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**THE AMBIENT HOMEOMORPHY OF AN INCOMPLETE
SUBSPACE OF INFINITE-DIMENSIONAL HILBERT SPACES**

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The pair (H, H_f) is studied from a topological point of view (where H is an infinite-dimensional Hilbert space and H_f is the linear span in H of an orthonormal basis), and a complete characterization is obtained of the images of H_f under homeomorphisms of H onto itself. As the characterization is topological and essentially local in nature, it is applicable in the context of Hilbert manifolds and provides a characterization of (H, H_f) -manifold pairs (M, N) (with M an H -manifold and N an H_f -manifold lying in M so that each coordinate chart f of M may be taken to be a homeomorphism of pairs $(U, U \cap N) \xrightarrow{f} (f(U), f(U) \cap H_f)$).

This implies that in the countably infinite Cartesian product of H with itself, the infinite (weak) direct sum of H_f with itself is homeomorphic to H_f (the two form such a pair), and that if K is a locally finite-dimensional simplicial complex equipped with the barycentric metric (inducing the Euclidean metric on each simplex) and if no vertex-star of K contains more than $\dim(H)$ vertices, then $(K \times H, K \times H_f)$ is an (H, H_f) -manifold pair.

These results are used in [10] to study H_f -manifolds much more intensively to obtain results previously available only for H -manifolds or in the case that H_f is separable, i.e., connected H_f -manifolds are homeomorphic to open subsets of H_f , homotopy-equivalent H_f -manifolds are homeomorphic, and there is an essentially unique completion of an H_f -manifold into an H -manifold, yielding an (H, H_f) -pair.

It should be remarked that this characterization has already been achieved for separable Hilbert spaces by R. D. Anderson [1] and by C. Bessaga and A. Pełczyński [5], and that the observations concerning (H, H_f) -manifold pairs have been made by T. A. Chapman [6, 7] in that case. (Chapman then proceeded to obtain most of the results of [10] in the separable case by methods which seem at the moment to be limited to separability.)

Throughout the discussion, X will denote some complete metric space, and $\mathcal{H}(X)$, the group of all homeomorphisms of X onto itself. The term "isotopy" ("isotopic") will be understood as an abbreviation for "invertible, ambient isotopy", that is, a map $F: X \times [0, 1] \rightarrow X$ such that the function $G: X \times [0, 1] \rightarrow X \times [0, 1]$ defined from F by setting $G(x, t) = (F(x, t), t)$ is a homeomorphism. (When an embedding

f of a subset of X into X is said to be isotopic to the identity, then, there will exist an extension g of f to an element of $\mathcal{H}(X)$ which is invertibly ambient isotopic to the identity.) If \mathcal{U} is a collection of open sets of X , a map f of a subset Y of X into X will be said to be *limited by* \mathcal{U} if for each point y of Y such that $y \neq f(y)$, there is a member of \mathcal{U} containing both. A homotopy $F: Y \times [0, 1] \rightarrow X$ will be said to be *limited by* \mathcal{U} if for each point y of Y such that $F(\{y\} \times [0, 1]) \neq \{y\}$, there is an element of \mathcal{U} containing $F(\{y\} \times [0, 1])$. If \mathcal{S} is a collection of subsets of X then \mathcal{S}^* will denote their union, and \mathcal{S} will be termed *normal* whenever there is an open cover \mathcal{U} of \mathcal{S}^* by mutually disjoint sets with the property that for each U in \mathcal{U} , $U \cap \mathcal{S}^* \in \mathcal{S}$. The letter N means the positive integers. Finally, if A is a subset of X and \mathcal{S} is a collection of subsets of X , then $\text{st}(A, \mathcal{S})$ denotes the star of A with respect to \mathcal{S} , that is, the union of all members of \mathcal{S} meeting A , and $\text{st}(\mathcal{S}) = \{\text{st}(S, \mathcal{S}) \mid S \in \mathcal{S}\}$. Also,

$$\text{st}^n(A, \mathcal{S}) = \text{st}(\text{st}^{n-1}(A, \mathcal{S}), \mathcal{S}),$$

and $\text{st}^n(S) = \text{st}(\text{st}^{n-1}(\mathcal{S}))$. All refinements used will be understood to be composed of open sets, and \mathcal{T} is a st^n -refinement of \mathcal{S} provided that $\text{st}^n(\mathcal{T})$ refines \mathcal{S} .

The first lemma is due to Anderson and Bing [2].

LEMMA 1. *Let $\{f_n\}_{n \in N}$ be a sequence of homeomorphisms of the complete metric space X onto itself, and let \mathcal{U} be any open cover of X . If $\{U_n\}_{n=0}^\infty$ is a collection of open covers of X such that $\text{st}^2(\mathcal{U}_0)$ refines \mathcal{U} and for each n in N \mathcal{U}_n is a star-refinement of \mathcal{U}_{n-1} of mesh less than $1/2^n$, then $\{f_n \circ \dots \circ f_1\}_{n \in N}$ converges (uniformly) to a member of $\mathcal{H}(X)$ which is limited by \mathcal{U} provided that for each n in N f_{n+1} is limited by \mathcal{U}_n and mesh*

$$(f_1^{-1} \circ \dots \circ f_n^{-1}(\mathcal{U}_n)) < 1/2^n.$$

Proof. Anderson and Bing proved that $\{f_n \circ \dots \circ f_1\}_{n \in N}$ converges uniformly to a member f of $\mathcal{H}(X)$. To verify that f is limited by \mathcal{U} , it is sufficient to observe that for each x in X and n in N , there is a $U(x, n)$ in \mathcal{U}_n containing both $f_n \circ \dots \circ f_1(x)$ and $f_{n+1} \circ \dots \circ f_1(x)$, and there is also a $U(x, 0)$ in \mathcal{U}_0 containing both x and $f_1(x)$. If $V(x, n)$ is an element of \mathcal{U}_{n-1} containing $\text{st}(U(x, n), \mathcal{U}_n)$ for each x and n , then x and $f_{n+1} \circ \dots \circ f_1(x)$ lie in

$$\begin{aligned} \bigcup_{m=0}^n U(x, m) &\subset \bigcup_{\substack{m=0 \\ n-2}}^{n-1} U(x, m) \cup V(x, n) \\ &\subset \bigcup_{m=0}^{n-1} U(x, m) \cup V(x, n-1) \subset \dots \subset U(x, 0) \cup V(x, 1) \\ &\subset \text{st}(U(x, 0), \mathcal{U}_0), \end{aligned}$$

so x and $f(x)$ must lie in the closure of $\text{st}(U(x, 0), \mathcal{U}_0)$, which is contained in $\text{st}^2(U(x, 0), \mathcal{U}_0)$, which lies in some member of \mathcal{U} .

LEMMA 2. *If \mathcal{U} is a collection of pairwise disjoint open subsets of X , then there is an open cover \mathcal{V} of \mathcal{U}^* , refining \mathcal{U} , with the property that if for each $U \in \mathcal{U}$ f_U is a homeomorphism of U onto itself which is limited by \mathcal{V} , then the function f defined by $f(x) = f_U(x)$, if $x \in U$, and $f(x) = x$, if $x \notin \mathcal{U}^*$, is a homeomorphism of X onto itself.*

Proof. Let $\mathcal{V} = \{V(x) = \{y \in X \mid d(y, x) < d(z, x)/2 \text{ for each } z \text{ in } X \setminus U\} \mid x \in U \in \mathcal{U}\}$, where $d(\cdot, \cdot)$ is the metric for X . Then for any points z of $X \setminus \mathcal{U}^*$, and y of X , $d(z, f(y)) \leq 3d(z, y)$, which establishes continuity. As f must be one-to-one and onto, and the same argument establishes the continuity of f^{-1} , f is a homeomorphism.

Let \mathcal{K} be an hereditary collection of closed subsets of X which is invariant under the action of $\mathcal{H}(X)$, that is, each closed subset of a member of \mathcal{K} is in \mathcal{K} and $f(K) \in \mathcal{K}$ if $K \in \mathcal{K}$ and $f \in \mathcal{H}(X)$. A set A in X will be termed \mathcal{K} -absorptive if for each open cover \mathcal{U} of a member K of \mathcal{K} and each member K' of \mathcal{K} contained in $K \cap A$, there is a homeomorphism f in $\mathcal{H}(X)$ which is limited by \mathcal{U} , is the identity on K' , and carries K into A . If f may always be chosen so that there is an isotopy from it to the identity which is limited by \mathcal{U} , then A will be called *strongly \mathcal{K} -absorptive*.

LEMMA 3. *If A is \mathcal{K} -absorptive (strongly \mathcal{K} -absorptive), L is an open subset of a member of \mathcal{K} , and \mathcal{U} is an open cover of L in X , then there is a member f of $\mathcal{H}(X)$ carrying L into A which is limited by \mathcal{U} (is isotopic to the identity by an isotopy limited by \mathcal{U}).*

Proof. As \mathcal{U}^* is an open subset of the complete metric space X , it may be given an equivalent metric under which it is itself complete, so Lemma 1 holds under the new metric. Let $\{V_n\}_{n \in N}$ be a sequence of open sets in X such that each contains its successor and $\bigcap_{n \in N} V_n = X \setminus \mathcal{U}^*$, and let \mathcal{W} be a refinement of \mathcal{U} which covers \mathcal{U}^* and has the property that any member of $\mathcal{H}(\mathcal{U}^*)$ which is limited by \mathcal{W} extends to an element of $\mathcal{H}(X)$ which is also limited by \mathcal{W} . If \mathcal{K}' is the collection of all members of \mathcal{K} which lie in \mathcal{U}^* , then from the definition of (strong) \mathcal{K} -absorptivity it is immediate that as a subset of \mathcal{U}^* , $A \cap \mathcal{U}^*$ is (strongly) \mathcal{K}' -absorptive. Using Lemma 1 and the fact that $L \setminus V_{n+1}$ contains $L \setminus V_n$ for all n in N and that both are in \mathcal{K}' , select a sequence $\{f_n\}_{n \in N}$ of members of $\mathcal{H}(\mathcal{U}^*)$ with $\{f_n \circ \dots \circ f_1\}_{n \in N}$ converging to a member of $\mathcal{H}(\mathcal{U}^*)$ which is limited by \mathcal{W} and such that for each n , f_n

carries $f_{n-1} \circ \dots \circ f_1(L \setminus V_n)$ into $A \cap \mathcal{U}^*$ and is the identity on

$$f_{n-1} \circ \dots \circ f_1(L \setminus V_{n-1}).$$

This may be done because each of the functions $f_n \circ \dots \circ f_1$ may be kept limited by \mathcal{W} , which ensures that they permute the elements of \mathcal{K}' . Extending the limit homeomorphism to all of X so that it is the identity off \mathcal{U}^* produces the desired member of $\mathcal{H}(X)$. (In the case that an isotopy is desired, and that A is strongly \mathcal{K} -absorptive, consider the cover $\mathcal{W}' = \{W \times [0, 1] \mid W \in \mathcal{W}\}$ of $\mathcal{U}^* \times [0, 1]$ and construct a level-preserving homeomorphism of \mathcal{U}^* which is limited by \mathcal{W}' , is the identity on $\mathcal{U}^* \times \{0\}$, and carried $L \times \{1\}$ into $A \times \{1\}$. The associated isotopy extends to X .)

A collection \mathcal{A} of members of K will be called a \mathcal{K} -complex if it may be expressed as a countable union $\bigcup_{n=0}^{\infty} \mathcal{A}_n$ of subsets of itself such that $\mathcal{A}^n = \bigcup_{m=0}^n \mathcal{A}_m^*$ is closed for each n and $\mathcal{A}[n] = \{A \setminus \mathcal{A}^{n-1} \mid A \in \mathcal{A}_n\}$ is normal for all n . (Here, $\mathcal{A}^{-1} = \emptyset$.) The set \mathcal{A}^* will be said to admit the structure of a \mathcal{K} -complex. If \mathcal{A}^* is (strongly) \mathcal{K} -absorptive, then it will be referred to as a (strong) \mathcal{K} -absorption base.

THEOREM 1. *If \mathcal{U} is an open cover of X and A^* and B^* are two (strong) \mathcal{K} -absorption bases in X , there is a homeomorphism f of X onto itself (an isotopy F of X), limited by \mathcal{U} , such that $f(A^*) = B^*(F(A^* \times \{1\})) = B^*$.*

Proof. Let $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ and $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$ be \mathcal{K} -complex structures for A^* and B^* respectively. As the construction of an isotopy in the strong case may be handled from the construction of a homeomorphism in the other case as was done in the previous proof, only the latter construction will be made here. It is quite simple. Since \mathcal{K} is invariant under the action of $\mathcal{H}(X)$, so is the collection of (strong) \mathcal{K} -absorption bases. A sequence $f_1, g_1, f_2, g_2, \dots$ of members of $\mathcal{H}(X)$ is to be chosen with $\{g_n^{-1} \circ f_n \circ \dots \circ g_1^{-1} \circ f_1\}_{n \in N}$ converging to an element f of $\mathcal{H}(X)$ which is limited by \mathcal{U} . Furthermore, $f_n(g_{n-1}^{-1} \circ \dots \circ f(\mathcal{A}^n))$ is to be a subset of \mathcal{B}^* , $g_n(\mathcal{B}^n)$ is to be a subset of $f_n \circ g_{n-1}^{-1} \circ \dots \circ g_1^{-1} \circ f_1(\mathcal{A}^*)$, f_n is to be the identity on $g_{n-1}^{-1} \circ f_{n-1} \circ \dots \circ g_1^{-1} \circ f_1(\mathcal{A}^{n-1}) \cup \mathcal{B}^{n-1}$, and g_n is to be the identity on $f_n \circ g_{n-1}^{-1} \circ \dots \circ g_1^{-1} \circ f_1(\mathcal{A}^n) \cup \mathcal{B}^{n-1}$. Then the limit homeomorphism f is limited by \mathcal{U} and $f(\mathcal{A}^*) = \mathcal{B}^*$. The selection of these homeomorphisms may be made inductively so as to satisfy the convergence criterion of Lemma 1 because for each n , $\mathcal{A}[n]$ and $\mathcal{B}[n]$ are normal and \mathcal{A}^{n-1} and \mathcal{B}^{n-1} are closed, so Lemmas 2 and 3 may be applied and the homeomorphisms constructed piecemeal on collections of pairwise disjoint open sets in X .

THEOREM 2. *If U is an open subset of X , A^* is a (strong) \mathcal{K} -absorption base for X , and \mathcal{K}' is the set of all members of \mathcal{K} contained in U , then $A^* \cap U$ is a (strong) \mathcal{K}' -absorption base for U .*

Proof. It has already been remarked that $A^* \cap U$ is (strongly) \mathcal{K}' -absorptive, so all that is necessary is to demonstrate that it admits the structure of a \mathcal{K}' -complex. If $A^* \cap U = \emptyset$, then $\mathcal{K}' = \{\emptyset\}$, and $A^* \cap U$ is a strong \mathcal{K}' -absorption base for U . Otherwise, let $\{V_n\}_{n \in N}$ be a collection of open sets with $X \setminus U \subset V_{n+1} \subset \bar{V}_{n+1} \subset V_n$ for each n , and with $\bigcap_{n \in N} V_n = X \setminus U$. Now, let

$$\mathcal{B}_{2n} = \bigcup_{m=0}^n \{A \setminus V_{2(n-m+1)} \mid A \in \mathcal{A}_m\}$$

and $\mathcal{B}_{2n+1} = \bigcup_{m=0}^{n+1} \{A \setminus V_{2(n-m+1)+1} \mid A \in \mathcal{A}_m\}$. If $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ is denoted by \mathcal{B} , it is apparent that \mathcal{B}^n is closed for each n . To see that $\mathcal{B}[n]$ is normal for each n , let $\{\mathcal{U}_n\}_{n \in N}$ be a collection of sets of mutually disjoint open sets of X with the property that \mathcal{U}_n^* contains $\mathcal{A}[n]^*$ and that for each U in \mathcal{U}_n , $U \cap \mathcal{A}[n]^* \in \mathcal{A}[n]$. Then define $\mathcal{W}_{2n} = \bigcup_{m=0}^n \{U \cap V_{2(n-m)+1} \setminus \bar{V}_{2(n-m+1)+1} \mid U \in \mathcal{U}_m\}$ and

$$\mathcal{W}_{2n+1} = \bigcup_{m=0}^{n+1} \{U \cap V_{2(n-m+1)} \setminus \bar{V}_{2(n-m+2)} \mid U \in \mathcal{U}_m\},$$

for each $n = 0, 1, \dots$. The collections \mathcal{W}_n are composed of pairwise disjoint open sets separating members of $\mathcal{B}[n]$, so \mathcal{B} is a \mathcal{K}' -complex. Since $\mathcal{B}^* = \mathcal{A}^* \cap U$, the proof is complete.

If $\{Y_n\}_{n \in N}$ is a collection of spaces, then $\prod_{n \in N} Y_n$ will denote their Cartesian product. If, for each n , $y_n \in Y_n$, then $\prod_{n \in N} (Y_n, y_n)$ will denote that subset of $\prod_{n \in N} Y_n$ composed of those points with n -th coordinate differing from y_n for at most finitely many n . Also, let \mathcal{C} be a class of spaces which is closed under the operations of taking closed subsets and of taking finite products, and for each space Y , let $\mathcal{C}(Y)$ denote the collection of images of members of \mathcal{C} under closed embeddings in Y .

THEOREM 3. *If $\{X_n\}_{n \in N}$ is a sequence of complete metric spaces and if, for each n , $\mathcal{A}(n)$ is a $\mathcal{C}(X_n)$ -complex, and x_n is a point of $\mathcal{A}(n)^*$, then $\prod_{n \in N} (\mathcal{A}(n)^*, x_n)$ admits the structure of a $\mathcal{C}(\prod_{n \in N} X_n)$ -complex.*

Proof. For each finite subset S of N , let f denote the natural injection of $\prod_{n \in S} X_n$ into $\prod_{n \in N} (X_n, x_n)$. Now, for each ordered n -tuple (m_1, \dots, m_n) of nonnegative integers, each of which is no greater than n , let $\mathcal{B}(n; m_1, \dots, m_n) = \{f(\prod_{i=1}^n A_i) \mid A_i \in \mathcal{A}(i)_{m_i}\}$. Order

the set of all these collections in such a manner that

$$\mathcal{B}(n; m_1, \dots, m_n) \geq \mathcal{B}(n'; m'_1, \dots, m'_n)$$

if $n \geq n'$ or if $n = n'$ and $m_j \geq m'_j$ for all j . The order selected will be isomorphic to the nonnegative integers, so index the \mathcal{B} 's by them in a manner consistent with the above requirements. Let $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$. For each n , \mathcal{B}_n^* is closed, so \mathcal{B}^* is, also. Thus, in order to check that \mathcal{B} is a $\mathcal{C}(\prod_{n \in N} X_n)$ -complex, it is only necessary to verify that $\mathcal{B}[i]$ is normal for each i . However, for n and (m_1, \dots, m_n) such that $\mathcal{B}_i = \mathcal{B}(n; m_1, \dots, m_n)$, and for B in \mathcal{B}_i , $B \setminus \mathcal{B}^{i-1} \subset B \setminus \bigcap_{j=1}^n \mathcal{A}(j)^{m_j-1}$, so if for each n in N and each nonnegative integer m , \mathcal{U}_m^n is an open cover of $\mathcal{A}(n)[m]^*$ in X_n by pairwise disjoint open sets U with the property that

$$U \cap \mathcal{A}(n)[m]^* \in \mathcal{A}(n)[m],$$

then $\mathcal{V}_i = \{\prod_{j=1}^n U_j \times \prod_{j=n+1}^{\infty} X_j \mid U_j \in \mathcal{U}_{m_j}^j \text{ for } j = 1, \dots, n\}$ is a cover of $\mathcal{B}[i]$ by mutually disjoint open sets of $\prod_{n \in N} X_n$ with the property that the intersection of each with $\mathcal{B}[i]^*$ is a member of $\mathcal{B}[i]$. Thus, each $\mathcal{B}[i]$ is normal and \mathcal{B} is a $\mathcal{C}(\prod_{n \in N} X_n)$ -complex. As it is immediate that $\mathcal{B}^* = \prod_{n \in N} (\mathcal{A}(n)^*, x_n)$, the theorem has been proved.

REMARK. It was tacitly assumed above that there were infinitely many X_n 's. Of course, the same proof works for a finite collection.

COROLLARY 1. *If, in the above, $\prod_{n \in N} (\mathcal{A}(n)^*, x_n)$ is (strongly) $\mathcal{C}(\prod_{n \in N} X_n)$ -absorptive, then it is a (strong) $\mathcal{C}(\prod_{n \in N} X_n)$ -absorption base.*

REMARK. It is clear from the definitions that if X and Y are homeomorphic, then any homeomorphism between them carries the $\mathcal{C}(X)$ -complexes to the $\mathcal{C}(Y)$ -complexes and the (strong) $\mathcal{C}(X)$ -absorption bases to the (strong) $\mathcal{C}(Y)$ -absorption bases.

From now on, \mathcal{C} will denote the class of all finite-dimensional compact metric spaces. The next lemma is an extension of Proposition 4.5 of [5] to the nonseparable case and to isotopies. It consists of combining Theorem 4.2 of [3] with the Bartle-Graves Theorem.

LEMMA 4. *If X is an infinite-dimensional Fréchet space and K is a compact subset of X , then for each open cover \mathcal{U} of K there is a second, \mathcal{V} , such that any embedding of K in X which is limited by \mathcal{V} is (invertibly ambient) isotopic to the identity by an isotopy which is limited by \mathcal{U} .*

Proof. For a real number (positive) r and a point x in a metric space, $N(x, r)$ will denote the open ball centered at x with radius r .

Let λ be a Lebesgue number of \mathcal{U} with respect to K , let $\mathcal{V}_1 = \{N(x, \lambda/3) \mid x \in K\}$, and, inductively, for $n > 1$, let

$$\mathcal{V}_n = \{N(x, \lambda/3^{n+5}) \mid x \in \mathcal{V}_{n-1}^*\}.$$

Now, let $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$. If f embeds K in X and is limited by \mathcal{V} , let Y be the closed linear span in X of the image of $F: K \times [0, 1] \rightarrow X$ defined by $F(x, t) = (1 - t)x + tf(x)$. Let $p_Y: X \rightarrow X/Y$ be the canonical projection, and let $q_Y: X/Y \rightarrow X$ be a right inverse for p_Y sending 0 to 0. (This is by the Bartle-Graves Theorem. For a proof see [11].) Now, the function $h_f: X/Y \times Y \rightarrow X$ defined by $h_f = q_Y p_1 + p_2$ is a homeomorphism, where p_1 and p_2 denote the projections onto the first and second factors, respectively.

From the definition of \mathcal{V} , it follows that for each element V of $\text{st}^4(\mathcal{V})$, $V + N(0, \lambda/3)$ is contained in some member of \mathcal{U} , where here “+” denotes the set of all sums of pairs of elements, one from the first set and one from the second. Letting W be a neighborhood of the origin in X/Y which q_Y carries into $N(0, \lambda/3)$, one sees that $h_f(W \times (\mathcal{V}^* \cap Y))$ lies in \mathcal{U}^* and, indeed, that $\{h_f(W \times V) \mid V \in \text{st}^4(\mathcal{V} \mid Y)\}$ refines \mathcal{U} . (Here, $\mathcal{V} \mid Y = \{V \cap Y \mid V \in \mathcal{V}\}$.)

Select a map $g: X/Y \rightarrow [0, 1]$ such that $g^{-1}(0) \supset (X/Y) \setminus W$ and $0 \in g^{-1}(1)$. Since Y is separable and $\mathcal{V}^* \cap Y$ is open in Y , [3] yields an isotopy $G: (\mathcal{V}^* \cap Y) \times [0, 1] \rightarrow \mathcal{V}^* \cap Y$ from the identity homeomorphism at $t = 0$ to an extension to $\mathcal{V}^* \cap Y$ of f at $t = 1$ which is limited by $\text{st}^4(\mathcal{V} \mid Y)$. Then $H: X \times [0, 1] \rightarrow X$ given by

$$H(x, t) = \begin{cases} h_f(p_Y(x), G(p_2 \circ h_f^{-1}(x), t \cdot g \circ p_Y(x))), & \text{if } x \in h_f(W \times (\mathcal{V}^* \cap Y)) \\ x & , \text{ if } x \notin h_f(W \times (\mathcal{V}^* \cap Y)) \end{cases}$$

is the desired isotopy.

Let H be an infinite-dimensional (real) Hilbert space, let E be a complete, orthonormal basis for H , and denote by H_f the collection of all (finite) linear combinations of members of E .

THEOREM 4. H_f is a strong $\mathcal{C}(H)$ -absorption base.

Proof. Two things must be shown, namely, that H_f admits the structure of a $\mathcal{C}(H)$ -complex and that it is strongly $\mathcal{C}(H)$ -adsorptive. To see the first, let \mathcal{A}_0 be the set of all integral linear combinations of members of E . For $n > 0$, let

$$\mathcal{Q}_n = \left\{ Q_n = \left\{ \sum_{m=1}^n t_m e_m \mid t_m \in [0, 1], m = 1, \dots, n \right\} \mid e_1, \dots, e_n \right.$$

are n distinct elements of E $\left. \right\}$,

and let $\mathcal{A}_n = \{A = Q_n + x \mid Q_n \in \mathcal{Q}_n, x \in \mathcal{A}_0\}$. It is readily seen that $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ is a $\mathcal{C}(H)$ -complex with $\mathcal{A}^* = H_f$.

By Lemma 4, in order to demonstrate that H_f is strongly $\mathcal{C}(H)$ -absorptive one must only show that for each member K of $\mathcal{C}(H)$, each open cover \mathcal{U} of K , and for each closed subset K' of $K \cap H_f$, there is an embedding f of K in H_f , limited by \mathcal{U} , which is the identity on K' . Since K is compact, there exists a Lebesgue number λ for \mathcal{U} with respect to K , so one must only find an embedding f of K in H_f which moves no point as much as λ and is the identity on K' . However, the total boundedness of K and the denseness in H of H_f lead to the existence of a sequence $\{e_i\}_{i \in N}$ in E and a sequence $\{n(i)\}_{i \in N}$ in N such that if p_i is the orthogonal projection of H onto the span of $\{e_j\}_{j=n(i-1)+1}^{n(i)}$, then $\|\sum_{i=1}^m p_i(x) - x\| < 2^{-m-2}\lambda$ for each $m \in N$ and $x \in K$. Also, since K is finite-dimensional, for each set S of $2\dim(K) + 2$ distinct elements of E , there is an embedding of K in the unit sphere (=elements of norm one) of the subspace spanned by S . Assume that for each i , $n(i) - n(i-1) \geq 2\dim(K) + 2$, and let f_i be an embedding of K in the unit sphere of the span of $\{e_j\}_{j=n(i-1)+1}^{n(i)}$. Now, let g map K into $[0, 1]$ such that $K' = g^{-1}(0)$, and for each i let h_i map $[0, 1]$ into $[0, 1]$ such that $h_i^{-1}(0) = [0, 1/n(3)]$ and $h_i^{-1}(1) = [1/n(2), 1]$ and for $i > 1$,

$$h_i^{-1}(0) = [1/n(i-1), 1] \cup [0, 1/n(i+2)]$$

and $h_i^{-1}(1) = [1/n(i+1), 1/n(i)]$. Finally, set

$$f(x) = \sum_{i \in N} (\max_{j \geq i} \{h_j \circ g(x)\}) p_i(x) + \sum_{i \in N} 2^{-i-1}\lambda \cdot h_i \circ g(x) f_{i+3}(x).$$

This is the desired embedding.

COROLLARY 2. *If \mathcal{U} is any collection of open sets of H and Y is any $\mathcal{C}(\mathcal{U}^*)$ -absorption base in \mathcal{U}^* , then there is an ambient, invertible isotopy of H onto itself which is limited by \mathcal{U} , is the identity at $t = 0$, and at $t = 1$ is a homeomorphism h_1 such that $h_1(Y) = \mathcal{U}^* \cap H_f$.*

Proof. Lemma 4 shows the equivalence of the concepts of $\mathcal{C}(\mathcal{U}^*)$ -absorption base and strong $\mathcal{C}(\mathcal{U}^*)$ -absorption base, Theorem 4 combined with Theorem 2 gives that $\mathcal{U}^* \cap H_f$ is also a strong $\mathcal{C}(\mathcal{U}^*)$ -absorption base, and Theorem 1 supplies the isotopy on \mathcal{U}^* limited by an open cover given by Lemma 2 which refines \mathcal{U} and has the property that any isotopy limited by it may be extended trivially to one on H .

COROLLARY 3. *Let $\{H_n\}_{n \in N}$ be an indexed, countably infinite collection of copies of H , and let Y be the subspace of $\prod_{n \in N} H_n$ consisting of all points with at most finitely many nonzero coordinates, each of which lies in the appropriate copy of H_f . Then Y is homeomorphic to H_f .*

Proof. It is easy to modify the proof of Theorem 4 to show that Y is $\mathcal{C}(\prod_{n \in N} H_n)$ -absorptive. If the copy of H_f in H_n is denoted by $(H_f)_n$, then $Y = \prod_{n \in N} ((H_f)_n, 0)$, so Corollary 1 applies to show that Y is a $\mathcal{C}(\prod_{n \in N} H_n)$ -absorption base. However, $\prod_{n \in N} H_n$ is homeomorphic to H by a theorem of Bessaga and Pełczyński [4], so by the remark following Corollary 1, Y may be embedded in H as a $\mathcal{C}(H)$ -absorption base. Corollary 2 now applies to finish the proof.

The above result is crucial to [10]. The next two results identify some simplicial complexes whose products with H_f are H_f -manifolds.

THEOREM 5. *If K is a metric simplicial complex and $K \times H$ is an H -manifold, then $K \times H_f$ is an H_f -manifold.*

Proof. By Theorem 3 (the remark after Theorem 3), $K \times H_f$ is a $\mathcal{C}(K \times H)$ -complex, since K is by definition a $\mathcal{C}(K)$ -complex. The strategy of the proof is to show that $K \times H_f$ is a $\mathcal{C}(K \times H)$ -absorption base, to embed $K \times H$ component-wise in H as open subsets (using a theorem of Henderson [8]) and then to use Corollary 2 to find a homeomorphism of the open subsets in question onto themselves throwing the images of $K \times H_f$ onto $H_f \cap$ (the open subsets). Thus, all that is necessary is to establish the $\mathcal{C}(K \times H)$ -absorptivity of $K \times H_f$. In fact, since for each vertex v of K , $\text{st}^0(v, K)$ —the open star of v in K —is a contractible open set, $\text{st}^0(v, K) \times H$ will be homeomorphic to H by [9], so all that is needed is to show that $\text{st}^0(v, K) \times H_f$ is $\mathcal{C}(\text{st}^0(v, K) \times H)$ -absorptive. Therefore, let X be a finite-dimensional compactum of $\text{st}^0(v, K) \times H$, let \mathcal{U} be an open cover of X in $\text{st}^0(v, K) \times H$ and let X' be a closed subset of $X \cap (\text{st}^0(v, K) \times H_f)$. Lemma 4 together with the fact that $\text{st}^0(v, K) \times H$ is homeomorphic to H establishes that it is sufficient to find an embedding of X in $\text{st}^0(v, K) \times H_f$ which is limited by \mathcal{U} , and is the identity on X' . Let λ be a Lebesgue number for \mathcal{U} with respect to X , and let p_H denote the projection of $K \times H$ onto H . As noted in the proof of Theorem 4, there exists a sequence $\{e_i\}_{i \in N}$ in E and another sequence $\{n(i)\}_{i \in N}$ in N such that $n(i) - n(i-1) \geq 2\dim(X) + 2$ for each i and $\|\sum_{i=1}^m p_i \circ p_H(x) - p_H(x)\| < 2^{-m-2}\lambda$ for each $m \in N$ and $x \in X$, the rest of the notation being as in the proof of Theorem 4. Constructing $f_0: X \rightarrow H_f$ by the same method as used in Theorem 4,

except for the substitution of $p_i \circ p_H$ for p_i , and setting $f = (p_K, f_0)$ produces the desired embedding, if p_K denotes the projection of $K \times H$ onto K .

COROLLARY 4. *If K is a metric, locally finite-dimensional, simplicial complex such that no vertex-star contains more vertices than $\dim(H)$, then $K \times H_f$ is an H_f -manifold.*

Proof. By Theorem 4 of [12], $K \times H$ is an H -manifold, so Theorem 5 applies. (This metric is assumed that in the abstract.)

Actually, if a pair (X, Y) of spaces, $Y \subset X$, is called a (H, H_f) -manifold pair provided that X is a paracompact H -manifold and there is an open cover \mathcal{U} of X by sets U for which there are open embeddings $f_U: U \rightarrow H$ such that $f_U(U \cap Y) = f_U(U) \cap H_f$, then the following have been established.

THEOREM 6. *The pair (X, Y) is a (H, H_f) -manifold pair if and only if Y is a $\mathcal{C}(X)$ -complex, X is an H -manifold, and the following weak $\mathcal{C}(X)$ -absorptivity condition is satisfied: For each finite-dimensional compactum C of X , each open cover \mathcal{U} of C , and each compact subset C' of $C \cap Y$, there is an embedding of C in Y which is limited by \mathcal{U} and extends the inclusion of C' . If (X, Z) is another (H, H_f) -manifold pair and \mathcal{V} is an open cover of X , then there is an isotopy of X , limited by \mathcal{V} , from the identity to a pair homeomorphism of (X, Y) onto (X, Z) .*

COROLLARY 5. *If (X, Y) and (X', Y') are (H, H_f) -manifold pairs, then $(X \times X', Y \times Y')$ is an (H, H_f) -manifold pair.*

COROLLARY 6. *If (X, Y) is an (H, H_f) -manifold pair and K is a metric, locally finite-dimensional, simplicial complex such that no vertex-star contains more than $\dim(H)$ vertices, then $(X \times K, Y \times K)$ is an (H, H_f) -manifold pair.*

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