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# FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

ERNEST L. STITZINGER

**In this paper the Lie algebra analogues to groups with property $E$ of Bechtell are investigated. Let $\mathfrak{X}$ be the class of solvable Lie algebras with the following property: if $H$ is a subalgebra of $L$, then $\phi(H) \subseteqq \phi(L)$ where $\phi(L)$ denotes the Frattini subalgebra of $L$; that is, $\phi(L)$ is the intersection of all maximal subalgebras of $L$. Groups with the analogous property are called $E$-groups by Bechtell. The class $\mathfrak{X}$ is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal $N$ of $L \in \mathfrak{X}$ be the Frattini subalgebra of $L$. Only solvable Lie algebras of finite dimension are considered here.**

The following notation will be used. We let $N(L)$ be the nil radical of $L$ and $S(L)$ be the socle of $L$; that is, $S(L)$ is the union of all minimal ideals of $L$. If $A$ and $B$ are subalgebras of $L$, let $Z_B(A)$ be the centralizer of $A$ in $B$. The center of $A$ will be denoted by $Z(A)$. If $[B, A] \subseteqq A$, we let $\mathrm{Ad}_A(B) = \{\mathrm{ad}\, b$ restricted to $A$; for all $b \in B\}$. $L'$ will be the derived algebra of $L$ and $L'' = (L')'$.

PROPOSITION 1. *Let $L$ be a Lie algebra such that $L'$ is nilpotent. Then the following are equivalent:*

(1) $\phi(L) = 0$.

(2) *$N(L) = S(L)$ and $N(L)$ is complemented by a subalgebra.*

(3) *$L'$ is abelian, is a semi-simple $L$-module and is complemented by a subalgebra.*

*Under these conditions, Cartan subalgebras of $L$ are exactly those subalgebras complementary to $L'$.*

*Proof*. Assume (1) holds. Nilpotency of $L'$ implies $\phi(L) \supseteqq L''$, so $L'$ is abelian and may be regarded as an $L/L'$-module. We may assume $L' = \sum \oplus V_\rho$, $V_\rho$ indecomposable $L/L'$-submodules. If $M$ is a maximal subalgebra of $L$ and if $V_\rho \nsubseteq M$, then $M \cap V_\rho$ is an ideal of $L$. If $S$ is an $L/L'$-submodule of $V_\rho$ properly contained between $M \cap V_\rho$ and $V_\rho$, then $M + S$ is a subalgebra of $L$ properly contained between $M$ and $L$, contradicting the maximality of $M$. Therefore $M$ contains all maximal submodules of $V_\rho$ for each $\rho$. Then $\phi(L) = 0$ implies the intersection of all maximal submodules of $V_\rho$ is zero for each $\rho$. If $V_1, \cdots, V_s$ are maximal submodules of $V_\rho$ with $V_1 \cap \cdots \cap V_s = 0$ and are minimal with respect to this property, we have $V = V_2 \cap \cdots V_s \neq 0$

and $V \cap V_1 = 0$ so that $V \oplus V_1 = V_\rho$, contradicting indecomposability. Therefore each $V_\rho$ is irreducible and $L'$ is a completely reducible $L/L'$-module and is also a completely reducible $L$-module. Since $L$ is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let $H$ be a Cartan subalgebra of $L$ and let $L_0$ and $L_1$ be the Fitting null and one component of $L$ with respect to $H$. Then $L = L_0 + L_1 = H + L_1 \subseteqq H + L'$ shows $L = H + L'$. We claim that $H \cap L' = 0$. If $H \cap L' \neq 0$, then, since $L'$ is abelian, $H$ is nilpotent and $L'$ is a completely reducible $L$-module, $L'$ is a sum of irreducible $H$-modules, $U_1, \cdots, U_\sigma$, such that for each $U_i$ $\underbrace{[\cdots[U_i, H]\cdots H]}_{k}=0$ for some $k$, hence $[U_i, H]=0$. Thus $[H, L' \cap H] = 0$. One sees that each $U_i$ is a central minimal ideal of $L$, and since $\phi(L) = 0$, $U_i$ is complemented by a maximal subalgebra $M$. Therefore $U_i$ is a one-dimensional direct summand of $L$, contradicting $U_i \subseteqq L'$. Hence $L' \cap H = 0$ and $H$ is a complement to $L'$ in $L$. Since $[H, H] \subseteqq H \cap L' = 0$, $H$ is abelian. Any minimal ideal not in $L'$ satisfies $[L, A] \subseteqq A \cap L' = 0$, so is central. Therefore $S(L) = L' + Z(L)$ and, since $H$ is a Cartan subalgebra, $Z(L) \subseteqq H$. Let $H_0$ be a complementary subspace to $Z(L)$ in $H$. One sees that $N(L) = L' + Z(L) + (N(L) \cap H_0) = S(L) + (N(L) \cap H_0)$. If $h$ is a nonzero element in $N(L) \cap H_0$, ad $h$ is nilpotent but not zero which implies $[V_\rho, h] = V_\rho$ for some $V_\rho \subseteqq L'$ and $\overbrace{[\cdots[V_\rho, h]\cdots h]}^{k} = 0$ for some $k$, a contradiction. Thus $S(L) = N(L)$ and $H_0$ is a complement. Consequently (1) implies (2).

Assume (2) holds and proceed by induction on the dimension of $L$. Since $L' \subseteqq N(L) = S(L)$ and minimal ideals are abelian, $L'$ is abelian. If every minimal ideal of $L$ is contained in $L'$, then $S(L) = L'$ and (3) follows. Therefore let $A$ be a minimal ideal of $L$ such that $A \nsubseteqq L'$. Hence $A \nsubseteqq \phi(L)$ and there exists a maximal subalgebra $M$ of $L$ such that $L = M + A$. Since $[L, A] \subseteqq A \cap L' = 0$, $A$ is central, hence one-dimensional. It follows that $L$ is the Lie algebra direct sum of $M$ and $A$. Since $M$ inherits the condition (2), $M$ satisfies (3) by induction. It now follows that $L$ also satisfies (3).

Assume (3) holds. Then $L'$ is a sum of minimal ideals of $L$, which we denote by $A_1, \cdots, A_k$, and $L = L' + H$, $H$ a subalgebra of $L$. Since $H' \subseteqq H \cap L' = 0$, $H$ is abelian. One sees that $L' = [L', H]$ and, consequently, $A_i = [A_i, H]$ for all $i$. Since $Z_{A_i}(H)$ is central in $L$, $Z_{A_i}(H)$ is an ideal in $L$ contained in $A_i$. Since $Z_{A_i}(H) \neq A_i$, $Z_{A_i}(H) = 0$. It follows that $H$ is its own normalizer, hence is a Cartan subalgebra of $L$. Now $H + A_1 + \cdots + \hat{A}_i + \cdots + A_k$ is a maximal subalgebra of $L$ since any containing algebra has a nonzero projection on $A_i$ which is ad $H$ stable, hence equal to $A_i$. Therefore $\phi(L) \subseteqq H$ and $\phi(L) \subseteqq H \cap L' = 0$. Hence (1) holds.

That complements to $L'$ are Cartan subalgebras is shown in (3) implies (1). That Cartan subalgebras are complements to $L'$ is shown in (1) implies (2). This completes the proof of Proposition 1.

THEOREM 1. *Let $L$ be a Lie algebra such that $L'$ is nilpotent and $\phi(L) = 0$. Then, for any subalgebra $M$ of $L$, $\phi(M) = 0$.*

*Proof.* Suppose $L' \subseteqq M$. Let $H$ be a complement to $L'$ in $L$, so $H \cap M$ is a complement to $L'$ in $M$. Since $L$ acts completely reducibly on $L'$ and $L'$ is abelian, $H$ acts completely reducibly on $L'$. Then, since $H$ is abelian, $H \cap M$ acts completely reducibly on $L'$, hence so does $M$. Therefore $L' = M' \oplus A$ for some ideal $A$ in $M$ where $M$ acts completely reducibly on $M'$ and $A + (H \cap M)$ is a complementary subalgebra of $M'$ in $M$. Thus by Proposition 1, $\phi(M) = 0$.

Suppose $L' \not\subseteqq M$. Since $M + L'$ falls in the preceeding case, we may assume $M + L' = L$. Since $L'$ is abelian, $L' \cap M$ is an ideal in $L$, $M/(L' \cap M)$ complements $L'/(L' \cap M) = (L/L' \cap M)'$ in $L/(L' \cap M)$ and $M/(L' \cap M)$ acts completely reducibly in $L'/(L' \cap M)$, $M/(L' \cap M)$ is a Cartan subalgebra of $L/(L' \cap M)$. Let $C$ be a Cartan subalgebra of $M$. By Lemma 4 of [1], $C$ is a Cartan subalgebra of $L$. Thus $C$ is a complement to $L'$ and $C + (L' \cap M) = M$ since $C \subseteqq M$. Hence $C$ is a complement to $L' \cap M$ in $M$. Since $M$ acts completely reducibly on $L' \cap M$ and $M' \subseteqq L' \cap M$, $M$ acts completely reducibly on $M'$, $L' \cap M = M' \oplus (L' \cap Z(M))$ and, since $Z(M) \subseteqq C$, $Z(M) \cap L' \subseteqq C \cap L = 0$. Therefore $C = M' = M$ and $C \cap M' = 0$. Now $M$ satisfies part (3) of Proposition 1, hence $\phi(M) = 0$.

If $L$ is a solvable Lie algebra it has been shown in [2] that $\phi(L)$ is an ideal of $L$. We look for a condition on the subalgebras of $L/\phi(L)$ which are necessary and sufficient that $L \in \mathfrak{X}$. In order to do this the following concept is introduced.

We shall say that a Lie algebra $L$ is the reduced partial sum of an ideal $A$ and a subalgebra $B$ if $L = A + B$ and for any subalgebra $C$ of $L$ such that $L = A + C$ and $C \subseteqq B$ then $C = B$. It is noted that if $A \not\subseteqq \phi(L)$, then there exists a $B \neq L$ such that $L$ is the reduced partial sum of $A$ and $B$. On the other hand, if $A \subseteqq \phi(L)$ and $L$ is the reduced partial sum of $A$ and $B$, then $B = L$.

LEMMA 1. *Let $L$ be the reduced partial sum of $A$ and $B$. Then $A \cap B \subseteqq \phi(B)$.*

*Proof.* Suppose $C = A \cap B \not\subseteqq \phi(B)$. Then $B$ contains a subalgebra $D$ such that $C + D = B$. Then $L = A + B = A + C + D = A + D$. This contradicts the minimality of $B$.

LEMMA 2. *Let $L$ be the reduced partial sum of $A$ and $B$. Then $\phi(L/A) \simeq A + \phi(B)/A$.*

*Proof.* Since $A \cap B \subseteqq \phi(B)$, $A \cap \phi(B) = A \cap B$. Since $L/A \simeq A + B/A \simeq B/A \cap B$, $\phi(L/A) \simeq \phi(B/A \cap B) \simeq \phi(B)/A \cap B = \phi(B)/A \cap \phi(B) \simeq A + \phi(B)/A$.

PROPOSITION 2. *The following are equivalent for the Lie algebra $L$:*

(1) $L \in \mathfrak{X}$.

(2) *For any subalgebra $H$ of $L/\phi(L)$, $\phi(H) = 0$.*

*Proof.* Let $L$ satisfy (1) and let $\pi: L \rightarrow L/\phi(L)$ be the natural homomorphism. Then $\phi(\pi(L)) = \pi(\phi(L)) = 0$. Let $\bar{W}$ be a subalgebra of $L/\phi(L)$ and let $W$ be the subalgebra of $L$ which contains $\phi(L)$ and corresponds to $\bar{W}$. Since $L$ satisfies (1), $\phi(W) \subseteqq \phi(L)$. If $\phi(W) = \phi(L)$, then $\phi(\pi(W)) = \pi(\phi(W)) = \pi(\phi(L)) = 0$. Suppose then that $\phi(W) \subset \phi(L)$. Then $W$ can be represented as a reduced partial sum $W = \phi(L) + K$. Let $T$ be a subalgebra of $W$ such that $T/\phi(L) \simeq \phi(W/\phi(L))$. If $T/\phi(L) \neq 0$, then $T = T \cap (\phi(L) + K) = (T \cap \phi(L)) + (T \cap K) = \phi(L) + (T \cap K)$. Consequently there exists an $x \in T \cap K$, $x \notin \phi(L)$. Since $\phi(K) \subseteqq \phi(L)$, $x \notin \phi(K)$ and there exists a maximum subalgebra $S$ of $K$ such that $x \notin S$. We claim that either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in $W$. Suppose $\phi(L) + S \neq W$ and let $J$ be a subalgebra of $W$ which contains $\phi(L) + S$. Then $S \subseteqq J \cap K$, so, by the maximality of $S$, either $J \cap K = S$ or $J \cap K = K$. If $J \cap K = S$, then $\phi(L) + S = \phi(L) + (J \cap K) = J \cap (\phi(L) + K) = J \cap W = J$. If $J \cap K = K$, then $J \supseteqq K$ and, since $J \supseteqq \phi(L)$, $J \supseteqq \phi(L) + K = W$, hence $J = W$. Consequently there exist no subalgebras of $W$ properly contained between $\phi(L) + S$ and $W$, hence either $\phi(L) + S = W$ or $\phi(L) + S$ is maximal in $W$. If $\phi(L) + S = W$, then $\phi(L) + K$ is not a reduced partial sum which is a contradiction. If $\phi(L) + S$ is maximal in $W$, then $\phi(L) + S/\phi(L) \supseteqq \phi(W/\phi(L)) \simeq T/\phi(L)$. Hence $T \subseteqq \phi(L) + S$. Since $S \subseteqq \phi(L) + S$ and $x \in T \cap K \subset T \subseteqq \phi(L) + S$, $K = \{S, x\} \subseteqq \phi(L) + S$. Then $W = \phi(L) + K \subseteqq \phi(L) + S \subseteqq W$ implies $\phi(L) + K$ is not a reduced partial sum, a contradiction. Hence $\phi(\bar{W}) = T/\phi(L) = 0$ and (2) is satisfied.

If $L/\phi(L)$ satisfies (2), then $\pi(\phi(H)) \subseteqq \phi(\pi(H)) = 0$ for every subalgebra $H$ of $L$. Then $\phi(H) \subseteqq \phi(L)$ for every subalgebra $H$ of $L$.

Combining Proposition 2 and Theorem 1 we have

THEOREM 2. *Let $L$ be a Lie algebra such that $L'$ is nilpotent. Then $L \in \mathfrak{X}$.*

THEOREM 3. *Let $L \in \mathfrak{X}$ and let $T$ be a Lie homomorphism of $L$.*

*Then* $T(\phi(L)) = \phi(T(L))$.

*Proof.* $T(\phi(L))$ is always contained in $\phi(T(L))$ by Proposition 1 in [6]. If $N =$ kernel $T \subseteqq \phi(L)$, then equality holds by Proposition 2 in [6]. Suppose $N \not\subseteq \phi(L)$. Let $L = N + K$ be a reduced partial sum. Using Lemma 2, $\phi(T(L)) = \phi(L/N) \simeq N + \phi(K)/N = T(\phi(K))$. Since $T(N + \phi(L)) = T(\phi(L)) \subseteqq \phi(T(L)) = \phi(L/N) = T(N + \phi(K))$, $N + \phi(L) \subseteqq N + \phi(K) \subseteqq N + \phi(L)$. Hence $N + \phi(L) = N + \phi(K)$ and $\phi(T(L)) = T(\phi(K)) = T(\phi(L))$.

THEOREM 4. *Let* $L \in \mathfrak{X}$. *Necessary conditions that an ideal* $N$ *of* $L$ *be the Frattini subalgebra of* $L$ *are that*

(1) $\phi(\mathrm{Ad}_N(L)) = \mathrm{Ad}_N(\phi(L))$.

(2) *There exists a subalgebra* $M$ *of* $L$ *such that* $M/N \simeq \mathrm{Ad}_N(L)/\mathrm{Ad}_N(\phi(L))$.

*Proof.* (1) Let $T$ be the mapping from $L$ into the derivation algebra of $N$ by $T(x) = \mathrm{ad}\, x$ restricted to $N$ for all $x \in L$. Then $T(\phi(L)) = \mathrm{Ad}_N(\phi(L)) = \phi(T(L)) = \phi(\mathrm{Ad}_N(L))$.

(2) Let $M = Z_L(\phi(L))$. Suppose that $M \not\subseteq \phi(L)$ and let $F = L/\phi(L)$ and $A = (M + \phi(L))/\phi(L)$. Since $\mathrm{Ad}_{\phi(L)}(L) \simeq L/M$ and $\mathrm{Ad}_{\phi(L)}(\phi(L)) \simeq \phi(L)/Z(\phi(L)) = \phi(L)/M \cap \phi(L) = (M + \phi(L))/M$, $F/A \simeq (L/\phi(L))/(M + \phi(L)/\phi(L)) \simeq L/(M + \phi(L)) \simeq (L/M)/((M + \phi(L))/M) \simeq \mathrm{Ad}_{\phi(L)}(L)/\mathrm{Ad}_{\phi(L)}(\phi(L))$. Since $\phi(F) = 0$, there exists a subalgebra $D$ in $F$ such that $F$ is the reduced partial sum of $A$ and $D$. Using Proposition 2 and Lemma 1, $A \cap D \subseteqq \phi(D) = 0$, hence $A \cap D = 0$. Let $E$ be the subalgebra of $L$ which contains $\phi(L)$ and corresponds to $D$. Then $E/\phi(L) \simeq D \simeq F/A \simeq \mathrm{Ad}_{\phi(L)}(L)/\mathrm{Ad}_{\phi(L)}(\phi(L))$. If $M \subseteqq \phi(L)$, then $\mathrm{Ad}_{\phi(L)}(L)/\mathrm{Ad}_{\phi(L)}(\phi(L)) \simeq (L/M)/(\phi(L)/Z(\phi(L))) = (L/M)/(\phi(L)/M \cap \phi(L)) = (L/M)/(\phi(L)/M) \simeq L/\phi(L)$.

Related to part (1) of Theorem 4 are the following results.

THEOREM 5. *Let* $L \in \mathfrak{X}$ *and let* $K$ *be an ideal of* $L$ *containing* $\phi(L)$. *Then* $\phi(\mathrm{Ad}_K(L)) \simeq \mathrm{Ad}_K(K)$ *if and only if* $K = \phi(L) + Z(K)$.

*Proof.* Let $T$ be the Lie homomorphism from $L$ into the derivation algebra of $K$ given by $T(x) = \mathrm{ad}\, x$ restricted to $K$ for each $x \in L$. Then $\phi(\mathrm{Ad}_K(L)) = \phi(T(L)) = T(\phi(L)) = \mathrm{Ad}_K(\phi(L)) \simeq \phi(L)/Z_{\phi(L)}(K) = \phi(L)/(Z(K) \cap \phi(L)) \simeq (\phi(L) + Z(K))/Z(K)$. If $\phi(L) + Z(K) = K$, then $\mathrm{Ad}_K(K) \simeq K/Z(K) = (\phi(L) + Z(K))/Z(K) \simeq \phi(\mathrm{Ad}_K(L))$. If $\phi(L) + Z(K) \subset K$, then $\mathrm{Ad}_K(K) \simeq K/Z(K) \supset (\phi(L) + Z(K))/Z(K) = \phi(\mathrm{Ad}_K(L))$.

THEOREM 6. *Let* $L \in \mathfrak{X}$ *and let* $A$ *be an ideal of* $L$ *contained in* $\phi(L)$. *Then* $\phi(\mathrm{Ad}_A(L)) \simeq \mathrm{Ad}_A(A)$ *if and only if* $\phi(L) = A + Z_{\phi(L)}(A)$.

*Proof.* If $\phi(L) = A + Z_{\phi(L)}(A)$, then $\mathrm{Ad}_A(A) = \mathrm{Ad}_A(\phi(L)) = T(\phi(L)) = \phi(T(L)) = \phi(\mathrm{Ad}_A(L))$.

Conversely, $\mathrm{Ad}_A(L) \simeq L/Z_L(A)$ and $Z_L(A) + A/Z_L(A) \simeq A/Z(A) = \mathrm{Ad}_A(A)$. Then $L/Z_L(A) + A \simeq \mathrm{Ad}_A(L)/\mathrm{Ad}_A(A)$ and $\phi(L/Z_L(A) + A) \simeq \phi(\mathrm{Ad}_A(L)/\mathrm{Ad}_A(A)) = \phi(\mathrm{Ad}_A(L))/\mathrm{Ad}_A(A) = 0$. Hence $\phi(L) \subseteqq Z_L(A) + A$ and $\phi(L) = Z_{\phi(L)}(A) + A$.

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