FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

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In this paper the Lie algebra analogues to groups with property $E$ of Bechtell are investigated. Let $\mathfrak{x}$ be the class of solvable Lie algebras with the following property: if $H$ is a subalgebra of $L$, then $\phi(H) \equiv \phi(L)$ where $\phi(L)$ denotes the Frattini subalgebra of $L$; that is, $\phi(L)$ is the intersection of all maximal subalgebras of $L$. Groups with the analogous property are called $E$-groups by Bechtell. The class $\mathfrak{x}$ is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal $N$ of $L \in \mathfrak{x}$ be the Frattini subalgebra of $L$. Only solvable Lie algebras of finite dimension are considered here.

The following notation will be used. We let $N(L)$ be the nil radical of $L$ and $S(L)$ be the socle of $L$; that is, $S(L)$ is the union of all minimal ideals of $L$. If $A$ and $B$ are subalgebras of $L$, let $Z_B(A)$ be the centralizer of $A$ in $B$. The center of $A$ will be denoted by $Z(A)$. If $[B, A] \subseteq A$, we let $\text{Ad}_A(B) = \{\text{ad } b \text{ restricted to } A; \text{ for all } b \in B\}$. $L'$ will be the derived algebra of $L$ and $L'' = (L')'$.

**Proposition 1.** Let $L$ be a Lie algebra such that $L'$ is nilpotent. Then the following are equivalent:

1. $\phi(L) = 0$.
2. $N(L) = S(L)$ and $N(L)$ is complemented by a subalgebra.
3. $L'$ is abelian, is a semi-simple $L$-module and is complemented by a subalgebra.

Under these conditions, Cartan subalgebras of $L$ are exactly those subalgebras complementary to $L'$.

**Proof.** Assume (1) holds. Nilpotency of $L'$ implies $\phi(L) \equiv L''$, so $L'$ is abelian and may be regarded as an $L/L'$-module. We may assume $L' = \sum \oplus V_\rho$, $V_\rho$ indecomposable $L/L'$-submodules. If $M$ is a maximal subalgebra of $L$ and if $V_\rho \nsubseteq M$, then $M \cap V_\rho$ is an ideal of $L$. If $S$ is an $L/L'$-submodule of $V_\rho$ properly contained between $M \cap V_\rho$ and $V_\rho$, then $M + S$ is a subalgebra of $L$ properly contained between $M$ and $L$, contradicting the maximality of $M$. Therefore $M$ contains all maximal submodules of $V_\rho$ for each $\rho$. Then $\phi(L) = 0$ implies the intersection of all maximal submodules of $V_\rho$ is zero for each $\rho$. If $V_1, \ldots, V_s$ are maximal submodules of $V_\rho$ with $V_1 \cap \cdots \cap V_s = 0$ and are minimal with respect to this property, we have $V = V_1 \cap \cdots \cap V_s \neq 0$.
and \( V \cap V_1 = 0 \) so that \( V \oplus V_1 = V_\rho \), contradicting indecomposability. Therefore each \( V_\rho \) is irreducible and \( L' \) is a completely reducible \( L/L' \)-module and is also a completely reducible \( L \)-module. Since \( L \) is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let \( H \) be a Cartan subalgebra of \( L \) and let \( L_0 \) and \( L_1 \) be the Fitting null and one component of \( L \) with respect to \( H \). Then \( L = L_0 + L_1 = H + L_1 \subseteq H + L' \) shows \( L = H + L' \). We claim that \( H \cap L' = 0 \). If \( H \cap L' \neq 0 \), then, since \( L' \) is abelian, \( H \) is nilpotent and \( L' \) is a completely reducible \( L \)-module, \( L' \) is a sum of irreducible \( H \)-modules, \( U_1, \ldots, U_\sigma \), such that for each \( U_i \), \( [U_i, H] = 0 \) for some \( \kappa \), hence \( [U_i, H] = 0 \).

Thus \( [H, L' \cap H] = 0 \). One sees that each \( U_i \) is a central minimal ideal of \( L \), and since \( \phi(L) = 0 \), \( U_i \) is complemented by a maximal subalgebra \( M \). Therefore \( U_i \) is a one-dimensional direct summand of \( L \), contradicting \( U_i \subseteq L' \). Hence \( L' \cap H = 0 \) and \( H \) is a complement to \( L' \) in \( L \). Since \( [H, H] \subseteq H \cap L' = 0 \), \( H \) is abelian. Any minimal ideal not in \( L' \) satisfies \( [L, A] \subseteq A \cap L' = 0 \), so is central. Therefore \( S(L) = L' + Z(L) \) and, since \( H \) is a Cartan subalgebra, \( Z(L) \subseteq H \). Let \( H_i \) be a complementary subspace to \( Z(L) \) in \( H \). One sees that \( N(L) = L' + Z(L) + (N(L) \cap H_0) = S(L) + (N(L) \cap H_0) \). If \( h \) is a nonzero element in \( N(L) \cap H_0 \), \( ad h \) is nilpotent but not zero which implies \( [V_\rho, h] = V_\rho \) for some \( V_\rho \subseteq L' \) and \( \cdots [V_\rho, h] \cdots h] = 0 \) for some \( \kappa \), a contradiction. Thus \( S(L) = N(L) \) and \( H_0 \) is a complement. Consequently (1) implies (2).

Assume (2) holds and proceed by induction on the dimension of \( L \). Since \( L' \subseteq N(L) = S(L) \) and minimal ideals are abelian, \( L' \) is abelian. If every minimal ideal of \( L \) is contained in \( L' \), then \( S(L) = L' \) and (3) follows. Therefore let \( A \) be a minimal ideal of \( L \) such that \( A \not\subseteq L' \). Hence \( A \not\subseteq \phi(L) \) and there exists a maximal subalgebra \( M \) of \( L \) such that \( L = M + A \). Since \( [L, A] \subseteq A \cap L' = 0 \), \( A \) is central, hence one-dimensional. It follows that \( L \) is the Lie algebra direct sum of \( M \) and \( A \). Since \( M \) inherits the condition (2), \( M \) satisfies (3) by induction. It now follows that \( L \) also satisfies (3).

Assume (3) holds. Then \( L' \) is a sum of minimal ideals of \( L \), which we denote by \( A_1, \ldots, A_\kappa \), and \( L = L' + H \), \( H \) a subalgebra of \( L \). Since \( H' \subseteq H \cap L' = 0 \), \( H \) is abelian. One sees that \( L' = [L', H] \) and, consequently, \( A_i = [A_i, H] \) for all \( i \). Since \( Z_{A_i}(H) \) is central in \( L \), \( Z_{A_i}(H) \) is an ideal in \( L \) contained in \( A_i \). Since \( Z_{A_i}(H) \neq A_i \), \( Z_{A_i}(H) = 0 \). It follows that \( H \) is its own normalizer, hence is a Cartan subalgebra of \( L \). Now \( H + A_1 + \cdots + A_\kappa \) is a maximal subalgebra of \( L \) since any containing algebra has a nonzero projection on \( A_i \) which is \( ad H \) stable, hence equal to \( A_i \). Therefore \( \phi(L) \subseteq H \) and \( \phi(L) \subseteq H \cap L' = 0 \). Hence (1) holds.
That complements to $L'$ are Cartan subalgebras is shown in (3) implies (1). That Cartan subalgebras are complements to $L'$ is shown in (1) implies (2). This completes the proof of Proposition 1.

**Theorem 1.** Let $L$ be a Lie algebra such that $L'$ is nilpotent and $\phi(L) = 0$. Then, for any subalgebra $M$ of $L$, $\phi(M) = 0$.

**Proof.** Suppose $L' \subseteq M$. Let $H$ be a complement to $L'$ in $L$, so $H \cap M$ is a complement to $L'$ in $M$. Since $L$ acts completely reducibly on $L'$ and $L'$ is abelian, $H$ acts completely reducibly on $L'$. Then, since $H$ is abelian, $H \cap M$ acts completely reducibly on $L'$, hence so does $M$. Therefore $L' = M' + A$ for some ideal $A$ in $M$ where $M$ acts completely reducibly on $M'$ and $A + (H \cap M)$ is a complementary subalgebra of $M'$ in $M$. Thus by Proposition 1, $\phi(M) = 0$.

Suppose $L' \not\subseteq M$. Since $M + L'$ falls in the preceding case, we may assume $M + L' = L$. Since $L'$ is abelian, $L' \cap M$ is an ideal in $L$, $M/(L' \cap M)$ complements $L'/((L' \cap M))$ in $L/(L' \cap M)$ and $M/(L' \cap M)$ acts completely reducibly in $L'/(L' \cap M)$, $M/(L' \cap M)$ is a Cartan subalgebra of $L/(L' \cap M)$. Let $C$ be a Cartan subalgebra of $M$. By Lemma 4 of [1], $C$ is a Cartan subalgebra of $L$. Thus $C$ is a complement to $L'$ and $C + (L' \cap M) = M$ since $C \subseteq M$. Hence $C$ is a complement to $L' \cap M$ in $M$. Since $M$ acts completely reducibly on $L' \cap M$ and $M' \subseteq L' \cap M$, $M$ acts completely reducibly on $M'$, $L' \cap M = M' + (L' \cap Z(M))$ and, since $Z(M) \subseteq C$, $Z(M) \cap L' \subseteq C \cap L = 0$. Therefore $C = M' = M$ and $C \cap M' = 0$. Now $M$ satisfies part (3) of Proposition 1, hence $\phi(M) = 0$.

If $L$ is a solvable Lie algebra it has been shown in [2] that $\phi(L)$ is an ideal of $L$. We look for a condition on the subalgebras of $L/\phi(L)$ which are necessary and sufficient that $L \in \mathcal{F}$. In order to do this the following concept is introduced.

We shall say that a Lie algebra $L$ is the reduced partial sum of an ideal $A$ and a subalgebra $B$ if $L = A + B$ and for any subalgebra $C$ of $L$ such that $L = A + C$ and $C \subseteq B$ then $C = B$. It is noted that if $A \not\subseteq \phi(L)$, then there exists a $B \not= L$ such that $L$ is the reduced partial sum of $A$ and $B$. On the other hand, if $A \subseteq \phi(L)$ and $L$ is the reduced partial sum of $A$ and $B$, then $B = L$.

**Lemma 1.** Let $L$ be the reduced partial sum of $A$ and $B$. Then $A \cap B \subseteq \phi(B)$.

**Proof.** Suppose $C = A \cap B \not\subseteq \phi(B)$. Then $B$ contains a subalgebra $D$ such that $C + D = B$. Then $L = A + B = A + C + D = A + D$. This contradicts the minimality of $B$. 

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*Note: This text is a self-contained excerpt from a mathematical paper discussing the Frattini subalgebras of a class of solvable Lie algebras.*
LEMMA 2. Let $L$ be the reduced partial sum of $A$ and $B$. Then $\phi(L/A) \simeq A + \phi(B)/A$.

Proof. Since $A \cap B \subseteq \phi(B), A \cap \phi(B) = A \cap B$. Since $L/A \simeq A + B/A \simeq B/A \cap B$, $\phi(L/A) \simeq \phi(B/A \cap B) \simeq \phi(B)/A \cap B = \phi(B)/A \cap \phi(B) \simeq A + \phi(B)/A$.

PROPOSITION 2. The following are equivalent for the Lie algebra $L$:

1. $L \in \mathfrak{x}$.
2. For any subalgebra $H$ of $L/\phi(L)$, $\phi(H) = 0$.

Proof. Let $L$ satisfy (1) and let $\pi: L \to L/\phi(L)$ be the natural homomorphism. Then $\phi(\pi(L)) = \pi(\phi(L)) = 0$. Let $\tilde{W}$ be a subalgebra of $L/\phi(L)$ and let $W$ be the subalgebra of $L$ which contains $\phi(L)$ and corresponds to $\tilde{W}$. Since $L$ satisfies (1), $\phi(W) \subseteq \phi(L)$. If $\phi(W) = \phi(L)$, then $\phi(\pi(W)) = \pi(\phi(W)) = \pi(\phi(L)) = 0$. Suppose then that $\phi(W) \subset \phi(L)$. Then $W$ can be represented as a reduced partial sum $W = \phi(L) + K$. Let $T$ be a subalgebra of $W$ such that $T/\phi(L) \simeq \phi(W/\phi(L))$. If $T/\phi(L) \neq 0$, then $T = T \cap (\phi(L) + K) = (T \cap \phi(L)) + (T \cap K) = \phi(L) + (T \cap K)$. Consequently there exists an $x \in T \cap K, x \notin \phi(L)$. Since $\phi(K) \subseteq \phi(L), x \in \phi(K)$ and there exists a maximum subalgebra $S$ of $K$ such that $x \notin S$. We claim that either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in $W$. Suppose $\phi(L) + S \neq W$ and let $J$ be a subalgebra of $W$ which contains $\phi(L) + S$. Then $S \subseteq J \cap K$, so, by the maximality of $S$, either $J \cap K = S$ or $J \cap K = K$. If $J \cap K = S$, then $\phi(L) + S = \phi(L) + (J \cap K) = J \cap (\phi(L) + K) = J \cap W = J$. If $J \cap K = K$, then $J \supseteq K$ and, since $J \supseteq \phi(L), J \supseteq \phi(L) + K = W$, hence $J = W$. Consequently there exist no subalgebras of $W$ properly contained between $\phi(L) + S$ and $W$, hence either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in $W$. If $\phi(L) + S = W$, then $\phi(L) + K$ is not a reduced partial sum which is a contradiction. If $\phi(L) + S$ is maximal in $W$, then $\phi(L) + S/\phi(L) \supseteq \phi(W/\phi(L)) \simeq T/\phi(L)$. Hence $T \supseteq \phi(L) + S$. Since $S \subseteq \phi(L) + S$ and $x \in T \cap K \subset T \subseteq \phi(L) + S, K = \{S, x\} \subseteq \phi(L) + S$. Then $W = \phi(L) + K \subseteq \phi(L) + S \subseteq W$ implies $\phi(L) + K$ is not a reduced partial sum, a contradiction. Hence $\phi(\tilde{W}) = T/\phi(L) = 0$ and (2) is satisfied.

If $L/\phi(L)$ satisfies (2), then $\pi(\phi(H)) \subseteq \phi(\pi(H)) = 0$ for every subalgebra $H$ of $L$. Then $\phi(H) \subseteq \phi(L)$ for every subalgebra $H$ of $L$.

Combining Proposition 2 and Theorem 1 we have

THEOREM 2. Let $L$ be a Lie algebra such that $L'$ is nilpotent. Then $L \in \mathfrak{x}$.

THEOREM 3. Let $L \in \mathfrak{x}$ and let $T$ be a Lie homomorphism of $L$. 
Then \( T(\phi(L)) = \phi(T(L)) \).

**Proof.** \( T(\phi(L)) \) is always contained in \( \phi(T(L)) \) by Proposition 1 in [6]. If \( N = \text{kernel } T \subseteq \phi(L) \), then equality holds by Proposition 2 in [6]. Suppose \( N \not\subset \phi(L) \). Let \( L = N + K \) be a reduced partial sum. Using Lemma 2, \( \phi(T(L)) = \phi(L/N) \cong N + \phi(K)/N = T(\phi(K)) \). Since 
\[
T(N + \phi(L)) = T(\phi(L)) \subseteq \phi(T(L)) = \phi(L/N) = T(N + \phi(K)), \quad N + \phi(L) \subseteq N + \phi(K) \subseteq N + \phi(L) \text{.}
\]
Hence \( N + \phi(L) = N + \phi(K) \) and \( \phi(T(L)) = T(\phi(K)) = T(\phi(L)) \).

**THEOREM 4.** Let \( L \in \mathfrak{X} \). Necessary conditions that an ideal \( N \) of \( L \) be the Frattini subalgebra of \( L \) are that

1. \( \phi(\text{Ad}_N(L)) = \text{Ad}_N(\phi(L)) \).
2. There exists a subalgebra \( M \) of \( L \) such that \( M/N \cong \text{Ad}_N(\phi(L)) \).

**Proof.** (1) Let \( T \) be the mapping from \( L \) into the derivation algebra of \( N \) by \( T(x) = \text{ad } x \) restricted to \( N \) for all \( x \in L \). Then 
\[
T(\phi(L)) = \text{Ad}_N(\phi(L)) = \phi(T(L)) \cong \phi(\text{Ad}_N(L)).
\]
(2) Let \( M = Z_L(\phi(L)) \). Suppose that \( M \not\subseteq \phi(L) \) and let \( F = L/\phi(L) \) and \( A = (M + \phi(L))/\phi(L) \). Since \( \text{Ad}_{\phi(L)}(L) \cong L/M \) and \( \text{Ad}_{\phi(L)}(\phi(L)) \cong \phi(L)/Z(\phi(L)) = \phi(L)/M \cap \phi(L) = (M + \phi(L))/M \), \( F/A \cong (L/\phi(L))/(M + \phi(L)/\phi(L)) \cong L/(M + \phi(L))/M \cong \text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L)). \)

Since \( \phi(F) = 0 \), there exists a subalgebra \( D \) in \( F \) such that \( F = D \) is the reduced partial sum of \( A \) and \( D \). Using Proposition 2 and Lemma 1, 
\[
\text{Ad}_N(\phi(L)) \cong D \cong F/A \cong \text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L)).
\]

Related to part (1) of Theorem 4 are the following results.

**THEOREM 5.** Let \( L \in \mathfrak{X} \) and let \( K \) be an ideal of \( L \) containing \( \phi(L) \). Then \( \phi(\text{Ad}_K(L)) \cong \text{Ad}_K(K) \) if and only if \( K = \phi(L) + Z(K) \).

**Proof.** Let \( T \) be the Lie homomorphism from \( L \) into the derivation algebra of \( K \) given by \( T(x) = \text{ad } x \) restricted to \( K \) for each \( x \in L \). Then 
\[
\phi(\text{Ad}_K(L)) = \phi(T(L)) = T(\phi(L)) = \text{Ad}_K(\phi(L)) \cong \phi(L)/Z(\phi(L)) = \phi(L)/(Z(K) \cap \phi(L)) \cong (\phi(L) + Z(K))/Z(K). \]
If \( \phi(L) + Z(K) = K \), then \( \text{Ad}_K(L) \cong K/Z(K) \cong (\phi(L) + Z(K))/Z(K) \cong \phi(\text{Ad}_K(L)) \). If \( \phi(L) + Z(K) \subset K \), then \( \text{Ad}_K(K) \cong K/Z(K) \cong (\phi(L) + Z(K))/Z(K) \cong \phi(\text{Ad}_K(L)) \).

**THEOREM 6.** Let \( L \in \mathfrak{X} \) and let \( A \) be an ideal of \( L \) contained in \( \phi(L) \). Then \( \phi(\text{Ad}_A(L)) \cong \text{Ad}_A(A) \) if and only if \( \phi(L) = A + Z_\phi(L)(A) \).
Proof. If \( \phi(L) = A + Z_{\phi(L)}(A) \), then \( \text{Ad}_A(A) = \text{Ad}_A(\phi(L)) = T(\phi(L)) = \phi(T(L)) = \phi(\text{Ad}_A(L)) \).

Conversely, \( \text{Ad}_A(L) \cong L/Z_L(A) \) and \( Z_L(A) + A/Z_L(A) \approx A/Z(A) \equiv \text{Ad}_A(A) \). Then \( L/Z_L(A) + A \equiv \text{Ad}_A(L)/\text{Ad}_A(A) \) and \( \phi(L/Z_L(A) + A) \approx \phi(\text{Ad}_A(L)/\text{Ad}_A(A)) = \phi(\text{Ad}_A(L))/\text{Ad}_A(A) = 0 \). Hence \( \phi(L) \equiv Z_L(A) + A \) and \( \phi(L) = Z_{\phi(L)}(A) + A \).

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