (i = 1, 2, 3)$, $G = V(\Delta, H_\delta)$, $M = (v \in G \mid v_{\delta_1} = v_{\delta_2} = 0\}$. Then the only $o$-ideal that contains $M$ is $G$, thus (2) holds. Let $a, b \in G$ such that $a_{\delta_1} = 1$, $a_{\delta_2} = a_{\delta_3} = 0$, $b_{\delta_2} = 1$, $b_{\delta_1} = b_{\delta_3} = 0$. The elements $a$ and $b$ are $p$-disjoint in $G$ and $a \notin M$, $b \notin M$, hence $M$ does not fulfil (3).

**4. Intersections and sums of two $p$-subgroups.** Another problem formulated in [2] is whether the intersection of two $p$-subgroups of a $p$-group $G$ must be a $p$-subgroup of $G$; there is remarked in [2] that this conjecture seems rather dubious. The answer to this problem is negative.

EXAMPLE 3. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 > \delta_3$, $\delta_2 > \delta_3$ and $\delta_1$, $\delta_2$ are incomparable. Let $H_{\delta_i} = E (i = 1, 2, 3)$, $G = V(\Delta, H_\delta)$. We write $v(\delta_i)$ instead of $v_{\delta_i}$. Let $c_i \neq 0$ $(i = 1, 2)$ be positive integers, $c_1 \neq c_2$. Denote

$$A_i = \{v \in G \mid v(\delta_3) = c_i[v(\delta_1) + v(\delta_2)]\}$$

$(i = 1, 2)$. Let $i \in \{1, 2\}$ be fixed. For proving that $A_i$ is a $p$-subgroup of $G$ we have to verify that to each $v \in A_i$ we can choose $a, b \in A_i$, $a \geqq 0$, $b \geqq 0$ such that $(*)$ holds and $v = a - b$. It is easy to verify that it suffices to consider the case when 0 and $v$ are uncomparable, hence we may assume $v(\delta_1) > 0$, $v(\delta_2) < 0$ (the case $v(\delta_1) < 0$, $v(\delta_2) > 0$ being analogous). Let $a, b \in G$,

$$a(\delta_1) = v(\delta_1),\ a(\delta_2) = 0,\ a(\delta_3) = c_i a(\delta_1)\ ,$$
$$b(\delta_1) = 0,\ b(\delta_2) = -v(\delta_2),\ b(\delta_3) = -c_i v(\delta_2)\ .$$

Then $a$ and $b$ have the desired properties, hence $A_i$ is a $p$-subgroup of $G$. Denote $C = A_1 \cap A_2$. If $v \in C$, we have

$$c_1[v(\delta_1) + v(\delta_2)] = v(\delta_3) = c_2[v(\delta_1) + v(\delta_2)]$$

and thus (since $c_1 \neq c_2$)$v(\delta_3) = 0$, $v(\delta_2) = -v(\delta_1)$. Therefore any element $v \in C$, $v \neq 0$ is incomparable with 0 and $C$ is not a $p$-subgroup of $G$.

The method used in this example can be employed for proving the following theorem:

THEOREM 2. *Let $\Delta$ be a partially ordered set and for each $\delta \in \Delta$ let $H_\delta \neq \{0\}$ be a linearly ordered group, $V = V(\Delta, H_\delta)$. If $V$ is not lattice ordered, then $V$ contains infinitely many pairs of $p$-subgroups $A_1$, $A_2$ such that $A_1 \cap A_2$ is not a $p$-subgroup of $V$.*

*Proof.* Assume that $V$ is not lattice ordered. Then $\Delta$ is no root system, hence there exist elements $\delta_1, \delta_2, \delta_3$ such that $\delta_1 > \delta_3$, $\delta_2 > \delta_3$ and $\delta_1$, $\delta_2$ are incomparable. Choose $e_i \in H_{\delta_i}$, $e_i > 0$ and let $c_1$, $c_2$ be positive integers, $c_1 \neq c_2$. Let $V_1 = \{v \in V \mid v_\delta = 0$ for each $\delta \notin \{\delta_1, \delta_2, \delta_3\}\}$,

$$A_i = \{v \in V_1 \mid v(\delta_1) = n_1 e_1,\ v(\delta_2) = n_2 e_2,\ v(\delta_3) = c_i(n_1 + n_2)e_3\}$$

where $n_1$ and $n_2$ run over the set of all integers $(i = 1, 2)$. Analogously as in Example 3 we can verify that $A_1$ and $A_2$ are $p$-subgroups of $V$. Let $v \in C = A_1 \cap A_2$. Then $c_1(n_1 + n_2) = c_2(n_1 + n_2)$, thus $n_2 = -n_1$ and $v(\delta_3) = 0$. Therefore no element of $C$ is strictly positive and $C$ is no $p$-subgroup of $G$. Since the positive integers $c_1 \neq c_2$ are arbitrary there exist enfinitely many such pairs $A_1$, $A_2$.

As a corollary, we obtain:

PROPOSITION 1. *Let* $V = V(\Delta, H_\delta)$, *where each* $H_\delta$ *is linearly ordered. Then the following conditions are equivalent*: (i) $V$ *is lattice ordered*; (ii) *if* $A$ *and* $B$ *are p-subgroups of* $V$, *then* $A \cap B$ *is a p-subgroup of* $V$ *as well.*

*Proof.* By Theorem 2 (ii) implies (i). Let $V$ be lattice ordered. Then a subgroup $A$ of $V$ is a $p$-subgroup of $V$ if and only if it is an 1-subgroup of $V$; since the intersection of two 1-subgroups is an 1-subgroup, (ii) is valid.

PROPOSITION 2. *Let* $\Delta$ *be a partially ordered set and for any* $\delta \in \Delta$ *let* $H_\delta \neq \{0\}$ *be a linearly ordered group. Assume that there exist* $\delta_1, \delta_2, \delta_3 \in \Delta$ *such that* $\delta_1 < \delta_3$, $\delta_2 < \delta_3$ *and* $\delta_1, \delta_2$ *are incomparable,* $V = V(\Delta, H_\delta)$. *Then there are infinitely many p-subgroups* $A$, $B$ *of* $V$ *such that* $A + B$ *is not a p-subgroup of* $V$.

*Proof.* Denote $V_1 = \{v \in V \mid v(\delta) = 0$ for each $\delta \notin \{\delta_1, \delta_2, \delta_3\}\}$ and let $c$ be a fixed positive integer, $e_i \in H_{\delta_i}$, $e_i > 0$ $(i = 1, 2, 3)$. Put

$$A = \{v \in V_1 \mid v(\delta_1) = ne_1,\ v(\delta_2) = -cne_2,\ v(\delta_3) = ne_3\}\ ,$$
$$B = \{v \in V_1 \mid v(\delta_1) = v(\delta_2) = 0,\ v(\delta_3) = ne_3\}$$

where $n$ runs over the set of all integers. $A$ and $B$ are linearly ordered subgroups of $V$, hence they are $p$-subgroups of $V$. The set $C = A + B$ is the system of all elements $v \in V_1$ such that

$$v(\delta_1) = n_1 e_1\ ,\quad v(\delta_2) = -cn_1 e_2\ ,\quad v(\delta_3) = n_2 e_3$$

where $n_1$, $n_2$ are arbitrary integers. Hence there is $g \in C$ satisfying

$$g(\delta_1) = e_1\ ,\quad g(\delta_2) = -ce_2\ ,\quad g(\delta_3) = 0\ .$$

If $g = a - b$, $a \in C$, $b \in C$, $a \geqq 0$, $b \geqq 0$, then $a \neq 0 \neq b$ (since $g \not> 0$, $g \not< 0$), thus $a(\delta_3) = b(\delta_3) \geqq e_3$. There exists $v \in V_1$ such that $v(\delta_3) = a(\delta_3)$, $v(\delta_1) < a(\delta_1)$ and $b(\delta_1)$, $v(\delta_2) < a(\delta_2)$ and $b(\delta_2)$. Thus $v < a$, $v < b$, but $2v \not< a$, $2v \not< b$. Therefore $a$ and $b$ are not $p$-disjoint in $G$ and $C$ is no $p$-subgroup of $G$.

One of the problems raised in [2] is affirmatively solved by

THEOREM 3. *Let $A$ be an o-ideal of $G$ and let $B$ be a p-subgroup of $G$. Then $A + B$ is a p-subgroup of $G$.*

*Proof.* Let us denote $G/A = \bar{G}$ and for any $t \in G$ write $t + A = \bar{t}$. Let $A + B = X$, $x \in X$. There are elements $a \in A$, $b \in B$ such that $x = a + b$ and since $B$ is a $p$-subgroup there exist $b_1, b_2 \in B$ such that $b = b_1 - b_2$ and $b_1$, $b_2$ are $p$-disjoint in $G$. Further $x = u - v$, $u$, $v \in G$, where $u$ and $v$ are $p$-disjoint in $G$. According to [2] $\bar{G}$ is a $p$-group and by [2], Proposition 2.2, $\bar{b}_1$ and $\bar{b}_2$ ($\bar{u}$ and $\bar{v}$) are $p$-disjoint in $\bar{G}$. Further we have

$$\bar{x} = \bar{b}_1 - \bar{b}_2 = \bar{u} - \bar{v} ,$$

hence if we apply (B) (§ 3) to the $p$-group $\bar{G}$ it follows that there exists $\bar{m} \in H(\bar{u}, \bar{v})$ fulfilling

$$\bar{b}_1 = \bar{u} + \bar{m} , \quad \bar{b}_2 = \bar{v} + \bar{m} .$$

Again, by Proposition 2.2 of [2], there is $m_1 \in \bar{m}$ such that $m_1 \in H(u, v)$. Thus according to (B) the elements $u_1 = u + m_1$ and $v_1 = v + m_1$ are $p$-disjoint in $G$ and $x = u_1 - v_1$. Since

$$u_1 \in \bar{u}_1 = \bar{u} + \bar{m}_1 = \bar{u} + \bar{m} = \bar{b}_1 = b_1 + A \subset A + B = X$$

and analogously $v_1 \in X$, the set $X$ is a $p$-subgroup of $G$.

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