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JAMES WILSON STEPP

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Let S be a locally compact Hausdorff semigroup which is a disjoint union of subgroups one of which is dense. If S the disjoint union of exactly two groups one of which is compact, then S has been completely described by K. H. Hofmann, and if S is the disjoint union of two subgroups where the dense subgroup G has the added property that it is abelian and G/G_0 is a union of compact groups, then S has been described in a previous paper of the author.

It is the purpose of this paper to consider S when each subgroup of S is a topological group when given the relative topology and G (the dense subgroup) has the added property that it is abelian and G/G_0 is a union of compact groups. In particular, we show how to reduce such a semigroup to a semigroup which is a union of real vector groups (§3). In §4 we give the structure of S under the added assumption that $E(S)$ is isomorphic to $E[(R^x)^n]$, where $(R^x)^n$ denotes the n-fold product of the nonnegative real numbers under multiplication.

2. Definitions and notations. If G is a topological group, G_0 will denote the identity component. Let $\mathscr{C}$ denote the full subcategory of the category of locally compact abelian groups whose objects G have the property that G/G_0 is a union of compact subgroups. Let $\mathscr{C}_c$ denote the full subcategory of $\mathscr{C}$ whose objects G_c have the property that G_c is a union of compact subgroups. If $G \in \mathscr{C}$, then by the structure theorem for locally compact abelian groups [2, p. 389] there is a real vector subgroup W of G such that $G/W \in \mathscr{C}_c$. If $W \cong R^n$, then $n = \dim G$ will be called the dimension of G. We will use the following properties of $\mathscr{C}$ and $\mathscr{C}_c$: P_1; for each G in $\mathscr{C}$ there is a unique subgroup $G_c \in \mathscr{C}_c$ such that G/G_c is a real vector group. P_2 [7]; if $\alpha: G \to W$ is a morphism in $\mathscr{C}$ with $\alpha(G)$ dense in W and if W is a real vector group, then there is morphism $\beta: W \to G$ in $\mathscr{C}$ such that $\alpha\beta = I_W$ (the identity morphism on W). P_3 [7]; if $\alpha: G \to H$ is a morphism in the category of locally compact abelian groups with $\alpha(G)$ dense in H and $G \in \mathscr{C}$, then $H \in \mathscr{C}$. Also, if G/G_0 is compact, then H/H_0 is compact.

Let $\mathscr{S}$ denote the category whose objects S are locally compact Hausdorff semigroups satisfying (i) S is a disjoint union of subgroups one of which is dense and (ii) each maximal subgroup of S is a member of $\mathscr{C}$, and whose morphisms are the continuous identity preserving homomorphisms. Let $\mathscr{R}$ denote the full subcategory of $\mathscr{S}$ whose objects R have the properties that (i) each maximal subgroup of R

is a real vector group and (ii) the minimial ideal of R exists and is compact (thus a zero for R).

Let $S \in \mathscr{S}$. Then we will use 1 to denote the identity for S. For each x in S let $H(x) = \{y \in S \mid yS = xS\}$. Since S is an abelian Clifford semigroup, each $H(x)$ is a maximal subgroup of S. Let $\delta: S \to E(S)$ be the function defined by $\delta(s)$ is the idempotent of S such that $H(s) = H(\delta(s))$. If $A \subseteqq S$, then $\bar{A}$ will denote the closure of A. Partially order $E(S)$ by $e \leqq f$ if and only if $ef = e$, and for each e and f in $E(S)$ let $(e, f) = \{a \in E(S) \mid e < a < f\}$. Let $Z = \{0, 1\}$ under multiplication, and let Z^n denote the n-fold product of n copies of Z. Finally, for a semigroup T we use $K(T)$ to denote the minimial ideal when it exists.

3. The purpose of this section is two fold. First we prove that each S in $\mathscr{S}$ splits into the direct product of two closed subsemigroups V and $\bar{W}$, where V is a real vector group and where $\bar{W} \in \mathscr{S}$ with the added property that $K(\bar{W}) \in \mathscr{C}_c$ (Proposition 3.5). Second we prove that there is a congruence ρ on S such that S/ρ is a locally compact Clifford semigroup with each H-class a real vector group and with $E(S) \cong E(S/\rho)$ (Theorem 3.11).

Throughout this section S will represent a fixed member of $\mathscr{S}$, and $E(S)^*$ will denote $E(S)\backslash\{1\}$.

LEMMA 3.1. *Let $e \in E(S)^*$. Then $H(e)$ is open in $S\backslash H(1)$ if and only if* $\dim H(e) = \dim H(1) - 1$.

Proof. By [7], if $H(e)$ is open in $S\backslash H(1)$, then $\dim H(e) = \dim H(1) - 1$.

Let $e \in E(S)$ with $\dim H(e) = \dim H(1) - 1$. Again by [7], if $f \in E(S)$ such that $e < f$, then $\dim H(e) < \dim H(f)$. Thus, since $\dim H(f) < \dim H(1)$ for all f in $E(S)^*$ [7], $(e, 1) = \varnothing$. Let $\psi: S \to eS$ be the morphism defined by $\psi(s) = es$. Since $H(e)$ is a topological group, $H(e)$ is open in $\overline{H(e)}$ [8] which is eS. Since ψ is continuous and since $H(e) = (S\backslash H(1)) \cap \psi^{-1}(H(e))$, it follows that $H(e)$ is open in $S\backslash H(1)$.

COROLLARY 3.2. *If $e \in E(S)^*$, then there is an f in $E(S)$ with $e < f$ and* $\dim H(e) = \dim H(f) - 1$.

Proof. Let $f \in E(S)$ with $e < f$ and $(e, f) = \varnothing$. Then $H(e) \subseteqq \overline{H(f)}$. Let $\psi: \overline{H(f)} \to e\overline{H(f)}$ morphism defined by $\psi(s) = es$. Since $(e, f) = \varnothing$, $H(e) = (\overline{H(f)}\backslash H(f)) \cap (\psi^{-1}(H(e)))$, and it follows that $H(e)$ is open in $\overline{H(f)}\backslash H(f)$. Thus, by Lemma 3.1, $\dim H(e) = \dim H(f) - 1$.

LEMMA 3.3. *A subgroup $H \in \mathscr{C}_c$ of S is closed in S.*

Proof. Let $g \in \bar{H}$. Since $H \subseteqq H(e)_c$ for some e in $E(S)$ and $g \in \overline{\delta(g)H}$, it follows that $g \in H(g)_c$. Thus there is a compact subgroup C of $H(g)_c$ with $g \in C$. Since $\{g^n\}_{n=1}^{\infty} \subseteqq C$ and C is compact, $\delta(g) \in \overline{\{g^n\}_{n=1}^{\infty}}$ [4, p. 15] which is a subset of $\bar{H}$; thus $\delta(g) \in \bar{H}$. By [7], there are no maximal subgroups of $\bar{H}$ which are topological other than $\bar{H}$; thus $\delta(g) = e$, and $\bar{H} \subseteqq H(e)$. Thus we need only show that H is a closed subgroup of $H(e)$, but this follows since H is a locally compact subgroup of a locally compact topological group.

PROPOSITION 3.4. *Let $e \in E(S)$, and let ψ be the map from S onto eS defined by $\psi(s) = es$. Then there are closed subgroups V and W of $H(1)$ with the following properties:*

(a) $\bar{W} = \psi^{-1}(H(e)_c)$,

(b) *V is a real vector group, and*

(c) *The morphism $m: V \times \bar{W} \rightarrow \psi^{-1}(H(e))$ defined by $m(v, w) = v \cdot w$ is an isomorphism.*

Proof. Let α be the natural map from $H(e)$ onto $H(e)/H(e)_c$, let Q be the corestriction of $\psi|_{H(1)}$ to $H(e)$, and let $\beta: H(e)/H(e)_c \rightarrow H(1)$ be a morphism in $\mathscr{C}$ such that $(\alpha Q)\beta$ is the identity map on $H(e)/H(e)_c$ [P_2]. Let $V = \beta(H(e)/H(e)_c)$, and let $W = Q^{-1}(H(e)_c)$. Then V and W are the desired closed subgroups of $H(1)$. The inverse of m is given by $s \mapsto ((\beta\alpha\psi)(s), [(\beta\alpha\psi)(s)]^{-1}s)$ which is clearly continuous. The theorem now follows.

PROPOSITION 3.5. *There are closed subgroups V and W of $H(1)$ with the following properties:*

(a) V is a real vector group,

(b) $K(\bar{W}) \in \mathscr{C}_c$, *and*

(c) *The morphism $m: V \times \bar{W} \rightarrow S$ defined by $m(v, w) = v \cdot w$ is an isomorphism.*

Proof. Again by [7], if $e \in E(S)^*$, then $\dim H(e) < \dim H(1)$. Thus there is an f in $E(S)$ with $\dim H(f) \leqq \dim H(e)$ for all e in $E(S)$. Since $\dim H(ef) \leqq \min\{\dim H(e), \dim H(f)\}$ with equality holding only for $e < f$ or $f \leqq e$, f is unique. The proposition now follows from Proposition 3.4 along with the observation that $S = \psi^{-1}(H(f))$ where $\psi: S \rightarrow fS$ is the morphism defined by $\psi(s) = sf$ for all s in S.

PROPOSITION 3.6. *If there is a s_0 in S with $H(s_0)_c$ compact, then $H(s)_c$ is compact for all s in S.*

Proof. From the structure theorem for locally compact abelian groups [2, p. 389] one can get that if $G \in \mathscr{C}_c$, then G_0 is compact. Thus for any s in S we have that $H(s)_c$ is compact if and only if $H(s)_c/(H(s)_c)_0$ is compact. But $H(s)_c/H(s)_c)_0$ is compact if and only if $H(s)/H(s)_0$ is compact. Therefore, by P_3 and since $\overline{H(1)} = S$, the theorem will follow if we can prove that $H(1)/H(1)_0$ is compact.

We do this by contradiction. That is, assume $H(1)/H(1)_0$ is not compact, and let $e \in E(S)$ satisfying the following:

(i) $H(e)/H(e)_0$ is compact,

(ii) $\delta(s_0) \leqq e$, and

(iii) if $f \in E(S)$ with $e < f$, then $H(f)/(f)_0$ is not compact.

By Corollary 3.2 and since $e \neq 1$, there is an f in $E(S)$ with $e < f$ and $\dim H(e) = \dim H(f) - 1$. Let $T = \overline{H(f)}$, and let $\psi\colon T \to eT$ be the morphism defined by $\psi(s) = se$. By Proposition 3.4, there is a real vector subgroup V of $H(f)$, a closed subgroup W of $H(f)$ with $\psi^{-1}(H(e)_c) = \bar{W}$, and a morphism $m\colon V \times \bar{W} \to \psi^{-1}(H(e))$ which is an isomorphism. Since $\bar{W}\backslash W = H(e)_c$ which is compact and by [3], W contains a compact subgroup C such that W/C is a real vector group. Thus $H_W(f)_c$ is compact. Since the corestriction of $m|_{V\times W}\colon V\times W \to H(f)$ is an isomorphism and V is a real vector group, it now follows that $H(f)_c$ is compact. This is the desired contradiction and the proof now follows.

SUBLEMMA. *Let e and f be elements of $E(S)$ with* $\dim H(e) = \dim H(f) + 1$ *and with $f < e$. If H is a subgroup of $H(e)$ with $H \in \mathscr{C}_c$, then fH is a closed subgroup of S.*

Proof. Let $g \in \overline{fH} \cap H(f)$. Since $H \in \mathscr{C}_c$, $fH \subseteqq H(f)_c$, and thus there is a compact subgroup C of $H(f)$ which is open relative to $H(f)_c$ and with $g \in C$. Let $\psi\colon \overline{H(e)} \to \overline{fH(e)}$ be the morphism defined by $\psi(s) = fs$. It follows from Proposition 3.4 and the fact that $H(f)$ is open in $\overline{H(e)}\backslash H(e)$ that $\psi^{-1}(C)$ is a locally compact semigroup which contains a dense group $\psi^{-1}(C) \cap H(e)$ whose complement C is compact. By [3], there is a unique compact subgroup C_1 of $\psi^{-1}(C) \cap H(e)$ and a one-parameter subgroup M of $\psi^{-1}(C) \cap H(e)$ such that $\psi^{-1}(C) = \bar{M}\cdot C_1$. Let $\{g_\alpha\}_{\alpha\in A}$ be a net in fH which converges to g. Since C is open in $H(f)_c$, there is a $\beta \in A$ such that if $\alpha \geqq \beta$, then $g_\alpha \in C$. For each $\alpha \in A$ with $\alpha \geqq \beta$ there is an $h_\alpha \in H$ with $g_\alpha = fh_\alpha$. It follows that each $h_\alpha \in C_1$, and therefore there is an h in $C_1 \cap H$ such that $fh = g$. Thus $\overline{fH} \subseteqq fH \subseteqq \overline{fH}$. We now have fH is a closed subgroup of $H(f)_c$, and therefore $fH \in \mathscr{C}_c$. The sublemma now follows by Lemma 3.3.

LEMMA 3.7. *If H is a subgroup of S with $H \in \mathscr{C}_c$ and if $f \in E(S)$, then fH is closed.*

Proof. Let $h \in H$; then $\delta(h)\cdot f \leqq \delta(h)$. If $\delta(h)f = \delta(h)$, then $fH = f\delta(h)H = \delta(h)H = H$ which is closed by Lemma 3.3. If $\delta(h)\cdot f < \delta(h)$, then there is a chain of idempotents $e_1 \cdots, e_{q+1}$ which is maximal with respect to the properties: (i) $e_1 = \delta(h)f$ and (ii) $e_{q+1} = \delta(h)$. Observe that since $e_1, \cdots, e_{q+1}$ is maximal, $\dim H(e_i) = \dim H(e_{i+1}) - 1$ for $i = 1, 2, \cdots, q$. If fH is not closed, then there is an integer p, $1 \leqq p \leqq q$ such that e_pH is not closed and $e_{p+1}H$ is closed. Since $e_pH = (e_p\cdot e_{p+1})H = e_p(e_{p+1}H)$ and since $e_{p+1}H \in \mathscr{C}_c$, e_pH is closed (sublemma). Thus e_pH is both closed and not closed which is impossible; thus it follows that fH must be closed.

Now that one has Lemma 3.7 it is easy to prove the following corollary.

COROLLARY 3.8. (i) *For each x in S, $xH(1)_c$ is closed.*

(ii) *If U is a nonempty compact subset of S, then $U\cdot H(1)_c$ is closed.*

THEOREM 3.9. *Let $R = \{(x, y) \in S \times S \mid xH(1)_c = yH(1)_c\}$. Then R is a congruence, and S/R is a locally compact semigroup with the following properties*:

(i) *If θ is the natural map from S onto S/R, then θ is an open map and $\theta\ (H(s)) \cong H(s)/(\delta(s)H(1)_c)$ for all s in S.*

(ii) *The corestriction of $\theta|_{E(S)}$ to $E(S/R)$ is an isomorphism.*

Proof. Clearly R is a congruence. Since $H(1)$ acts as a group of homeomorphisms on S and since $\theta^{-1}(\theta(A)) = A\cdot H(1)_c$ for all $A \neq \varnothing$, it follows that θ is an open map. Since θ is an open map, S/R is locally compact and also multiplication is continuous. We now show S/R is Hausdoff. Let $x, y \in S$ with $xH(1)_c \neq yH(1)_c$. Since $yH(1)_c$ is closed (Corollary 3.8) and since S is a locally compact (thus regular) Hausdorff space, there is a compact neighborhood N_x of x with $N_x \cap yH(1)_c = \varnothing$. Thus $y \notin N_x\cdot H(1)_c$ which is closed by Corollary 3.8, and using the fact that S is regular we obtain a compact neighborhood N_y of y with $N_y \cap (N_x\cdot H(1))_c = \varnothing$. It follows that $(N_y\cdot H(1)_c) \cap (N_x\cdot H(1)_c) = \varnothing$, and thus S/R is Hausdorff. This completes the proof.

REMARK. We wish to point out that each maximal subgroup of S/R is connected, and thus $H(\theta(s))_c$ is compact for each s in S.

LEMMA 3.10. *Let $T \in \mathscr{S}$ with $K(T)$ compact. Then for each nonnegative integer n there is a T_n in $\mathscr{S}$ and a surmorphism α_n: $T \rightarrow T_n$ in $\mathscr{S}$ satisfying*:

(a) *The corestriction of $\alpha_n|_{E(S)}$ to $E(T_n)$ is an isomorphism.*

(b) *If $x \in T$ with $\dim H(x) \leqq n$, then $\alpha_n(H(x)) = H(\alpha_n(x)) \cong H(x)/H(x)_c$.*

(c) *If* $x \in T$ *with* $\dim H(x) > n$, *then the corestriction of* $\alpha|_{H(x)}$ *to* $H(\alpha(x))$ *is an isomorphism.*

Proof. The proof is by induction. Let $R_0 = \{(x, y) \mid x = y$ or $x \in K(T)$ and $y \in K(T)\}$. Clearly R_0 is a congruence, and since $K(T)$ is compact, it follows that T/R_0 is a locally compact semigroup. Let α_0 be the natural map from T onto $T/R_0 = T_0$. Then, clearly, α_0 and T_0 satisfy (a)-(c) for $n = 0$.

Let k be a nonnegative integer such that there is a $T_k \in \mathscr{S}$ and a surmorphism $\alpha_k \colon T \to T_k$ satisfying (a)-(c). If $k \geqq \dim H(1)$, then let $T_{k+1} = T_k$ and $\alpha_{k+1} = \alpha_k$. Then T_{k+1} and α_{k+1} satisfy (a)-(c). If $k < \dim H(1)$, let $A = \{e \in E(T_k) \mid \dim H(e) = k + 1\}$, and let $\hat{T}_k = \{x \in T_k \mid x \in \overline{H(e)}$ for some e in $A\}$. For each e in A let $\psi_e \colon S \to eS$ be the morphism defined by $\psi_e(s) = es$. Then $\psi_e^{-1}(H(e)) \cap \hat{T}_k = H(e)$, and thus each $H(e)$ is open relative to $\hat{T}_k$. Let $R_{k+1} = \{(x, y) \in T_k \times T_k \mid x = y$ or $\delta(x) = \delta(y) \in A$ and $x \in yH(\delta(y))_c\}$. It is easy to show that R_{k+1} is a congruence. By Proposition 3.6 and since $K(T_k) = \{0\}$, each $H(e)_c$ is compact. Since each $H(e)_c$ is compact and since each $H(e)$ with $e \in A$ is open in $\hat{T}_{k+1}$, it follows that T_k/R_{k+1} is a locally compact semigroup. Let $T_{k+1} = T_k/R_{k+1}$ and $\alpha_{k+1} = \eta\alpha_k$, where η is the natural map from T_k onto T_k/R_{k+1}. Then $T_{k+1} \in \mathscr{S}$ and $\alpha_{k+1} \colon T \to T_{k+1}$ is a surmorphism satisfying (a)-(c) for $n = k + 1$. The theorem now follows by induction.

THEOREM 3.11. *Let* $S \in \mathscr{S}$. *Then there is a* $T \in \mathscr{S}$ *and a surmorphism* $\alpha \colon S \to T$ *in* $\mathscr{S}$ *satisfying*:

(i) *The corestriction of* $\alpha|_{E(S)}$ *onto* $E(T)$ *is an isomorphism.*

(ii) *Each H-class of* T *is a real vector group.*

Proof. By Proposition 3.5, there is an isomorphism $\beta \colon S \to V \times \bar{T}$ where V is a real vector group and where $\bar{T} \in \mathscr{S}$ with $K(\bar{T}) \in \mathscr{C}_c$. By first applying Theorem 3.9 and then Lemma 3.10 for $n = \dim H(1)$ one can obtain a surjective morphism $\beta_1 \colon \bar{T} \to T_n$ which preserves the structure of $E(\bar{T})$ and where the H-class of T_n are real vector groups. Let $T = V \times T_n$ and $\alpha \colon S \to V \times T_n$ be the map defined by $\alpha(s) = (p_{r_1}(\beta(s)), \beta_1(p_{r_2}(\beta(s)))$. Then clearly T and $\alpha \colon S \to T$ satisfy the conditions of the theorem.

4. Let $\mathscr{S}_1$ denote the full subcategory of $\mathscr{S}$ whose objects S have the property that $E(S) \cong Z^q$ for some nonnegative integer q. In this section we characterize the objects in $\mathscr{S}_1$. The fact that there are objects in $\mathscr{S}$ that are not in $\mathscr{S}_1$ is demonstrated by J. G. Horne, Jr., in [6]. However, if $S \in \mathscr{S}$ with $\dim H(1) \leqq 2$, then it is shown

that $S \in \mathscr{S}_1$.

Let R_+ denote the multiplicative group of positive real numbers, and recally that R^x denotes the multiplicative semigroup of nonnegative real numbers.

LEMMA 4.1. *Let E be a Hausdoff topological space which is the disjoint union of $R_+ \times R^x$ and a singleton set $\{w\}$, where $R_+ \times R^x$ has the product topology. If $\{w\} \cup (R_+ \times \{0\})$ is homeomorphic to R^x with $w \in \overline{(0, 1] \times \{0\}}$, then E is not locally compact at w.*

Proof. We assume E is locally compact at w and show that this assumption leads to the conclusion that R^x is compact. Let U be an open neighborhood of w with $\bar{U}$ compact. Then $\bar{U}\backslash U$ is a compact subset of $R_+ \times R^x$. Since $w \cup (R_+ \times \{0\})$ is homeomorphic to R^x with $\overline{(0, 1] \times \{0\}} = ((0, 1] \times \{0\}) \cup \{w\}$, there is an a in R_+ with $\{(x, 0) \mid 0 < x < a\} \subseteqq U$. For each b in R_+ with $0 < b < a$ either $\{b\} \times R^x \subseteqq U$ or $(\{b\} \times R^x) \cap (\bar{U}\backslash U) \neq \emptyset$. To see this, assume $(\{b\} \times R^x) \cap (\bar{U}\backslash U) = \emptyset$. Then $\{b\} \times R^x$ is the disjoint union of the two relatively open sets $(E/\bar{U}) \cap (\{b\} \times R^x)$ and $U \cap (\{b\} \times R^x)$. Since $\{b\} \times R^x$ is connected and $\{b\} \times R^x \cap U \neq \emptyset$, $(E\backslash\bar{U}) \cap (\{b\} \times R^x) = \emptyset$ and hence $\{b\} \times R^x \subseteqq U$.

We now prove there is a $r_0 < a$ in R_+ satisfying; if $b \in R_+$ and $b \leqq r_0$, then $\{b\} \times R^x \subseteqq U$. If this were not the case, then by the above there would exists a sequence $\{b_n\}_{n=1}^{\infty}$ in R_+ such that $\{b_n, 0)\}_{n=1}^{\infty}$ converges to w, and each $(\{b_n\} \times R^x) \cap (\bar{U}\backslash U) \neq \emptyset$. For each positive integer n let x_n be an element of R^x such that $(b_n, x_n) \in \bar{U}\backslash U$. Since $\bar{U}\backslash U$ is a compact subset of $R_+ \times R^x$, the sequence $\{(b_n, x_n)\}_{n=1}^{\infty}$ has a cluster point (b, x). Thus $\{(b_n, 0)\}_{n=1}^{\infty}$ converges to w and clusters to $(b, 0)$ which is impossible. Thus we now can conclude that there is a r_0 in R_+ such that if $b \in R_+$ with $b \leqq r_0$, then $\{b\} \times R^x \subseteqq U$. We point out at this point that if $b \in R_+$ and $b \leqq r_0$, then $\overline{\{b\} \times R^x} = \{w\} \cup \{b\} \times R^x$.

For each l in R_x, $\{(r, l) \mid r_0 \leqq r\}$ is connected, and $(r_0, l) \in U$. Thus a similar argument to the one above proves there is an l_0 in R^x such that if $l \geqq l_0$ then $\{(r, l) \mid r_0 \leqq r\} \subseteqq U$. Similarly, there is a $t_0 \in R_+$ with $t_0 \geqq r_0$ and such that if $t \in R_+$ with $t \geqq t_0$, then $\{(t, l) \mid 0 \leqq l \leqq l_0\} \subseteqq U$. Let $B = [r_0, t_0] \times [0, l_0]$ which is a compact subset of $R_+ \times R^x$. It is easy to show that $E\backslash B \subseteqq U$ and thus $E = \bar{U} \cup B$ and is compact. In particular, $(R \times \{0\}) \cup \{w\}$ is compact and homeomorphic to R^x. This is the desired contradiction.

THEOREM 4.2. *If S is a member of $\mathscr{R}$ with* $\dim H(1) = 2$, *then* $E(S) \cong Z^2$.

Proof. Since $S \in \mathscr{R}$, S has a zero. By Corollary 3.2, there is an element f in $E(S)$ with $\dim H(f) = 1$.

Case 1. There is only one f in $E(S)$ with $\dim H(f) = 1$. That is, $E(S) = \{0, e, 1\}$. By [7], $S\backslash\{0\} \cong R_+ \times R^x$. By [5] and since $\overline{R_+ \times \{0\}} = (R_+ \times \{0\}) \cup \{0\}$, $\overline{R \times \{0\}}$ is homeomorphic to R^x. By applying Lemma 4.1 we have S is not locally compact at $\{0\}$. Thus Case 1 is impossible.

Case 2. There are exactly two idempotents e_1 and e_2 with $\dim H(e_1) = \dim H(e_2) = 1$. Clearly in this case $E(S) \cong Z^2$.

Case 3. There are at least three idempotents e_1, e_2, e_3 with $\dim H(e_i) = 1$. Let P_1, and P_2 be one-parameter subgroups of $H(1)$ with $\bar{P}_i = P_i \cup \{e_i\}$ (Proposition 3.5). Let $\{s_\alpha\}$ be a net in $H(1)$ which converges to e_3. Since $S\backslash H(1)$ is an ideal, $\{s_\alpha^{-1}\}$ does not have a cluster point. Since $H(1) = P_1 \cdot P_2$, there are nets $\{s_{1\alpha}\} \subseteqq P_1$ and $\{s_{2\alpha}\} \subseteqq P_2$ such that $s_{1\alpha} \cdot s_{2\alpha} = s_\alpha$ for all α. By [3] and since $\{s_\alpha^{-1}\}$ does not have a cluster point, either $\{s_{1\alpha}\}$ clusters to e_1 and $\{s_{2\alpha}^{-1}\}$ clusters to e_2 or $\{s_{1\alpha}^{-1}\}$ clusters to e_1 and $\{s_{2\alpha}\}$ clusters to e_2. But the former implies $e_1 = e_3 \cdot e_2$, and the latter implies $e_2 = e_1 \cdot e_3$. Since $e_3 \cdot e_2 = 0$ and $e_1 \cdot e_3 = 0$, either $e_1 = 0$ or $e_2 = 0$. This is the desired contradiction. Thus Case 3 is impossible.

LEMMA 4.3. *Let S be a member of $\mathscr{R}$ with* $\dim H_S(1) \geqq 2$, *and let $e \in E(S)$ with $H(e) \cong R_+$. Then there is an f in $E(S)$, such that* $\dim H(f) = \dim H_S(1) - 1$ *and that* $ef = 0$.

Proof. By Corollary 3.2 and since $\dim H_S(1) \geqq 2$, there is an idempotent e_1 in S such that $e < e_1$ and that $\dim H(e_1) = \dim H(e) + 1 = 2$. Let $T = \overline{H(e_1)}$, then T is a member of $\mathscr{R}$ and $\dim H_T(1) = 2$. Thus, by applying Theorem 4.2 to T one observes that there is an f in $E(T)^*$ (and thus in $E(S)^*$) such that $f \neq 0$ and that $ef = 0$. Let f be a maximal such idempotent with respect to $f \neq 0$ and $ef = 0$.

Claim. $\dim H(f) = \dim H_S(1) - 1$. If this were not the case, then applying Corollary 3.2 two time we observe there are idempotents f_1 and f_2 such that $f < f_1 < f_2$ and $\dim H(f) = \dim H(f_1) - 1 = \dim H(f_2) - 2$. By applying Proposition 3.4 to $\overline{H(f_2)}$ we observe there is a subsemigroup $R \subseteqq \overline{H(f_2)}$ such that $K(R) = \{f\}$ and $\dim H_R(1) = 2$. By Theorem 4.2, there is an idempotent f_3 in $E(R)^*$ (and thus $E(S)^*$) such that $f_3 \neq f$ and $f_3 \cdot f_1 = f$. But since f_1 and f_3 are elements of $E(S)^*$ which are larger that f, $ef_1 \neq 0$ and $ef_3 \neq 0$. In fact, since $ef_1 \leqq e$ and $ef_3 \leqq e$, $ef_1 = ef_2 = e$. However, $0 = ef = e(f_1 f_3) = ef_3 = e$, and this is the desired contradiction. Therefore, f is maximal in $E(S)^*$. From the proof of Lemma 3.1, we have that f maximal in $E(S)^*$ implies $\dim H(f) = \dim H(1) - 1$. For the remainder of this paper we will use the following notation. If $S \in \mathscr{S}$ and $e \in E(S)$, then $\psi_e \colon S \to eS$ is the morphism defined by $\psi_e(s) = es$ for all s in S.

We omit the proof of the next lemma since the proof is straight forward.

LEMMA 4.4. (i) *If f and e are element of $E((R^x)^n)$ with* $\dim H(f) = 1, \dim H(e) = n - 1$ *and if* $\psi_f^{-1}(f) \cap \psi_e^{-1}(e) = \{1\}$, *then the morphism* $m: \psi_f^{-1}(f) \times \psi_e^{-1}(e) \to (R^x)^n$, *defined by* $m(s, t) = st$, *is an isomorphism.*

(ii) *If* $e \in E(S)$ *with* $\dim H(e) = p$, *then* (a) $\psi_e^{-1}(e) \cong (R^x)^{n-p}$ *and* (b) $\psi_e[(R^x)^n] \cong (R^x)^p$.

LEMMA 4.5. *If* $\alpha: (R^x)^n \to (R^x)^n \in \mathscr{R}$ *is a surmorphism with* $\alpha(E(R^x)^n) = E((R^x)^n)$, *then* α *is an isomorphism.*

Proof. The proof is by induction on $\dim H(1)$. The lemma is trivially true for $n = 0$. If $n = 1$, then $\alpha(R_+)$ is a dense connected subgroup of R^x and thus $\alpha(R_+) = R_+$. By [2, p. 84], $\alpha|_{R_+}: R_+ \to R_+$ is an isomorphism, and thus it follows that α is bijective. We show α is a closed map. Let A be a closed subset of R^x. If $A \subseteq R_+$, then there is an r in R_+ with $[0, r] \cap A = \varnothing$. Thus $\alpha(A)$ is closed in R_+ and $[0, f(r)) \cap \alpha(A) \subseteq [0, f(r)] \cap \alpha(A) = \varnothing$. Since $[0, f(r))$ is open in R^x, $0 \notin \overline{\alpha(A)}$, and thus it follows that $\overline{\alpha(A)} = \alpha(A)$. If $0 \in A$, then either $A = R^x$ or there is an r in R_+ with $r \notin A$. If $A = R^x$, then clearly $\alpha(A)$ is closed. If there is an r in R_+ with $r \notin A$, then $A = ([0, r] \cap A) \cup ([r, \infty) \cap A)$. We now have

$$\begin{aligned}\alpha(A) &= \alpha([0, r] \cap A) \cup ([r, \infty) \cap A)] \\ &= \alpha([0, r] \cap A) \cup \alpha([r, \infty) \cap A)\ .\end{aligned}$$

Since $[0, r] \cap A$ is compact, $\alpha([0, r] \cap A)$ is compact, thus closed, and by the first case $\alpha([r, \infty) \cap A)$ is closed. We now have α is a closed bijection and thus an isomorphism.

Let n be an integer larger than 1 such that the lemma is true for all nonnegative integers less than n. Let S denote $(R^x)^n$, and define $\hat{\alpha}: E(S) \to E(S)$ by $\hat{\alpha}(e) = \alpha(e)$ for all e in $E(S)$. Since $\hat{\alpha}$ is bijective and since $E(S)$ is finite, $\hat{\alpha}$ is an isomorphism. For each e in $E(S)$ define $\psi_e: S \to eS$ by $\psi_e(s) = es$ for all s in S. Let $e_1 = (0, 1, 1, \cdots, 1)$ and $e_2 = (1, 0, 0, \cdots, 0)$, and let $A = \psi_{e_1}^{-1}(e_1)$ and $B = \psi_{e_2}^{-1}(e_2)$. Then $A \cong R^x$, $B \cong (R^x)^{n-1}$ and $e_1 \cdot e_2 = 0$. Define $F: A \times B \to S$ by $F(a, b) = ab$; then, by Lemma 4.4i, F is an isomorphism. Let $f_1 = \alpha(e_1)$ and $f_2 = \alpha(e_2)$. We now show $\alpha(A) = \psi_{e_1}^{-1}(f_1) \cong R^x$, $\alpha(B) = \psi_{f_2}^{-1}(f_2) \cong (R^x)^{n-1}$, and $\alpha(A) \cap \alpha(B) = \{1\}$. From which it will follow by Lemma 4.4i that the morphism $G: \alpha(A) \times \alpha(B) \to S$, defined by $G(a, b) = ab$, is an isomorphism. Let $A_1 = \psi_{f_1}^{-1}(f_1)$ and $A_2 = \psi_{f_2}^{-1}(f_2)$. Since $\hat{\alpha}$ is an isomorphism, $\hat{\alpha}$ preserves the less than order on $E(S)$; thus $\dim H(f_1) = \dim H(e_1) = n - 1$

and $\dim H(f_2) = \dim H(e_2) = 1$. Therefore, $A_1 \cong R^x$ and $A_2 \cong (R^x)^{n-1}$ (Lemma 4.4iia). If $A_1 \cap A_2 \neq \{1\}$, then either $f_1 \in A_1 \cap A_2$ or there is an element $g \in H(1) \cap A_1 \cap A_2$ with $g \neq 1$. Since $f_1 \cdot f_2 = \alpha(e_1) \cdot \alpha(e_2) = \alpha(e_1 e_2) = \alpha(0) = 0$, $f_1 \notin A_2$, and thus there is a $g \in H(1) \cap A_1 \cap A_2$ with $g \neq 1$. Since $A_1 \cong R^x$ either $\{g^n\}_{n=1}^{\infty}$ converges to f_1 or $\{(g^{-1})^n\}_{n=1}^{\infty}$ converges to f_1 [3]. But both imply $f_1 \in A_2$ which is impossible by the above. Thus $A_1 \cap A_2 = \{1\}$. Clearly, $\alpha(A) \subseteqq A_1$. Let $t \in A_1$. Since $\alpha(S) = \alpha(A \cdot B) = \alpha(A) \cdot \alpha(B)$, there is an element $a \in \alpha(A)$ and $b \in \alpha(B)$ such that $t = ab$. It follows that $f_1 = f_1 t = f_1 a \cdot b = f_1 b$ which implies $b \in A_1$. But $\alpha(B) \subseteqq B_1$ and $B_1 \cap A_1 = \{1\}$; thus $b = \{1\}$. The proof that $\alpha(B) = B_1$ is similar and will therefore be omitted. We now have the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S \\ {\scriptstyle F^{-1}}\downarrow & & \uparrow{\scriptstyle G} \\ A \times B & \xrightarrow{\alpha|_A \times \alpha|_B} & \alpha(A) \times \alpha(B) \, . \end{array}$$

By the inductive hypothesis, $\alpha|_A : A \to \alpha(A)$ and $\alpha|_B : B \to \alpha(B)$ are isomorphisms. The lemma now follows.

LEMMA 4.6. *Let X, Y and Z be Hausdorff spaces and assume $F: X \times Y \to Z$ is a continuous surjection. If there are continuous surjections $\alpha: Z \to X$* and *$\beta: Z \to Y$ such that the diagram*

$$\begin{array}{ccccc} & & X \times Y & & \\ & {\scriptstyle P_{r_1}}\swarrow & \downarrow{\scriptstyle F} & \searrow{\scriptstyle P_{r_2}} & \\ X & \xleftarrow{\alpha} & Z & \xrightarrow[\beta]{} & Y \end{array}$$

is commutative, then F is a homeomorphism.

Proof. The inverse of F is given by $z \mapsto (\alpha(z), \beta(z))$ which is clearly continuous.

THEOREM 4.7. *If S is an object in both $\mathscr{R}$ and $\mathscr{S}_1$, then $S \cong (R^x)^n$ where $n = \dim H_S(1)$.*

Proof. The proof is by induction on $\dim H(1)$. The claim for $\dim H(1) = 1$ is proven in [5]. Let n be an integer larger than 1 such that the claim is true for all positive integers less than n. Let e be an idempotent with $e > 0$ and $eS \cong R^x$ (Corollary 3.2). By Lemma 4.3 there is an idempotent f with $f \neq 0$, $\dim H(f) = n - 1$ and $ef = 0$. Let $A = \psi_f^{-1}(f)$ and $B = \psi_e^{-1}(e)$. Then by the inductive hypothesis,

$A \cong R^x$ and $B \cong (R^x)^{n-1}$. Also $\psi_e^{-1}(H(e)) \cong H(e) \times B \cong R_+ \times B$ (Proposition 3.4). Now define a morphism $F: A \times B \to S$ by $F(a, b) = ab$. Observe that $\psi_e(F(a, b)) = eab = ea$ and $\psi_f(F(a, b)) = fb$. We now show $S = A \cdot B$. Since $E(S) \cong Z^n$ it follows that $E(S) = E(A) \cdot E(B)$. Let $s \in S$; then $\delta(s) = e_1 \cdot f_1$ for some $e_1 \in E(A)$ and $f_1 \in E(B)$. Also, $s = \delta(s) \cdot g$ for some $g \in H_S(1)$. Since $H_A(1) \cap H_B(1) = \{1\}$ (see proof that $A_1 \cap B_1 = \{1\}$ in Lemma 4.5), $g = a \cdot b$ for some $a \in H_A(1)$ and $b \in H_B(1)$. Thus $s = \delta(s)g = \delta(s)ab = e_1 f_1 ab = (e_1 a)(f_1 b) \in A \cdot B$. Clearly, $\psi_e(A) = eA \subseteqq eS$. Let $t \in eS$; then $t = ea \cdot b$ for some $a \in A$ and $b \in B$. Thus $t = eab = eb \cdot a = ea$ and hence $eA = eS$. By Lemma 4.5, $\psi_e|_A: A \to eS$ is an isomorphism. Similarly it can be shown that $fB = fS$ and thus, by Lemma 4.5, $\psi_f|_B: B \to fS$ is an isomorphism. We now have the following diagram

$$\begin{array}{ccccc} R^x \cong A & \xleftarrow{P_{r_1}} & A \times B & \xrightarrow{P_{r_2}} & B \cong (R^x)^{n-1} \\ \downarrow{\scriptstyle \psi_e|_A} & & \downarrow & & \downarrow{\scriptstyle \psi_f|_B} \\ eS & \xleftarrow{\psi_e} & S & \xrightarrow{\psi_f} & fS \end{array}$$

which can be reduced to

$$\begin{array}{ccccc} & & A \times B & & \\ & \swarrow{\scriptstyle P_{r_1}} & \downarrow{\scriptstyle F} & \searrow{\scriptstyle P_{r_2}} & \\ A & \xleftarrow[(\psi_e^{-1}|_A)\circ\psi_e]{} & S & \xrightarrow[(\psi_f^{-1}|_B)\circ\psi_f]{} & B\,. \end{array}$$

Thus by Lemma 4.6, F is an isomorphism, and the theorem now follows by induction.

DEFINITION. An object S in $\mathscr{S}$ is an *H-semigroup* if (i) $H_S(1) \cong R_+$ and (ii) $K(S)$ is compact.

LEMMA 4.8. *Let S be a object in $\mathscr{S}_1$ having the added properties that* (i) *$H_S(1)$ is a real vector group of dimension n and* (ii) *$K(S)$ is compact. Then there are subsemigroups $S_1, \cdots, S_n$ of S which are H-semigroups, the morphism $m: \times_{i=1}^n S_i \to S$ defined by $m((s_1, \cdots, s_n)) = s_1 \cdot s_2 \cdot \cdots \cdot s_n$ is a surmorphism which preserves the H-class structure of $\times_{i=1}^n S_i$,* and also *m induces an isomorphism on the groups of units. Further, for each i there is an idempotent e_i with* $\dim H(e_i) = n - 1$ *and $S_i = \psi_{e_i}^{-1}(H(e_i)_c)$.*

Proof. Since $E(S) \cong Z^n$, there are exactly n-idempotents $e_1, \cdots, e_n$ in S with $\dim H(e_i) = n - 1$. By Proposition 3.4 and since $H_S(1)$ is a real vector group, each $\psi_{e_i}^{-1}(H(e_i)_c)$, is an H-semigroup. Let $S_i = \psi_{e_i}^{-1}(H(e_i)_c)$, and let $F: S \to (R^x)^n$ be a surmorphism which preserves the H-class structure of S (Proposition 3.11 then Theorem 4.7). Since F preserves the H-class structure of S, $\dim H(e_i) = \dim H(F(e_i)) = n - 1$ for $i = 1, 2, \cdots, n$ and, also, $F(S_i) = \psi_{e_i^1}^{-1}(H(e_i^1)) \cong R^x$ for $i = 1, 2, \cdots, n$, where $e_i^1 = F(e_i)$. Using the structure of $(R^x)^n$ we know $\psi_{e_i^1}^{-1}(F(e_i^1)) \cong R^x$ if and only if there is an integer $j(i), 1 \leqq j(i) \leqq n$ such that

$$P_{r_{j(i)}} \mid \psi_{e_i^1}^{-1}(F(e_i^1)) \colon \psi_{e_i^1}^{-1}(F(e_i^1)) \twoheadrightarrow R^x$$

is an isomorphism. For each $i, i = 1, 2, \cdots, n$ let $\pi_i\colon S_i \rightarrow S_i/K(S_i)$ be the natural map where $S/K(S_i)$ denotes the Rees quotient semigroup. Since each $K(S_i)$ is compact [3], π_i is a closed map. Thus for each i there is a bijective morphism $\beta_i\colon S_i \twoheadrightarrow R^x$ such that the following diagram commutes

$$\begin{array}{ccc} S_i & \xrightarrow{P_{r_{j(i)}} \circ F|S_i} & R^x \\ {\scriptstyle \pi_i}\downarrow & \nearrow {\scriptstyle \beta_i} & \\ S/K(S_i) & & \end{array} .$$

By Lemma 4.5 each β_i is an isomorphism. Since each $K(S_i)$ is compact, it is easy to show that a net $\{g_\alpha\}_{\alpha \in A} \subseteqq S_i$ has a cluster point if and only if $\{\pi_i(g_\alpha)\}_{\alpha \in A}$ has a cluster point. Thus it follows that $\{g_\alpha\}_{\alpha \in A} \subseteqq S_i$ has a cluster point if and only if $\{P_{r_{j(i)}}(F(g_\alpha))\}_{\alpha \in A}$ has a cluster point.

Let $x \in S$ and let $\{g_\alpha\}_{\alpha \in A}$ be a net in $H_S(1)$ which converges to x. Then for each α there are elements $g_i(\alpha) \in S_i$ $i = 1, 2, \cdots, n$ such that $g_\alpha = g_1(\alpha) \cdot g_2(\alpha) \cdot \cdots \cdot g_n(\alpha)$. Since $P_{r_{j(i)}}F(g_i(\alpha)) = P_{r_{j(i)}}(F(g_\alpha))$ for $i = 1, 2, 3, \cdots, n$ and since $P_{r_{j(i)}}(F(g_\alpha))$ has a cluster point and by the above, each $\{g_i(\alpha)\}_{\alpha \in A}$ has a cluster point. Clearly, we can choose a subnet $\{g_\alpha\}_{\alpha \in B}$ such that each $\{g_i(\alpha)\}_{\alpha \in B}$ converges. It now follows that $x \in m(\times_{i=1}^n S_i)$. Clearly, m induces an isomorphism on the groups of units.

THEOREM 4.9. *Let $S \in \mathscr{S}_1$. Then $S \cong T \times \boldsymbol{R}^n$ for a suitable n and where T is an object in $\mathscr{S}_1$ satisfying the following: There are subsemigroups $S_1, \cdots, S_n$ of T with each S_i an H-semigroup and a surmorphism $m\colon H_T(1)_c \times (\times_{i=1}^n S_i) \twoheadrightarrow T$ which preserves the H-class structure and which induces an isomorphism on the groups of units. Further, there are surmorphisms $G_1\colon S \twoheadrightarrow (R^x)^n$* and *$G_2\colon H_T(1)_c \times (\times_{i=1}^n S_i) \twoheadrightarrow (R^x)^n$ such that the following diagram is commutative*

$$\begin{array}{ccc} H_T(1)_c \times (\times_{i=1}^m S_i) & \xrightarrow{\ m\ } & T \\ & {\scriptstyle G_2}\searrow \quad \swarrow {\scriptstyle G_1} & \\ & (R^x)^n & \end{array}$$

Proof. By Proposition 3.5, $S \cong T \times \boldsymbol{R}^m$ for a suitable choice of m, where $T \in \mathscr{S}$ with $K(T) \in \mathscr{C}_c$. Since $E(S) \cong Z^n$ for some n and since $E(S) \cong E(T)$, $T \in \mathscr{S}_1$. Using Lemma 3.1 and Corollary 3.2, it is easy to see that $\dim H_T(1) = n$. Since $E(S) \cong Z^n$, there are exactly n idempotents $e_1, \cdots, e_n$ such that $\dim H(E_i) = n - 1$. For each e_i let C_i be a compact subgroup of $H(e_i)_c$ which is open relative to $H(e_i)_c$. It follows from Proposition 3.4 and the fact that each $H(e_i)$ is open in $T\backslash H_T(1)$, that each $\psi_{e_i}^{-1}(C_i)$ is a locally compact semigroup which contains a dense group whose complement is compact. Since each $\psi_{e_i}^{-1}(C_i) \in \mathscr{S}$ and by [7], there is a one-parameter subgroup $P_i \subseteqq \psi_{e_i}^{-1}(C_i) \cap H_T(1)$ such that $\bar{P}_i \cap C_i \neq \emptyset$. For each i let $S_i = \bar{P}_i$; then each S_i an H-semigroup. Let $m\colon H_T(1)_c \times (\times_{i=1}^n S_i) \to T$ be a morphism defined by $m(g, s_1, \cdots, s_n) = g \cdot s_1 \cdot s_2 \cdot \cdots \cdot s_n$ and let $m_1\colon \times_{i=1}^n S_i \to T$ be the morphism defined by $m_1(s) = m(1, s)$ for all s in $\times_{i=1}^n S_i$.

Let T/R be the semigroup constructed as in Theorem 3.9 and let $F\colon T \to T/R$ be the natural map. Since F preserves the H-class structure, $\dim H(F(e_i)) = n - 1$ for each i. Since for each i $F(K(S_i))$ is a compact ideal for $F(P_i)$, $\overline{F(P_i)} = F(P_i) \cup F(K/S_i))$ [5]; thus $F(S_i) = \overline{F(P_i)}$. Also, $H(F(e_i))_c$ is a compact ideal for $F(P_i)$; thus $F(S_i) = \overline{F(P_i)} = F(P_i) \cup H(F(e_i))_c$. It now follows from Lemma 4.8 that $F(m_1(\times_{i=1}^n S_i)) = T/R$ and thus $m_i(\times_{i=1}^n S_i) \cdot H_T(1) = T$. Therefore, m is a surmorphism.

Let $T_1 = \overline{m_1(\times_{i=1}^n S_i)}$. Since $E(T) = E(m_i(\times_{i=1}^n S_i)) \cong Z^n, E(T) \cong Z^n$, and thus it follows that $\dim H_{T_1}(1) = n$. Let $F_1\colon T_1 \to T_1/R_1$ be the natural map where T_1/R_1 is the semigroup guaranteed by Theorem 3.9. Let $H_1 = H_{T_1}/R_1(1)$. Then H_1 is an n-dimensional vector group with $\overline{F_1(m_1(\times_{i=1}^n P_i))} = H_1$. Thus by P_2 there is a morphism $\beta\colon H_1 \to \times_{i=1}^n P_i$ such that $F_1 m_1 \beta = I_{T_1/R_1}$. It follows that the inverse of $F_1|_{m_1}(\times_{i=1}^n P_i)$ is the corestriction of $m_1\beta$ to $m_1(\times_{i=1}^n P_i)$. Thus $m_1(\times_{i=1}^n P_i)$ is a locally compact subgroup $H_{T_1}(1)$ and thus closed. Therefore, it follows that the corestriction of $m_1 | \times_{i=1}^n P_i\colon \times_{i=1}^n P_i \to \times_{i=1}^n P_i$ is an isomorphism. Since $H_T(1) = m_i(\times_{i=1}^n P_i) \cap H_T(1)_c$ and $m_1(\times_{i=1}^n P_i) \cap H_T(1)_c = \{1\}$, it now easily follows that m induces an isomorphism on the group of units.

The remainder of the proof follows directly from Theorem 3.11 and Theorem 4.7.

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HOUSTON, TEXAS

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