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In this paper we find a class $\mathscr{C}$ of Abelian groups with the property that if a group A is a direct sum of groups in the class $\mathscr{C}$, then any two direct sum decompositions of A have isomorphic refinements. The class $\mathscr{C}$ includes those groups which are complete and Hausdorff in their natural topology and also the torsion-complete p-groups.

All groups in this paper are Abelian groups, additively written. The *natural topology* (or Z-topology) is defined on a group G by taking as neighborhoods of 0 the subgroups nG, for nonzero integers n. A group G is called *Hausdorff* if it is Hausdorff in this topology, or, equivalently, if $\bigcap nG = 0$ (where n ranges over all nonzero integers). G has *bounded order* if for some nonzero integer n, $nG = 0$. We will frequently use Prufer's theorem that a group of bounded order is a direct sum of cyclic groups [9, Th. 6]. The groups which are complete and Hausdorff in the natural topology are exactly the reduced algebraically compact groups in the terminology of [9]. A p-group is *torsion-complete* if it is Hausdorff and it is the maximal torsion subgroup of its completion in the natural topology (which in this case is the same as the p-adic topology).

We use the symbol $\sum$ for the direct sum of a family of groups, $A \oplus B$ for the direct sum of the groups A and B (either abstractly or as a subgroup of another group), and $A + B$ for the ordinary sum of two subgroups of a group (not necessarily a direct sum). If a group G has two direct sum decompositions, $G = \sum_{i \in I} A_i = \sum_{j \in J} B_j$, we say that these decompositions are isomorphic if there is a bijective mapping $\phi: I \rightarrow J$, such that $A_i \cong B_{\phi(i)}$ for all $i \in I$, and we say that the second decomposition is a *refinement* of the first if each B_j is contained in one of the A_i.

A group B has the *exchange property* if for any group A, if $A = B' \oplus C = \sum_{i \in I} D_i$, with $B \cong B'$, then there are subgroups $D'_i \subseteq D_i$ such that $A = B' \oplus \sum_{i \in I} D'_i$. If this holds in every case where the index set I is finite, then B is said to have the *finite exchange property*. It is not known whether these two properties are equivalent. The exchange property has been exploited for the study of infinite direct sum decompositions by P. Crawley and B. Jónsson in [4].

DEFINITION. An Abelian group G is in the class $\mathscr{C}$ if it satisfies the following three conditions:

(i) G is Hausdorff;

(ii) G has the finite exchange property;

(iii) If $f: G \to M$ is a homomorphism of G into a Hausdorff group M and $M = \sum_{i \in I} M_i$ then there is a finite subset $J \subseteq I$ and a decomposition of G, $G = G_1 \oplus G_2$, where G_1 is of bounded order and every nonzero element of G_2 has a nonzero multiple whose image under f is in $\sum_{i \in J} M_i$.

The main result of § 4 below is that complete Hausdorff groups are in $\mathscr{C}$. Torsion-complete p-groups are also in $\mathscr{C}$ since Crawley and Jónsson showed [4, Lemma 11.4] that they have the exchange property, and property (iii) is easy to check directly (using, for example, the completeness of the socle in the p-adic topology and applying the Baire category theorem as in § 4 below).

There are many other examples of groups in $\mathscr{C}$. Crawley proved [3, Lemma 3.5] that for p-groups properties (i) and (ii) above imply (iii) (his condition appears weaker than (iii) but is actually equivalent) so any Hausdorff p-group with the finite exchange property is in $\mathscr{C}$. He also constructs in [3] a class of "stiff" p-groups which are in $\mathscr{C}$, but which are not torsion-complete. For other examples, we remark that if G is a Hausdorff group whose maximal torsion subgroup T is a stiff p-group and if G/T is divisible of finite rank, then G is a mixed group in $\mathscr{C}$. Finite rank pure subgroups of the p-adic integers (for any prime p) are examples of torsionfree groups which are not complete but which are in $\mathscr{C}$ (see Proposition 1 and the proof of Proposition 4 in [14]).

We will need two important additional properties of $\mathscr{C}$.

LEMMA 1. *If G is in $\mathscr{C}$, so is any summand of G. If $G_i (i = 1, \cdots, n)$ are in $\mathscr{C}$, so is $G_1 \oplus \cdots \oplus G_n$.*

LEMMA 2. *If $G \in \mathscr{C}$, any two finite direct sum decompositions of G have isomorphic refinements.*

Lemma 1 is obvious except perhaps for property (ii) for which see [4, Lemma 3.10]. Lemma 2 is immediate from the finite exchange property. The groups in $\mathscr{C}$ actually have the exchange property (not just the finite exchange property). For a proof we refer to [3, Lemma 3.6], only remarking that one must use our Lemmas 6 and 7 below instead of Crawley's 3.2 and 3.3. We will not need this result.

We will state our main results for abstract classes of groups, since the class $\mathscr{C}$ is not the only class of groups for which these theorems can be proved.

THEOREM 1. *Let $\mathscr{D}$ be a class of groups such that*

(i) *Summands and finite direct sums of groups in $\mathscr{D}$ are in $\mathscr{D}$,*

(ii) *If $G \in \mathscr{D}$ then any two finite direct sum decompositions of G have isomorphic refinements, and*

(iii) *If $G \in \mathscr{D}$, and $f: G \to M$ is a homomorphism, where $M = \sum_{i \in I} M_i$, and the M_i are all in $\mathscr{D}$, then there is a finite subset $J \subseteqq I$ and a decomposition of G, $G = G_1 \oplus G_2$, where G_1 is of bounded order and every nonzero element of G_2 has a nonzero multiple whose image under f is in $\sum_{i \in J} M_i$.*

Then if A is any group which is a direct sum of groups in the class $\mathscr{D}$, any two decompositions of A into summands in the class $\mathscr{D}$ have isomorphic refinements.

THEOREM 2. *Let $\mathscr{D}$ be a class of Abelian groups satisfying the hypotheses of Theorem 1, and such that the elements of $\mathscr{D}$ have the finite exchange property. Then if a group A is a direct sum of groups in the class $\mathscr{D}$, any summand of A is also a direct sum of groups in the class $\mathscr{D}$.*

Sections 2 and 3 below are devoted to the proofs of these theorems.

COROLLARY. *If $\mathscr{D}$ is a class of groups satisfying the conditions of Theorem 2 and A is a direct sum of groups in the class $\mathscr{D}$ then any two direct sum decompositions of A have isomorphic refinements. In particular, this applies to the class $\mathscr{C}$, (by definition and Lemmas 1 and 2) so (specializing further) if A is a direct sum of complete Hausdorff groups or of torsion-complete p-groups, then any two direct sum decompositions of A have isomorphic refinements.*

The results of this paper for torsion-free and mixed groups are entirely new, but the corresponding questions for p-groups have a considerable history. Reinhold Baer completely solved the problem for countable p-groups in 1935 [1]. Kulikov proved in [11] that if an Abelian p-group is a direct sum of cyclic groups, then any two direct sum decompositions of the group have isomorphic refinements, thus generalizing one of the results obtained by Baer in the countable case. Kulikov also defined torsion-complete p-groups in [11] and showed that any two direct sum decompositions of a torsion-complete p-group have isomorphic refinements. E. Enochs, in work based partly on earlier work of Kolettis [10], proved in [5] the special case of Theorem 1 involving direct sums of torsion-complete p-groups. Our proof of Theorem 1 was motivated by his paper. Crawley generalized this result in [3], replacing torsion-complete p-groups by Hausdorff p-groups with the finite exchange property. In both of these cases one still needs to prove the corresponding special case of Theorem 2. This has previously been done only in the special case of direct sums of torsion-complete p-groups—for countable sums by Irwin, Richman and Walker [7], and in general by P. Hill [6] and the author [12], independently.

We close this introduction with some examples to illustrate the limitations of our results. Numerous examples of groups without the isomorphic refinement property are known, due to Baer [1] (for countable p-groups), Jónsson [8] (for torsion-free groups), and Corner and Crawley [2] (for Hausdorff p-groups). On the other hand, there are groups not in the class $\mathscr{C}$ for which a theorem such as ours should be provable. If G is a p-group such that the subgroup G^1 of elements of infinite height is torsion-complete and not zero, and such that G/G^1 is stiff (in the sense of [3]) then G has the exchange property and any two direct sum decompositions of G have isomorphic refinements, but G is not Hausdorff and therefore is not in $\mathscr{C}$. Possibly the class $\mathscr{C}$ could be enlarged by omitting condition (i) and suitably altering condition (iii).

1. Lemmas on pure subgroups and projections.

We recall that if A is an Abelian group and B a subgroup, then B is *pure* in A if $nB = B \cap nA$ for all integers n. We define the *p-height* (denoted h_p) of an element x by setting $h_p(x) = n$ if $x = p^n y$ for some y but $x \neq p^{n+1}z$ for any z, and $h_p(x) = \infty$ if x is divisible by all powers of p. By the *height* of x we mean the function associating to each prime p the number $h_p(x)$. Clearly a subgroup B is pure in A if and only if the heights computed with respect to B and with respect to A are the same.

Most of the lemmas that follow are generalizations to mixed groups of well-known and widely used results for p-groups. The first two, for example, are generalizations of Lemmas 12 and 7 of [9].

LEMMA 3. *If M is a group and N a pure subgroup such that for every $x \in M$ ($x \neq 0$) there is an integer n such that $nx \neq 0$ and $nx \in N$, then $N = M$.*

REMARK. This lemma is not true for modules over an arbitrary integral domain. For instance, if D is a divisible R-module which is not injective, and E is the injective envelope of D, then D is pure in E and every nonzero element of E has a nonzero multiple in D, but $E \neq D$.

Proof. By a finite reduction process, it suffices to show that if p is a prime and $px \in N$, then $x \in N$. If $px = 0$ this is trivial by hypothesis, since some nonzero multiple of x must be in N. Otherwise, $px = py$ for some $y \in N$ (by the purity of N) and $p(x - y) = 0$ so either $x = y$ (and we are done) or $x - y$ is of order p and hence in N, and $x = (x - y) + y$ is a sum of elements in N.

LEMMA 4. *Let M be a group and N a subgroup and say that for all $x \in N$ $(x \neq 0)$ there is an integer n such that $nx \neq 0$, and nx has the same height in M and in N. Then N is pure.*

Proof. By a finite reduction we need only show that if px has the same height in N and M then so does x. Since x and px have the same q-height for all primes q, $q \neq p$, we need only consider p-height and we must show that if $x = p^n y$ for some $y \in M$ then there is a $z \in N$ with $x = p^n z$. If $px = 0$ the result is trivial since in this case, if $nx \neq 0$ then nx and x have the same height. We therefore assume $px \neq 0$.

Suppose $x = p^n y$, $y \in M$. Then $px = p^{n+1}y$ and by hypothesis, $px = p^{n+1}z_0$ for some $z_0 \in N$. Then $p(p^n z_0 - x) = 0$ so either $x = p^n z_0$ (and we are done) or $p^n z_0 - x$ is of order p and hence has the same height in M as in N. Since it is divisible by p^n in M (since $x = p^n y$) it is also divisible by p^n in N, so that there is a $z_1 \in N$ with $p^n z_1 = p^n z_0 - x$, that is, $x = p^n(z_0 - z_1)$, proving the result.

DEFINITION. If two pure subgroups A and B of a group M have the property that each nonzero element of A has a nonzero multiple in B and each nonzero element of B has a nonzero multiple in A, then A and B are said to be *essentially linked.*

LEMMA 5. *If M is a group with pure subgroups A and B which are essentially linked, such that A is a summand ($M = A \oplus A'$), then B is also a summand and $M = B \oplus A'$.*

Proof. The conclusion is equivalent to the statement that the projection $\theta: M \to A$ carries B isomorphically onto A. θ restricted to B is clearly injective since any nonzero element of B has a nonzero multiple which is left fixed by θ. We next note that $\theta(B)$ is a pure subgroup of A, by Lemma 4, since if $\theta(b)$ is in $\theta(B)$, then for some integer n, $n\theta(b) = nb \neq 0$, and $n\theta(b)$ has the same height in $\theta(B)$ as in B (since θ restricted to B is injective) and the same height in B as in A (since A and B are pure). Finally, any nonzero element in A has a nonzero multiple in $\theta(B)$, so $A = \theta(B)$ by Lemma 3.

LEMMA 6. *If M is a group with pure, essentially linked subgroups A and B, both summands (say $M = A \oplus A' = B \oplus B'$), then also $M = A \oplus B' = B \oplus A'$.*

Proof. Apply Lemma 5 twice.

LEMMA 7. *If A and B are summands of a group M and every nonzero element of B has a nonzero multiple in A, then the projection of B into A carries B isomorphically onto a summand of A.*

Proof. Let the projection into A be θ. $\theta(B)$ is pure by Lemma 4. By Lemma 5 therefore, $\theta(B)$ is a summand with the same complement as B.

LEMMA 8. *If M is a group and n a positive integer, we can decompose $M = A \oplus A'$ where $nA = 0$ and any nonzero element of A' has a nonzero multiple in nA'. Furthermore if $G = B \oplus B'$ is another such decomposition then $A \cong B$ and $A' \cong B'$.*

Proof. Choose A to be a subgroup of M maximal with respect to the properties that A is pure and $nA = 0$. A pure, bounded-order subgroup is a summand [9, Th. 7], so we can decompose $M = A \oplus A'$. We must show that any nonzero element of A' has a nonzero multiple in nA'. If the element has infinite order the result is trivial, and otherwise it has a nonzero multiple of prime order, so it will suffice to show that if $x \in A'$, $x \neq 0$, and $px = 0$, for some prime p, then x is divisible by n. This is equivalent to showing that $h_p(x) \geqq k$, where p^k is the highest power of p dividing n. If this were not the case, there would be an element $y \in A'$ with $p^m y = x$, where $h_p(x) = m$ and $m < k$. By Lemma 4, the subgroup $[y]$ generated by y would be pure, and hence a summand of A'. $A \oplus [y]$ would then be a pure subgroup satisfying $n(A \oplus [y]) = 0$, contradicting the maximality of A.

To prove the final statement, note that $nM = nA' = nB'$, so A' and B' are essentially linked, so by Lemma 6, $M = A \oplus B' = B \oplus A'$, which implies that $A \cong B$ and $A' \cong B'$.

LEMMA 9. *Let M be a group, and $M = A \oplus B \oplus C = A' \oplus D \oplus E$, and suppose that A and A' are essentially linked, and every nonzero element of D has a nonzero multiple in $A \oplus B$, and that π is the projection of M onto B from the first decomposition. Then $\pi(D)$ is a summand isomorphic to D, and the subgroups $A \oplus \pi(D)$ and $A' \oplus D$ are essentially linked.*

Proof. Let θ be the projection of M onto $B \oplus C$. By Lemma 6, $M = A \oplus D \oplus E$, so $\theta(D)$ is a summand of $B \oplus C$. By the condition on elements of D, every nonzero element of $\theta(B)$ has a nonzero multiple in B. By Lemma 7 therefore, $\pi(D) = \pi(\theta(D))$ is a summand of B, and it is clearly isomorphic to D. Note that $\pi(D)$ and $\theta(D)$ are essentially linked by construction. To prove the last statement of the lemma, it suffices to show that any nonzero element of D has a nonzero multiple in $A \oplus \pi(D)$ and that any nonzero element of $\pi(D)$ has a nonzero multiple in $A' \oplus D$. For the first, we note that $D \subseteq A \oplus \theta(D)$, and any element of $A \oplus \theta(D)$ has a nonzero multiple in $A \oplus \pi(D)$. For the second, let x be a nonzero element of $\pi(D)$ and nx a nonzero multiple which is in $\theta(D)$. Then $nx = a + d$, where $a \in A$ and $d \in D$. If $a = 0$ we are done. Otherwise there is a nonzero multiple ra of a in A'. Certainly $rnx = ra + rd$ is in $A' \oplus D$, and, since $ra \neq 0$, $rnx \neq 0$ since $ra \in A'$, and $rd \in D$ and $A \cap D = 0$.

2. Proof of Theorem 1. We begin with three remarks which we will need to refer to.

(2.1). Hypothesis (iii) of Theorem 1 can be strengthened by adding the condition that none of the finite set of summands $M_i (i \in J)$ are of bounded order. To see this, let n be a positive integer such that $nM_i = 0$ for all of the $M_i (i \in J)$ which are of bounded order. This is possible since there are only a finite number of them. Let G, G_1 and G_2 be as in the statement of condition (iii) and use Lemma 8 to decompose G_2 so that $G_2 = G_2^* \oplus G_2'$, where $nG_2^* = 0$ and every nonzero element in G_2' has a nonzero multiple in nG_2'. We now let $G_1' = G_1 \oplus G_2^*$, so that $G = G_1' \oplus G_2'$, where G_1' is again of bounded order, and every nonzero element of G_2' has a nonzero multiple whose image under f is in the sum of those $M_i (i \in J)$ not of bounded order.

(2.2). If G is in the class $\mathscr{D}$ then any two direct sum decompositions of G have isomorphic refinements. For if $G = \sum_{i \in I} A_i = \sum_{j \in J} B_j$, then by condition (iii) of Theorem 1 there is a positive integer n such that $nA_i = nB_j = 0$ for all but a finite number of the i's and j's. We now apply Lemma 8 to each of the summands A_i and B_j, using this integer n, and obtain decompositions

$$A_i = A_i^* \oplus A_i', \; B_j = B_j^* \oplus B_j'$$

where $nA_i^* = nB_j^* = 0$ for all $i \in I$, $j \in J$, and any nonzero element of A_i' or B_j' has a nonzero multiple in nG. If $A = \sum_{i \in I} A_i^*$, $A' = \sum_{i \in I} A_i'$ and B and B' are defined similarly, then $G = A \oplus A' = B \oplus B'$. These decompositions satisfy the conditions of Lemma 8, so $A \cong B$ and

$A' \cong B'$. These four groups have decompositions inherited from the original decompositions of G. The decompositions of A and B are finite, so by hypothesis (ii) of Theorem 1 they have isomorphic refinements. A' and B' are of bounded order and hence are direct sums of cyclic groups, so their decompositions have isomorphic refinements by Kulikov's theorem ([11] or [9, Exercise 34]). Putting these results together, we have the required isomorphic refinements of the original decompositions.

(2.3). Applying (2.2) several times, it is easy to see that if G is a group which is decomposed in two ways as a direct sum of groups in the class $\mathscr{D}$, $G = \sum_{\gamma \in \Gamma} C_\gamma = \sum_{\lambda \in \Lambda} D_\lambda$, and if these decompositions have isomorphic refinements, then if $G = \sum_{i \in I} A_i$ is a refinement of the first decomposition, and $G = \sum_{j \in J} B_j$ is a refinement of the second, then these two decompositions also have isomorphic refinements.

We now outline the rest of the proof of Theorem 1. Suppose $G = \sum_{i \in I} A_i = \sum_{j \in J} B_j$ where the A_i and B_j are groups of the class $\mathscr{D}$. We regroup the summands A_i into finite sets, setting $C_\gamma = \sum_{i \in I_\gamma} A_i$, where γ is an ordinal in some initial segment of the ordinal numbers ($\gamma < \lambda$), and the I_γ we construct will be disjoint and their union will be I. We similarly group the summands B_j defining $D_\gamma = \sum_{j \in J_\gamma} B_j$, where J_γ is a finite subset of J, and the J_γ are disjoint sets which together include at least all elements $j \in J$ for which B_j is not of bounded order.

We will then have

$$G = \sum_{\gamma < \lambda} C_\gamma = \Big(\sum_{\gamma < \lambda} D_\gamma\Big) \oplus \sum_{j \in J_*} B_j$$

where J_* is the set of all $j \in J$ not contained in any of the J_γ, and all of the $B_j (j \in J_*)$ are of bounded order. We will construct isomorphic refinements of these decompositions, which will prove Theorem 1 by (2.3). We will decompose the C's and D's as follows:

$$C_\gamma = C_\gamma^1 \oplus C_\gamma^2 \oplus C_\gamma^3 \quad D_\gamma = D_\gamma^1 \oplus D_\gamma^2 \oplus D_\gamma^3$$

and we will have by construction

(1) $D_\gamma^1 \cong C_\gamma^1$

(2) $D_\gamma^2 \cong C_{\gamma+1}^2$ where $C_\gamma^2 = 0$ if $\gamma = 0$ or γ is a limit ordinal.

(3) D_γ^3 and C_γ^3 are of bounded order.

(4) $\sum_{\gamma < \lambda} (D_\gamma^1 \oplus D_\gamma^2)$ and $\sum_{\gamma < \lambda} (C_\gamma^1 \oplus C_\gamma^2)$ are essentially linked.

We now note that in the above situation, the theorem is proved, since by (4) and Lemma 6,

$$G = \sum_{\gamma < \lambda} (C_\gamma^1 \oplus C_\gamma^2) \oplus \sum_{\gamma < \lambda} D_\gamma^3 \oplus \sum_{j \in J^*} B_j$$

so that $\sum_{\gamma<\lambda} D_\gamma^3 \oplus \sum_{j \in J_*} B_j \cong \sum_{\gamma<\lambda} C_\gamma^3$, and since these last groups are direct sums of finite cyclic groups, we can get isomorphic refinements by Kulikov's theorem. Hence our pairing in formulas (1) and (2) above and this remark together prove the theorem.

We now construct the subgroups C_γ, D_γ and their decompositions to satisfy (1), (2), (3), and (4). We say the process is completed up to k if

(a) for $n \leqq k$ (ordinal numbers) the finite sets I_n of indices are chosen, and for $n < k$ the sets J_n are chosen.

(b) for $n < k$ the D_n and C_n decompose as above and the summands D_n^i, C_n^i satisfy the statements (1), (2), (3) where they apply.

(c) C_k is chosen and C_k^2, a summand such that $C_k^2 \cong D_{k-1}^2$ if $k-1$ exists, and $C_k^2 = 0$ if k is a limit ordinal.

(d) $\sum_{n<k} (C_n^1 \oplus C_n^2) \oplus C_k^2$ and $\sum_{n<k} (D_n^1 \oplus D_n^2)$ are essentially linked.

Now let the induction hypothesis be that this has been done for all $k < \gamma$, and do it for γ. If γ is a limit ordinal the process is trivial. Take C_γ to be any summand A_i not previously included in $C_k (k < \gamma)$, and set $C_\gamma^2 = 0$. I_γ is the single chosen index i. (If no A_i remain then we are done, for no B_j can remain except possibly groups of bounded order, since by the previous argument, if we let K be the sum of the remaining summands B_j, we have $\sum_{k<\gamma} D_k^3 \oplus K \cong \sum_{k<\gamma} C_k^3$, a direct sum of finite cyclic groups, and by condition (iii), any element of $\mathscr{D}$ which is a direct sum of cyclic groups is necessarily of bounded order.)

If we are not at a limit ordinal, we change notation and assume that the process has been carried out for γ and do it for $\gamma + 1$. We are given C_γ and C_γ^2. Let C_γ^* be a complement to C_γ^2 in C_γ. Let $\sum^\gamma B_j$ be the sum of those summands B_j not in D_k for any $k < \gamma$.

We now apply condition (iii) of Theorem 1 to the subgroup C_γ^* and its natural inclusion mapping into G, using the decomposition

$$G = \sum_{k<\gamma} (D_k^1 \oplus D_k^2) \oplus \sum_{k<\gamma} D_k^3 \oplus \sum{}^\gamma B_j \,.$$

We obtain a decomposition $C_\gamma^* = C_\gamma^1 \oplus C_\gamma^3$, where C_γ^3 is of bounded order, and also a finite subset J_γ of J disjoint from all of the J_k, $k < \gamma$, such that if $D_\gamma = \sum_{j \in J_\gamma} B_j$ then any nonzero element of C_γ^1 has a nonzero multiple in

$$\sum_{k<\gamma} (D_k^1 \oplus D_k^2) \oplus D_\gamma \,.$$

Hence we have used remark (2.1) to eliminate summands of the form D_k^3, $k < \gamma$. We now apply Lemma 9, where A and A' (in the terminology of that lemma) are $\sum_{k<\gamma} (D_k^1 \oplus D_k^2)$ and $\sum_{k<\gamma} (C_k^1 \oplus C_k^2) \oplus C_\gamma^2$, B is D_γ, and D is C_γ^1. We obtain a summand D_γ^1 of D_γ which is isomorphic to C_γ^1, and such that the subgroups

$$\sum_{k<\gamma} (C_k^1 \oplus C_k^2) \oplus C_\gamma^1 \oplus C_\gamma^2$$

and $\sum_{k<\gamma} (D_k^1 \oplus D_k^2) \oplus D_\gamma^1$ are essentially linked.

We now apply the same process in the other direction, choosing $I_{\gamma+1}$, $C_{\gamma+1}$, D_γ^2, D_γ^3, $C_{\gamma+1}^2$ exactly as we choose J_γ, D_γ, C_γ^1, C_γ^3, and D_γ^1, respectively. The proof is exactly the same, thus completing the induction and the proof of Theorem 1.

3. Proof of Theorem 2. Suppose we have

$$M = \sum_{j \in J} M_j = A \oplus B ,$$

where the M_j are groups of the class $\mathscr{D}$. We group these, as in the proof of Theorem 1, defining summands N_i, where each N_i is the sum of a finite number of the M_j. The indices i of the N_i will form an initial segment $(i < \lambda)$ of the ordinal numbers and the N_i will be constructed by transfinite induction, so that we will have $M = \sum_{i<\lambda} N_i$. For each i we will also construct summands A_i, B_i of A and B respectively, where A_i and B_i are in the class $\mathscr{D}$ and we will set $C_i = A_i \oplus B_i$. By construction the C_i will be independent, by which we mean that the subgroup generated by them is their direct sum. We will decompose the N_i as follows

$$N_i = N_i^1 \oplus N_i^2 \oplus N_i^3$$

where N_i^3 is of bounded order and $N_i^2 = 0$ if $\mathrm{i} = 1$ or i is a limit ordinal. We then regroup and decompose again, so that we will have

$$N_i^1 \oplus N_{i+1}^2 = P_i \oplus P_i^3$$

where P_i^3 is of bounded order (the superscript 3 will always mean this). Finally, the subgroups $\sum_{i<\lambda} P_i$ and $\sum_{i<\lambda} C_i$ will be essentially linked, so that by Lemma 5, $\sum_{i<\lambda} C_i$ is actually a summand of M.

Let us first show that when this construction has been carried out we will have proved the theorem. By Lemma 6 the summands $\sum_{i<\lambda} P_i$ and $\sum_{i<\lambda} C_i$ interchange and we have

$$M = \sum_{i<\lambda} C_i \oplus \sum_{i<\lambda} (P_i^3 \oplus N_i^3)$$

where the second term is a direct sum of finite cyclic groups. Now $\sum_{i<\lambda} C_i = \sum_{i<\lambda} A_i \oplus \sum_{i<\lambda} B_i$, and since $\sum_{i<\lambda} C_i$ is a summand of M, so are $\sum_{i<\lambda} A_i$ and $\sum_{i<\lambda} B_i$. Hence $\sum_{i<\lambda} A_i$ and $\sum_{i<\lambda} B_i$ are summands of A and B respectively, and we have $A = \sum_{i<\lambda} A_i \oplus A^*$, $B = \sum_{i<\lambda} B_i \oplus B^*$, and $A^* \oplus B^* \cong \sum_{i<\lambda} (P_i^3 \oplus N_i^3)$ (since both are complements to $\sum_{i<\lambda} C_i$) and since this is a direct sum of cyclic groups, so are A^* and B^* by

Kulikov's theorem. Any cyclic summand of M is in $\mathscr{D}$ (by hypothesis (i) of Theorem 1) since it is actually contained in (and therefore a summand of) the sum of a finite number of the original summands M_j. Hence $A = A^* \oplus \sum_{i<\lambda} A_i$ is a direct sum of groups in the class $\mathscr{D}$, which is what we wanted to prove.

To complete the proof, we must carry out the construction of the subgroups N_i, C_i and P_i in the way outlined above. We say the construction has been carried out for k if for $i \leqq k$, N_i is chosen and for $i < k$, C_i, A_i, and B_i are chosen (all belonging to the class $\mathscr{D}$), where A_i and B_i are summands of A and B respectively, $C_i = A_i \oplus B_i$, and all these are chosen so that

(a) $N_i = N_i^1 \oplus N_i^2 \oplus N_i^3 (i < k)$ with N_i^3 of bounded order and $N_i^2 = 0$ if $i = 1$ or a limit ordinal,

(b) N_k has a summand N_k^2 which is zero if k is 1 or a limit ordinal.

(c) For $i < k$, $N_k^1 \oplus N_{i+1}^2 = P_i \oplus P_i^3$ where P_i^3 is of bounded order.

(d) The C_i are independent and $\sum_{i<k} C_i$ is a pure subgroup of M.

(e) $\sum_{i<k} C_i$ and $\sum_{i<k} P_i$ and essentially linked, so that, in particular, $\sum_{i<k} C_i$ is a summand of M by Lemma 5.

We now suppose that this has been done for all $k < \gamma$ and do it for γ. Suppose first that γ is 1 or γ is a limit ordinal. We let N_γ be one of the remaining M_j (if any remain) and set $N_\gamma^2 = 0$ (as we must). Note that this choice guarantees that the process eventually terminates with the choice of all of the M_j. Conditions (a) and (c) are trivially verified, having already been assumed for $i < \gamma$, and (b) is immediate from our definition of N_γ^2. For (d), it is clear that the $C_i \, (i < \gamma)$ certainly are independent and their direct sum is a pure subgroup, since it is an ascending union of pure subgroups. For (e), we note that $\sum_{i<\gamma} C_i$ and $\sum_{i<\gamma} P_i$ are essentially linked, and since $\sum_{i<\gamma} P_i$ is a summand, we can apply Lemma 5 to show that $\sum_{i<\gamma} C_i$ is also a summand. This completes the induction in this case.

Suppose, then, we are not at a limit ordinal. For convenience we assume that the construction has been carried out for γ and do it for $\gamma + 1$. Say $N_\gamma = N_\gamma^2 \oplus N_\gamma^*$, and let the projections to A and B respectively be θ_A and θ_B. We can decompose M in three ways.

$$M = \sum_{i<\gamma} C_i \oplus A_\gamma^* \oplus B_\gamma^* \tag{1}$$

where A_γ^* is a complement in A of $\sum_{i<\gamma} A_i$, and B_γ^* is a complement in B of $\sum_{i<\gamma} B_i$.

$$M = \sum_{i<\gamma} P_i \oplus N_\gamma^* \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus \sum{}^{\gamma} M_j \tag{2}$$

where $\sum^{\gamma} M_j$ denotes the sum of those M_j not chosen to be in N_i for any i, $i \leqq \gamma$. Since $\sum_{i<\gamma} C_i$ and $\sum_{i<\gamma} P_i$ are essentially linked we also have

$$M = \sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus \sum^{\gamma} M_j \,. \tag{3}$$

We now apply condition (iii) of Theorem 1 to the group N_γ^* and the two homomorphisms θ_A and θ_B (applying the condition twice), using the decomposition (3) above. We obtain a decomposition

$$N_\gamma^* = N_\gamma^1 \oplus N_\gamma^3$$

where N_γ^3 is of bounded order, and there are a finite number of the summands M_j not included in any N_i for $i \leqq \gamma$, such that if $L_{\gamma+1}$ is the sum of this finite number of subgroups, then any nonzero element of N_γ^1 has a nonzero multiple whose images under θ_A and θ_B are both in

$$\sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus L_{\gamma+1} \,.$$

Hence we have used remark (2.1) in order to eliminate summands of the form P_i^3 or N_i^3.

Now let π be the natural projection of M onto $A_\gamma^* \oplus B_\gamma^*$ from decomposition (1). We have immediately

$$\sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus L_{\gamma+1} = \sum_{i<\gamma} C_i \oplus \pi(N_\gamma^* \oplus L_{\gamma+1}) \,. \tag{4}$$

We let $K = \sum_{i<\gamma} C_i \oplus N_\gamma^* \oplus L_{\gamma+1}$. Note also that

$$K \cap (A_\gamma^* \oplus B_\gamma^*) = \pi(N_\gamma^* \oplus L_{\gamma+1}) \,. \tag{5}$$

Now $\pi(N_\gamma^* \oplus L_{\gamma+1})$ is isomorphic to $N_\gamma^* \oplus L_{\gamma+1}$ and is therefore in $\mathscr{D}$ (since summands and finite direct sums of elements of $\mathscr{D}$ are in $\mathscr{D}$) and therefore has the finite exchange property, so that

$$A_\gamma^* \oplus B_\gamma^* = \pi(N_\gamma^* \oplus L_{\gamma+1}) \oplus A_\gamma^{**} \oplus B_\gamma^{**}$$

where $A_\gamma^{**} \subseteqq A_\gamma^*$, $B_\gamma^{**} \subseteqq B_\gamma^*$. We have natural decompositions

$$A_\gamma^* = A_\gamma^{**} \oplus D_\gamma^A,\ B_\gamma^* = B_\gamma^{**} \oplus D_\gamma^B$$

where the groups D_γ^A, D_γ^B can be identified as follows:

$$D_\gamma^A = A_\gamma^* \cap (\pi(N_\gamma^* \oplus L_{\gamma+1}) \oplus B_\gamma^{**})$$

$$D_\gamma^B = B_\gamma^* \cap (\pi(N_\gamma^* \oplus L_{\gamma+1}) \oplus A_\gamma^{**}) \,.$$

Note that the above formulas and statement (5) imply that

$$(6) \qquad K \cap A_\gamma^* \subseteqq D_\gamma^A, \text{ and } K \cap B_\gamma^* \subseteqq D_\gamma^B .$$

We let $D_\gamma = D_\gamma^A \oplus D_\gamma^B$, and we claim that any nonzero element of N_γ^1 has a nonzero multiple in $\sum_{i<\gamma} C_i \oplus D_\gamma$. By the original definition of N_γ^1, if $x \in N_\gamma^1$ and $x \neq 0$, then x has a nonzero multiple nx such that if $nx = y + z$, with $y \in A$ and $z \in B$, then y and z are in K. We will show that y is in $\sum_{i<\gamma} C_i \oplus D_\gamma$, and the proof for z will be the same. We have $y = a_1 + a_2$, where $a_1 \in \sum_{i<\gamma} A_i$ and $a_2 \in A_\gamma^*$. Since $a_1 \in \sum_{i<\gamma} C_i$, it will be enough to show that $a_2 \in D_\gamma$. Since y and a_1 are both in K, so is a_2, so $a_2 \in A_\gamma \cap K$, and thus is in D_γ by formula (6).

We have now shown that any nonzero element of N_γ^1 has a nonzero multiple in $\sum_{i<\gamma} C_i \oplus D_\gamma$. We apply Lemma 9 to obtain a summand D_γ^1 of D_γ such that the subgroups $\sum_{i<\gamma} P_i \oplus N_\gamma^1$ and $\sum_{i<\gamma} C_i \oplus D_\gamma^1$ are essentially linked.

Let D_γ^* be a complement to D_γ^1 in D_γ. As usual, we cannot handle all of D_γ^*, so we apply condition (iii) of Theorem 1 again, with respect to the decomposition

$$M = \sum_{i<\gamma} P_i \oplus N_\gamma^* \oplus L_{\gamma+1} \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus \sum{}^{\prime\prime} M_j .$$

where we use the notation $\sum^{\prime\prime} M_j$ to denote the sum of those M_j not chosen to be in N_i for any $i \leqq \gamma$ or in $L_{\gamma+1}$. We obtain a decomposition $D_\gamma^* = D_\gamma^2 \oplus D_\gamma^3$, where D_γ^3 is of bounded order, and there are a finite number of summands M_j from the term $\sum^{\prime\prime} M_j$ such that if we let $N_{\gamma+1}$ be the sum of $L_{\gamma+1}$ and this additional set of summands, then any nonzero element of D_γ^2 has a nonzero multiple in

$$\sum_{i<\gamma} P_i \oplus N_\gamma^1 \oplus N_{\gamma+1} .$$

Applying Lemma 9 again, (where this time the subgroups corresponding to the A and A' of that lemma are $\sum_{i<\gamma} P_i \oplus N_\gamma^1$ and $\sum_{i<\gamma} C_i \oplus D_\gamma^1$ respectively), we obtain a summand $N_{\gamma+1}^2$ of $N_{\gamma+1}$ such that the subgroups $\sum_{i<\gamma} C_i \oplus D_\gamma^1 \oplus D_\gamma^2$ and $\sum_{i<\gamma} P_i \oplus N_\gamma^1 \oplus N_{\gamma+1}^2$ are essentially linked.

Unfortunately, $D_\gamma^1 \oplus D_\gamma^2$ cannot be the C_γ we need for our induction since it is not necessarily the sum of its A and B components. We return then to D_γ, and compare decompositions. We have

$$D_\gamma = D_\gamma^A \oplus D_\gamma^B = D_\gamma^1 \oplus D_\gamma^2 \oplus D_\gamma^3 ,$$

where $nD_\gamma^3 = 0$ for some positive integer n. Applying Lemma 8 (using this integer n) we obtain decompositions

$$D_\gamma^A = A_\gamma \oplus A_\gamma^3,\ D_\gamma^B = B_\gamma \oplus B_\gamma^3$$

where A_γ^3 and B_γ^3 are of bounded order and every nonzero element of A_γ and B_γ has a nonzero multiple in $D_\gamma^1 \oplus D_\gamma^2$. Let

$$C_\gamma = A_\gamma \oplus B_\gamma \ .$$

Let σ be the projection onto $N_\gamma^1 \oplus N_{\gamma+1}^2$ from the decomposition

$$M = \sum_{i<\gamma} P_i \oplus (N_\gamma^1 \oplus N_{\gamma+1}^2) \oplus \sum_{i<\gamma} (P_i^3 \oplus N_i^3) \oplus N_\gamma^3 \oplus N_{\gamma+1}^* \oplus \sum^{\gamma+1} M_j \ ,$$

where $N_{\gamma+1}^*$ is a complement to $N_{\gamma+1}^2$ in $N_{\gamma+1}$. Since $\sum_{i<\gamma} C_i$ and $\sum_{i<\gamma} P_i$ are essentially linked, and $\sum_{i<\gamma} C_i \oplus D_\gamma^1 \oplus D_\gamma^2$ and $\sum_{i<\gamma} P_i \oplus N_\gamma^1 \oplus N_{\gamma+1}^2$ are also essentially linked, we know that σ takes $D_\gamma^1 \oplus D_\gamma^2$ isomorphically onto $N_\gamma^1 \oplus N_{\gamma+1}^2$. Let $P_\gamma = \sigma(C_\gamma)$ and $P_\gamma^3 = \sigma(A_\gamma^3 \oplus B_\gamma^3)$. We then have

$$N_\gamma^1 \oplus N_{\gamma+1}^2 = P_\gamma \oplus P_\gamma^3 \ ,$$

where P_γ^3 is of bounded order. We now apply Lemma 9 once more, where the A, A', and D of that lemma correspond to $\sum_{i<\gamma} P_i$, $\sum_{i<\gamma} C_i$, and C_γ respectively, and we see that the subgroups $\sum_{i<\gamma} P_i \oplus P_\gamma$ and

$$\sum_{i<\gamma} C_i \oplus C_\gamma$$

are essentially linked. It is also clear that C_γ is in $\mathscr{D}$ since it is isomorphic to P_γ and P_γ is a summand of $N_\gamma \oplus N_{\gamma+1}$, which in turn is a direct sum of a finite number of groups in the class $\mathscr{D}$. We therefore have completed our induction and the proof of Theorem 2.

4. Complete Abelian groups. For any Abelian group A there is a natural homomorphism

$$A \rightarrow \varprojlim A/nA$$

where the limit is taken over the nonzero integers n ordered by divisibility. The inverse limit is denoted $\hat{A}$ and it is the Hausdorff completion of A with respect to the uniform structure defined by taking as neighborhoods of zero the subgroups nA ($n \neq 0$). The mapping $A \rightarrow \hat{A}$ is injective if and only if A is Hausdorff. We remark that the homomorphism $A \rightarrow \hat{A}$ induces an isomorphism $A/nA \rightarrow \hat{A}/n\hat{A}$, so that the image of A is a pure subgroup of $\hat{A}$ and the Z-topology on $\hat{A}$ agrees with the topology induced (by the completion process) from the Z-topology of A. The group A is complete and Hausdorff if and only if $A = \hat{A}$.

Note that a subgroup B of A is pure if and only if for all integers n, $n \neq 0$, the natural homomorphism $B/nB \rightarrow A/nA$ is injective. B is dense in A (with respect to the Z-topology) if and only if for all nonzero integers n, the natural homomorphism $B/nB \rightarrow A/nA$ is surjective.

LEMMA 10. *If B is a pure dense subgroup of a group A and f is a homomorphism from B into a complete Hausdorff group C then f extends in one and only one way to a homomorphism from A to C.*

This follows from standard inverse limit or topological arguments. (From the topological point of view, one needs to observe that any homomorphism between two groups is continuous in the Z-topology and that the Z-topology on a pure subgroup B agrees with the topology induced from the Z-topology on A.)

If A is any group, we let A^1 be the subgroup of A consisting of those elements divisible by all integers n. The proof of the following lemma is an elementary computation.

LEMMA 11. *If B is a subgroup of a group A, then the closure of B is the inverse image in A of $(A/B)^1$. In particular, B is closed if and only if A/B is Hausdorff, and B is dense in A if and only if A/B is divisible.*

For any prime p, we denote by Z_p the ring of rational numbers which can be written as fractions with denominators prime to p, and for any group A, we let $A_p = A \otimes Z_p$, regarded as a Z_p-module. A_p is the localization of A at the prime p. If A_p^1 is the submodule of A_p consisting of all elements divisible by all powers of p then we define the *Hausdorff localization*, A_p^* of A by $A_p^* = A_p/A_p^1$. We have natural homomorphisms $\phi_p: A \to A_p$, and hence a natural homomorphism

$$\phi: A \to \prod_p A_p^* .$$

If A is Hausdorff, this imbeds A as a pure, dense subgroup of $\prod_p A_p^*$. This proves the following lemma.

LEMMA 12. *If C is a complete Hausdorff group then the natural homomorphism $C \to \prod_p C_p^*$ is an isomorphism.*

To exploit this lemma, we need some results about modules over the rings Z_p. The results are actually valid for modules over any discrete valuation ring. A subset X of a Z_p-module is a *pure independent subset* if the elements are independent and the submodule $[X]$ generated by X is a pure submodule. A submodule B of M is a *basic submodule* if it is pure, dense, and a direct sum of cyclic modules. By [9, Lemma 21] any maximal pure independent subset generates a basic submodule, and it is trivial to verify that if X is a pure independent subset then X is maximal if and only if $[X]$ is dense (or equivalently, $M/[X]$ is divisible). The next lemma is a refinement of [9, Th. 23].

LEMMA 13. *If M is a Z_p-module and C a pure submodule which is complete and Hausdorff, X a maximal pure independent subset of C, and Y a set disjoint from X such that $X \cup Y$ is a maximal pure independent subset of M, then $M = C \oplus D$, where D is the closure of the submodule generated by Y.*

Proof. Define a function f on the set $X \cup Y$ by $f(x) = x$ if $x \in X$ and $f(y) = 0$ if $y \in Y$. This extends to a homomorphism of the basic submodule generated by $X \cup Y$, which can be regarded as a homomorphism of $[X \cup Y]$ into C. By Lemma 10, this extends to a homomorphism of M into C, which we also call f. Since f is the identity on $[X]$ and $[X]$ is dense in C, f is a projection onto C. If D is the kernel of the projection then D is closed since C is Hausdorff. To show that D is the closure of Y, we remark that $M/[X \cup Y]$ is divisible and $M/[X \cup Y] \cong C/[X] \oplus D/[Y]$, so $D/[Y]$ is divisible, which implies that Y is dense in D by Lemma 11.

LEMMA 14. *If M is a Z_p-module with torsion submodule T, and X is a subset of M, and X_0 and X_1 are the subsets of X consisting of the elements of finite and infinite order respectively, then X is a maximal pure independent subset if and only if X_0 is a maximal pure independent subset of T and X_1 is mapped bijectively onto a basis of the Z/pZ-vector space $M/(T + pM)$.*

Proof. Let X be a maximal pure independent subset of M and let $C = [X_1]$. Then the natural homomorphism $C/pC \to M/(T + pM)$ is an isomorphism by the proof of Lemma 21 of [9], and certainly X_0 is a maximal pure independent subset of T, which proves half of the lemma. Conversely, if the condition above is satisfied, and $\sigma: M \to M/T$ is the natural map, then σ takes X_1 bijectively onto a maximal pure independent subset of M/Y by [13, Lemma 3]. The submodule B generated by $\sigma(X_1)$ is therefore free, so

$$\sigma^{-1}(B) = T \oplus [X_1] .$$

It follows immediately that X is an independent set. Also, since B is pure in M/T, $\sigma^{-1}(B)$ is pure in M, and since $[X_0]$ is a pure submodule of the summand T, $[X]$ is a pure submodule of M. Finally, $M/[X]$ is clearly divisible, since $T/[X_0]$ and $M/(T + [X_1])$ are both divisible.

LEMMA 15. *Let M be a Z_p-module, Y a maximal pure independent subset of M, and X a pure independent subset of M. Then there* *is a subset Z of Y, disjoint from X, such that $X \cup Z$ is a maximal pure independent subset of M.*

Proof. This result was proved for p-groups in [4, Lemma 10.12]. We therefore know that if X_0 and X_1 are the sets of elements of finite order and infinite order respectively in X and Y_0 and Y_1 are the corresponding subsets of Y, then there is a subset Z_0 of Y_0, disjoint from X_0, such that $X_0 \cup Y_0$ is a maximal pure independent subset of the torsion submodule T of M. If ϕ is the natural map of M onto $M/(T + pM)$, (where T is the maximal torsion submodule of M), then ϕ takes Y_1 bijectively onto a Z/pZ-basis for $M/(T + pM)$, and X_1 bijectively onto an independent subset of $M/(T + pM)$. There is therefore a subset Z_1 of Y_1, disjoint from X_1, such that ϕ takes $X_1 \cup Z_1$ bijectively onto a basis for $M/(T + pM)$. This proves the lemma, setting $Z = Z_0 \cup Z_1$.

THEOREM 3. *A complete Hausdorff group has the exchange property.*

Proof. Let A be a group and C a complete Hausdorff summand of A, and say $A = \sum_{i \in I} D_i$. We will show that there are subgroups $D_i' \subseteqq D_i$ with

$$A = C \oplus \textstyle\sum_{i \in I} D_i' .$$

We first prove the theorem in the local, Hausdorff case. Suppose that A, C, and the D_i are all Z_p-modules. Suppose in addition that A is Hausdorff. Let X be a maximal pure independent subset of C and Y_i a maximal pure independent subset of D_i. By Lemma 15, we can extend X to a maximal pure independent subset of A by adding elements from the sets Y_i. Let the added sets be $Y_i' \subseteqq Y_i$, and let Z be the union of the sets Y_i' (so that $X \cup Z$ is a maximal pure independent subset of A). By Lemma 14, if E is the closure of the subgroup generated by Z, then $A = C \oplus E$. We let $E_i = E \cap D_i$, and we claim that $E = \sum_{i \in I} E_i$. Since A is Hausdorff, D_i is closed, so E_i is also closed. Since the E_i are in different summands, $\sum_{i \in I} E_i$ is closed, and it contains Z, so $E = \sum_{i \in I} E_i$ as desired. Hence, $A = C \oplus \sum_{i \in I} E_i$, proving the exchange theorem in this case.

We now prove the theorem in general. If $A = \sum_{i \in I} D_i$ then $A_p^* = \sum_{i \in I} (D_i)_p^*$ and C_p^* is a Hausdorff complete summand of A_p^*. By the previous case, there are submodules $E_i(p) \subseteqq (D_i)_p^*$ such that

$$A_p^* = C_p^* \oplus \sum_{i \in I} E_i(p) .$$

This means that there is a projection $g_p \colon A_p^* \to C_p^*$ such that

$$(1) \qquad \operatorname{Ker}(g_p) = \sum_{i \in I} \ker(g_p) \cap (D_i)_p^* \,.$$

What we need to prove is that there is a projection $f: A \to C$ such that

$$(2) \qquad \operatorname{Ker}(f) = \sum_{i \in I} \operatorname{Ker}(f) \cap D_i \,.$$

The definition is now clear. Let

$$g: \prod_p A_p^* \to \prod_p C_p^*$$

be the homomorphism induced by the mappings $g_p: A_p^* \to C_p^*$, let ϕ be the natural homomorphism

$$\phi: A \to \prod_p A_p^*$$

with coordinate mappings ϕ_p, and let

$$\sigma: \prod_p C_p^* \to C$$

be the inverse of the isomorphism of Lemma 15. Let $f = \sigma g \phi$. To prove that (2) holds, we need only check that if $x \in A$ and $x = \sum x_i$ in the decomposition $\sum_{i \in I} D_i$ then if $f(x) = 0$, we also have $f(x_i) = 0$. If $f(x) = 0$, then $g_p(\sum \phi_p(x_i)) = 0$ for each prime p. By (1), this implies that $g_p(\phi_p(x_i)) = 0$ for each prime p, which shows that $f(x_i) = 0$ as desired. This proves that (2) holds, and if we define

$$E_i = \operatorname{Ker}(f) \cap D_i \,,$$

then we have

$$A = C \oplus \sum_{i \in I} E_i \,.$$

This completes the proof of Theorem 3.

COROLLARY. *A complete Hausdorff group is in the class $\mathscr{C}$.*

Proof. Condition (1) is immediate and condition (ii) is contained in Theorem 3. For condition (iii), we suppose that C is a complete Hausdorff group and $f: C \to M$ a homomorphism of C into a Hausdorff group M which is a direct sum, $M = \sum_{i \in I} M_i$. We first remark that it will suffice to show that there is a finite subset $J \subseteqq I$, such that for some nonzero integer n, $f(nC) \subseteqq \sum_{i \in J} M_i$. For in this case we apply Lemma 8 to obtain a decomposition $C = C_1 \oplus C_2$, where $nC_1 = 0$ and any nonzero element of C_2 has a nonzero multiple in nC_2., whose image under f is therefore in $\sum_{i \in I} M_i$.

We assume first that the decomposition of M is countable, $M =$

$\sum_{i=1}^{\infty} M_i$. The subgroups $f^{-1}(\sum_{i=1}^{m} M_i)$ are closed subgroups of C whose union is all of C, so by the Baire category theorem, for some integer m, $f^{-1}(\sum_{i=1}^{m} M_i)$ contains a neighborhood of 0, namely nC, for some nonzero integer n.

If the result were false in the general case (with an arbitrarily large index set I) and if the mapping f and the group M in fact provided a counterexample, then we could find a sequence of integers $n_j (j = 1, 2, \cdots)$, a sequence of elements x_j of C, and a sequence of distinct summands of the original family, which we simply write N_j, such that x_j is divisible by n_j and $f(x_j)$ has a nonzero coordinate in N_j. If we let N_0 be the direct sum of all of the summands M_i not in our chosen list, then we have a decomposition $M = \sum_{j=0}^{\infty} N_j$ which provides another counterexample, this time with a countable number of summands. Since this has been shown to be impossible, the corollary is proved.

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