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In a locally symmetric Riemannian manifold the scalar curvature is constant and each k-th covariant derivative of the Riemannian curvature tensor vanishes. In this note, we show that if the covariant derivatives of the Riemannian curvature tensor satisfy some algebraic conditions at each point, then the Riemannian manifold is locally symmetric.

Let R be the Riemannian curvature tensor of a Riemannian manifold M^m with a positive-definite metric tensor g. Manifolds and tensors are assumed to be of class C^∞ unless otherwise stated. We denote by ∇ the Riemannian connection defined by g. For tangent vectors X and Y, we consider $R(X, Y)$ as a derivation of the tensor algebra at each point. A conjecture by K. Nomizu [4] is that $R(X, Y)\cdot R = 0$ on a complete and irreducible manifold $M^m (m \geqq 3)$ implies $\nabla R = 0$, that is, M^m is locally symmetric. Here we consider some additional conditions.

For an integer k and tangent vectors $V_k, \cdots, V_1$ at a point p of M^m, we adopt a notation:

$$\begin{aligned}(\nabla_V^k R) &= (V_k, V_{k-1}, \cdots, V_1; \nabla^k R) \\ &= (V_k^t V_{k-1}^s \cdots V_1^r \nabla_t \nabla_s \cdots \nabla_r R^a_{bcd}) ,\end{aligned}$$

where V_k^t, etc., are components of V_k, etc., and $\nabla_t \nabla_s \cdots \nabla_r R^a_{bcd}$ are components of the k-th covariant derivative $\nabla^k R$ of R in local coordinates.

PROPOSITION 1. *Let $M^m (m \geqq 3)$ be a real analytic Riemannian manifold. Assume that*
(1.0) *the restricted holonomy group is irreducible,*
(1.1) $R(X, Y)\cdot R = 0$,
(1.2) $R(X, Y)\cdot(\nabla_V^k R) = 0$ *for* $k = 1, 2, \cdots$.
Then M^m is locally symmetric.

Here we note that condition (1.0) means that it holds at some, hence every, point and condition (1.1), and (1.2), mean that for any point p and for any tangent vectors $X, Y, V_k, \cdots, V_1$ at p, they hold.

PROPOSITION 2. *Let $M^m (m \geqq 3)$ be a Riemannian manifold. Assume* (1.1) *and* (1.2) *and that*

(1.0)′ *the infinitesimal holonomy group is irreducible at every point. Then* M^m *is locally symmetric.*

Propositions 1 and 2 are essentially related to the following results.

PROPOSITION 3. *Let* $M^m(m \geqq 3)$ *be a Riemannian manifold. Assume that the restricted holonomy group* H^0 *(the infinitesimal holonomy group* H'*, resp.) is irreducible, and* R *is invariant by* H^0 *(*H'*, resp.). Then* M^m *is locally symmetric.*

PROPOSITION 3′. (*J. Simons* [5], p. 233) *Let* M^m $(m \geqq 3)$ *be an irreducible Riemannian manifold. Assume that* R *is invariant by the holonomy group* H*. Then* M^m *is locally symmetric.*

Proposition 3 is a generalization of a result by A. Lichnerowicz ([2], p. 11), which contains an assumption of compactness. We remark here that condition (1.2) is equivalent to

$$(1.2)' \qquad R(X, Y)\cdot(\nabla_{V_k}\nabla_{V_{k-1}} \cdots \nabla_{V_1}R) = 0 \text{ for } k = 1, 2, \cdots ,$$

where X, Y, V_k, $\cdots$, V_1 are vector fields on M^m.

With respect to Nomizu's conjecture and the above propositions we have

THEOREM 4. *Let* M^m $(m \geqq 3)$ *be a Riemannian manifold. Assume that*

(i) *the scalar curvature* S *is constant,*
(ii) $R(X, Y)\cdot R = 0$,
(iii) $R(X, Y)\cdot\nabla_V R = 0$,
(iv) $R(X, Y)\cdot(X, V; \nabla^2 R) = 0$,
(or (iv)′ $R(X, Y)\cdot\nabla_X\nabla_V R = 0$ *for vector fields*).

Then M^m *is locally symmetric.*

THEOREM 5. *Let* M^m $(m \geqq 3)$ *be a Riemannian manifold. Assume that*

(i) *the Ricci curvature tensor* R_1 *is parallel;* $\nabla R_1 = 0$,
(ii) $R(X, Y)\cdot R = 0$,
(iii) $R(X, Y)\cdot\nabla_V R = 0$.

Then M^m *is locally symmetric.*

In Theorems 4 and 5, if $m = 2$, then $\nabla R_1 = 0$ implies $\nabla R = 0$.

In Theorem 5, if M^m is compact, (iii) can be dropped (A. Lichnerowicz [2], or K. Yano [6], p. 222).

In §2 we reduce proofs of Propositions 1 and 2 to that of Proposition 3, and next we reduce proofs of Propositions 3 and 3′ to that of Theorem 4. In §3 we prove Theorems 4 and 5.

2. Holonomy algebras. Conditions (1.1) and (1.2) imply that

$$[R(X, Y), (\nabla_V^k R)(A, B)] = (\nabla_V^k R)(R(X, Y)A, B) + (\nabla_V^k R)(A, R(X, Y)B) \tag{2.1}$$

for $k = 0, 1, \cdots$, where $\nabla^0 R$ means R, and $[T, T']$ for linear transformations T, T' means $TT' - T'T$.

Now we show

LEMMA 2.1. *The condition* (2.1) *implies*

$$[(\nabla_W^j R)(X, Y), (\nabla_V^k R)(A, B)] = (\nabla_V^k R)((\nabla_W^j R)(X, Y)A, B) + (\nabla_V^k R)(A, (\nabla_W^j R)(X, Y)B) \tag{2.2}$$

for $j, k = 0, 1, 2, \cdots$. *And* (2.1) *is equivalent to*

$$[(\nabla_W^j R)(X, Y), R(A, B)] = R((\nabla_W^j R)(X, Y)A, B) + R(A, (\nabla_W^j R)(X, Y)B) \tag{2.3}$$

for $j = 0, 1, 2, \cdots$.

Proof. We prove (2.2) by induction in j and by tensor calculus in local coordinates. By (2.1), (2.2) holds for $(j, k) = (0, k)$, $k = 0, 1, 2, \cdots$. Assume that (2.2) holds for $(j - 1, k)$, $(j - 2, k)$, $\cdots$, $(0, k)$, $k = 0, 1, 2, \cdots$. Then, denoting by $\nabla_t \nabla_s \cdots \nabla_r R^p{}_{uxy}$ the j-th covariant derivative of R and by $\nabla_f \cdots \nabla_e R^u{}_{qab}$ the k-th covariant derivative of R, we show

$$\begin{aligned}&\nabla_t \nabla_s \cdots \nabla_r R^p{}_{uxy} \nabla_f \cdots \nabla_e R^u{}_{qab} - \nabla_f \cdots \nabla_e R^p{}_{uab} \nabla_t \nabla_s \cdots \nabla_r R^u{}_{qxy} \\ &\quad = \nabla_f \cdots \nabla_e R^p{}_{qvb} \nabla_t \nabla_s \cdots \nabla_r R^v{}_{axy} + \nabla_f \cdots \nabla_e R^p{}_{qav} \nabla_t \nabla_s \cdots \nabla_r R^v{}_{bxy} .\end{aligned} \tag{2.4}$$

In fact, we have

$$\begin{aligned}&\nabla_t \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_f \cdots \nabla_e R^u_{qab} - \nabla_f \cdots \nabla_e R^p_{uab} \nabla_t \nabla_s \cdots \nabla_r R^u_{qxy} \\ &\quad = \nabla_t (\nabla_s \cdots \nabla_r R^p_{uxy} \nabla_f \cdots \nabla_e R^u_{qab}) \\ &\qquad - \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_t \nabla_f \cdots \nabla_e R^u_{qab} \\ &\qquad - \nabla_t (\nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy}) \\ &\qquad + \nabla_t \nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy} \\ &\quad = \nabla_t (\nabla_f \cdots \nabla_e R^p_{qvb} \nabla_s \cdots \nabla_r R^v_{axy} + \nabla_f \cdots \nabla_e R^p_{qav} \nabla_s \cdots \nabla_r R^v_{bxy}) \\ &\qquad - \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_t \nabla_f \cdots \nabla_e R^u_{qab} \qquad \text{(by (2.2) for } (j-1, k)) \\ &\qquad + \nabla_t \nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy} \\ &\quad = \nabla_f \cdots \nabla_e R^p_{qvb} \nabla_t \nabla_s \cdots \nabla_r R^v_{axy} + \nabla_f \cdots \nabla_e R^p_{qav} \nabla_t \nabla_s \cdots \nabla_r R^v_{bxy}\end{aligned}$$

$$+ (\nabla_t \nabla_f \cdots \nabla_e R^p_{uab} \nabla_s \cdots \nabla_r R^u_{qxy} - \nabla_s \cdots \nabla_r R^p_{uxy} \nabla_t \nabla_f \cdots \nabla_e R^u_{qab})$$
$$+ (\nabla_t \nabla_f \cdots \nabla_e R^p_{qub} \nabla_s \cdots \nabla_r R^v_{axy} + \nabla_t \nabla_f \cdots \nabla_e R^p_{qav} \nabla_s \cdots \nabla_r R^v_{bxy}) .$$

The second and third terms vanish by (2.2) for $(j - 1, k + 1)$. Therefore we have (2.4).

Similarly we can show that (2.3) implies (2.2), including (2.1).

By the theory of holonomy groups (cf. A. Nijenhuis [3]), the set of linear transformations

$$R(X, Y),\ (\nabla_W R)(X, Y),\ (\nabla^2_W R)(X, Y),\ \cdots \tag{2.5}$$

for $X, Y, W_1, \cdots \in M_p$, the tangent space to M at p of M, spans a Lie algebra h'_p called the infinitesimal holonomy algebra at p. h'_p generates the infinitesimal holonomy group H'_p which is a subgroup of the local holonomy group $H^*_p = H^0_p(U)$. Clearly H^*_p is a subgroup of the restricted holonomy group H^0_p. If a Riemannian manifold is real analytic we have $H' = H^* = H^0$.

The condition (2.3) implies that

$$[T, R(A, B)] = R(TA, B) + R(A, TB) \tag{2.6}$$

for any $T \in h'_p$. This says that R is invariant by T. Therefore, for any element $\alpha \in H'_p$ we have

$$\alpha R(A, B)C = R(\alpha A, \alpha B)\alpha C \qquad \text{for } A, B, C \in M_p . \tag{2.7}$$

Thus, we have reduced proofs of Propositions 1 and 2 to proof of Proposition 3.

Since (2.7) or (2.6) is equivalent to (2.1), condition (2.7) implies conditions (ii), (iii) and (iv) of Theorem 4. Consequently, if we show that, under the conditions in Proposition 3 (3′, resp.), the scalar curvature S is constant, then Proposition 3 (3′, resp.) will follow from Theorem 4.

Let E_i, $1 \leqq i \leqq m$, be an orthonormal basis at p. Then the Ricci curvature tensor R_1 is given by

$$R_1(X, Y) = \sum_i g(R(X, E_i)Y, E_i) .$$

Since R is invariant by H' or H^0 or H, we have $R_1(X, Y) = R_1(\alpha X, \alpha Y)$ for any $\alpha \in H'$, or H^0 or H. Since H' or H^0 or H is irreducible, we have some real number λ so that $R_1 = \lambda g$ at p. Because p is an arbitrary point of M and $m \geqq 3$, λ is constant on M, and hence $S = m\lambda$ is constant.

3. Proofs of Theorems 4 and 5.

To prove theorems it suffices to show two propositions below.

Proposition 3.1. *On* M^m $(m \geqq 3)$ *assume that*

(i) *the scalar curvature* S *is constant,*

(ii) $(R(X, Y)\cdot R)(X, V) = 0$,

(iii) $(R(X, Y)\cdot \nabla_V R)(X, Y)V = 0$,

(iv) $(R(X, Y)\cdot \nabla_V R_1)(V, X) = 0$,

(v) $(R(X, Y)\cdot (X, V; \nabla^2 R_1)(V, Y) = 0$,

(or (v)′ $(R(X, Y)\cdot \nabla_X \nabla_V R_1)(V, Y) = 0$ *for vector fields*).

Then we have $\nabla R = 0$.

Proof. Let $\{E_i\}$ be an orthonormal basis at p of M. Put $X = E_x$, $Y = E_y$, $V = E_v$ in (iii) and take a sum on x, y, v. Then we have

$$R^{irxy}\nabla_v R_r{}^v{}_{xy} - R^{rvxy}\nabla_v R^i{}_{rxy} - R^{rx}{}_{xy}\nabla_v R^{iv}{}_r{}^y - R^{ry}{}_{xy}\nabla_v R^{ivx}{}_r = 0 .$$

The third and fourth terms vanish. We apply the second Bianchi identity to the first two terms;

$$\begin{aligned} R^{irxy}(-\nabla_x R_r{}^v{}_{yv} - \nabla_y R_r{}^v{}_{vx}) &= -2R^{irxy}\nabla_y R_{rx} , \\ -R^{rvxy}(-\nabla_i R_{rvxy} - \nabla_r R_{vixy}) &= R^{rvxy}\nabla_i R_{rvxy} + R^{rvxy}\nabla_r R_{vixy} \\ &= R^{rvxy}\nabla_i R_{rvxy} + R^{rvxy}\nabla_v R_{irxy} . \end{aligned}$$

Therefore, we have

$$-4R^{irxy}\nabla_y R_{rx} + R^{rvxy}\nabla^i R_{rvxy} = 0 . \tag{3.1}$$

Likewise, (iv) implies that

$$R^{rvx}{}_y\nabla_v R_{rx} + R^{rx}{}_{xy}\nabla_v R^v_r = 0 . \tag{3.2}$$

And (v) implies that

$$R^{rvxy}\nabla_x\nabla_v R_{ry} + R^r{}_y{}^{xy}\nabla_x\nabla_v R^v_r = 0 . \tag{3.3}$$

For (v)′ we assume that E_i are local vector fields such that $(\nabla E_i)_p = 0$ and $\{E_i\}$ forms an orthonormal basis at p. Then we have the same (3.3).

Since $\nabla_v R^v_r = (1/2)\nabla_r S = 0$, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} R^{rvxy}\nabla_x\nabla_v R_{ry} &= 0 , \\ R^{rvxy}\nabla_i R_{rvxy} &= 0 . \end{aligned}$$

On the other hand, in a Riemannian manifold generally we have

$$\begin{aligned} \nabla^h\nabla_h(R_{ijkl}R^{ijkl}) &= 2(\nabla_h R_{ijkl}\nabla^h R^{ijkl}) \\ &\quad + 8R^{ijkl}\nabla_i\nabla_k R_{jl} + 4R^{ijkl}B^h_{jkl,hi} , \end{aligned} \tag{3.4}$$

where $B^i_{jkl,ab}X^aY^b$ are components of $R(X, Y)\cdot R$ (A. Lichnerowicz [2], p. 10). Since (ii) is equivalent to $B^h_{jkl,hi} = 0$, we have $\nabla_h R_{ijkl} = 0$.

Proposition 3.2. *On* M^m $(m \geqq 3)$ *assume that*

(i) $\nabla R_1 = 0$,
(ii) $(R(X, Y)\cdot R)(X, V) = 0$,
(iii) $(R(X, Y)\cdot \nabla_V R)(X, Y) = 0$.
Then we have $\nabla R = 0$.

Proof. We have (3.1) by (iii). Then we have $\nabla_h(R_{ijkl}R^{ijkl}) = 0$. Therefore, (ii) and (3.4) show $\nabla R = 0$.

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