FINITE LINEAR GROUPS OF DEGREE SEVEN. II

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The determination of finite groups which can be represented as a group of $7 \times 7$ matrices irreducible over the complex numbers is finished in this paper. To simplify the cases, the matrices are assumed unimodular and the groups are primitive. The groups discussed here are essentially simple and have orders $7^3 \cdot 3^3 \cdot 2^\alpha$. The theory of groups with a prime to the first power in the group order and of course the representation of degree seven are used heavily in the determination.

This paper is the third in a series of papers discussing linear groups, the first two being [24, 25]. We shall prove the following result.

THEOREM I. Suppose $G$ has an irreducible complex representation $X$ of degree 7 which is faithful, primitive and unimodular. Suppose, further, that $G$ has an abelian 7-Sylow subgroup. Then, by [5, 4A], $Z$, the center of $G$, has order 1 or 7 and $G = G_1 \times Z$ for a subgroup $G_1$ of $G$. We prove that $G_1$ is one of the following groups. Let $|G_1|$ be the cardinality of $G_1$.

I. $G_1 \cong \text{PSL}(2,13)$

II. $G_1 \cong \text{PSL}(2,8)$

III. $G_1 \cong A_8$

IV. $G_1 \cong \text{PSL}(2,7)$

V. $G_1 \cong U_4(3)$

VI. $G_1 \cong S_6(2)$

VII. An extension of III, IV, V by an automorphism of order 2 or an extension of II by an automorphism of order 3. For III it is $S_6$; for IV it is induced by $\text{PGL}(2,7)$. For V and II it is induced by field automorphisms. For V it is $G_2(2)$.

This result together with [25, Th. 4.1] determines the linear groups of degree 7. The proof is in several parts. Rather than use the notation $G$, we assume $G$ is as stated in the theorem and assume $Z = e$. Set $|G| = g$, the order of $G$. By [5, 4A] we know $|G| = 7 \cdot g_5$ where $7 \not| g_5$. Let $|G| = g = 7 \cdot 5^\alpha \cdot 3^3 \cdot 2^\alpha$. We use the notation of [24, 25]. Thus $\chi$ is the character of $X$. As $\chi$ is of zero 7-defect, $\chi(\xi) = 0$ where $\xi$ is an element of order 7 [7, Th. 1]. This means the eigenvalues are all distinct and so by [5, 3F] $C(P) = P$ where $P$
is a 7-Sylow group. The characters of $G$ satisfy many properties described in [5, §8]. These are different depending on the value $s = |N(P)}/P|$. The possible values are 2, 3, or 6. The value $s = 1$ is impossible as by Burnside's theorem there would be a normal 7-complement contradicting the primitivity of $X$. The cases 2, 3, 6 are treated separately. The case $s = 6$ is by far the most difficult. It is treated first (§2–§5) because some of the ideas are used for the case $s = 3$ (§7). However much of the treatment for $s = 3$ (§7) and all of the treatment for $s = 2$ (§6) is independent of the earlier sections and can be read independently.

If there are primes higher than 7 occurring in $g = |G|$, $G$ is case I by [17, 5, 2D]. In the remaining cases we assume $g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. We know by [5, 3E and 25, 2.6] that $a \leq 6$, $b \leq 8$, $c \leq 10$. A flow chart for the order of the elimination is given at the end.

As in [24 or 25], some notation is standard. Thus if $K$ is a subset of $G$, $C(K)$ and $N(K)$ are the centralizer and normalizer of $N$. If $\gamma \in G$ we define $N(\gamma) = N(\langle \gamma \rangle)$. Also set $C(K) \cap K = Z(K)$. Let $|K|$ denote the cardinality of $K$. We have set $|G| = g$. We mention Theorem 2.1 of [25] which says that a 5-Sylow group of $G$ is abelian. We label the principal $p$-block $B_q(p)$ for any prime $p$ dividing $g$.

2. Preliminary properties of $\chi$ when $s = 6$. We first assume $s = 6$, $G = G'$. By [5, 8A], $G$ is simple. As in [5, §8] the results of [2] apply to $G$. There are seven characters $\chi_i$, $i = 0, 1, \cdots, 6$, in $B_q(7)$ of $G$. Their degrees $x_i$, $i = 0, 1, \cdots, 6$ are congruent to $\pm 1$ (mod 7). We set $\chi_i(\xi) = \delta_i$, $i = 0, 1, \cdots, 6$. Here $\delta_i = \pm 1$, $\xi$ is an element of order 7. The degree equation for $B_q(7)$ is $\sum_{i=1}^6 \delta_i x_i = 0$ [2, Th. 11 or 5, §8]. We assume $\chi_0$ is the identity character.

If $\chi$ is the character of degree 7 corresponding to $X$ we have

$$\chi \bar{\chi} = \chi_0 + \sum_{i=1}^6 a_i \chi_i + \eta.$$  

Here $\eta$ is a sum of irreducible characters of $G$ of zero 7-defect. There must be some $i$ with $a_i \neq 0$, $i = 1, 2, \cdots, 6$ for which $\delta_i = -1$. This is because $\chi(\xi) = 0$ and so $\chi \bar{\chi}(\xi) = 0$. The possible values of $x_i$ are 6, 20, 27, and 48. This means $G$ must have a character, say $\chi_\nu$ of degree 6, 20, 27, or 48. We will consider each of these cases individually in this and later sections. The case $x_i = 6$ is easily eliminated by considering the restriction of $\chi$ to $N(\xi)$. As this analysis will be needed in later sections, we include more than is necessary here.

Let $N = N(\xi)$. We are assuming $|N| = 42 = 7 \cdot 6$. Let $\tau$ be an
element of order 6 in \( N \). The character table is as follows where \( \epsilon = e^{2\pi i/3} \).

**Table I. Character table for \( N = N(\zeta) \).**

<table>
<thead>
<tr>
<th>element</th>
<th>1</th>
<th>( \xi )</th>
<th>( \tau )</th>
<th>( \tau^3 )</th>
<th>( \tau^4 )</th>
<th>( \tau^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi_0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \psi_1 )</td>
<td>1</td>
<td>1</td>
<td>( -\epsilon )</td>
<td>( \epsilon^2 )</td>
<td>( -1 )</td>
<td>( \epsilon )</td>
</tr>
<tr>
<td>( \psi_2 )</td>
<td>1</td>
<td>1</td>
<td>( \epsilon^2 )</td>
<td>( \epsilon )</td>
<td>1</td>
<td>( \epsilon^2 )</td>
</tr>
<tr>
<td>( \psi_3 )</td>
<td>1</td>
<td>1</td>
<td>( -1 )</td>
<td>1</td>
<td>( -1 )</td>
<td>1</td>
</tr>
<tr>
<td>( \psi_4 )</td>
<td>1</td>
<td>1</td>
<td>( \epsilon )</td>
<td>( \epsilon^2 )</td>
<td>1</td>
<td>( \epsilon )</td>
</tr>
<tr>
<td>( \psi_5 )</td>
<td>1</td>
<td>1</td>
<td>( -\epsilon^2 )</td>
<td>( \epsilon )</td>
<td>( -1 )</td>
<td>( \epsilon^2 )</td>
</tr>
<tr>
<td>( \psi_6 )</td>
<td>6</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If \( \chi_j \) is a character of \( G \) of degree 6 \( \chi_j \mid N = \psi_6 \). However the eigenvalues of the representation corresponding to \( \psi_6(\tau) \) are 1, \( -1 \), \( -\epsilon \), \( -\epsilon^2 \), \( \epsilon \), \( \epsilon^2 \). This means the determinant is \(-1\). The representation \( X_j \) corresponding to \( \chi_j \) cannot be unimodular and so \( G \neq G' \). As we are assuming in this section that \( G' = G \) we can assume there are no characters \( \chi_j \) of degree 6.

For later use we require some further results. The restriction of \( \chi \) to \( N \) must contain \( \psi_6 \) and \( \psi_j \) \( 0 \leq j \leq 5 \) as constituents. In order that \( X(\tau) \) be unimodular, \( j = 3 \). This gives

\[
\chi \mid N = \psi_3 + \psi_6.
\]

Let \( P_2(\chi) \) be the character corresponding to the symmetric tensors of rank 2 for \( X \) and \( C_2(\chi) \) be the character corresponding to the skew symmetric tensors of rank 2. Their restrictions to \( N \) are as follows

\[
P_2(\chi) \mid N = 3\psi_6 + \psi_1 + \psi_5 + \psi_3 ,
\]

\[
C_2(\chi) \mid N = 4\psi_6 + 2\psi_0 + \psi_2 + \psi_4 .
\]

We obtain similar results for a character \( \chi_j \) of degree 8 or 20. Here \( \chi_j \mid N \) will have two linear constituents. The character \( \psi_6 \) of degree six will appear once if \( \chi_j \) has degree 8 and three times if \( \chi_j \) has degree 20. As the representation corresponding to \( \chi_j \) is unimodular there are three possibilities for the two linear constituents. These are \( \psi_6 + \psi_3, \psi_1 + \psi_2, \psi_4 + \psi_5 \). If \( \chi_j \) has degree 8 this gives

\[
(\text{i}) \quad \chi_j \mid N = \psi_6 + \psi_0 + \psi_3
\]

\[
(\text{ii}) \quad \chi_j \mid N = \psi_0 + \psi_1 + \psi_3
\]

\[
(\text{iii}) \quad \chi_j \mid N = \psi_0 + \psi_4 + \psi_5 .
\]
If $\chi_j$ has degree 20 the corresponding result is

(i) $\chi_j | N = 3\psi_0 + \psi_0 + \psi_3$

(2.5) (ii) $\chi_j | N = 3\psi_0 + \psi_1 + \psi_2$

(iii) $\chi_j | N = 3\psi_0 + \psi_4 + \psi_5$.

If $\chi_j$ is real the only possibilities are (2.4)(i) or (2.5)(i).

The following lemma will be needed several times.

**Lemma 2.1.** Let $Q$ be a 5-Sylow group of $G$. If a character $\mu$ of $G$ is not real and $\mu \mid Q$ is not rational, there are at least four distinct nonreal conjugates of $\mu$.

**Proof.** Let $K$ be a splitting field for $G$ containing $\lambda_i = e^{2\pi i/20}$. Let $K_i$ be the subfield of $K$ containing any $r$-th roots of 1 lying in $K$ where $r$ runs over the primes in $g$ other than 5. We may pick $\sigma \in G(K/K_i)$ the Galois group of $K$ over $K_i$ so that $\sigma(\lambda_i) = (\lambda_i)^i$. Suppose $\mu, \mu^\sigma, \mu^\tau, \mu^\pi$ are not all distinct. As there are no elements of order $5^j$ in $G[5,3B]$, $\mu \mid Q \neq \mu^\sigma \mid Q$ by hypothesis. Also $\mu \neq \mu^\tau$ by hypothesis. This means $\mu^\pi = \mu^\sigma$. In particular $\mu^\sigma \mid Q = \mu^\tau \mid Q$. Let $\eta = \sigma^9$. We have $\mu^{\sigma^9} \mid Q = \mu^\pi \mid Q = \mu^\sigma \mid Q$. This implies $\mu^\sigma \mid Q = \mu \mid Q$ a contradiction. We see $\mu, \mu^\sigma, \mu^\tau, \mu^\pi$ are all distinct. Clearly none can be real. This proves the lemma.

Several times we will need to study the case in which $Q$ is cyclic of order 5. That is $g = 7^\cdot 5^\cdot 3^3 \cdot 2^3$. The results of [2] can be applied for $p = 5$. Let $\pi$ be an element of order 5. By Burnside’s theorem $C(\pi) = \langle \pi \rangle \times V$. As there are no elements of order $5^3 \cdot 7$, $|V| = 3^2 2^4$. Let $|N(\pi)/C(\pi)| = w$. As $G$ has no normal 5-complement $w$ is 2 or 4 by Burnside’s theorem. Each 5 block contains $e$ nonexceptional characters and $4/e$ exceptional characters where $e$ is 1, 2, or 4. Let $B_j(5)$ be the 5-block containing $\chi$. If $\chi$ is nonexceptional there are two possibilities for $\chi \mid C(\pi)$ as can be seen from close inspection of [2, II, Th. 1]

(2.6) $\chi \mid \langle \pi \rangle \times V = \theta + \sum_{i=1}^{4} \lambda^i \phi$

where $\theta$ is of degree 2 with $\theta(\pi) = 2$, $\lambda$ is the linear character of $\langle \pi \rangle$ such that $\lambda(\pi) = e^{2\pi i/6}$, $\phi$ is a linear character of $V$.

(2.7) $\chi \mid \langle \pi \rangle \times V = \varphi_1 + \varphi_2 + \sum_{i=1}^{4} \lambda^i \phi$

where $\lambda, \phi$ are as in (2.6) and $\varphi_1, \varphi_2$ are distinct linear characters of $c(\pi)$ conjugate in $N(\pi)$. Also $\varphi_i(\pi) = 1, i = 1, 2$. In the case (2.6)
V is nonabelian; in the case (2.7) V is abelian. Suppose \( \chi \) is exceptional. If \( s = 4 \) and \( e = 1 \) or 2 an examination of [2, II, Th. 1] shows \( \chi \) cannot have degree 7. If \( e = 4 \) any of the characters in \( B_7(5) \) can be chosen exceptional and so \( \chi \) can be chosen nonexceptional. In any case \( \chi \) satisfies (2.6) or (2.7). If \( s = 2 \), there are at least two characters \( \chi \) and \( \chi' \) of degree 7 in \( B_7(5) \). By [7, p. 579] \( \chi \chi' \) has a constituent in \( B_7(5) \). As \( \chi \neq \chi' \) this constituent is not \( x_0 \) and so \( B_7(5) \) has a nontrivial character whose degree is at most 49. By [2] there are three characters in \( B_7(5) \) besides \( x_0 \). Their degrees must be congruent to \( \pm 1 \) or 0 mod 7. By examining the possibilities one sees the smallest such degree equation for \( B_7(5) \) is \( 1 + 63 = 64 \). This means \( \chi \) cannot be exceptional and so \( \chi \) must satisfy (2.6) or (2.7).

3. The case \( \chi \chi = \chi_0 + \chi_1 \), \( \deg \chi_i = 48 \), \( g = 7 \cdot 5^a \cdot 3^b \cdot 2^c \). In this section we consider the case IB where \( \chi \chi = \chi_0 + \chi_1 \), \( \deg \chi_i = 48 \). We still assume \( s = 6 \). This case is eliminated by first showing \( \chi \) is rational when restricted to 5-Sylow group and so \( a \leq 1 \). The case \( a = 1 \) is eliminated by finding \( C(\pi) \) where \( \pi \) is an element of order 5. The case \( a = 0 \) is then eliminated using some results in [6].

Suppose \( \chi \) is not rational. In particular let \( \sigma \) be an element of the Galois group of a splitting field \( K \) for \( G \) over the rationals for which \( (\chi)^\sigma \neq \chi_i \). Clearly \( \chi \chi^\sigma = \chi_0 + \chi_1 \). This implies \( \chi \chi^\sigma \chi^\sigma \) has \( \chi_0 \) as a constituent with multiplicity 1. This means \( \chi \chi^\sigma \) of degree 49 is irreducible giving a contradiction. We see \( \chi \) is rational. Also \( \chi \chi = \chi_0 + \chi_1 \), must be rational.

Let \( \pi \) be an element of order 5. Suppose \( \chi | \langle \pi \rangle = \sum_{i=1}^t b_i \lambda_i \) where the \( \lambda_i \), \( i = 1, 2, \ldots, t \) are distinct linear characters of \( \langle \pi \rangle \), \( b_i \neq 0 \). Certainly \( t \leq 5 \) and \( \sum_{i=1}^t b_i = 7 \). If \( c = \sum_{i=1}^t (b_i)^2 \) we have \( \chi \chi^\sigma(\pi) = c - b \) where \( c + 4b = 49 \). This means \( c \equiv 1 \pmod{4} \). If the numbers \( \{b_1, \ldots, b_t\} \) are arranged in decreasing order the following possibilities occur:

\[ \{3, 1, 1, 1, 1\}; \{2, 2, 2, 1\}; \{4, 2, 1\}; \{3, 2, 2\}; \{6, 1\}; \{5, 2\}; \{4, 3\}. \]

Now let \( \lambda \) be the linear character of \( \langle \pi \rangle \) with \( \lambda(\pi) = e^{2\pi i/b} \). Suppose \( \chi | \langle \pi \rangle = \sum_{i=0}^t a_i \lambda^i \). This means

\[
\chi \chi | \langle \pi \rangle = \left( \sum_{i=0}^t a_i \lambda^i \right) \left( \sum_{i=0}^t a_i \lambda^{-i} \right) = \sum_{i=0}^t \left( \sum_{j=0}^i a_j a_{j-i} \right) \lambda^i.
\]

As \( \chi \chi \) is rational we obtain

\[
(3.1) \quad \sum_{j=0}^i a_j a_{j-i} = b , \quad i = 1, 2, 3, 4.
\]
The nonzero entries among \{a_0, a_1, \ldots, a_4\} are the values \{b_1, \ldots, b_4\} given in the above paragraph possibly rearranged. A routine check of these possibilities shows \{3, 1, 1, 1, 1\} to be the only set for which (3.1) can be satisfied. The checking is facilitated by noting that the first two integers \(a_0, a_1\) can be picked arbitrarily from \{b_1, \ldots, b_4, 0, \ldots, 0\} without changing the form of (3.1). Here the bracketed integers are completed with zeroes to give 5 terms. The case \{4, 2, 1\} or \{3, 2, 2\} can be chosen as \(\delta_1, \delta_2, \delta_3\). This is impossible for any of the choices of \(\delta_1, \delta_2, \delta_3\). The case \{2, 2, 2\} can be taken as \(\delta_1, \delta_2, \delta_3\). Equation (3.1) gives \(\lambda_1 + \lambda_2 = \lambda_3\). This is impossible for \(\lambda_1, \lambda_2, \lambda_3\). The case \{2, 2, 2\} can be taken as \(\delta_1, \delta_2, \delta_3\). Equation (3.1) cannot be satisfied. In the cases \{b_1, b_2\} we consider \(\delta_1, \delta_2, \delta_3\). Equation (3.1) cannot be satisfied. In the case \{3, 1, 1, 1, 1\} the unimodularity of \(X\) gives \(\chi(\pi) = 2\). Suppose \(\pi\) has order 5. As \(\pi^2\) has order 5 the constituents of \(\chi(\pi)\) are \(\{\lambda_1^{15}, \lambda_2^{15}, \lambda_3^{15}, \lambda_4^{15}\}\) where \(\lambda(\pi^2) = e^{\pi^2/5}\). Let the linear constituents of \(\chi(\pi)\) be \(\{\lambda_1, \lambda_2, \lambda_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}\) where \(\varepsilon_i^5 = \lambda_i^5\). Suppose \(\lambda_i = \lambda_j, i \neq j, i, j = 1, 2, 3\). This means \(\varepsilon_i\) and \(\varepsilon_j\) are equal and so all twenty conjugates appear with equal multiplicity at least 2 in \(\chi(\pi)\). The following thirteen linear characters also appear in \(\chi(\pi)\): \(\varepsilon_i\), \(i = 1, 2, 3, 4\); \(\lambda_j, \lambda_k, j, k = 1, 2, 3\). None of these are conjugate to \(\varepsilon_i\), \(\varepsilon_j\) and so we have too many constituents. This means \(\lambda_1, \lambda_2, \lambda_3\) are all distinct. Also \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\) are all distinct as their fifth powers are distinct. This means the trivial character of \(\pi\) occurs seven times. The number of conjugates of any nontrivial character is 4 or 20 and hence divisible by 4. However 4 \(\not|\) 42. This means there are no elements of order 5 in \(G\).

We have shown \(\chi\) is rational on a 5-Sylow group. In particular, by [21] \(a \leq 1\). We now consider this case, case IB(ii) of the flow chart. There are two possibilities for \(\chi(\pi)\), (2.6) and (2.7). In case (2.7), \(C(\pi)\) is abelian as \(\chi(\pi)\) has seven linear constituents. In case (2.6), \(V\) is nonabelian as \(\chi(\pi)\) has a constituent of degree 2.

We may also consider the restriction \(\chi_i(\pi)\). Here \(\chi_i(\pi) = 3\). The only possibility [2, II, Th. 1] is \(\chi_i(\pi) = \theta \cdot \lambda_i\), deg \(\theta = 3\). This means \(V\) is nonabelian as \(\theta\) is irreducible. In particular, case (2.7) above does not occur. The character \(\theta\) is rational as \(\chi_i\) is rational. As \(\chi(\pi) = \lambda_0 + \chi_i\) we have \(\theta \cdot \theta_i = \theta_0 + \theta\) where \(\theta_0\) is the trivial character of \(V\). \(\chi(\pi) = \theta_0 + \sum_{i=0}^{4} \lambda_i \cdot \varphi\).

We know \(\chi(\pi) = \theta_1 + \delta \varphi\). If \(\varphi\) is in the kernel of \(\theta_1\), \((\varphi(R))^5 = 1\) by the unimodularity of \(X\). This means \(\varphi(R) = 1\) and so \(\theta_1\) is faithful. Let \(V_s\) be a 3-Sylow group of \(V\). As \(\theta_1\) is faithful of degree 2, \(V_s\) must be abelian. We see \(X(\pi)\) has at most 3 distinct linear characters and so \(|V_s| \leq 3^2 [5, 3D]\). Let \(V_2\) be a 2-Sylow group of \(V\). As
$V_2$ has a faithful representation of degree 2 there is an abelian subgroup $A$ of index 2. As $X | A$ has at most 3-distinct linear characters, $|A| \leq 2^3$ [5, 3D]. Here $V_2$ is nonabelian if and only if $|V_2| = 2^3$. In this case an involution in $Z(V_2)$ must satisfy $\theta_1(J) = -2$. Clearly $\varphi(J) = 1$. In particular $J \in Z(V)$.

We now consider $\theta$. Let $K$ be the kernel of $\theta$. As $\theta_1 \vartheta_1 = \theta_0 + \theta$, if $R \in K$, $|\theta_1(R)| = 2$. In particular $R \in Z(V)$. Suppose there is an element $R$ of order 3 in $K$. As $\theta$ is faithful and rational on $V/K$ $|V| = 3^2 \cdot 2^4$. Also $\theta_1(R) = 2u$ or $2u^2$ where $u = e^{2\pi i/3}$. We can assume $\theta_1(R) = 2u$ by taking $R^3$ if necessary. By the unimodularity of $X$, $\varphi(R) = u^2$. If there is an element $J \in F/K$ which is nonabelian if and only if $|V_2| = 2^3$. In this case an involution in $Z(V_2)$ must satisfy $\theta_1(J) = -2$. Clearly $\varphi(J) = 1$ and Blichfeldt’s theorem is violated for $X$ and $JR$. This means $\delta = 2$. As $V/K$ has a representation of degree 3, $\delta = 2$. If $V_2$ is elementary abelian there is an element $J$ for which $\theta_1(J) = -2$. We see $V_2$ is cyclic. This means there is a normal 2-complement. However, in this case there can be no character of degree 3 by Ito’s theorem [12, 53.18]. This shows there are no elements of order 3 in $K$.

As $V/K$ has a rational character of degree 3 $|V| = 3 \cdot 2^3$. Here $V$ has characters of degree 3 and 2. As $3^2 + 2^4 > 12$ we see $|V| = 3 \cdot 2^2$. This means $V_2$ is nonabelian and so there is an involution in $Z(V)$ for which $\theta_1(J) = -2$. Let $T$ be an element of order 3 in $V$. If $\theta_1(T) = u + \bar{u}$ where $u = e^{2\pi i/3}$ the element $TJ$ would contradict Blichfeldt’s theorem. This means $\theta_1(T) = 1 + u$ or $1 + \bar{u}$. We see $\vartheta_1 \neq \theta_1$. By the unimodularity of $X$, $\chi(T) = 1 + 6u$. We see $T$ is not conjugate to $T^{-1}$ in $G$. There must be a normal three complement to $\langle T \rangle$ in $V$ and so the number of linear characters of $V$ is a multiple of three.

The characters obtained so far have degrees 1, 1, 1, 3, 2, 2. There must be one further character of degree 2. As $\theta_1 \vartheta_1 = \theta_0 + \theta$ has two irreducible constituents, $\theta_1 \vartheta_1 = P_2(\theta_1) + C_2(\theta_1)$ must have two irreducible constituents. These are the characters corresponding to the symmetric tensors of rank 2, $P_2(\chi_1)$, and the skew symmetric tensors of rank 2, $C_2(\chi_1)$. We see $P_2(\theta_1) = \theta$. Similarly $\chi_1 = P_2(\chi) + C_2(\chi)$. As $\chi_1 = 1 + \chi_1$, $P_2(\chi)$ and $C_2(\chi)$ are irreducible. Clearly $\chi_1 v = \theta_1 \vartheta_1 (v) = (\theta + C_2(\theta_1))(v)$, where $v \in V$. Let $\psi = P_2(\chi)$. Clearly $\psi(v) = \theta(v)$. This means $\psi$ is in the same 5-block as $\chi_1$. Denote this 5-block by $B_5$. Evaluating $\psi(T)$ we find $\psi(T) = (1 + 6u^2 + (1 + 6u^2))/2 = -5 + 15u^2$. This means $\psi \neq \bar{\psi}$. However $\psi(v) = \overline{\psi(v)}$ for $v \in V$ and so $\psi \in B_5$. We show case I B(ii) is impossible by showing the block $B_5$ cannot be completed without giving a contradiction.

There cannot be two exceptional characters in $B_5$ or there would be too many characters. Here $\psi, \bar{\psi}$ cannot be the exceptional characters as $28 \equiv 48 \pmod{5}$. This means there are two missing
characters with degrees $R$ and $S$. There are two cases for the degree
equation $104 + R = S$ and $104 = R + S$. There must be one more
character from $B_q(7)$ and one whose degree is not divisible by 3. If
the degree of the character from $B_q(7)$ is not divisible by 3 it must
be 8, 64, or 512. The only solution is $48 + 28 + 28 + 8 = 112$. If
the character of degree 8 is denoted by $\chi_6$, $\chi_6 | V = \theta + 5\zeta$ where $\zeta$
is linear. As $\theta(J) = 3$ we see $\zeta(J) = 1$. This means $J$ is in the
kernel of $\chi_6$ contradicting the simplicity of $G$. This means $R$ or $S$
is of the form $7 \cdot 2^a$ where $7 \cdot 2^a \equiv \pm 2 \pmod{5}$. The degree of the
character in $B_q(7)$ is divisible by 3. We see $B = 2, 4, 6, 8, 10, 7 \cdot 2^a \equiv 1$
(mod 3). We need only consider the cases $7 \cdot 2^a \equiv 3 \pmod{5}$. The
values are 28, 448, 7168. There are no solutions. This case is there-
fore impossible and we can assume $g = 7 \cdot 3^b \cdot 2^c$.

We now begin Case IB (iii). We assume $s = 6$, $g = 7 \cdot 3^b \cdot 2^c$. We
know $b \leq 8$. By Sylow’s theorem $b = 1, 3, 5, 7$. There must be a
character of degree 27 or 729 and one of degree 8,64, or 512.

If a character of degree 729 occurs it must be in a 3-block con-
taining 3 characters of degree 729, for if not there would be a
character of degree at least $6 \cdot 729$. The degree equation would then
be $1 + 729 + 729 + 729 + \left\{\begin{array}{l} 8 \\ 64 \\ 512 \end{array}\right\} = 48 + \chi_6$. There is no solution.

There must be a character $\chi_3$ of degree 27. Let $g = 7 \cdot 3^b \cdot 2^c$. Sylow’s theorem gives $b = 3, 5, 7$. Suppose first $b = 3$. Then $c = 3, 6, 9$. If $c = 3$ or 6, $g < 20,000$. All simple groups of order at
most 20,000 are listed in [19] and none have this order. If $c = 9$
the result [6, 1H] is contradicted as $2^a \geq 12 \cdot 3^3 = 324$. When $b = 5$
or 7 [6, 1L] can be applied to show there is no character of degree
512. As there must be a character of degree $2^a$ it must be 8 or 64.
Each of these cases can be eliminated with a routine elimination of
degree equations using block separation and Schur’s theorem [21].
We do not include the details.

4. The case $\chi_\chi = \chi_0 + \chi_1 + \cdots$, deg $\chi_0 = 20$, $g = 7 \cdot 5^a \cdot 3^b \cdot 2^c$. In
this section we consider the case IC where $\chi_\chi = \chi_0 + \chi_1 + \cdots$, deg $\chi_1 = 20$. The case is eliminated by first showing $a = 1$. This case is
eliminated by considering $\chi | C(\pi)$ where $\pi$ is an element of order 5.
The relations (2.1)-(2.7) are used.

We begin with a preliminary discussion regarding the tree for
the prime 7 [2]. The character $\chi$ is a principal 7-indecomposable
and so $\chi_\chi$ is a sum of principal 7-indecomposables [9]. There is
exactly one principal indecomposable containing $\chi_0$ as a constituent.
This is $\chi_0 + \chi_1$ where $\chi_1$ is adjacent to $\chi_0$ on the tree. This means
$\chi_0 + \chi_1$ is a constituent of $\chi_\chi$. We have already eliminated the case
$x_j = 6$ or $x_j = 48$. This means $x_j$ is 20 or 27. We are assuming $x_1 = 20$. If $x_j = 27$, $\chi_\bar{\bar{X}}$ would have two linear constituents contradicting the simplicity of $G$ or the irreducibility of $\chi$. This means $x_j = 20$.

Suppose $j \neq 1$. This would mean $\chi_\bar{\bar{X}} = \chi_0 + \chi_i + \chi_j + \mu$. Here $\mu$ would be of degree 8. Clearly $\chi_i$ is real as it is adjacent to $\chi_0$ on the stem. Also $\mu$ is real as it is the only constituent of $\chi_\bar{\bar{X}}$ of degree 8 and $\chi_\bar{\bar{X}}$ is real. Using (2.4) and (2.5) we see $\chi_0 + \mu + \chi_j | N$ has $\psi_0$ as constituent with multiplicity 3. However (2.2) and (2.3) imply $\chi_\bar{\bar{X}} | N$ has $\psi_0$ as constituent with multiplicity 2. This means $j = 1$ and so $\chi_i$ is adjacent to $\chi_0$ on the stem.

Let $\chi_\bar{\bar{X}} = \chi_0 + \chi_i + \mu$. As above $\chi_i$ is real. From (2.2)-(2.5) we see

$$\mu | N = 4\psi_0 + \psi_1 + \psi_5 + \psi_2 + \psi_4.$$ 

In particular $\psi_0$ and $\psi_2$ are not constituents of $\mu | N$. This means by (2.1) that $\mu$ has no irreducible constituent of degree 7. By (2.5), $\mu$ has no real constituent of degree 20. As $\mu$ itself is real it can have no constituent of degree 20. By (2.4), $\mu$ has no real constituent of degree 8. If there is a nonreal constituent its conjugate also appears as $\mu$ is real. This leaves a remaining constituent of degree at most 12 which is impossible as no $\chi_i$ has degree 6. This means $\mu$ is irreducible or has two constituents of degree 14.

We can now show $\chi_i$ is rational. Suppose there is some element $\sigma$ of the Galois group of $K$ such that $\chi_i \neq \chi_i$. Then $(\chi_\bar{\bar{X}})^\sigma = (\chi_i)^{(\bar{\bar{X}})^\sigma} = \chi_0 + \chi_i + \mu^\sigma$. As $\mu^\sigma$ has no constituent of degree 20, $\chi_i$ must be adjacent to $\chi_0$ on the stem. However this means $\chi_i = \chi_i^\sigma$.

We now show that $\chi_i$ is not in the principal 5-block $B_{s}(5)$. In fact we show that a character $\eta$ of degree 20 in $B_{s}(5)$ must be irrational when restricted to a 5-Sylow group $Q$. Suppose not. If $\pi$ has order 5, $\eta(\pi) = -5$. If there is an elementary abelian subgroup of $Q$ of order $5^5$ summing the character $\eta$ over the subgroup gives $(-5)(24) + 20 < 0$ giving a contradiction. If $Q$ has order $5^5$, $\eta$ has 5-defect 1 and so $\eta \in B_{s}(5)$. This gives the result as there are no elements of order $5^5$ in $G$. Let $B_{s}(5)$ be the 5-block containing $\chi_i$.

We will now assume $a \geq 2$. This is Case IC(i) in the flow chart. We have

$$\chi_\bar{\bar{X}} = \chi_0 + \chi_i + \mu.$$ 

Again, let $Q$ be a 5-Sylow group. As $a \geq 2$, $\chi | Q$ cannot be rational by Schur [21]. Let $\sigma$ be an element of the Galois group of a splitting field $K$ which fixes all $p$-th roots of unity for primes $p$ other than 5. Set
Here the $\zeta_i$ are irreducible characters of $G$. Let 4.2 be written in terms of $\zeta_i$ as

\[(4.4) \quad \chi \chi = \sum b_i \zeta_i.\]

We see $\chi \chi$ and $\chi^* \chi$ are equal on 5-regular element. This means for $B$ any 5-block

\[(4.5) \quad \sum_{i \in B} a_i \zeta_i(\rho) = \sum_{i \in B} b_i \zeta_i(\rho)\]

for any 5-regular $\rho$.

We apply (4.5) with $B = B_5(5)$, the 5-block containing $\chi_i$. The character $\chi_i$ appears with multiplicity one on the right hand side. There is possibly one second character of degree 14 appearing with no zero coefficient on the right hand side. The degree is therefore 20 or 34. In particular it is congruent to $-1 \pmod{7}$. This means the left hand side must contain a character $\chi_j$ of degree 20 or 27. As (4.3) is a sum of principal 7-indecomposables there is a character $\chi_k$ whose degree is congruent to 1 $\pmod{7}$. Its degree must be 8 or 15. Also $\chi_j$ and $\chi_k$ are adjacent on the tree.

Suppose $x_k = 8$. It follows from the discussion in the above paragraph that $\chi_k \notin B_5(5)$. As $N$ is 5-regular we may use (4.5), (4.1), and (2.4) to see $\chi_k$ cannot be real. If $\chi_k | Q$ is not rational where $Q$ is a 5-Sylow group of $G$, then Lemma 2.1 gives four nonreal conjugates of $\chi_j$. The degree equation for $B_5(7)$ must now be

\[1 + 8 + 8 + 8 + 8 = 20 + 13.\]

This is impossible and so $\chi_k | Q$ is rational. By Schur's theorem [21], $a \leq 2$. As we are assuming $a \geq 2$ we have $a = 2$. Also $B_5(5)$ is of defect 1 and so the right hand side of (4.5) is $\chi_i$. This means $\chi_i$ has degree 20 and $\chi_i = \chi_j$ for 5-regular elements.

Suppose $j \neq 1$ and so $\chi_j \neq \chi_i$. If $\chi_j$ is not real the degree equation becomes

\[1 + 8 + 8 + 43 = 20 + 20 + 20\]

which is impossible. If $\chi_j$ is real it is on the stem. The stem has 5 characters giving

\[
\begin{align*}
\chi_k & \quad \chi_i \quad \chi_j \quad \chi_l \\
\bullet & \quad \circ \quad \bullet \quad \circ \\
& \quad \chi_j \quad \chi_i \\
& \bullet \quad \circ
\end{align*}
\]
The character $\chi_t$ has degree 8 or 15. This is impossible as $8 + 8 + 8 > 20$ and so $2x_k + x_l > x_j$. This means $j = 1$ and so $\chi_j = \chi_t$.

As $\chi_j = \chi_t$ and there are no characters with degree smaller than 8, the tree must be

![Diagram]

The 7-modular constituents of $\chi_t$ are therefore of degree 1, 8, 8, and 3. By [23] the constituent of degree 3 is realizable in GF(7). However $5^3 \not| \text{GL}(3,7)$ giving a contradiction. This means $x_k = 15$.

Suppose now $\chi_k$ has degree 15. Let $\chi_k$ be in the 5-block $B_5(5)$. We assume first $B_5(5) \neq B_5(5)$. Apply equation (4.5) with $B = B_5(5)$. A character of degree 15 cannot be fitted into the sum (4.5) over $B_5(5)$ as that sum is of degree 20 or 34. This means $B_5(5) \neq B_5(5)$.

The possible sums of degree over $B_5(5)$ on the right of (4.5) are 14 and 28. However 14 < 15. If the sum is 28, (4.3) must have a linear constituent giving a contradiction. This means $\chi_k \in B_5(5)$.

Because $\chi_k \in B_5(5)$, $\chi_k(\pi)$ must be irrational for any element $\pi$ of order 5. This means $\chi_k | Q$ is irrational where $Q$ is a 5-Sylow group of $G$. If $\chi_i$ is not real Lemma 2.1 gives four nonreal conjugates of $\chi_k$. The degree equation is impossible. Therefore $\chi_k$ is real. If $\chi_k^\sigma$ is a conjugate of $\chi_k$ a similar argument with $\chi^\sigma$ shows $\chi_k^\sigma$ is real. There is at least one such $\chi_i \neq \chi_k$.

Assume first $\chi_j = \chi_t$. As $\chi_k$ and $\chi_k^\sigma$ are real they are on the stem. The stem contains at least five characters. Also $\chi_k$ is adjacent to $\chi_t$ on the stem. This forces a branch at $\chi_i$. The tree must be

![Diagram]

Clearly $x_i = 8$ and $G$ has a 7-modular representation of degree 3. This contradicts $a \geq 2$ and implies $\chi_j \neq \chi_t$.

We now eliminate this case using the tree and the degree equation. The tree has at least 5 real characters as $\chi_0, \chi_t, \chi_j, \chi_k, \chi_k^\sigma$ are all real. A branch at $\chi_i$ implies a 7-modular character of degree 3 which is a contradiction. This means the character $\chi_t$ adjacent to $\chi_i$ other than $\chi_0$ must have degree at least 19 and so cannot be $\chi_k$ or $\chi_k^\sigma$. In
turn the stem must have 7 characters. If \( \chi_j \) has degree 27 the configuration

\[
\begin{array}{ccc}
\chi^i & \chi^j & \chi^k \\
3 & 12 & 15
\end{array}
\]

implies \( \chi^i \) has a 7-modular constituent of degree 3 again giving a contradiction as \( a \geq 2 \) [23]. It follows that \( x_j = 20 \). The degree equation is

\[
1 + 15 + 15 + x_i = 20 + 20 + x_m
\]

where \( x_m \) is the character adjacent to \( \chi^i \) above. As \( \chi_i \) is a constituent of \( \chi_i \chi_i = \chi_i \chi_i \) its degree is at most \( 20 \cdot 21/2 \). The only such degrees are 36, 50, 64, 120, and 162. There is only one solution

\[
1 + 15 + 15 + 36 = 20 + 20 + 27
\]

This is eliminated by 5-block separation using \( B_2(5) \). We have completed all cases where \( a \geq 2 \) which is part IC(i) of the flow chart.

We now consider the case in which \( a = 1 \). This is IC(ii) of the flow chart. We use the results (2.6) and (2.7). The case is eliminated by a careful examination of the decompositions of \( \chi \chi \) and \( \chi \chi \) and their restrictions to \( C(\pi) \). The results of [2] provide contradictions for each of the possibilities for decompositions of \( \chi \chi \) and \( \chi \chi \).

We know from (2.6) and (2.7) that \( \chi \) is 5-rational, that is \( \chi \) lies in the field of \( g/5\text{th} \) roots of unity. This can also be shown using (4.5). We know that \( \chi \chi = \chi_o + \chi_i + \mu \). From (4.1) we know that \( \mu \) is either irreducible or has two constituents \( \mu_1 \) and \( \mu_2 \) of degree 14. Also (4.1) shows \( \mu_i \mid N \neq \mu_2 \mid N \). As \( \chi \) is 5-rational so is \( \mu \). As \( N \) consists of 5-regular elements, \( \mu_1 \) and \( \mu_2 \) are 5-rational also when \( \mu \) is reducible. This means that the constituents of \( \chi \chi \) of full 5-defect are all 5-rational and consequently nonexceptional for \( p = 5 \). [2].

Let \( S_o \) be the character of \( \langle \pi \rangle \) defined by \( S_0(e) = 5, S_0(\pi) = 0 \). In case (2.6) let \( \gamma = \theta \), in case (2.7) let \( \gamma = \varphi_1 + \varphi_2 \). Then by (2.6) or (2.7), \( \chi \mid C(\pi) = \gamma + \varphi S_\sigma \). Let \( \gamma \xi = \xi_o + \xi_i \) where \( \xi_o \) is the trivial character of \( C(\pi) \). This means

\[
(4.6) \quad \chi \chi \mid C(\pi) = \xi_o + \xi_i + \gamma \varphi S_o + \varphi \xi_o + 5\varphi \varphi S_o.
\]

Assume now \( \mu \) is reducible. By [2, II, Th. 1] set \( \mu_i \mid C(\pi) = \pm \psi_i + S_\sigma \) where \( \psi_i \) is a sum of \( \tau_i \) irreducible characters \( \theta_i, j = 1, 2, \ldots, \tau_i \) containing \( \pi \) in their kernel. The \( S_\sigma \) are \( \gamma_i S_o \) where \( \gamma_i \) is a character of \( V \). The \( \theta_i, j = 1, 2, \ldots, \tau_i \) are conjugate in \( N(\pi) \) and so \( \tau_i = 1, 2, 4 \). Using 4.6 and 4.2 we see \( \pm \psi_1 \pm \psi_2 = \xi_i \). If \( \psi_1 \neq \psi_2 \) both signs must be plus. Interchanging \( \psi_1 \) with \( \psi_2 \) if necessary we may assume \( \psi_1 \) has degree 1, \( \tau_1 = 1 \). But then degree \( \mu_i \equiv 1 \) (mod 5) giving a contradiction. If \( \psi_1 = \psi_2 \) then both signs must be equal.
As $\xi_1$ has odd degree this is impossible.

This contradiction implies that $\mu$ must be irreducible. We may set $\mu | C(\pi) = \pm \psi + S$ as before. This time $\psi = \xi_1$. As $\psi$ is a sum of $\tau$ irreducible characters $\theta^i$ of $C(\pi)$ with the same degree and $\tau = 1, 2, 4$, we see $\tau = 1$ and $\theta^i$ has degree 3. In particular $V$ is nonabelian and so case (2.6) holds. This gives $\theta \bar{\theta} = \xi_0 + \theta^i$.

The same technique can be applied to $\chi \chi$. We have first that $\chi \chi = P_2(\chi) + C_2(\chi)$. Using (2.2), (2.3), (2.4) and (2.5) it can be seen that the constituents of $\chi \chi$ are all of zero 7-defect. As $\chi \chi$ has three distinct irreducible constituents so does $\chi \chi$. Using (2.2) and (2.3) we see that either $P_2(\chi)$ has two distinct constituents of degree 14 and $C_2(\chi)$ is irreducible or $P_2(\chi)$ is irreducible and $C_2(\chi)$ has constituents of degrees 14 and 7. Let these be $\eta_1, \eta_2, \eta_3$. We may again check using (2.2), (2.3), (2.4) that they are 5-rational.

We also have $\theta \bar{\theta} = P_2(\theta) + C_2(\theta)$. Also $P_2(\theta)$ and $C_2(\theta)$ are irreducible as $\theta \bar{\theta} = \xi_0 + \theta^i$. As before we may set $\psi_i | C(\pi) = \pm \psi_i + S_i$ where the $\psi_i$ are sums of characters conjugate in $N(\pi)$ each with $\pi$ in the kernel. We have

$$\pm \psi_1 \pm \psi_2 \pm \psi_3 = P_2(\theta) + C_2(\theta).$$

If the $\psi_i$, $i = 1, 2, 3$ are all distinct there would be three constituents on the right. It is easy to see that we may assume $\psi_1 = \psi_2$ and the signs are opposite. Therefore $\psi_3 = P_2(\theta) + C_2(\theta)$ which is impossible as $P_2(\theta)$ is irreducible of degree 3 and $\psi_3$ is a sum of irreducible characters of the same degree. This contradiction finishes this section.

5. The case $\chi \chi = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 27$, $g = 7 \cdot 5^s \cdot 3^t \cdot 2^r$. In this section we consider the case ID where $\chi \chi = \chi_0 + \chi_1 + \cdots$, $\deg \chi_1 = 27$. Here $s = 6$. There is exactly one group of this form $S_6(2)$ of order $7 \cdot 5^s \cdot 3^t \cdot 2^r$. Cases are eliminated by first showing $\chi$ is rational when restricted to a 3-Sylow group. This is done much as in § 4 where it was shown that $\chi$ restricted to a 5-Sylow group was rational. Here the character $\chi_1$ of degree 27 cannot be in the principal 3-block by [8]. We can therefore use relations like (4.5) for the 3-block containing $\chi_1$. Once it is known that $\chi$ restricted to a 3-Sylow group is rational, the value of $b$ is at most 4 by Schur [21]. As $\chi_1$ is of degree 27, $b \geq 3$. The two cases $b = 3$ and $b = 4$ are treated separately. For $b = 3$, the generalized decomposition numbers for $\chi$ on $C(\tau^2)$ are examined where $\tau^2$ is of order 3 and normalizes a 7-Sylow group. These lead to a contradiction. For $b = 4$, the 3-Sylow group is determined explicitly. The character $\chi_1$ is of 3-defect 1. The various possibilities for the tree are eliminated except of course the one
leading to $S_6(2)$. In this case it is shown there is an involution $J$ for which $\chi(J) = -5$. Adjoining $-I$ to the matrices $X(G)$ gives a group generated by reflections. These groups are all known and we obtain $S_6(2)$.

The analysis here is much longer than in preceding sections and we do not give all the details. Where arguments are similar to earlier arguments they are not repeated. In eliminating cases, the various known techniques involving block separation, cyclic defect groups, etc., are used implicitly. Consequently, only the most troublesome cases are treated.

Let $\chi = \chi_0 + \chi_1 + \mu$. If $\mu$ contained a constituent of 7-defect 1 its degree would by 6, 8, or 20. We have ruled out a degree 6 in § 2. As $\mu$ has degree 21, $\mu$ has no constituents of 7-defect 1. As in § 4, $\chi_0 + \chi_1$ is a principal 7-indecomposable and so $\chi_1$ adjoins $\chi_0$ on the stem. In particular $\chi_1$ is real. As in § 4, $\chi_1$ is rational.

We now consider the case in which $\chi \mid S$ is not rational. Here $S$ is a 3-Sylow group of $G$. This is part ID(i) of the flow chart. Let $B_3(3)$ be the 3-block containing $\chi$. By [8] it is not of full defect. This means that $B_3(3) \neq B_3(3)$ where $B_3(3)$ is the principal 3-block. Also the degrees of the characters in $B_3(3)$ are all divisible by 3. Let $\sigma$ be an element of the Galois group of $K$ which fixes all roots of unity except 3rd roots of unity and for which $\chi \mid S \neq \chi \mid S$. This will mean $\chi^\sigma \chi$ and $\chi \chi$ are equal on 3-regular elements. Let

$$\chi^\sigma \chi = \sum b_i \zeta_i \tag{5.2}$$

Let $\mu = \sum c_i \zeta_i$. We have for 3-regular elements $\rho$

$$\chi_i(\rho) + \sum c_i \zeta_i(\rho) = \sum b_i \zeta_i(\rho) \tag{5.3}$$

where the sum $\sum b_i$ is taken only over the characters $\zeta_i$ in $B_3(3)$. If some $c_i \neq 0$ appearing in (5.3) the degree $\zeta_i$ must be 21 as such degrees must be divisible by 3. This means some constituent of (5.2) is linear giving a contradiction. Therefore the degree of $\sum b_i \zeta_i$ is 27. There must be a $b_i \neq 0$ appearing in (5.3) for which the degree is congruent to $-1 \pmod{7}$. It can only be 27. This gives for 3-regular elements $\rho$

$$\chi_i(\rho) = \zeta_i(\rho) \tag{5.3}'$$

where $\zeta_i$ is an irreducible character of $G$ in $B_3(3)$. As in § 4 there are two cases (i) $\zeta_i = \chi_i$ and (ii) $\zeta_i \neq \chi_i$.

In either case there must be exactly one further character $\zeta_\xi$ in $B_3(7)$ appearing in (5.2). Its degree must be 8 or 15. As $\chi^\sigma \chi$ must be a sum of principal indecomposables $\zeta_i + \zeta_\xi$ must be a sum of
principal indecomposables and so $ζ_2$ is adjacent to $ζ_1$ on the tree.

It is now possible to eliminate each of these cases by careful analysis of the tree. The method is routine using block separation, properties of the tree, and the degree equation. Lemma 2.1 and Schur’s theorem [21] are used when $ζ_2$ is of degree 8 and nonreal. We do not give any further details of this enumeration.

We now treat the cases in which $χ | S$ is rational. Again $S$ is a 3-Sylow group. By Schur’s theorem [21] the order of $S$, 3^3, is at most 3^3. As $χ_i$ has degree 27 we have either $b = 3$ or 4. We will first treat the case $b = 3$, ID (ii) of the flow chart. We then treat the case $b = 4$, ID (iii) of the flow chart.

Assume then that $|S| = 3^3$. We will show first that $S$ is nonabelian. Suppose first $S$ is abelian. Let $χ | S = \sum \lambda_i$, where the $\lambda_i$ are linear characters of $S$. Suppose there is an element $T$ of order 9 in $S$. There must be an $i$ for which $\lambda_i(T)$ is a primitive 9-th root of unity. As $χ$ is rational all six conjugates must appear amongst the $\lambda_i$, $i = 1, 2, \cdots , 7$. For the remaining character $\lambda_j$ we have $\lambda_j(T) = 1$ by the unimodularity of $X$. There can be no element of $S$ independent from $T$. As there are no elements of order 27 in $G$ [5, 3B] we see $|S| \leq 9$. Therefore $S$ must be elementary abelian. We can write $χ | S = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6$ where $\lambda_6$ is the trivial character. We know $χ_i$ of degree 27 appears as a constituent in $χ_S$. As $χ_i | S$ is the character of the regular representation of $S$, we see all linear characters of $S$ appear as constituents of $χ_S | S$. Checking with the characters $\lambda_1, \lambda_2, \lambda_3$ shows that $χ_S | S$ cannot contain 27 distinct linear characters as constituents. This means $S$ is nonabelian.

As $χ | S$ is faithful it must contain a nonlinear constituent. As $S$ is a 3-group its degree must be 3. Let $μ$ be this nonlinear character and let $U$ be the representation corresponding to it. Here $U$ must be faithful as any proper quotient group of a group of order 27 is abelian. There is an element $R$ in $Z(S)$ for which $μ(R) = 3u$ where $u = e^{2πi/3}$. Here $U(R) = uI_3$ where $I_3$ is the $3 \times 3$ identity matrix. This means $μ$ is nonreal. As $χ | S$ is rational $μ$ must also appear as a constituent. We see $χ | S = μ + \bar{μ} + \lambda_1$ where $\lambda_1$ is linear.

The constituents of $χ | S$ are all distinct and so $|C(S)|$ is divisible by the primes 3 and 7 only by [5, 3F]. As $7 \nmid |C(S)|$, we see $C(S) = Z(Q)$. This means that the principal 3-block $B_3(3)$ is the only 3-block of full defect [3, 16D]. We know $Z(S)$ has order 3. Let $⟨R⟩ = Z(S)$. Then $χ(R) = 3u + 3\bar{u} + \lambda_1(R) = -3 + \lambda_1(R)$. Clearly $\lambda_1(R) = 1$ and so $χ(R) = -2$. In particular, if $S$ is a 3-element and $χ(S) \neq -2$, then $S$ is not in the center of any 3-Sylow group and so $3^3 \nmid |C(S)|$.

We will apply this to the element $τ^2$ of order 3 given in Table
I, §2. Here $\tau^2 \in N(\xi)$ where $\xi$ is an element of order 7. We may assume $\tau^2 \in S$. By (2.1), $\chi(\tau^2) = 1$ and so $3^3 \nmid |C(\tau^2)|$. Let $|C(\tau^2)| = 3^3 \cdot c_0$ where $3 \nmid c_0$. Here $|C(\tau^2)/\langle \tau^2 \rangle| = 3c_0$. For the sake of simplicity we will replace $\tau^2$ with $T$. This case will be eliminated by considering the generalized decomposition numbers for $T$.

Let $C = C(\tau^2)$, $\bar{C} = C/<\tau^2>$. We know $|C| = 3^3 \cdot c_0$, $|\bar{C}| = 3c_0$. If $b$ is a 3-block of full defect of $C$ there is a corresponding block $\bar{b}$ of full defect for $\bar{C}$. The modular characters of $b$ all have $T$ in their kernel and can be considered as modular characters of the block $\bar{b}$ of $\bar{C}$. If $\bar{C}_i$ is the Cartan matrix for $\bar{b}$, the Cartan matrix, $C_i$, for $b$ is $3\bar{C}_i$, [4, p. 154].

Any 3-block of full defect of $\bar{C}$ has a cyclic defect group of order 3. The theory of such defect blocks can readily be applied [2]. There are two cases, (a) and (b). In case (a) there is one modular character and three ordinary characters of the same degree. The Cartan matrix is (3). In case (b) there are two modular characters and three ordinary characters. If $f_1$ and $f_2$ are the degrees of the modular characters, the degrees of the ordinary characters are $f_1 + f_2$, $f_1$. Also $f_1 \equiv f_2 \pmod{3}$. The Cartan matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

We apply these results to the principal 3-block $b_0(3)$ of $\bar{C}$. We first show case (a) is impossible. Suppose, then, $\bar{C}$ had one modular character in $b_0(3)$. Then the principal 3-block $b_0(3)$ of $C$ has one modular character which is of course the trivial character $\varphi_0$. We know that $\chi \in B_0(3)$ as there is only one 3-block of full defect. Results on generalized decomposition numbers in [3, 4] show $\chi(TS) = d_0\varphi_0(S) = d$ where $S \in C(T)$ and $S$ is 3-regular. We may pick $J = \tau^i$ of order 2. From (2.1) we see $\chi(T) = d = 1$. Therefore $d = 1$. However from (2.1) we see also $\chi(TJ) = -1$. This would mean $d = -1$ giving a contradiction. This shows case (a) does not occur.

This means case (b) occurs. There are two modular constituents of $C(T)$ in $b_0(3)$. One is $\varphi_0$. Let $\varphi_1$ be the second. Let $J = \tau^i$. If $d_0$ and $d_1$ are the decomposition numbers for $\chi$ we obtain

$$\begin{align*}
\chi(T) &= d_0 + d_1\varphi_1(e) = 1 \\
\chi(TJ) &= d_0 + d_1\varphi_1(J) = -1.
\end{align*}$$

(5.4)

Subtracting we find $d_0(\varphi_1(e) - \varphi_1(J)) = 2$. As $J$ is an involution $\varphi_1(e) - \varphi_1(J)$ is an even integer. Therefore $d_1 = 1$, $\varphi_1(e) - \varphi_1(J) = 2$. Equations (5.4) become

$$\begin{align*}
d_0 + \varphi_1(e) &= 1 \\
d_0 + \varphi_1(J) &= -1.
\end{align*}$$

(5.4')

This means $d_0$ is a rational integer. As $d_0 d_0 \leq 6$ and $\varphi_1(e) \equiv 1 \pmod{3}$
we see \( \varphi(e) = 1, \ d_0 = 0 \). This shows that for any character \( \chi \) of
degree 7 the decomposition numbers are \( d_0 = 0, \ d_1 = 1 \).

We are now in a position to analyze the decomposition matrix if
there are at least four characters of degree 7. Suppose then that
there are four characters of degree 7. Let \( D \) be the nonzero rows
of the decomposition matrix of \( B_s(3) \) with respect to \( T \). We know
\( \text{tr} D = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} \). The entries of \( D \) are in \( \mathbb{Z}[\rho] \) where \( \rho = e^{2\pi i/3} \). We
let the first column correspond to \( \varphi_0 \), the second to \( \varphi_1 \). By a small
amount of trial and error, we find there is one possibility to within
permutations of the rows and changes in sign. This is

\[
D = \begin{bmatrix}
1 & 0 \\
\pm 1 & \pm 1 \\
\pm 2 & \pm 1 \\
0 & 1 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\]

This shows there is exactly one character whose degree is not
congruent to zero (mod 3) other than \( \chi_0 \) and the four characters of
degree 7. This is the character corresponding to the second row.
As \( B_s(3) \) is the only 3-block of full defect, these six characters are
the only ones whose degrees are not divisible by 3. This new
character must therefore be in \( B_s(7) \) or the degree equation could
not be satisfied.

We are now in a position to obtain a contradiction if \( a \geq 2 \) where
\( g = 7 \cdot 5^a \cdot 3^t \cdot 2^u \). Certainly if \( a \geq 2 \), \( \chi \mid Q \) is irrational by Schur’s
theorem. Here \( Q \) is a 5-Sylow group of \( G \). Suppose \( \chi \) is not real.
By Lemma 2.1 there are at least four conjugates of \( \chi \). The conclu-
sions of the above paragraph apply. Let \( P_i(\chi) = \sum a_i \eta_i \) where \( \eta_i \) are
irreducible characters of \( G \), \( P_i(\chi) \) is the character corresponding to
the symmetric tensors of rank two for \( X \). As \( \chi \neq \overline{\chi} \) none of the \( \eta_i \)
are linear. By (2.3) no character of degree 7 can be a constituent of
\( P_i(\chi) \). As 28 is not divisible by 3, \( P_i(\chi) \) is reducible by the above
paragraph. In fact it is impossible to write \( P_i(\chi) \) as a sum of
characters satisfying the above paragraph. This means \( \chi \mid Q \) is real.

Suppose \( a \geq 3 \) where \( g = 7 \cdot 5^a \cdot 3^t \cdot 2^u \). By [25, Th. 3.1], \( \chi \mid Q \) is
not real. The above paragraph applies and eliminates this case.
Therefore \( a \leq 2 \). Sylow’s theorem gives \( a = 2 \) or 0.

We first treat the case \( a = 2 \). By the paragraphs above we
know \( \chi \) is real. Therefore \( \chi \overline{\chi} = \chi \chi = P_i(\chi) + C_i(\chi) \) where \( P_i(\chi) \) is the
character corresponding to the symmetric tensors of rank two. As
χ₁ is a constituent of χχ we see by considering degrees that \( P_2(χ) = χ₀ + χ₁ \). As χ₀ and χ₁ are rational this means \( P_2(χ) \) is rational. Using the formula for \( P_2(χ) \) we have that \( χ²(R) + χ(R²) \) is rational for any \( R \in G \). We prove a lemma regarding this situation. We assume \( χ \mid Q \) is real.

**Lemma 5.1.** If \( P_2(χ) \mid Q \) is rational, \( Q \) is cyclic. Here \( Q \) is the 5-Sylow group of \( G \), \( P_2(χ) \) is the character associated with the symmetric tensors.

**Proof.** Let \( π \) be an element of order 5. Let \( σ \) be an automorphism of \( R[λ] \) mapping \( λ \) to \( λ^3 \) where \( λ = e^{πi/5} \), \( R \) — rationals. We have \( χ^σ(π) = χ(π^2), \ χ^σ(π) = χ(π) \). Also

\[
(χ^σ(π) + χ(π^2))^σ = χ^σ(π) + χ(π^2),
\]

\[
(χ^σ(π) + χ(π)) = χ^σ(π) + χ(π),
\]

\[
(χ^σ - χ)(χ^σ + χ - 1)(π) = 0.
\]

This implies \( χ^σ(π) = χ(π) \) or \( (χ^σ + χ - 1)(π) = 0 \). Assume first \( (χ^σ + χ)(π) = 1 \). This will be true also for \( π, π^2, π^3, π^4 \). Therefore

\[
(χ^σ + χ)(π + π^2 + π^3 + π^4 + e)
\]

is 18 giving a contradiction. Therefore \( χ^σ(π) = χ(π) \). In particular \( χ \) is rational when restricted to elements or order 5 in \( Q \). As \( Q \) is abelian it must be cyclic by Schur’s theorem [21]. This completes the proof of the lemma.

As there are no elements of order 125, \( |Q| ≤ 25 \). Let \( π \) be a generator such that \( (π)^3 = π \). We know \( χ(π) = 2 \). Suppose

\[
χ \mid Q = λ₁ + λ₂ + λ₃ + λ₄ + λ₅,
\]

where the \( λᵢ \) are linear characters of \( Q \), \( λ₅ \) is the trivial character. Only two of \( λ₁, λ₂, λ₃ \) can represent \( π \), by a primitive 25-th root of unity. This means there are at least five conjugates of \( χ \) contradicting the above paragraphs.

We have shown then that \( g = 7 \cdot 3^4 \cdot 2^e \). Sylow’s theorem gives \( e = 3, 6, 9 \). The cases 3, 6 are well within the known range under 20,000 [19] and there are no simple groups with these orders. If \( e = 9 \) we again mention [6, 1H] which shows \( 2^e ≤ 12 \cdot 3^4 = 324 \).

This eliminates all cases for which \( b = 3 \), Case ID(ii) of the flow chart. We begin now Case ID(iii) of the flow chart in which \( b = 4 \). We show there is exactly one group of this form \( S₄(2) \).

We are considering groups whose orders are \( 7 \cdot 5 \cdot 3^4 \cdot 2^e \). Sylow’s theorem gives \( a \equiv 1 \pmod{2} \). As \( a ≤ 6 \) we have \( a = 1, 3, \text{ or } 5 \). We will first show that \( a ≠ 5 \).
Suppose then that \( a = 5 \). Let \( Q \) be a 5-Sylow group. We know from [25, Th. 2.1] that \( Q \) is abelian. By [25, Corollary 2.8] an elementary abelian subgroup of \( Q \) has order at most \( 5^4 \). In particular \( Q \) is not elementary abelian. Let \( \pi \) be an element of order 25.

We know \( \chi_i \) is a rational character of degree 27. Let

\[
\chi_i \mid Q = \sum_{i=1}^{27} \lambda_i
\]

where the \( \lambda_i \), \( i = 1, 2, \cdots, 27 \) are linear characters of \( Q \). As \( \chi_i \) is faithful there is an \( i \) such that \( \lambda_i(\pi) \) is a primitive 25th root of 1. Each of the 20 conjugates of \( \lambda_i \) must appear as constituents in \( \chi_i \mid Q \) as \( \chi_i \) is rational. Let \( K \) be the kernel of \( \lambda_i \). As there are no elements of order 125 in \( Q \), \( K \) has order 125 and \( Q \cong K \times \langle \pi \rangle \).

Clearly \( K \) is in the kernel of each of the conjugates of \( \lambda_i \). If \( \lambda_1, \ldots, \lambda_7 \) are the characters \( \lambda_i \) not conjugate to \( \lambda_i \) we have \( y = \sum_{i=1}^{7} \lambda_i \) a faithful rational representation of \( K \). As \( K \) has order \( 5^4 \) this is impossible by Schur's theorem [21]. We have proved that \( a \neq 5 \) and so \( a \) must be 1 or 3.

The character \( \chi_i \) of degree 27 is in a 3-block of defect 1. Let \( B_i(3) \) be the 3-block containing \( \chi_i \). As the defect group is cyclic of order 3 we may apply results in [13]. There are exactly three characters in \( B_i(3) \). The three characters in \( B_i(3) \) may all be of degree 27, in which case two are nonreal, or there will be two degrees \( y_1, y_2 \) besides 27 such that \( 27 + y_1 = y_2 \). The degrees \( y_1 \) and \( y_2 \) will be divisible by \( 3^4 \) but not \( 3^4 \). By checking the various possibilities for the degrees of representations of the groups we are considering we find exactly one possibility. This is \( y_1 = 189, y_2 = 216 \).

Block separation has been used in this elimination. This means that \( B_i(7) \) contains characters with degrees 1, 27, 216 or characters with degrees 1, 27, 27, 27. In the latter case the tree has a branch.

It is now possible to eliminate all but eleven possible degree equations by straightforward techniques as described earlier. We do not include the details but list the degree equations not eliminated.

1. \( 1 + 15 + 120 + 512 = 27 + 216 + 405 \)
2. \( 1 + 64 + 36 + 162 = 27 + 216 + 20 \)
3. \( 1 + 120 + 120 + 162 = 27 + 216 + 160 \)
4. \( 1 + 960 + 120 + 162 = 27 + 216 + 1000 \)
5. \( 1 + 64 + 64 + 162 = 48 + 27 + 216 \)
6. \( 1 + 8 + 120 + 162 = 27 + 216 + 48 \)
7. \( 1 + 512 + 288 + 162 = 27 + 216 + 720 \)
8. \( 1 + 120 + 162 = 27 + 216 + 20 + 20 \)
9. \( 1 + 120 = 27 + 27 + 27 + 20 + 20 \)
10. \( 1 + 120 + 120 = 27 + 27 + 27 + 160 \)
11. \( 1 + 960 + 120 = 27 + 27 + 27 + 1000 \).
We now separate the cases $g = 7 \cdot 5 \cdot 3^4 \cdot 2^c$ and $g = 7 \cdot 5^2 \cdot 3^4 \cdot 2^c$. We begin with $g = 7 \cdot 5 \cdot 3^4 \cdot 2^c$ and show that all degree equations are impossible except (1) and that this leads to $S_6(2)$.

By Sylow's theorem $c = 3, 6, 9$. The only possible degree equations for $c = 3$ are (8) and (9) as the remaining equations contain degrees divisible by 16. Equations (8) and (9) can be eliminated using the two characters of degree 20 which must be in a 2-block of defect 1. However on the tree they are off the stem and so they must be complex conjugates. This is impossible. This leaves the two cases $g = 7 \cdot 5 \cdot 3^4 \cdot 2^c$ and $g = 7 \cdot 5^2 \cdot 3^4 \cdot 2^c$. We will do the latter case completely to obtain $S_6(2)$. The case $g = 7 \cdot 5 \cdot 3^4 \cdot 2^c$ can be done similarly.

We assume then that $g = 7 \cdot 5 \cdot 3^4 \cdot 2^c$ or $7 \cdot 5^2 \cdot 3^4 \cdot 2^c$. The results and notation of §2 for groups of order $7 \cdot 5 \cdot 3^4 \cdot 2^c$ will be used, in particular equation (2.6) and (2.7). Let $\pi$ be of order 5, $C(\pi) = \langle \tau \rangle \times V$. We know $|V| = 3 \cdot 2^c$. Let $V_3$ be a 3-Sylow group of $V$ and $V_5$ a 5-Sylow group of $V$. The restriction $\chi | V_3$ has linear constituents by (2.6), (2.7) and so $V_3$ is abelian. By (2.6), (2.7), $X | V_3$ has at most 3-distinct linear characters. Using [5, 3D] we see $|V_3| \leq 3^r$. In particular $r \leq 2$. If $V_3$ is abelian, the variety of $\chi | V_3$ is at most 3 and so $s \leq 2$. In case 2.7 this is always the case. In the case 2.7, $V_3$ at any rate has a subgroup of index two which is abelian with variety at most 3. In this case $s \leq 3$. If $V_3$ is in fact nonabelian $|V_3| = 2^s$, and $\theta(J) = -2$ where $J \in Z(V_3)$.

We also know that $\chi | V_3$ is rational. Therefore there can be no element of order 9 in $V_3$ as the variety is at most 3. Furthermore if $T$ is of order 3, the eigenvalues of $X(T)$ must be $\lambda, \bar{\lambda}, 1, 1, 1, 1$ where $\lambda = e^{2\pi i / 3^r}$. In case (2.6) then $\varphi(T) = 1$, $\theta(T) = -1$; in case (2.7), $\varphi(T) = 1$, $(\varphi_1 + \varphi_2)(T) = -1$. In particular $3^r | |V|$. If there is an involution $J$ which commutes with $T$, then $X(J)$ will have either six or two eigenvalues $-1$. In particular, the eigenvalues of $X(TJ)$ will be $1, 1, 1, 1, -\lambda, -\bar{\lambda}$ or $-1, -1, -1, -1, -1, \lambda, -\bar{\lambda}$ for $T$ or $T^s$. In each case Blichfeldt's theorem [1, p. 96] is contradicted. This means that $V$ cannot contain an element of order 6. In particular if $|V|$ is divisible by 24 this is the case. In case (2.7), $V$ is abelian and so $|V|$ is not divisible by six.

We now consider the case $g = 7 \cdot 5 \cdot 3^4 \cdot 2^c$. Let $w = |N(\pi)/C(\pi)|$. If $w = 2$ by Sylow's theorem $|V| \equiv 2$ (mod 5). This means $|V| = 2$ or 12. If $|V| = 2$, $V$ is abelian and so case (6.7) applies. However here $\varphi_1$ must equal $\varphi_2$ giving a contradiction. If $|V| = 12$, $V$ must be nonabelian and so case (2.6) applies. Therefore $V$ has an irreducible representation of degree 2 and so there is an element of order 6 in

---

1 The variety is defined in [1] or in [24, Introduction].
This shows \( w \neq 2 \).

Suppose \( g = 7 \cdot 5 \cdot 3 \cdot 2^4 \) and \( w = 4 \). In this case Sylow's theorem gives \( |V| \equiv 1 \) (mod 5). This shows \( |V| = 1 \) or 6. If \( |V| = 1 \) there would be one 5-block of full defect and so no character of degree 7. Therefore \( |V| = 6 \). As there can be no element of order 6 in \( V \), \( V \) must be isomorphic to \( S_3 \) as there are only two nonisomorphic groups of order six. Case (2.6) applies.

The group \( V \) is generated by an element \( T \) of order 3 and an involution \( J \) such that \( J T J = T^{-1} \). There are three irreducible characters of \( V \); \( \eta_0, \eta_1, \theta \). Here \( \eta_0 \) is the trivial character; \( \eta_1(T) = 1 \), \( \eta_1(J) = -1 \), \( \eta_1(e) = 1 \); \( \theta(T) = -1 \), \( \theta(J) = 0 \), \( \theta(e) = 2 \). There are therefore three 5-blocks of full defect: \( B_0(5) \) corresponding to \( \eta_0 \), \( B_1(5) \) corresponding to \( \eta_1 \), and \( B_2(5) \) corresponding to \( \theta \). In each case \( e = 4 \) and so each block consists of 5 ordinary characters. The degrees of characters in \( B_0(5) \) and \( B_1(5) \) are congruent to \( \pm 1 \) (mod 5). In \( B_2(5) \) they are congruent to \( \pm 2 \) (mod 5).

Suppose there is a character, say \( \chi_2 \), of degree 162. Then \( \chi_2 \) is in \( B_2(5) \) and so \( \chi_2(\pi T) = \pm 1 \). As \( \chi_2 \) is of full 3-defect this is a contradiction [7, p. 579]. This shows that cases 2, 3, \( \cdots \), 8 are impossible.

The cases 9, 10, 11 can be eliminated by block separation as \( \chi_0 \) is the only possible character in \( B_0(7) \cap B_0(5) \).

This leaves (1) as the only possible degree equation remaining. The degrees in \( B_2(5) \) are so far 7, 27, 512. If \( \chi \) is not rational the degree equation would be \( 7 + 7 + 27 + 512 = 553 = 7 \cdot 79 \) which is absurd. Therefore \( \chi \) is rational.

It is now possible to show \( G \cong S_6(2) \). Let \( J \) be an involution in \( V \). As \( \theta(J) = 0 \), \( \varphi(J) = -1 \) by the unimodularity of \( X(J) \). Therefore \( X(J) \) has six eigenvalues \(-1\) and one eigenvalue 1. As \( G \) is simple the conjugates of \( J \) generate \( G \).

We consider the group \( \tilde{G} = G \times Z_2 \) and a representation \( \tilde{X} \) of \( \tilde{G} \) given by \( \tilde{X}(a, b) = X(a)\eta(b) \) where \( a \in G \), \( b \in Z_2 \). Here \( \eta \) is the nontrivial character of \( Z_2 \), \( Z_2 \) is the group of order 2. As \( \tilde{X} \) is rational and of odd degree it can be written in the real field [22]. We may assume the matrices are orthogonal. If \( Z_2 = \langle J \rangle \), \( \tilde{X}(JJ) \) is a reflection in \( R^7 \). The same is true of any conjugate. The group generated by these conjugates is \( \tilde{G} \) as can be quickly checked. This group is then a group generated by reflections in \( R^7 \). These groups have all been classified as Weyl groups of certain Dynkin diagrams containing seven elements [10, 11, 26]. These are \( A_7, B_7, D_2, E_7 \). The only group with the correct order is the Weyl group of \( E_7 \). It is known that the Weyl group of \( E_7 \) has a subgroup of index 2 isomorphic to \( S_6(2) \) and that it has a complex irreducible representation of degree 7 [14].
The case \( g = 7 \cdot 5 \cdot 3^4 \cdot 2^6 \) can be handled similarly. All degree equations can be eliminated.

We now proceed to the case \( g = 7 \cdot 5^3 \cdot 3^4 \cdot 2^2 \). Sylow's theorem gives \( c = 4, 7, 10 \). The cases 1, 2, 6, 7, 8, 9 can be eliminated by a routine use of the established techniques. For example in cases 1 and 7 the character of degree 512 implies \( c = 10 \). However the character of degree 512 is then of 2-defect 1 and so must occur with multiplicity two. In cases 2, 8, and 9 the characters of degree 20 cannot be in \( B_c(5) \) and the cases are eliminated by 5-block separation. In case 6 there is a rational representation of degree 8 contradicting [21].

The remaining cases 3, 4, 5, 10, 11 will be eliminated by examining a 3-Sylow group of \( G \). It will be shown there is a self centralizing element \( \pi_9 \) of order 9. As \( 5^5 \mid g \) we know that \( \chi \mid P_8 \) is not real by [25, Th. 3.1]. By Lemma 2.1 there are at least four conjugates of \( \chi \). The cases can be eliminated by showing it is impossible to complete the \( \pi_9 \) column of the character table. The decompositions of \( \chi \bar{\chi} \) and \( \chi \chi \) together with equations (2.1)-(2.3) will be used. As no groups arise we sometimes only sketch the arguments.

We first show there is a self centralizing element of order 9. Let \( T \) generate the defect group of \( B_9(3) \). Then, as \( B_9(3) \) is not of defect 0, \( \chi_9(T) \neq 0 \) by [3]. As \( g \cdot \chi_9(T) \mid C(T) \mid 27 \) is an algebraic integer \( 3^4 \mid |C(T)| \). Therefore \( Q \) is nonabelian where \( Q \) is a 3-Sylow group of \( G \). As earlier using the fact that \( \chi \mid Q \) is rational we obtain \( \chi \mid Q = \mu + \bar{\mu} + \lambda_0 \) where \( \mu \) has degree three. The representation \( U \) corresponding to \( \mu \) can be written in monomial form.

There must be an abelian subgroup of order 27. Let \( M \) be any abelian subgroup of order 27. Let \( \chi \mid M = \sum_{i=1}^7 \zeta_i \). Suppose there is an element of order 9 in \( M \). If \( \zeta_i \) represents it faithfully all six conjugates must appear in \( \chi \mid M \). Therefore \( \zeta_i \) is faithful and \( M \) is cyclic. This is a contradiction and shows \( M \) is elementary abelian.

The matrices \( U(M) \) may be picked as all diagonal matrices of the form

\[
\begin{bmatrix}
\rho_1 & 0 & 0 \\
0 & \rho_2 & 0 \\
0 & 0 & \rho_3
\end{bmatrix}, \quad (\rho_i)^3 = 1.
\]

If \( R \notin M \) we may choose a basis so that

\[
U(R) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & C
\end{bmatrix}.
\]
As $U(R^3) = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}$ we have $C^3 = 1$. We may choose $R$ so that $C = 1$ or $\rho$ where $\rho = e^{2\pi i n}$. Picking $R$ with $C = \rho$ gives an element $\pi_9$ of order 9. The eigenvalues of $\chi(\pi_9)$ are distinct and so $|C(\pi_9)|$ is a 3-group by $[5, 3F]$ and the fact $7 \nmid |C(\pi_9)|$. If $3^3 \mid |C(\pi_9)|$ there would be an abelian subgroup of order 27 which was not elementary. This means $\pi_9$ is self centralizing. We have also shown $\chi(\pi_9) = 1$.

The value of $\chi_0$ and the four characters of degree 7 on $\pi_9$ is 1. Let $\eta_1, \ldots, \eta_4$ be the remaining characters of $G$ for which $\eta_i(\pi_9) \neq 0$. Let $\eta_i(\pi_9) = b_i$. The orthogonality relations give

\[ \sum_{i=1}^4 b_i \bar{b}_i = 4 \]
\[ \sum_{i=1}^4 b_i \eta_i(e) = -29. \]  

There must be at least one character from $B_4(7)$ amongst the $\eta_i$ and in the 5 remaining cases such degrees can be given explicitly. In case 5 the characters of degree 64 must be amongst the $\eta_i$. In cases 3 and 10 the character of degree 160 and in cases 4 and 11 the character of degree 1000 must be one of the $\eta_i$. From the tree in each case it can be seen the value is real and except for case 5 is rational and so the value $b_i$ can be obtained by its congruence mod 3.

We now show there are at least two $\eta_i$ occurring as constituents of $\chi\chi$. As $\chi\chi$ has at least three constituents the same is true of $\chi\chi$. This implies $P_2(\chi)$ or $C_2(\chi)$ must be reducible. From (2.2) and (2.3) we see that $P_3(\chi)$ has no constituents of degree 7 and $C_3(\chi)$ has at most one. If $P_3(\chi)$ is reducible there are two constituents of degree 14. As $P_3(\chi)(\pi_9)$ is 1 they are not both equal. If $C_3(\chi)$ is reducible there is one constituent of degree 7 and one of degree 14. In any case there are at least two new constituents. Their values on $\pi_9$ can be readily evaluated. In each case it is now impossible to satisfy (5.5). This completes the final case in Section ID of the flow chart.

6. The case $s = 2$. In this section we consider the case $s = 2$. There is one group $\text{PSL}(2,8)$, of order 504, of this kind, case II in Theorem I. This is section II of the flow chart. There are five characters in $B_4(7)$, $\chi_0$ the trivial character, $\chi_i$ of degree congruent to $\pm 1 \pmod{7}$ and three exceptional characters $\chi_i^*, \chi_i^0, \chi_i^5$ whose degrees are congruent to $\pm 2 \pmod{7}$. There are two possible degree equations

(a) $1 + x_i = x_i$ or (b) $1 + x_i = x_i^5$

where $x_i = \text{degree } \chi_i$, $x_i^5 = \text{degree } \chi_i^5$.

If $\chi$ is the character of degree 7 set

\[ (6.1) \chi\chi = \chi_0 + a\chi_i + b(\chi_i^* + \chi_i^0 + \chi_i^5) + \eta \]
where \( \gamma \) has constituents of zero 7-defect. Here \( a \) and \( b \) are non-negative integers. As the degree of \( \gamma \) is 49 and the degree of \( \gamma \) is divisible by 7 it is clear that either \( a \neq 0 \) and \( x_1 = -1 \) (mod 7) or \( b \neq 0 \) and \( x_2 = 2 \) (mod 7). This means that \( x_1 = 6, 20, 27, 48 \) or \( x_2 = 2, 9, 16 \). The only degree equations possible for \( G \) are then

(a) \( 1 + 5\gamma_1 = 6 \), (b) \( 1 + 1\gamma_2 = 2 \), (c) \( 1 + 8\gamma_3 = 9 \), (d) \( 1 + 15\gamma_4 = 16 \).

The first case (a) is impossible by [5] or [17]. In case (b) \( G' \) has a normal 7-complement and is not simple [5]. It can also be eliminated by [1] or [2]. In case (c) 2-block and 3-block separation imply \( g = 7\cdot 5\cdot 3^2 \cdot 2^2 \). As there is a rational character of degree 8 \( a \leq 2 \) by Schur [21]. Sylow's theorem gives \( a = 0 \). There is one simple group \( PSL(2,8) \) of order 504 [19]. It has a representation of degree 7 by [15]. One can also work out the character table quite easily.

In case (d) the character \( \gamma_1 \) of degree 15 is rational. A 5-Sylow group is abelian. Therefore \( \gamma_1 \) cannot be in \( B_0(5) \) as \( \chi(S)/15 = 1 \) (mod 5) for any 5-element \( S \). This would imply \( \chi(S) = 15 \) or \( \chi(S) = -10 \). Neither are possible in \( G \). This argument is similar to one in §4 where a character of degree 20 was involved. If \( \gamma_1 \notin B_0(5) \), 5-block separation implies \( g = 7\cdot 5\cdot 3^2 \cdot 2^2 \). However the four characters \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \) are all in \( B_0(5) \). By [2, Th. 11] there must be one further character in \( B_0(5) \) of degree 49. This is a contradiction and completes this section.

7. The case \( s = 3 \). We now consider the case \( s = 3, G = G' \). Thus \( G \) is simple by [5, 8A]. This is section III of the flow chart. In this case \( B_0(7) \) contains \( \chi_0 \), two characters \( \chi_1, \chi_2 \) whose degrees are congruent to \( \pm 1 \) (mod 7), and two exceptional characters \( \chi_3, \chi_4 \) whose degrees are congruent to \( \pm 3 \) (mod 7). If \( x_i = \text{degree} \chi_i \), \( x_0 = \text{degree} \chi_0 \) for \( i = 1, 2 \) the degree equation becomes one of

\[
\begin{align*}
(a) & \quad 1 + x_1 = x_0 + x_2 & \quad x_1 = -1 \pmod{7}, \quad x_2 = 1 \pmod{7}, \\
(b) & \quad 1 + x_0 = x_1 + x_2 & \quad x_1, x_2 = -1 \pmod{7}, \\
(c) & \quad 1 + x_1 + x_2 = x_0 & \quad x_1 = x_2 = 1 \pmod{7}.
\end{align*}
\]

(7.1)

As in §§2 and 6 we have

\[
(7.2) \quad \gamma \overline{\chi} = \chi_0 + a_1 \chi_1 + a_2 \chi_2 + b(\chi_3 + \chi_4) + \eta
\]

where again \( a_1, a_2, b \) are nonnegative integers and the constituents of \( \eta \) are of zero 7-defect. We may assume that either (i) \( a_1 \neq 0, x_1 = -1 \pmod{7} \) or (ii) \( b \neq 0, x_2 = 3 \pmod{7} \). We may also assume that \( \chi_1 \) or \( \chi_4 \) adjoins \( \chi_0 \) on the stem as \( \gamma \overline{\chi} \) is a sum of principal indecomposables [9]. The possibilities for \( x_1 \) in (i) are 6, 20, 27, 48. The possibilities for \( x_2 \) in (ii) are 3, 10, 24. It is clear that case c in (7.1)
is impossible. This means the tree contains four real characters all on the stem.

If there is a character of degree 3, then $G \cong PSL(2,7)$ by [1, 2, or 17]. Also, $PSL(2,7)$ has a character of degree 7 by [15]. Alternatively, the character table for $PSL(2,7)$ can be quickly worked out. This is Case IV of Theorem I.

We will show that when $x_1 = 6 \ G = PSL(2,7)$; when $x_1 = 20, G \cong A_8$, when $x_1 = 27, G \cong U_4(3)$. The remaining cases can all be eliminated. The methods are similar to those of earlier chapters but in general much simpler because there are only two missing degrees in the degree equation. The details will not all be given.

Suppose that $x_1 = 6$ and $\chi_1$ adjoins $\chi_0$ on the stem. This means that $\chi_1 \chi_0$ contains $\chi_2$ as a constituent in case (a) or $\chi_0$ as a constituent in case (b) by an argument involving the tree. In particular $\chi_2$ or $\chi_0$ has degree at most 21 as $(\chi_0)^2$ has constituents of degrees 15 and 21 corresponding to the symmetric and skew symmetric tensors. The possible degree equations are

$$(i) \ 1 + 8 = 6 + 3 \quad \text{and} \quad (ii) \ 1 + 15 = 6 + 10.$$  

Case (i) is again by [1 or 2], $PSL(2,7)$.

In case (ii), $\chi_1$ is in $B_0(5)$ by 5-block separation. [As $\chi_1$ is rational, $5^5 \nmid g$ and $3^5 \nmid g$ by [21]. This means $g = 7 \cdot 5 \cdot 3^2 \cdot 2^2; \ b \leq 4$. As in §2 we apply the results in [2, II, Th. 1] for the prime 5 this time with $\chi$ replaced by $\chi_1$. Let $C(\pi) = \langle \pi\rangle \times V$ where $\pi$ is a 5-element. Then $\chi_1 | V = \varphi_0 + 5\varphi_1$ where $\varphi_0$ is the trivial character of $V$, $\varphi_1$ is a linear character of $V$. It is immediate from the unimodularity of the representation corresponding to $\chi_1$ that $\varphi_1 = \varphi_0$. This means $|V| = 1$. In particular there is only one 5-block of full defect by [3] and so $\chi \in B_0(5)$. This means $B_0(5)$ contains $\chi_0, \chi_1, \chi$ and a fourth character $\chi^*$ conjugate to $\chi$.

As there are no elements of order 5·2 or 5·3 block separation applies to $B_0(5)$. If $b = 1$, $B_0(5) \cap B_0(3)$ contains three characters $\chi_0, \chi, \chi^*$. This contradicts the theory of cyclic 3-defect groups. Therefore $b \geq 2$. By block separation $B_0(5) \cap B_0(3) = B_0(5)$. In particular $\chi_1 \in B_0(3)$ and so $b \geq 3$ [12, 90.19]. Let $\pi_3$ be an element of order 3 in the center of a 3-Sylow group. As $\chi_1$ is rational, $\chi_1(\pi_3) = 3, 0, -3$. As $\chi_1 \in B_0(3)$, $\chi_1(\pi_3) = -3$. This implies $\chi(\pi_3) = \chi^*(\pi_3) = -2$. Computing $a(\pi, \pi, \pi_3)$ [6–3.1, 3.2] now gives a negative value and so a contradiction. This case is therefore impossible.

Suppose $x_0 = 10$ and $\chi_1$ adjoins $\chi_0$ on the stem. Using arguments involving the tree we see $x_1 \leq 100$. The only degree equations possible are


(i) $1 + 15 = 6 + 10$  
(ii) $1 + 36 = 10 + 27$.

Case (i) is eliminated as in the above paragraph. In case ii, $\chi_i$ is rational when restricted to a 5-Sylow group and so $5^2 \not| g$. The case $g = 7 \cdot 5^3 \cdot 3^2 \cdot 2$ is eliminated by 5-block separation on $\chi_i$ and $\chi_i^3$. The case $7 \cdot 5^3 \cdot 2^3$ is eliminated by 5-block separation with $B_5(5)$.

If $\chi_i = 48$ or $x_i = 24$ an argument similar to that of § 3 applies to give $g = 7 \cdot 3^4 \cdot 2^2$. This case can be eliminated as there must be a degree congruent to ±1 of the form $2^b$ or $3^c$. These few cases can all be easily eliminated.

There are two cases remaining, $x_i = 20$ and $x_i = 27$ with $\chi_i$ adjacent to $\chi_o$ on the stem. We can apply the same tree arguments to see that $\chi_i^3$ or $\chi_i$ has degree at most 210 if $x_i = 20$ and at most 378 if $x_i = 27$. There are only a few possibilities and all but the following three can be easily eliminated using techniques already discussed.

(i) $1 + 64 = 20 + 45$

(ii) $1 + 32 = 27 + 6$

(iii) $1 + 50 = 27 + 24$.

In case (ii) the order is $7 \cdot 5^3 \cdot 3^2 \cdot 2^2$ by block separation. As $\chi_i$ has degree 6 and is rational $a \leq 1$ by [21]. By Sylow's theorem $a = 0$. There is exactly one simple group with this order, $U_3(3)$, [19]. It is known to have a representation of degree 7. For example, a character table is given in [16].

In case (i), $\chi_i$ of degree 20 is rational. As in § 4, $\chi_i$ cannot be in $B_5(5)$. Block separation now gives $g = 7 \cdot 5^3 \cdot 3^2 \cdot 2$. This means $b = 2$ or 8. For $b = 2$ the order is 20160. There are two known simple groups of this order $A_9$ and $PSL(3,4)$. Only $A_9$ has a character of degree 7. The character table can be completed in a routine way to the character table of $A_9$. By [20], $G$ must be $A_9$. The case $b = 8$ can be eliminated by closely examining $C(\pi)$ and noting $|V| = 2$, 12, or 72.

The third case is, curiously enough, quite troublesome. We do not give full details but just sketch the argument. Block separation on the characters of degrees 50 and 27 gives $g = 7 \cdot 5^3 \cdot 3^2 \cdot 2$. Sylow's theorem gives $c = 3, 6, 9$.

If $\chi$ is real $P_3(\chi) = \chi_0 + \chi_i$ and so $P_3(\chi)$ is rational. By Lemma 5.1 a 5-Sylow group is cyclic. This case can now be eliminated using [13]. This shows $\chi$ is not real. By Lemma 2.1 there are at least four conjugates of $\chi$.

As in § 5 it can be shown that the 3-Sylow group is nonabelian. Let $T$ be an element of order 3 not in the center. The decomposition numbers for $C(T)$ can be analyzed. The analysis is not as easy as in § 5 as there is no involution inverting a 7-element. As in § 5...
there are two possible Cartan matrices (9) and $\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$. Using $\chi_0$, $\chi_2$, and the four characters of degree 7 and carefully analyzing the possible decomposition matrices this case can be eliminated.

8. The case $G \neq G'$. We now discuss the case in which $G \neq G'$. This is case IV of the flow chart. It is shown in [5] that $G'$ is a simple group and so $X(G')$ is one of the representations already obtained. The candidates for $G'$ are $\text{PSL}(2,7)$, $\text{PSL}(2,8)$, $A_8$ and $U_3(3)$. $G'$ could not be $S_6(2)$ as $|N(P)/C(P)| = 6$ in this case and so any extension would have an element in $C(\xi)$ where $\xi$ is an element of order 7. For the same reason $|G:G'| = 2$ if $G'$ is $\text{PSL}(2,7)$, $A_8$ or $U_3(3)$; $|G:G'| = 3$ if $G'$ is $\text{PSL}(2,8)$.

If $\alpha$ is an element of $G$ not in $G'$ the map $\theta_a: \xi \rightarrow \xi^\alpha$, $\xi \in G'$ is an automorphism of $G'$. As $\alpha$ cannot commute with an element of order 7, $\theta_a$ cannot be the identity automorphism. It cannot be an inner automorphism as no element $\alpha \eta$, $\eta \in G'$, can commute with an element of order 7. Therefore $\theta_a$ is an outer automorphism. The automorphisms of our groups are all known. A very readable account without proofs using the fact they are all Chevalley groups can be found in [9].

In the case of $\text{PSL}(2,8)$, $(\theta_a)^3$ is an inner automorphism as $\alpha^3 \in G'$. By taking $\theta_a \eta$ where $\eta$ is an inner automorphism we can assume $(\theta_a)^3 = \theta_a$ as in $G$ a 7-element is self centralizing. We see then that $G$ is the semidirect product of $G$ by $\langle \theta_a \rangle$. If $A$ is the automorphism group of $\text{PSL}(2,8)$, $I(A)$ the inner automorphism group, then $A/I(A)$ has an element of order 3 generated by a field automorphism. There are seven such extensions all isomorphic. We may assume then the extension is induced by a field automorphism. From the character table [15], $\text{PSL}(2,8)$ has four characters of degree 7 and so one must lift.

In the remaining case $(\theta_a)^2$ is an inner automorphism as $\alpha^2 \in G'$. In each of the remaining cases there is exactly one element of order 2 in $A/I(A)$. It must be $\theta_a$. By taking an element $\alpha \eta$ instead of $\alpha$ we can assume $(\theta_a)^2 = \theta_a$ the identity automorphism. In this case $\alpha^2$ must commute with all elements of $G'$ and so must be $e$. This shows that $G$ is uniquely determined as the semidirect product of $G'$ and $\langle \alpha \rangle$ with the automorphism $\theta_a$.

In each of the groups $\text{PSL}(2,7)$, $A_8$, and $U_3(3)$, there is exactly one rational character of degree 7. This means $\theta_a$ must leave it fixed and so the character of degree 7 can be lifted to $G$.

In the case of $\text{PSL}(2,7)$ there is certainly only one representation of degree 7 as the sum of the squares of the degrees in $B_2(7)$ is $168 - 7^3$. The element $\alpha$ can be taken $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ in the usual
matrix form of $PSL(2,7)$.

In the case of $A_8$ one may check the character table [18] to see there is only one representation of degree 7. However, this is not necessary because $S_8$ is known to have an irreducible unimodular representation of degree 7 and so $G$ is $S_8$.

For the final case $U_3(3)$ one must check a character table [16]. Here there are three representations of degree 7, two of them are conjugate and the third rational. The automorphism is a field automorphism. The group $G$ must be the group $G_2(2)$ as $G_2(2)$ cannot be $Z_2 \times G'$ or it would have an automorphism of order 2 [9].

**Flow Chart.** $|G| = g = 7 \cdot 5^s \cdot 3^2 \cdot 2^e$.

I. $s = 6$. $\chi = \chi_0 + \sum_{i=1}^{5} a_i \chi_i + \eta$. $G = G'$.

A(§ 2). One of $\chi_i, i = 1, 2, \ldots, 6$ has degree 6.

B(§ 3). Some $a_i = 1, x_i = 48$. We assume $a_i = 1, x_i = 48$.

(i) $\chi$ irrational on a 5-Sylow group.

(ii) $g = 7 \cdot 5 \cdot 3^2 \cdot 2^e$.

(iii) $g = 7 \cdot 3^2 \cdot 2^e$.

C(§ 4). Some $a_i = 1, x_i = 20$. We assume $a_i = 1, x_i = 20$.

(i) $a \geq 2$.

(ii) $a = 1$.

D(§ 5). Some $a_i = 1, x_i = 27$. We assume $a_i = 1, x_i = 27$.

(i) $\chi$ restricted to a 3-Sylow group is irrational.

(ii) $b = 3$.

(iii) $b = 4$. (This case gives $S_4(2)$).

II. (§ 6). $s = 2$ $G = G'$. (This case gives $PSL_2(8)$.)

III. (§ 7). $s = 3$ $G = G'$. (This case gives $PSL_2(7), A_8$, and $U_3(3)$.)

IV. (§ 8). $G \neq G'$. (This is VII of Theorem I).

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