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ON SUBGROUPS OF A PSEUDO LATTICE ORDERED GROUP

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The purpose of this note is to investigate some problems raised in a recent paper of Conrad and Teller concerning o -ideals and p -subgroups in an abelian pseudo lattice ordered group.

The concept of a pseudo lattice ordered group ("p-group") has been introduced by Conrad [1]. In recent papers by Teller [5] and Conrad and Teller [2] there is developed a systematic theory of p -groups. Let G be an abelian p -group. In § 3 it is proved that if M is a subgroup of G such that $\{a, b\} \cap M \neq \emptyset$ for any pair of p -disjoint elements $a, b \in G$, then M contains a prime o -ideal; this generalizes a result from [2]. In § 4 we prove that the intersection of two p -subgroups of a p -group G need not be a p -subgroup of G . Moreover, if Δ is a partially ordered set and for each $\delta \in \Delta$ $H_\delta \neq \{0\}$ is a linearly ordered group, then for the mixed product $G = V(\Delta, H_\delta)$ the following conditions are equivalent: (i) for any two p -subgroups A, B of G their intersection $A \cap B$ is a p -subgroup of G as well; (ii) G is an l -group. If A is an o -ideal of a p -group G and B is a p -subgroup of G , then $A + B$ is a p -subgroup of G .

2. Preliminaries. Let G be a partially ordered group. G is a Riesz group (cf. Fuchs [3], [4]) if it is directed and if from $a_i, b_j \in G$, $a_i \leq b_j$ ($i, j = 1, 2$) it follows that there exists $c \in G$ satisfying $a_i \leq c \leq b_j$ ($i, j = 1, 2$). G is a p -group (cf. [1] and [5]) if it is Riesz and if each $g \in G$ has a representation $g = a - b$ such that $a, b \in G$, $a \geq 0$, $b \geq 0$ and

$$(*) \quad x \in G, x \leq a, x \leq b \implies nx \leq a, nx \leq b$$

for any positive integer n .

Throughout the paper G denotes an abelian p -group. Elements $a, b \in G$, $a \geq 0$, $b \geq 0$ satisfying (*) are called p -disjoint. A subgroup M of G is a p -subgroup, if for each $m \in M$ there are elements $a, b \in M$ such that a, b are p -disjoint in G and $m = a - b$. A subgroup C of G is an o -ideal, if it is directed and if $0 \leq g \leq c \in C$, $g \in G$ implies $g \in C$. Let $O(G)$ be the system of all o -ideals of G (partially ordered by the set inclusion). An o -ideal C of G is called prime, if G/C is a linearly ordered group. For any pair a, b of p -disjoint elements $H(a, b)$ denotes the subgroup of G generated by the set

$$\{0 \leq m \in G \mid m \leq a, m \leq b\}.$$

Then $H(a, b) \in O(G)$ (cf. [2]).

Let Δ be a partially ordered set and let $H_\delta \neq \{0\}$ be a linearly ordered group for each $\delta \in \Delta$. Let $V = V(\Delta, H_\delta)$ be the set of all Δ -vectors $v = (\dots, v_\delta, \dots)$ where $v_\delta \in H_\delta$, for which the support $S(v) = \{\delta \in \Delta \mid v_\delta \neq 0\}$ contains no infinite ascending chain. An element $v \in V$, $v \neq 0$ is defined to be positive if $v_\delta > 0$ for each maximal element $\delta \in S(v)$. Then ([2], Th. 5.1) V is a p -group; V is an 1-group if and only if Δ is a root system (i.e., $\{\delta \in \Delta \mid \delta \geq \gamma\}$ is a chain for each $\gamma \in \Delta$).

3. Subgroups containing a prime o -ideal. The following assertion has been proved in [2] (Proposition 4.3):

(A) For $M \in O(G)$, the following are equivalent: (1) M is prime; (2) the o -ideals of G that contain M form a chain; (3) if a and b are p -disjoint in G , then $a \in M$ or $b \in M$.

Further it is remarked in [2] that each subgroup M of G fulfilling (3) is a p -subgroup and any subgroup containing a prime o -ideal satisfies (3); then it is asked whether a subgroup M of a p -group G satisfies (3) if and only if it contains a prime o -ideal (a similar assertion is known to be valid for lattice ordered groups). We shall prove that the answer is positive.

We need the following propositions (cf. [2] and [5]):

(B) Let $g = a - b \in G$ where a and b be p -disjoint. Then $g = x - y$, where x and y are p -disjoint, if and only if $x = a + m$ and $y = b + m$ for some $m \in H(a, b)$.

(C) If a and b are p -disjoint, then na and nb are p -disjoint for any positive integer n and $H(a, b) = H(na, nb)$ ([2], Proposition 3.1).

LEMMA 1. *Let M be a subgroup of G fulfilling (3) and let a, b be p -disjoint elements in G . Then $H(a, b) \subset M$.*

Proof. Let $h \in H(a, b)$. According to (3) we may assume without loss of generality that $a \in M$. Suppose (by way of contradiction) that $h \notin M$. Then $a + h \notin M$, hence by (B) $b + h \in M$, and analogously $b - h \in M$, thus $2b \in M$. Further $2a + h \notin M$ and therefore according to (C) and (B) $2b + h \in M$, which implies $h \in M$.

LEMMA 2. *Let M be a subgroup of G satisfying (3) and let $X = \{X_i\}$ be the system of all o -ideals of G such that $X_i \subset M$. Then the system X has a largest element.*

Proof. Let Y be the subgroup of G generated by the set $\bigcup X_i$.

Then $Y \subset M$ and Y is the supremum of the system $\{X_i\}$ in the lattice \mathcal{S} of all subgroups of G . Since $O(G)$ is a complete sublattice of \mathcal{S} ([2], Th. 2.1), $Y \in O(G)$ and thus $Y \in X$.

Let H be the subgroup of G generated by the set $\bigcup H(a, b)$ where a, b runs over the system of all p -disjoint pairs of elements in G . Since each set $H(a, b)$ is an o -ideal ([2]), $H = \bigvee H(a, b)$ (a and b p -disjoint in G) where \bigvee denotes the supremum in the lattice $O(G)$. According to Lemma 1 $H \subset M$ whenever the subgroup M of G satisfies (3).

For any $u, v \in G$, $u \leq v$, the interval $[u, v]$ is the set

$$\{x \in G \mid u \leq x \leq v\}.$$

LEMMA 3. *Let M be a subgroup of G satisfying (3) and let N be the largest o -ideal of G that is contained in M . Let $g \in G$, $g > 0$. Then*

$$[0, g] \subset M \implies g \in N.$$

Proof. According to Lemma 2 the largest o -ideal N in M exists. Assume that $g \in G$, $g > 0$, $[0, g] \subset M$. The set

$$Z = \bigcup_{n=1}^{\infty} [-ng, ng]$$

is clearly an o -ideal in G . Let $z \in Z$, hence $z \in [-ng, ng]$ for a positive integer n . This implies $0 \leq y \leq 2ng$ where $y = z + ng$. Since G is a Riesz group, according to [3, p. 158, Th. 27] there are elements $g_1, \dots, g_{2n} \in G$, $0 \leq g_i \leq g$ such that $y = g_1 + \dots + g_{2n}$. Thus $g_i \in M$, therefore $y \in M$ and $Z \subset M$. Now we have $Z \subset N$ and so $g \in N$.

LEMMA 4. *Let M be a subgroup of G fulfilling (3) and let N be the largest o -ideal of G contained in M . Then G/N is a linearly ordered group.*

Proof. Assume (by way of contradiction) that G/N is not linearly ordered. According to Lemma 1 $H \subset N$, hence by [2], Theorem 4.1 G/N is a lattice ordered group. Thus there exist elements $X, Y \in G/N$ such that $X \wedge Y = \bar{0}$, $X > \bar{0}$, $Y > \bar{0}$ ($\bar{0}$ being the neutral element of G/N). From [2] (Proposition 2.2, (ii)) it follows that there are elements $x \in X$, $y \in Y$ such that x and y are p -disjoint in G and hence $x \in M$ or $y \in M$. Clearly $x \notin N$, $y \notin N$ and thus according to Lemma 3 there exist elements $x_1, y_1 \in G$ such that

$$0 < x_1 \leq x, \quad 0 < y_1 \leq y, \quad x_1 \notin M, \quad y_1 \notin M.$$

Then in G/N we have $\bar{0} < x_1 + N \leq x + N = X$, $\bar{0} < y_1 + N \leq y + N = Y$, whence

$$(x_1 + N) \wedge (y_1 + N) = \bar{0}.$$

Thus by using repeatedly [2], Proposition 2.2, we can choose elements $x_2 \in x_1 + N$, $y_2 \in y_1 + N$ such that x_2 and y_2 are p -disjoint in G . Therefore (without loss of generality) we may assume $x_2 \in M$ and this implies $x_1 \in x_1 + N = x_2 + N \subset M$, a contradiction. The proof is complete.

THEOREM 1. *Let M be a subgroup of a p -group G . Then (3) \Rightarrow (2) and the condition (3) is equivalent to (1') M contains a prime o -ideal.*

Proof. According to Lemma 4 (3) \Rightarrow (1'). By [2] (1') \Rightarrow (3). Assume that M is a subgroup of G fulfilling (3). Let K_1, K_2 be o -ideals of G such that $M \subset K_1 \cap K_2$. Let N have the same meaning as in Lemma 4. Since $N \subset M$,

$$K_1 \subset K_2 \iff K_1/N \subset K_2/N.$$

K_1/N and K_2/N are o -ideals of G/N and G/N is linearly ordered, hence $K_1/N \subset K_2/N$ or $K_2/N \subset K_1/N$; therefore (2) holds.

If M is an o -ideal of G satisfying (3), then by Theorem 1 M contains a prime o -ideal N ; according to [2] (Corollary 1 to the Induced Homomorphism Theorem) G/M is isomorphic to $(G/N)/(M/N)$ and hence (G/N being linearly ordered) G/M is a linearly ordered group and M is prime. Thus it follows from Theorem 1 that (3) \Rightarrow (1) for $M \in O(G)$ (cf. (A)).

Let us remark that if M is a subgroup of G fulfilling (3) then M need not contain any nonzero o -ideal that is a lattice; further (3) is not implied by (2).

EXAMPLE 1. Let B be an infinite Boolean algebra that has no atoms and put $\mathcal{A} = \{b \in B \mid b \neq 0\}$. For each $\delta \in \mathcal{A}$ let $H_\delta = E$ where E is the additive group of all integers with the natural order, $G = V(\mathcal{A}, H_\delta)$. Let $M = \{v \in G \mid v_1 = 0\}$ (by 1 we denote the greatest element of B). Then M is a prime o -ideal of G , hence M satisfies (3) and M contains no lattice ordered o -ideal different from $\{0\}$.

EXAMPLE 2. Let $\mathcal{A} = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 < \delta_3$, $\delta_2 < \delta_3$ and δ_1, δ_2 are incomparable. Put $H_{\delta_i} = E$ ($i = 1, 2, 3$), $G = V(\mathcal{A}, H_\delta)$, $M = \{v \in G \mid v_{\delta_1} = v_{\delta_2} = 0\}$. Then the only o -ideal that contains M is G , thus (2) holds. Let $a, b \in G$ such that $a_{\delta_1} = 1$, $a_{\delta_2} = a_{\delta_3} = 0$, $b_{\delta_2} = 1$, $b_{\delta_1} = b_{\delta_3} = 0$. The elements a and b are p -disjoint in G and $a \notin M$, $b \notin M$, hence M does not fulfil (3).

4. **Intersections and sums of two p -subgroups.** Another problem formulated in [2] is whether the intersection of two p -subgroups of a p -group G must be a p -subgroup of G ; there is remarked in [2] that this conjecture seems rather dubious. The answer to this problem is negative.

EXAMPLE 3. Let $\Delta = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 > \delta_3$, $\delta_2 > \delta_3$ and δ_1, δ_2 are incomparable. Let $H_{\delta_i} = E(i = 1, 2, 3)$, $G = V(\Delta, H_\delta)$. We write $v(\delta_i)$ instead of v_{δ_i} . Let $c_i \neq 0$ ($i = 1, 2$) be positive integers, $c_1 \neq c_2$. Denote

$$A_i = \{v \in G \mid v(\delta_3) = c_i[v(\delta_1) + v(\delta_2)]\}$$

($i = 1, 2$). Let $i \in \{1, 2\}$ be fixed. For proving that A_i is a p -subgroup of G we have to verify that to each $v \in A_i$ we can choose $a, b \in A_i$, $a \geq 0$, $b \geq 0$ such that (*) holds and $v = a - b$. It is easy to verify that it suffices to consider the case when 0 and v are incomparable, hence we may assume $v(\delta_1) > 0$, $v(\delta_2) < 0$ (the case $v(\delta_1) < 0$, $v(\delta_2) > 0$ being analogous). Let $a, b \in G$,

$$\begin{aligned} a(\delta_1) &= v(\delta_1), \quad a(\delta_2) = 0, \quad a(\delta_3) = c_1 a(\delta_1), \\ b(\delta_1) &= 0, \quad b(\delta_2) = -v(\delta_2), \quad b(\delta_3) = -c_2 v(\delta_2). \end{aligned}$$

Then a and b have the desired properties, hence A_i is a p -subgroup of G . Denote $C = A_1 \cap A_2$. If $v \in C$, we have

$$c_1[v(\delta_1) + v(\delta_2)] = v(\delta_3) = c_2[v(\delta_1) + v(\delta_2)]$$

and thus (since $c_1 \neq c_2$) $v(\delta_3) = 0$, $v(\delta_2) = -v(\delta_1)$. Therefore any element $v \in C$, $v \neq 0$ is incomparable with 0 and C is not a p -subgroup of G .

The method used in this example can be employed for proving the following theorem:

THEOREM 2. *Let Δ be a partially ordered set and for each $\delta \in \Delta$ let $H_\delta \neq \{0\}$ be a linearly ordered group, $V = V(\Delta, H_\delta)$. If V is not lattice ordered, then V contains infinitely many pairs of p -subgroups A_1, A_2 such that $A_1 \cap A_2$ is not a p -subgroup of V .*

Proof. Assume that V is not lattice ordered. Then Δ is no root system, hence there exist elements $\delta_1, \delta_2, \delta_3$ such that $\delta_1 > \delta_3$, $\delta_2 > \delta_3$ and δ_1, δ_2 are incomparable. Choose $e_i \in H_{\delta_i}$, $e_i > 0$ and let c_1, c_2 be positive integers, $c_1 \neq c_2$. Let $V_1 = \{v \in V \mid v_\delta = 0 \text{ for each } \delta \in \{\delta_1, \delta_2, \delta_3\}\}$,

$$A_i = \{v \in V_1 \mid v(\delta_1) = n_1 e_1, \quad v(\delta_2) = n_2 e_2, \quad v(\delta_3) = c_i(n_1 + n_2) e_3\}$$

where n_1 and n_2 run over the set of all integers ($i = 1, 2$). Analo-

gously as in Example 3 we can verify that A_1 and A_2 are p -subgroups of V . Let $v \in C = A_1 \cap A_2$. Then $c_1(n_1 + n_2) = c_2(n_1 + n_2)$, thus $n_2 = -n_1$ and $v(\delta_3) = 0$. Therefore no element of C is strictly positive and C is no p -subgroup of G . Since the positive integers $c_1 \neq c_2$ are arbitrary there exist enfnitely many such pairs A_1, A_2 .

As a corollary, we obtain:

PROPOSITION 1. *Let $V = V(\Delta, H_\delta)$, where each H_δ is linearly ordered. Then the following conditions are equivalent: (i) V is lattice ordered; (ii) if A and B are p -subgroups of V , then $A \cap B$ is a p -subgroup of V as well.*

Proof. By Theorem 2 (ii) implies (i). Let V be lattice ordered. Then a subgroup A of V is a p -subgroup of V if and only if it is an 1-subgroup of V ; since the intersection of two 1-subgroups is an 1-subgroup, (ii) is valid.

PROPOSITION 2. *Let Δ be a partially ordered set and for any $\delta \in \Delta$ let $H_\delta \neq \{0\}$ be a linearly ordered group. Assume that there exist $\delta_1, \delta_2, \delta_3 \in \Delta$ such that $\delta_1 < \delta_3$, $\delta_2 < \delta_3$ and δ_1, δ_2 are incomparable, $V = V(\Delta, H_\delta)$. Then there are infinitely many p -subgroups A, B of V such that $A + B$ is not a p -subgroup of V .*

Proof. Denote $V_1 = \{v \in V \mid v(\delta) = 0 \text{ for each } \delta \notin \{\delta_1, \delta_2, \delta_3\}\}$ and let c be a fixed positive integer, $e_i \in H_{\delta_i}$, $e_i > 0$ ($i = 1, 2, 3$). Put

$$\begin{aligned} A &= \{v \in V_1 \mid v(\delta_1) = ne_1, v(\delta_2) = -cne_2, v(\delta_3) = ne_3\}, \\ B &= \{v \in V_1 \mid v(\delta_1) = v(\delta_2) = 0, v(\delta_3) = ne_3\} \end{aligned}$$

where n runs over the set of all integers. A and B are linearly ordered subgroups of V , hence they are p -subgroups of V . The set $C = A + B$ is the system of all elements $v \in V_1$ such that

$$v(\delta_1) = n_1e_1, \quad v(\delta_2) = -cn_1e_2, \quad v(\delta_3) = n_2e_3$$

where n_1, n_2 are arbitrary integers. Hence there is $g \in C$ satisfying

$$g(\delta_1) = e_1, \quad g(\delta_2) = -ce_2, \quad g(\delta_3) = 0.$$

If $g = a - b$, $a \in C$, $b \in C$, $a \geq 0$, $b \geq 0$, then $a \neq 0 \neq b$ (since $g \not\geq 0$, $g \not\leq 0$), thus $a(\delta_3) = b(\delta_3) \geq e_3$. There exists $v \in V_1$ such that $v(\delta_3) = a(\delta_3)$, $v(\delta_1) < a(\delta_1)$ and $b(\delta_1)$, $v(\delta_2) < a(\delta_2)$ and $b(\delta_2)$. Thus $v < a$, $v < b$, but $2v \not< a$, $2v \not< b$. Therefore a and b are not p -disjoint in G and C is no p -subgroup of G .

One of the problems raised in [2] is affirmatively solved by

THEOREM 3. *Let A be an o -ideal of G and let B be a p -subgroup of G . Then $A + B$ is a p -subgroup of G .*

Proof. Let us denote $G/A = \bar{G}$ and for any $t \in G$ write $t + A = \bar{t}$. Let $A + B = X$, $x \in X$. There are elements $a \in A$, $b \in B$ such that $x = a + b$ and since B is a p -subgroup there exist $b_1, b_2 \in B$ such that $b = b_1 - b_2$ and b_1, b_2 are p -disjoint in G . Further $x = u - v$, $u, v \in G$, where u and v are p -disjoint in G . According to [2] \bar{G} is a p -group and by [2], Proposition 2.2, \bar{b}_1 and \bar{b}_2 (\bar{u} and \bar{v}) are p -disjoint in \bar{G} . Further we have

$$\bar{x} = \bar{b}_1 - \bar{b}_2 = \bar{u} - \bar{v},$$

hence if we apply (B) (§ 3) to the p -group \bar{G} it follows that there exists $\bar{m} \in H(\bar{u}, \bar{v})$ fulfilling

$$\bar{b}_1 = \bar{u} + \bar{m}, \quad \bar{b}_2 = \bar{v} + \bar{m}.$$

Again, by Proposition 2.2 of [2], there is $m_1 \in \bar{m}$ such that $m_1 \in H(u, v)$. Thus according to (B) the elements $u_1 = u + m_1$ and $v_1 = v + m_1$ are p -disjoint in G and $x = u_1 - v_1$. Since

$$u_1 \in \bar{u}_1 = \bar{u} + \bar{m}_1 = \bar{u} + \bar{m} = \bar{b}_1 = b_1 + A \subset A + B = X$$

and analogously $v_1 \in X$, the set X is a p -subgroup of G .

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Johan Aarnes, Edward George Effros and Ole A. Nielsen, <i>Locally compact spaces and two classes of C^*-algebras</i>	1
Allan C. Cochran, R. Keown and C. R. Williams, <i>On a class of topological algebras</i>	17
John Dauns, <i>Integral domains that are not embeddable in division rings</i>	27
Robert Jay Daverman, <i>On the number of nonpiercing points in certain crumpled cubes</i>	33
Bryce L. Elkins, <i>Characterization of separable ideals</i>	45
Zbigniew Fiedorowicz, <i>A comparison of two naturally arising uniformities on a class of pseudo-PM spaces</i>	51
Henry Charles Finlayson, <i>Approximation of Wiener integrals of functionals continuous in the uniform topology</i>	61
Theodore William Gamelin, <i>Localization of the corona problem</i>	73
Alfred Gray and Paul Stephen Green, <i>Sphere transitive structures and the triality automorphism</i>	83
Charles Lemuel Hagopian, <i>On generalized forms of aposynthesis</i>	97
J. Jakubík, <i>On subgroups of a pseudo lattice ordered group</i>	109
Cornelius W. Onneweer, <i>On uniform convergence for Walsh-Fourier series</i>	117
Stanley Joel Osher, <i>On certain Toeplitz operators in two variables</i>	123
Washek (Vaclav) Frantisek Pfeffer and John Benson Wilbur, <i>On the measurability of Perron integrable functions</i>	131
Frank J. Polansky, <i>On the conformal mapping of variable regions</i>	145
Kouei Sekigawa and Shûkichi Tanno, <i>Sufficient conditions for a Riemannian manifold to be locally symmetric</i>	157
James Wilson Stepp, <i>Locally compact Clifford semigroups</i>	163
Ernest Lester Stitzinger, <i>Frattini subalgebras of a class of solvable Lie algebras</i>	177
George Szeto, <i>The group character and split group algebras</i>	183
Mark Lawrence Teply, <i>Homological dimension and splitting torsion theories</i>	193
David Bertram Wales, <i>Finite linear groups of degree seven. II</i>	207
Robert Breckenridge Warfield, Jr., <i>An isomorphic refinement theorem for Abelian groups</i>	237
James Edward West, <i>The ambient homeomorphy of an incomplete subspace of infinite-dimensional Hilbert spaces</i>	257
Peter Wilker, <i>Adjoint product and hom functors in general topology</i>	269
Daniel Eliot Wulbert, <i>A note on the characterization of conditional expectation operators</i>	285