SUFFICIENT CONDITIONS FOR A RIEMANNIAN MANIFOLD TO BE LOCALLY SYMMETRIC

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In a locally symmetric Riemannian manifold the scalar curvature is constant and each $k$-th covariant derivative of the Riemannian curvature tensor vanishes. In this note, we show that if the covariant derivatives of the Riemannian curvature tensor satisfy some algebraic conditions at each point, then the Riemannian manifold is locally symmetric.

Let $R$ be the Riemannian curvature tensor of a Riemannian manifold $M^m$ with a positive-definite metric tensor $g$. Manifolds and tensors are assumed to be of class $C^\infty$ unless otherwise stated. We denote by $\nabla$ the Riemannian connection defined by $g$. For tangent vectors $X$ and $Y$, we consider $R(X, Y)$ as a derivation of the tensor algebra at each point. A conjecture by K. Nomizu [4] is that $R(X, Y)\cdot R = 0$ on a complete and irreducible manifold $M^m(m \geq 3)$ implies $\nabla R = 0$, that is, $M^m$ is locally symmetric. Here we consider some additional conditions.

For an integer $k$ and tangent vectors $V_k, \ldots, V_i$ at a point $p$ of $M^m$, we adopt a notation:

$$\left(\nabla^k R\right) = (V_k, V_{k-1}, \ldots, V_i; \nabla^k R) = (V_i V_{k-i} \cdots V_i \nabla_i \nabla_s \cdots \nabla_r R^e_{bde}),$$

where $V_i$, etc., are components of $V_k$, etc., and $\nabla_i \nabla_s \cdots \nabla_r R^e_{bde}$ are components of the $k$-th covariant derivative $\nabla^k R$ of $R$ in local coordinates.

**Proposition 1.** Let $M^m(m \geq 3)$ be a real analytic Riemannian manifold. Assume that

(1.0) the restricted holonomy group is irreducible,
(1.1) $R(X, Y)\cdot R = 0$,
(1.2) $R(X, Y)\cdot (\nabla^k R) = 0$ for $k = 1, 2, \ldots$.

Then $M^m$ is locally symmetric.

Here we note that condition (1.0) means that it holds at some, hence every, point and condition (1.1), and (1.2), mean that for any point $p$ and for any tangent vectors $X, Y, V_s, \ldots, V_i$ at $p$, they hold.

**Proposition 2.** Let $M^m(m \geq 3)$ be a Riemannian manifold. Assume (1.1) and (1.2) and that
(1.0)' the infinitesimal holonomy group is irreducible at every point. Then \( M^m \) is locally symmetric.

Propositions 1 and 2 are essentially related to the following results.

**Proposition 3.** Let \( M^m (m \geq 3) \) be a Riemannian manifold. Assume that the restricted holonomy group \( H^\circ \) (the infinitesimal holonomy group \( H' \), resp.) is irreducible, and \( R \) is invariant by \( H^\circ \) (\( H' \), resp.). Then \( M^m \) is locally symmetric.

**Proposition 3'.** (J. Simons [5], p. 233) Let \( M^m (m \geq 3) \) be an irreducible Riemannian manifold. Assume that \( R \) is invariant by the holonomy group \( H \). Then \( M^m \) is locally symmetric.

Proposition 3 is a generalization of a result by A. Lichnerowicz ([2], p. 11), which contains an assumption of compactness. We remark here that condition (1.2) is equivalent to

\[
(1.2)' \quad R(X, Y) \cdot (\nabla_{V_k} \nabla_{V_{k-1}} \cdots \nabla_{V_1} R) = 0 \quad \text{for} \quad k = 1, 2, \cdots ,
\]

where \( X, Y, V_k, \ldots, V_1 \) are vector fields on \( M^m \).

With respect to Nomizu's conjecture and the above propositions we have

**Theorem 4.** Let \( M^m (m \geq 3) \) be a Riemannian manifold. Assume that

(i) the scalar curvature \( S \) is constant,
(ii) \( R(X, Y) \cdot R = 0 \),
(iii) \( R(X, Y) \cdot \nabla_r R = 0 \),
(iv) \( R(X, Y) \cdot (X, V; \nabla^2 R) = 0 \),
(or \( iv' \) \( R(X, Y) \cdot \nabla_X \nabla_R = 0 \) for vector fields).

Then \( M^m \) is locally symmetric.

**Theorem 5.** Let \( M^m (m \geq 3) \) be a Riemannian manifold. Assume that

(i) the Ricci curvature tensor \( R_i \) is parallel; \( \nabla R_i = 0 \),
(ii) \( R(X, Y) \cdot R = 0 \),
(iii) \( R(X, Y) \cdot \nabla_r R = 0 \).

Then \( M^m \) is locally symmetric.

In Theorems 4 and 5, if \( m = 2 \), then \( \nabla R_i = 0 \) implies \( \nabla R = 0 \).
In Theorem 5, if \( M^m \) is compact, (iii) can be dropped (A. Lichnerowicz [2], or K. Yano [6], p. 222).

In §2 we reduce proofs of Propositions 1 and 2 to that of Proposi-
tion 3, and next we reduce proofs of Propositions 3 and 3' to that of Theorem 4. In §3 we prove Theorems 4 and 5.

2. Holonomy algebras. Conditions (1.1) and (1.2) imply that

\[(R(X, Y), (\nabla^a R)(A, B)] = (\nabla^a R)(R(X, Y)A, B) + (\nabla^a R)(A, (\nabla^a R)(X, Y)B)
\]

for \(k = 0, 1, \ldots\), where \(\nabla^a R\) means \(R\), and \([T, T']\) for linear transformations \(T, T'\) means \(TT' - T'T\).

Now we show

**Lemma 2.1.** The condition (2.1) implies

\[(\nabla^a R)(X, Y), (\nabla^a R)(A, B)] = (\nabla^a R)((\nabla^a R)(X, Y)A, B) + (\nabla^a R)(A, (\nabla^a R)(X, Y)B)
\]

for \(j, k = 0, 1, 2, \ldots\). And (2.1) is equivalent to

\]

for \(j = 0, 1, 2, \ldots\).

**Proof.** We prove (2.2) by induction in \(j\) and by tensor calculus in local coordinates. By (2.1), (2.2) holds for \((j, k) = (0, k), k = 0, 1, 2, \ldots\). Assume that (2.2) holds for \((j - 1, k), (j - 2, k), \ldots, (0, k), k = 0, 1, 2, \ldots\). Then, denoting by \(\nabla_1 \nabla_2 \cdots \nabla_j R_{xyz}\) the \(j\)-th covariant derivative of \(R\) and by \(\nabla_f \cdots \nabla_x R_{qab}\) the \(k\)-th covariant derivative of \(R\), we show

\[\nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} = \nabla_1 \nabla_2 \cdots \nabla_f R_{xyz} - \nabla_1 \nabla_2 \cdots \nabla_x R_{qab} \nabla_1 \nabla_2 \cdots \nabla_j R_{qxy} \]

In fact, we have

\[
\begin{align*}
\nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} & = \nabla_1(\nabla_2 \cdots \nabla_j R_{xyz}) \\
& \quad - \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& \quad - \nabla_1(\nabla_2 \cdots \nabla_j R_{xyz}) \\
& \quad + \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& = \nabla_1(\nabla_2 \cdots \nabla_j R_{xyz}) \\
& \quad - \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& \quad - \nabla_1(\nabla_2 \cdots \nabla_j R_{xyz}) \\
& \quad + \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& \quad \text{(by (2.2) for } (j - 1, k) \text{)} \\
& \quad + \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& = \nabla_1(\nabla_2 \cdots \nabla_j R_{xyz}) \\
& \quad - \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& \quad - \nabla_1(\nabla_2 \cdots \nabla_j R_{xyz}) \\
& \quad + \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& \quad \text{(by (2.2) for } (j - 1, k) \text{)} \\
& \quad + \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz} \\
& = \nabla_1 \nabla_2 \cdots \nabla_j R_{xyz}
\end{align*}
\]
The second and third terms vanish by (2.2) for \((j - 1, k + 1)\). Therefore we have (2.4).

Similarly we can show that (2.3) implies (2.2), including (2.1).

By the theory of holonomy groups (cf. A. Nijenhuis [3]), the set of linear transformations
\[
R(X, Y), (\nabla_y R)(X, Y), \ldots
\]
for \(X, Y, W \in M_p\), the tangent space to \(M\) at \(p\) of \(M\), spans a Lie algebra \(h'_p\) called the infinitesimal holonomy algebra at \(p\). \(h'_p\) generates the infinitesimal holonomy group \(H'_p\) which is a subgroup of the local holonomy group \(H^\sharp_p = H^\sharp_p(U)\). Clearly \(H^\sharp_p\) is a subgroup of the restricted holonomy group \(H^\flat_p\). If a Riemannian manifold is real analytic we have \(H^\flat = H^\sharp = H^\flat\).

The condition (2.3) implies that
\[
[T, R(A, B)] = R(TA, B) + R(A, TB)
\]
for any \(T \in h'_p\). This says that \(R\) is invariant by \(T\). Therefore, for any element \(\alpha \in H'_p\) we have
\[
\alpha R(A, B)C = R(\alpha A, \alpha B)\alpha C \quad \text{for} \ A, B, C \in M_p.
\]
Thus, we have reduced proofs of Propositions 1 and 2 to proof of Proposition 3.

Since (2.7) or (2.6) is equivalent to (2.1), condition (2.7) implies conditions (ii), (iii) and (iv) of Theorem 4. Consequently, if we show that, under the conditions in Proposition 3 (3', resp.), the scalar curvature \(S\) is constant, then Proposition 3 (3', resp.) will follow from Theorem 4.

Let \(E_i, 1 \leq i \leq m\), be an orthonormal basis at \(p\). Then the Ricci curvature tensor \(\mathcal{R}\) is given by
\[
\mathcal{R}(X, Y) = \sum_i g(R(X, E_i)Y, E_i).
\]
Since \(R\) is invariant by \(H'_p\) or \(H^\flat\) or \(H\), we have \(\mathcal{R}(X, Y) = R(\alpha X, \alpha Y)\) for any \(\alpha \in H'_p\) or \(H^\flat\) or \(H\). Since \(H'_p\) or \(H^\flat\) or \(H\) is irreducible, we have some real number \(\lambda\) so that \(\mathcal{R} = \lambda g\) at \(p\). Because \(p\) is an arbitrary point of \(M\) and \(m \geq 3\), \(\lambda\) is constant on \(M\), and hence \(S = m\lambda\) is constant.

3. Proofs of Theorems 4 and 5. To prove theorems it suffices to show two propositions below.
PROPOSITION 3.1. On $M^m$ ($m \geq 3$) assume that

(i) the scalar curvature $S$ is constant,
(ii) $(R(X, Y) \cdot R)(X, V) = 0$,
(iii) $(R(X, Y) \cdot \nabla_{R}) (X, Y) V = 0$,
(iv) $(R(X, Y) \cdot \nabla_{R}) (V, X) = 0$,
(v) $(R(X, Y) \cdot (X, V) ; \nabla R)(V, Y) = 0$,
(or (v) $'(R(X, Y) \cdot \nabla_{X} R) (V, Y) = 0$ for vector fields).

Then we have $\nabla R = 0$.

Proof. Let $\{E_i\}$ be an orthonormal basis at $p$ of $M$. Put $X = E_x$, $Y = E_y$, $V = E_v$ in (iii) and take a sum on $x, y, v$. Then we have

$$R^{xyz}_{\sigma \tau} \nabla_{x} R^{\sigma}_{yz} - R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zx} - R^{xyz}_{\sigma \tau} \nabla_{z} R^{\sigma}_{xy} = 0 .$$

The third and fourth terms vanish. We apply the second Bianchi identity to the first two terms;

$$R^{xyz}_{\sigma \tau}(-\nabla_{x} R^{\sigma}_{yz} - \nabla_{y} R^{\sigma}_{zx}) = -2R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zx} ,$$

$$-R^{xyz}_{\sigma \tau}(-\nabla_{v} R^{\sigma}_{zy} - \nabla_{z} R^{\sigma}_{xy}) = R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zy} + R^{xyz}_{\sigma \tau} \nabla_{z} R^{\sigma}_{xy}$$

$$= R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zy} + R^{xyz}_{\sigma \tau} \nabla_{z} R^{\sigma}_{xy} .$$

Therefore, we have

(3.1) $-4R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zx} + R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zy} = 0 .$

Likewise, (iv) implies that

(3.2) $R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zx} + R^{xyz}_{\sigma \tau} \nabla_{z} R^{\sigma}_{xy} = 0 .$

And (v) implies that

(3.3) $R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zy} + R^{xyz}_{\sigma \tau} \nabla_{z} R^{\sigma}_{xy} = 0 .$

For (v)' we assume that $E_i$ are local vector fields such that $(\nabla R)_{p} = 0$ and $\{E_i\}$ forms an orthonormal basis at $p$. Then we have the same (3.3).

Since $\nabla_{y} R^{\sigma}_{xy} = (1/2) \nabla_{y} S = 0$, by (3.1), (3.2) and (3.3), we have

$$R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zx} = 0 ,$$

$$R^{xyz}_{\sigma \tau} \nabla_{y} R^{\sigma}_{zy} = 0 .$$

On the other hand, in a Riemannian manifold generally we have

(3.4) $\nabla^{h} \nabla_{h} (R_{ijkl} R^{ijkl}) = 2(\nabla_{h} R_{ijkl} \nabla^{h} R^{ijkl})$

$$+ 8R^{ijkl}_{\sigma \tau} \nabla_{h} R^{\sigma}_{ij} + 4R^{ijkl}_{\sigma \tau} B^{h}_{ijkl} ,$$

where $B_{ijkl}^{h}$ are components of $R(X, Y) \cdot R$ (A. Lichnerowicz [2], p. 10). Since (ii) is equivalent to $B_{ijkl}^{h} = 0$, we have $\nabla_{h} R_{ijkl} = 0$.

PROPOSITION 3.2. On $M^m$ ($m \geq 3$) assume that
Then we have \( \nabla R = 0 \).

Proof. We have (3.1) by (iii). Then we have
\[
\nabla_b(R_{ijkl} R^{ijkl}) = 0.
\]
Therefore, (ii) and (3.4) show \( \nabla R = 0 \).

REFERENCES


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NIIGATA UNIVERSITY
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