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The well known natural equivalence $[R\times S, T] \cong [R, T^S]$, valid in the category of sets and set mappings, can be derived in various ways in the category of topological spaces and continuous maps, provided suitable topologies are introduced on the product set $R \times S$ and on the set of all continuous maps from S to T. In this paper we will show how to construct topologies of this kind. The ordinary product topology on $R \times S$ and the compact-open topology on T^S will be given their proper setting in this context.

Given the category of topological spaces and continuous maps, we shall write Con (A, B) for the set of morphisms from space A to space B, $A \times B$ for the product of the carrier sets of A and B. If R, S, T are three topological spaces, suppose topologies have been fixed on $R \times S$ and on Con (S, T). $[R \times S, T]$ and $[R, \mathrm{Con}\,(S, T)]$ will denote the sets of continuous maps from $R \times S$ to T and from R to Con (S, T), respectively. (No topologies introduced on these sets.)

As in the category of sets and set mappings, there is a naturally given function Φ from $[R \times S, T]$ to $[R, \mathrm{Con}\,(S, T)]$ defined by

$$f \in [R \times S, T];\ (\Phi f)(r/s) = f(r, s) \qquad (r \in R,\ s \in S)\ .$$

Its inverse is the function Ψ from $[R, \mathrm{Con}(S, T)]$ to $[R \times S, T]$, given by

$$g \in [R, \mathrm{Con}\,(S, T)];\ (\Psi g)(r, s) = g(r/s) \qquad (r \in R,\ s \in S)\ .$$

The problem to be investigated in this paper is the following: what topologies on $R \times S$ and on Con (S, T) will make the couple (Φ, Ψ) a natural equivalence, i.e., will make the functors $-\times S$ and Con $(S, -)$ adjoint functors?

The best known example is probably the use of the product topology on $R \times S$ and of the compact-open topology on Con (S, T), restricting S to locally compact Hausdorff spaces. Starting from this standard situation, two lines of attack on the general problem have been opened in the literature. One can start with the product topology on $R \times S$ and look for conditions on Con (S, T), or else one starts at the other end by using the compact-open topology for Con (S, T), looking for suitable topologies on $R \times S$. (See [1], [2], [3]; the authors do not use categorical language and their aims are somewhat different from ours).

In this paper we shall use a different approach. For the space $\mathrm{Con}(S, T)$ a class K of topologies is chosen generalizing the class of set-open topologies of [1]. Requirements for $\mathrm{Con}(S, T)$ and for $R \times S$ to be functors and for (Φ, Ψ) to be a natural equivalence will on the one hand reduce K to a subclass of admissible topologies on $\mathrm{Con}(S, T)$ and will on the other hand force $R \times S$ to carry topologies uniquely determined by the topologies on $\mathrm{Con}(S, T)$. (The word "admissible" is used in its ordinary sense, not as in [1]).

By a suitable choice of topologies a natural equivalence $[R \times S, T] \cong [R, \mathrm{Con}(S, T)]$ can always be established in more than one way, irrespective of the nature of the spaces R, S, T. There is even a minimal and a maximal nontrivial solution to this problem (given the class K of topologies on $\mathrm{Con}(S, T)$). This will be the content of §'s 2, 3 and 4.

The remaining part of the paper is concerned with the role of the compact-open topology on $\mathrm{Con}(S, T)$ and of the product topology on $R \times S$ in this context. One of the admissible topologies on $\mathrm{Con}(S, T)$ is determined by the compact sets of S; it turns out to be equal to the compact-open topology only in case S satisfies a condition which does not seem to be easily reducible to familiar properties of topological spaces, but holds for well-known classes of spaces, e.g., Hausdorff spaces. Given an admissible topology on $\mathrm{Con}(S, T)$ and the corresponding one on $R \times S$, if the latter is required to be the product topology the space S has to satisfy a local condition related to local compactness.

2. Topologies on $\mathrm{Con}(S, T)$ and on $R \times S$. Notation: R, S, T will always denote topological spaces. Given S, the letter $\mathfrak{S}$ will be used to describe the space of open sets of S as well as the corresponding lattice. $\mathfrak{F}, \mathfrak{G}, \cdots$ will be used for filters on $\mathfrak{S}$, $\mathscr{F}, \mathscr{G}, \mathscr{M}, \mathscr{K}, \cdots$ for families of such filters. If K is any subset of S, $\mathfrak{F}(K)$ will denote the filter of all open sets of S containing K; $\mathfrak{S} = \mathfrak{F}(\varnothing)$ counts as a filter. Finally, Z will stand for a Sierpinski space, i.e., a space consisting of two points z_1, z_2 and with $\varnothing, \{z_1\}, \{z_1, z_2\}$ as its open sets.

Let $\mathscr{F}$ be a family of filters on $\mathfrak{S}$ containing the filter $\mathfrak{S}$ itself. For any $\mathfrak{F} \in \mathscr{F}$ and any open set U of a space T we define

$$\langle \mathfrak{F}, U \rangle = \{f \in \mathrm{Con}(S, T) : f^{-1}U \in \mathfrak{F}\} .$$

By requiring the family of all these sets to be an open subbasis one introduces a topology $\tau(\mathscr{F})$ on $\mathrm{Con}(S, T)$. If all filters of $\mathscr{F}$ are of the form $\mathfrak{F}(K)$, $\tau(\mathfrak{F})$ will be a set-open topology in the sense of [1].

Let $\mathfrak{F}, \mathfrak{G} \in \mathscr{F}$; obviously, $\langle \mathfrak{F}, U \rangle \cap \langle \mathfrak{G}, U \rangle = \langle \mathfrak{F} \cap \mathfrak{G}, U \rangle$, where $\mathfrak{F} \cap \mathfrak{G}$ is the intersection (meet) of the filters $\mathfrak{F}$ and $\mathfrak{G}$. Hence there is no loss of generality by assuming $\mathscr{F}$ to contain all finite intersections of its members; this will be tacitely understood in the sequel.

Next, look at each $\mathfrak{F} \in \mathscr{F}$ as a subset of the space $\mathfrak{S}$ and introduce a topology on $\mathfrak{S}$ by requiring $\mathscr{F}$ to be an open basis.

To determine a topology on $R \times S$ we introduce a special notation. Let $A \subset R \times S$; for each $r \in R$, let ϕr denote the set of all $s \in S$ such that $(r, s) \in A$. The function ϕ from R to the power set of S completely describes A and we shall write $A = [\phi]$.

Suppose topologies have been defined on $\mathfrak{S}$ and on Con (S, T) by means of a family $\mathscr{F}$ of filters on $\mathfrak{S}$, while $R \times S$ carries a topology yet to be specified. Consider the transformations Φ and Ψ introduced in §1. For Φ and Ψ to be natural transformations, the following conditions are obviously necessary: given $f \in [R \times S, T]$, $g \in [R, \mathrm{Con}\,(S, T)]$, $r \in R$, the mappings $\Phi f: R \to \mathrm{Con}\,(S, T)$, $(\Phi f)(r/-): S \to T$ and $\Psi g: R \times S \to T$ must be continuous. We shall say that Φ and Ψ must *preserve continuity.*

THEOREM 1. *Φ and Ψ preserve continuity if and only if the following holds: a subset $[\phi]$ of $R \times S$ is open in the chosen topology if and only if ϕ is a continuous map from R to $\mathfrak{S}$.*

Proof. Assume the conditions on the open sets of $R \times S$. Let U be open in T, $f \in [R \times S, T]$, $r \in R$. Define ϕ by $[\phi] = f^{-1}U$. It is easy to see that for any $\mathfrak{F} \in \mathscr{F}$, $(\Phi f)^{-1}\langle \mathfrak{F}, U\rangle = \phi^{-1}\mathfrak{F}$ and $((\Phi f)(r/-))^{-1}U = \phi r$. Φ preserves continuity if and only if $\phi^{-1}\mathfrak{F}$ is open in R and ϕr is open in S, which is just our assumption. Similarly, for $g \in [R, \mathrm{Con}\,(S, T)]$ define ϕ' by $[\phi'] = (\Psi g)^{-1}U$. Ψ preserves continuity if and only if $[\phi']$ is open in $R \times S$. But $\phi' r = (g(r/-))^{-1}U$ and $\phi'^{-1}\mathfrak{F} = g^{-1}\langle \mathfrak{F}, U\rangle$. Thus, ϕ' is a continuous map from R to $\mathfrak{S}$ and $[\phi']$ is open by assumption.

Suppose Φ, Ψ preserve continuity and let $T = Z$. If $[\phi]$ is any open set of $R \times S$, construct $f: R \times S \to Z$ by putting $f(r, s) = z_1$ for $(r, s) \in [\phi]$, $f(r, s) = z_2$ for $(r, s) \notin [\phi]$. Obviously, $f \in [R \times S, Z]$. Choosing $\{z_1\}$ as the set U and repeating the argument used above we see that ϕ is a continuous map from R to $\mathfrak{S}$. Conversely, let $\phi: R \to \mathfrak{S}$ be continuous. Construct $g: R \to \mathrm{Con}\,(S, Z)$ by $g(r/s) = z_1$ for $s \in \phi r$, $g(r/s) = z_2$ for $s \notin \phi r$. We must show $g \in [R, \mathrm{Con}\,(S, Z)]$. Since $(g(r/-))^{-1}\{z_1\} = \phi r$, we have $g(r/-) \in \mathrm{Con}(S, T)$. For any $\mathfrak{F} \in \mathscr{F}$ and U open in Z, $g^{-1}\langle \mathfrak{F}, U\rangle$ can only be equal to $\emptyset$, R or $\phi^{-1}\mathfrak{F}$. Hence g is continuous. Repeating the argument used to show that Ψ preserves continuity we see that $[\phi]$ is open in $R \times S$.

Theorem 1 restricts the system of filters $\mathscr{F}$ which may be used to define a topology on Con (S, T), because the family of subsets $[\phi]$ of $R \times S$ described by Theorem 1 must satisfy the axioms of a topology. Actually, three of these axioms hold for any family $\mathscr{F}$. The sets $\emptyset$ and $R \times S$ are open in $R \times S$, because they correspond to the constant functions from R to $\mathfrak{S}$ with $\emptyset$ and S as values, respectively.

Let $\phi_1, \phi_2: R \to \mathfrak{S}$ be two continuous functions and define $\phi_1 \cap \phi_2$ by $(\phi_1 \cap \phi_2)r = \phi_1 r \cap \phi_2 r$, the intersection on the right to be taken in S. Obviously, $[\phi_1] \cap [\phi_2] = [\phi_1 \cap \phi_2]$, hence $\phi_1 \cap \phi_2$ must be continuous. But this is always the case, since for any filter $\mathfrak{F}$ of $\mathscr{F}$ we have $(\phi_1 \cap \phi_2)^{-1}\mathfrak{F} = \{r \in R: \phi_1 r \cap \phi_2 r \in \mathfrak{F}\} = \{r \in R: \phi_1 r \in \mathfrak{F} \text{ and } \phi_2 r \in \mathfrak{F}\} = \phi_1^{-1}\mathfrak{F} \cap \phi_2^{-1}\mathfrak{F}$, which is open in $R \times S$. There remains the union axiom. For any family $\phi_i: R \to \mathfrak{S}(i \in I)$ of continuous maps define $\cup \phi_i$ by $(\cup \phi_i)r = \cup \phi_i r(i \in I)$. Because $\cup [\phi_i] = [\cup \phi_i]$, $\cup \phi_i$ must be continuous. We investigate this requirement.

A filter $\mathfrak{F}$ of $\mathfrak{S}$ will be called *compact*, if for any system $A_i(i \in I)$ of open subsets of S, whenever $\cup A_i \in \mathfrak{F}(i \in I)$, there is a finite subset $K \subset I$ such that $\cup A_i \in \mathfrak{F}(i \in K)$. If $\mathfrak{F}$ and $\mathfrak{G}$ are compact, so is $\mathfrak{F} \cap \mathfrak{G}$. If $\mathfrak{F}$ is of the form $\mathfrak{F}(K)$, it is a compact filter if and only if K is a compact subset of S. Filters of this kind will be called *compactly generated.*

A family of filters $\mathscr{F}$ may satisfy the following condition:

(A) For any two open subsets A_1, A_2 of S and for any $\mathfrak{F} \in \mathscr{F}$, if $A_1 \cup A_2 \in \mathfrak{F}$, there are filters $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathscr{F}$ such that $A_1 \in \mathfrak{G}_1$, $A_2 \in \mathfrak{G}_2$, $\mathfrak{G}_1 \cap \mathfrak{G}_2 \subset \mathfrak{F}$.

THEOREM 2. *$\cup \phi_i(i \in I)$ is a continuous map from R to $\mathfrak{S}$, for any R, if and only if $\mathscr{F}$ is a system of compact filters satisfying* (A).

Proof. Let $\phi_i: R \to \mathfrak{S}(i \in I)$ be a family of continuous maps. For $\mathfrak{F} \in \mathscr{F}$, define the set $F = (\cup \phi_i)^{-1}\mathfrak{F} = \{r \in R: \cup(\phi_i r) \in \mathfrak{F}(i \in I)\}$. F must be shown to be open in R.

Assume $\mathfrak{F}$ compact; for each $r \in F$ there will be a finite subset $K_r \subset I$ such that $\cup(\phi_i r) \in \mathfrak{F}(i \in K_r)$. Write $F_r = \{s \in R: \cup(\phi_i s) \in \mathfrak{F}(i \in K_r)\}$. By definition, $F \subset \cup F_r(r \in F)$. On the other hand, for any $s \in F_r$ we have $\cup(\phi_i s) \subset \cup(\phi_j s)(i \in K_r, j \in I)$ and because $\mathfrak{F}$ is a filter, $\cup(\phi_j s) \in \mathfrak{F}$, i.e., $s \in F$. Hence $F = \cup F_r$ and to show F open it will be sufficient to consider only finite families of continuous maps ϕ_i. As the general finite case follows by induction, two functions ϕ_1, ϕ_2 will suffice.

$\phi_1 \cup \phi_2$ is continuous if and only if the set $G = (\phi_1 \cup \phi_2)^{-1}\mathfrak{F} = \{r \in R: \phi_1 r \cup \phi_2 r \in \mathfrak{F}\}$ is open in R for any $\mathfrak{F} \in \mathscr{F}$. Assume $\mathscr{F}$ to satisfy condition (A). To any $r \in G$ there correspond filters $\mathfrak{G}_r, \mathfrak{H}_r \in \mathscr{F}$ such that $\phi_1 r \in \mathfrak{G}_r$, $\phi_2 r \in \mathfrak{H}_r$, $\mathfrak{G}_r \cap \mathfrak{H}_r \subset \mathfrak{F}$. We claim $G = \cup(\phi_1^{-1}\mathfrak{G}_r \cap \phi_2^{-1}\mathfrak{H}_r)(r \in G)$. By definition, G is contained in the right-hand set. Let s be an element of $\phi_1^{-1}\mathfrak{G}_r \cap \phi_2^{-1}\mathfrak{H}_r$, for some $r \in R$. Then

$$\phi_1 s \in \mathfrak{G}_r,\ \phi_2 s \in \mathfrak{H}_r,\ \phi_1 s \cup \phi_2 s \in \mathfrak{H}_r \cap \mathfrak{G}_r \subset \mathfrak{F}\ ,$$

hence $s \in G$. This proves equality and shows G to be open.

For the converse, let $\mathfrak{F} \in \mathscr{F}$ and let $A_i(i \in I)$ be a family of open sets of S such that $\cup A_i \in \mathfrak{F}(i \in I)$. Consider the space $R = \mathfrak{S}^I$, equipped with the product topology and write $\pi_i\colon R \to \mathfrak{S}(i \in I)$ for the canonical projections. An open basis of R consists of finite intersections of sets $\pi_i^{-1}\mathfrak{G}$, where $\mathfrak{G} \in \mathscr{F}$. The functions π_i are continuous. By hypothesis, the same is true for $\cup \pi_i(i \in I)$. Thus the set $H = \{u \in R\colon \cup(\pi_i u) \in \mathfrak{F}\ (i \in I)\}$ is open in R and each point of H belongs to a basic open set contained in H.

Define $a \in R$ by $\pi_i a = A_i$ for all $i \in I$. Then $a \in H$ and there are a finite set $J \subset I$ as well as filters $\mathfrak{H}_j \in \mathscr{F}$ such that $a \in \cap \pi_j^{-1}\mathfrak{H}_j \subset H$ $(j \in J)$. This implies $\pi_j a \in \mathfrak{H}_j$ for all $j \in J$. Define $b \in R$ by $\pi_j b = A_j$ $(j \in J)$, $\pi_i b = \varnothing(i \in I - J)$. For $j \in J$, $\pi_j b = \pi_j a$, thus $b \in \cap \pi_j^{-1}\mathfrak{H}_j$. Since $\cap \pi_j^{-1}\mathfrak{H}_j$ is a subset of H, $b \in H$ and $\cup \pi_i b = \cup A_j \in \mathfrak{F}(i \in I, j \in J)$. This shows $\mathfrak{F}$ to be compact.

Finally, let I be the set $\{1, 2\}$, otherwise using the same notation as above. The element $a \in R = \mathfrak{S} \times \mathfrak{S}$, defined by $\pi_1 a = A_1$, $\pi_2 a = A_2$ will again satisfy $a \in \pi_1^{-1}\mathfrak{H}_1 \cap \pi_2^{-1}\mathfrak{H}_2 \subset H$ for two appropriately chosen filters $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathscr{F}$; thus $A_1 \in \mathfrak{H}_1$, $A_2 \in \mathfrak{H}_2$. Let the set C be a member of $\mathfrak{H}_1 \cap \mathfrak{H}_2$ and consider the element $c \in R$, defined by $\pi_1 c = \pi_2 c = C$. Because $c \in \pi_1^{-1}\mathfrak{H}_1 \cap \pi_2^{-1}\mathfrak{H}_2 \subset H$, we have $\pi_1 c \cup \pi_2 c = C \in \mathfrak{F}$. Since this holds for any such C, $\mathfrak{H}_1 \cap \mathfrak{H}_2 \subset \mathfrak{F}$ and $\mathscr{F}$ satisfies condition (A).

A system of compact filters satisfying (A) will be called an *adjoining system*. Suppose all filters of a system $\mathscr{F}$ are compactly generated. Our next theorem describes condition (A) in this case.

For any subset X of S define X^* to be the intersection of all open sets containing X. Obviously, $(X_1 \cup X_2)^* = X_1^* \cup X_2^*$; if K is compact in S, the same is true for K^*, because any open cover of K^* is also an open cover of K, hence contains a finite subcover which must be above K^*.

We shall call a compact set K for which $K = K^*$, *fully compact*. All compact sets of a topological space are fully compact if and only if the space is T_1.

Let $\mathfrak{F}(K)$ be a compactly generated filter; it is an immediate consequence of our definition that $\mathfrak{F}(K) = \mathfrak{F}(K^*)$. Moreover, $\mathfrak{F}(K_1^*) \cap \mathfrak{F}(K_2^*) = \mathfrak{F}((K_1 \cup K_2)^*)$ and $\mathfrak{S} = \mathfrak{F}(\varnothing) = \mathfrak{F}(\varnothing^*)$. One may therefore assume that the compact sets generating the filters of the given family $\mathscr{F}$ are all fully compact.

We need one more fact, easily seen to hold: if K and L are compact sets in S, then $\mathfrak{F}(K) \subset \mathfrak{F}(L)$ if and only if $L^* \subset K^*$.

THEOREM 3. *Let $\mathscr{F}$ be a family of compactly generated filters on $\mathfrak{S}$, and let $\mathfrak{K}$ denote the family of fully compact sets of S which generate the filters of $\mathscr{F}$. Then $\mathscr{F}$ satisfies condition (A) if and only if the following holds: if A_1, A_2 are open sets of S and if $K \in \mathfrak{K}$ is such that $K \subset A_1 \cup A_2$, there are $K_1, K_2 \in \mathfrak{K}$ with $K_1 \subset A_1$, $K_2 \subset A_2$,*

$K \subset K_1 \cup K_2$.

Theorem 3 is an immediate consequence of the facts established above.

3. **Functorial requirements.** Given an adjoining system $\mathscr{F}$ on $\mathfrak{S}$, Theorems 1 and 2 show that $\mathscr{F}$ uniquely determines topologies on Con (S, T) as well as on $R \times S$ such that Φ and Ψ preserve continuity. (Φ, Ψ) determine a natural equivalence between $[R \times S, T]$ and $[R, \mathrm{Con}\,(S, T)]$ if, given the space S, Con (S, T) and $R \times S$ are (object mappings of) functors in T and R, respectively. This will now be shown.

Let T, T' be topological spaces, $p: T \rightarrow T'$ a continuous map. As usual, the morphism mapping $C(p)$: Con $(S, T) \rightarrow$ Con (S, T') of the functor Con $(S, -)$ will be defined by $C(p)f = pf$, for any $f \in \mathrm{Con}\,(S, T)$. It must be shown that $C(p)$ is continuous with respect to the topologies defined on Con (S, T) and on Con (S, T'). For U' open in T' and for $\mathfrak{F} \in \mathscr{F}$ we have $C(p)^{-1}\langle\mathfrak{F}, U'\rangle = <\mathfrak{F}, p^{-1}U'>$, which implies continuity of $C(p)$.

Similarly, let R, R', $q: R \rightarrow R'$ be given, q continuous. The morphism mapping $P(q): R \times S \rightarrow R' \times S$ of the functor $- \times S$ is defined by $P(q)(r, s) = (qr, s)$, for any $(r, s) \in R \times S$. We show that $P(q)$ is continuous with respect to the topologies chosen for $R \times S$ and $R' \times S$. Let $[\phi']$ be open in $R' \times S$ and define $\phi: R \rightarrow S$ by $\phi = \phi' q$. It is easy to see that $P(q)^{-1}[\phi'] = [\phi]$, which is open in $R \times S$ by Theorem 2.

The variety of adjoining systems for a given space S depends of course on the nature of this space. To obtain a categorically significant simultaneous choice of adjoining systems $\mathscr{F}(S)$ for each space S we now require Con (S, T) and $R \times S$ to be functors in S as well. To investigate this requirement we have to define two operations on systems $\mathscr{F}$ and list their properties.

Let $\mathscr{F}$ be a family of filters on $\mathfrak{S}$ and let $\mathfrak{F}_i \in \mathscr{F}$, $i \in I$. In general, $\cup \mathfrak{F}_i (i \in I)$ will not be a filter. We want to adjoin to $\mathscr{F}$ all unions of its members which are themselves filters. Hence we define $\overline{\mathscr{F}}$ to be the family of all filters on $\mathfrak{S}$ which can be written as unions of filters belonging to $\mathscr{F}$. If all filters of $\mathscr{F}$ are compact, the same holds for $\overline{\mathscr{F}}$, and if $\mathscr{F}$ satisfies condition (A) of §2, so does $\overline{\mathscr{F}}$. Consequently, if $\mathscr{F}$ is adjoining, then $\overline{\mathscr{F}}$ is also adjoining. $\overline{\mathscr{F}}$ can actually be larger than $\mathscr{F}$. The following example will incidentally prove the existence of compact, not compactly generated filters.

Let the space S consist of the set of natural numbers N together with an element $w \notin N$. Write $S_0 = N$, $S_n = N - \{0, 1, 2, \cdots, n-1\}$ $(n \geqq 1)$. Open sets of S shall be the sets S_n, $\{w\}$, $S_n \cup \{w\}$ and the empty set. Write $\mathfrak{F}(n)$ for the filter generated by the compact set $\{n\}$ on $\mathfrak{S} (n \in N)$. Then $\mathfrak{F} = \cup \mathfrak{F}(n) (n \in N)$ is a filter, as is easily seen.

If it were compactly generated by a fully compact set $K \subset S$, the intersection of all its members would be K. But this intersection is the empty set $\varnothing$, while $\mathfrak{F} \neq \mathfrak{F}(\varnothing) = \mathfrak{S}$, since $\{w\} \notin \mathfrak{F}$. Hence for $\mathscr{F} = \{\mathfrak{F}(n): n \in N\}$, $\mathscr{F}$ is properly contained in $\overline{\mathscr{F}}$.

THEOREM 4. *Let $\mathscr{F}, \mathscr{G}$ be two (not necessarily adjoining) families of filters on a space $\mathfrak{S}$, and let $\tau(\mathscr{F}, T)$, $\tau(\mathscr{G}, T)$ be the corresponding topologies on the space* Con (S, T). *Writing $\sigma \leqq \tau$, if the topology τ is finer than the topology σ, we have $\tau(\mathscr{G}, T) \leqq \tau(\mathscr{F}, T)$ for all T if and only if $\mathscr{G} \subset \overline{\mathscr{F}}$.*

COROLLARY. *$\tau(\mathscr{F}, T) = \tau(\overline{\mathscr{F}}, T)$ for all T.*

Proof. Let $T = Z$, a Sierpinski space, and let $\mathfrak{G} \in \mathscr{G}$. The set $\langle \mathfrak{G}, \{z_1\} \rangle$ is not empty, since the constant function $f: S \to Z$ with value z_1 belongs to it. On the other hand $\langle \mathfrak{G}, \{z_1\} \rangle = \text{Con}(S, Z)$ if and only if $\mathfrak{G} = \mathfrak{S}$. Suppose $\mathfrak{G} \neq \mathfrak{S}$ and assume $\tau(\mathscr{G}, Z) \leqq \tau(\mathscr{F}, Z)$. Then $\langle \mathfrak{G}, \{z_1\} \rangle = \cup \langle \mathfrak{F}_i, \{z_1\} \rangle (i \in I)$, where $\mathfrak{F}_i \in \mathscr{F}$. We want to show $\mathfrak{G} = \cup \mathfrak{F}_i$. For $A \in \mathfrak{G}$, define $f \in \text{Con}(S, Z)$ by $fa = z_1 (a \in A)$, $fa = z_2 (a \in S - A)$. Because $f \in \langle \mathfrak{G}, \{z_1\} \rangle$, there is an $i \in I$ such that $f \in \langle \mathfrak{F}_i, \{z_1\} \rangle$, which implies $A \in \mathfrak{F}_i$ and $\mathfrak{G} \subset \cup \mathfrak{F}_i$. By a similar argument, $\cup \mathfrak{F}_i \subset \mathfrak{G}$. This proves $\mathscr{G} \subset \overline{\mathscr{F}}$.

The converse is an easy consequence of the definitions. The corollary is implied by the fact that $\mathscr{F}$ is a basis for the topology $\tau(\mathscr{F}, T)$.

Let S, S' be two topological spaces and let $q: S \to S'$ be continuous. For any filter $\mathfrak{F}$ on $\mathfrak{S}$ define the subset $q'\mathfrak{F}$ of $\mathfrak{S}'$ by $q'\mathfrak{F} = \{A' \in \mathfrak{S}': q^{-1}A' \in \mathfrak{F}\}$. $q'\mathfrak{F}$ is a filter on $\mathfrak{S}'$, and if $\mathfrak{F}$ is compact, so is $q'\mathfrak{F}$; this follows easily from the algebraic properties of q^{-1}. If $\mathfrak{F} = \mathfrak{F}(K)$, then $q'\mathfrak{F}(K) = \mathfrak{F}(qK)$; if, therefore, $\mathfrak{F}$ is compactly generated, so is $q'\mathfrak{F}$.

Given a family $\mathscr{F}$ of filters on $\mathfrak{S}$, let us write $q'\mathscr{F}$ for the family of filters $q'\mathfrak{F}$, $\mathfrak{F} \in \mathscr{F}$. If $\mathscr{F}$ satisfies condition (A) of §2, the same is true for $q'\mathscr{F}$. This again is an immediate consequence of the definitions. Together with the facts noted above we have: if $\mathscr{F}$ is an adjoining system, then $q'\mathscr{F}$ is also adjoining.

Returning to the investigation of the functorial requirements, let T be a fixed topological space and consider Con (S, T) as the object mapping of the (contravariant) functor Con $(-, T)$. For a continuous $q: S \to S'$, define the morphism mapping $D(q): \text{Con}(S', T) \to \text{Con}(S, T)$ by $D(q)f' = f'q$ $(f' \in \text{Con}(S', T))$. The topologies on the two spaces Con (S, T) and Con (S', T) are assumed to be given by adjoining systems $\mathscr{F}$ and $\mathscr{F}'$, respectively. $D(q)$ must be continuous, i.e., for each $\mathfrak{F} \in \mathscr{F}$ and for each open subset $U \subset T$, the set $D(q)^{-1} \langle \mathfrak{F}, U \rangle$ has to be open in Con (S', T). It is easy to see that $D(q)^{-1} \langle \mathfrak{F}, U \rangle = \langle q'\mathfrak{F}, U \rangle$, which is a basic open set of the topology defined by $q'\mathscr{F}$ on Con (S', T). Thus, this topology must be coarser than the given one. Conversely, if this is true, $D(q)$ will be continuous. Theorem 4 now implies

THEOREM 5. *If for each space* Con (S, T), T *fixed, a topology is defined by the choice of an adjoining system* $\mathscr{F}(S)$ *on the corresponding space* S, *then* Con $(-, T)$ *together with the mapping* D *is a functor if and only if for each continuous map* $q: S \to S'$ *the relation* $q'\mathscr{F}(S) \subset \overline{\mathscr{F}(S')}$ *holds.*

We shall describe this relation briefly by "$\mathscr{F}(S)$ is functorial".

If adjoining systems have been chosen according to Theorem 5, then $R \times S$ with the topology induced by $\mathscr{F}(S)$ is also, for fixed R, (the object mapping of) a functor. For any continuous $q: S \to S'$ define the morphism mapping $Q(q): R \times S \to R \times S'$ by $Q(q)(r, s) = (r, qs)$ $((r, s) \in R \times S)$. $Q(q)$ is continuous. To see this, let $[\phi']$ be open in $R \times S'$, where $\phi': R \to \mathfrak{S}'$. The inverse q^{-1} of q induces a mapping $q^{-1}: \mathfrak{S}' \to \mathfrak{S}$; define $\phi: R \to \mathfrak{S}$ by $\phi = q^{-1}\phi'$. Then $Q(q)^{-1}[\phi'] = [\phi]$, as is easy to see. For any $\mathfrak{F} \in \mathscr{F}(S)$, we have $\phi^{-1}\mathfrak{F} = \phi'^{-1}(q'\mathfrak{F})$. By assumption, $q'\mathfrak{F}$ is open in the space $\mathfrak{S}'$ with respect to the topology given by $\mathscr{F}(S')$. This shows $\phi^{-1}\mathfrak{F}$ to be open in $\mathfrak{S}$, ϕ to be continuous, $[\phi]$ to be open in $R \times S$ and finally $Q(q)$ to be continuous.

4. Minimal and maximal adjoining systems. Let the variable E range over the finite subsets of a topological space S. By $\mathscr{E}$ we shall denote the family of all filters $\mathfrak{F}(E)$. $\mathscr{E}$ is adjoining: every $\mathfrak{F}(E)$ is compactly generated, and if A, B are open sets in S with $A \cup B \in \mathfrak{F}(E)$ for some E, then $A \in \mathfrak{F}(A \cap E)$, $B \in \mathfrak{F}(B \cap E)$, $\mathfrak{F}(A \cap E) \cap \mathfrak{F}(B \cap E) = \mathfrak{F}((A \cup B) \cap E) = \mathfrak{F}(E)$. It is also easy to see that $\mathscr{E}(S)$ is functorial: for any continuous $q: S \to S'$, $q'\mathfrak{F}(E) = \mathfrak{F}(qE)$, hence $q'\mathscr{E}(S) \subset \mathscr{E}(S') \subset \overline{\mathscr{E}(S')}$.

Before stating the next theorem we need a preliminary discussion. Let S, S' be two topological spaces. Given a point $t \in S'$, write q_t for the constant map $q_t: S \to S'$ with value t. If $\mathfrak{F}$ is a filter on $\mathfrak{S}$, then $q_t'\mathfrak{F} = \mathfrak{F}(\{t\})$, provided $\mathfrak{F} \neq \mathfrak{S}$; otherwise, $q_t'\mathfrak{S} = \mathfrak{S}'$. This follows from $q_t^{-1}A' = S(t \in A' \subset S')$, $q_t^{-1}A' = \varnothing (t \notin A')$.

Consider the system of filters $\mathscr{T}$ on $\mathfrak{S}$, consisting of the filter $\mathfrak{S} = \mathfrak{F}(\varnothing)$ alone. It is a trivial fact that $\mathscr{T}$ is adjoining and $\mathscr{T}(S)$ is functorial. One can prove a slightly stronger result: if $\mathscr{F}(S)$ is functorial and $\mathscr{F}(S_0) = \mathscr{T}(S_0)$ for some space S_0, then $\mathscr{F}(S) = \mathscr{T}(S)$ for all spaces S. Let $t \in S_0$ (we exclude the empty space); for any space S, consider the constant function $q_t: S \to S_0$. Since $\mathscr{F}(S)$ is functorial, we must have $q_t'\mathscr{F}(S) \subset \overline{\mathscr{T}(S_0)} = \mathscr{T}(S_0)$, hence $q_t'\mathfrak{F} = \mathfrak{S}_0$ for all $\mathfrak{F} \in \mathscr{F}(S)$. By our discussion above, this forces $\mathfrak{F} = \mathfrak{S}$ and

$\mathscr{F}(S) = \mathscr{T}(S)$. Of course, the choice of $\mathscr{T}(S)$ leads to a trivial solution of our original problem: Con (S, T) carries the indiscrete, $R \times S$ the discrete topology.

THEOREM 6. *$\mathscr{E}(S)$ is a minimal adjoining system for any space S, in the following sense: if $\mathscr{F}(S)$ is a functorial choice of adjoining systems different from $\mathscr{T}(S)$, then $\mathscr{E}(S) \subset \overline{\mathscr{F}(S)}$.*

Proof. Given S, let $s \in S$ and consider the constant map $q_s: S \rightarrow S$. By assumption, there is a filter $\mathfrak{F} \in \mathscr{F}(S)$ with $\mathfrak{F} \neq \mathfrak{S}$. Since $q'_s\mathfrak{F} = \mathfrak{F}(\{s\})$, Theorem 5 implies $\mathfrak{F}(\{s\}) \in \overline{\mathfrak{F}}$ and $\mathscr{E} \subset \overline{\mathscr{F}}$.

The topology induced by $\mathscr{E}$ on Con (S, T) is equal to the subspace topology with respect to the space T^S of all functions from S to T, carrying the ordinary product topology. In keeping with the term "set-open" a name sometimes used for this topology is "*point-open*"; it is called p-topology in [3], usually also the topology of pointwise convergence.

The topology on $R \times S$ corresponding to $\mathscr{E}$ can be described in different ways. Suppose $r: R \rightarrow \mathfrak{P}S$ is a mapping from R to the power set of S, $s: S \rightarrow \mathfrak{P}R$ a mapping from S to the power set of R. We shall call r, s *reciprocal,* if for any $v \in S$, $s(v) = \{u \in R: v \in r(u)\}$, or equivalently, if for any $u \in R$, $r(u) = \{v \in S: u \in s(v)\}$. Obviously, to each r there corresponds exactly one reciprocal s, and conversely. If r maps R to points of S, s is simply the inverse mapping r^{-1}.

Suppose r is a map from R to $\mathfrak{S}$, the space of open subsets of S. We shall call r and its reciprocal s, *topologically reciprocal,* if the same is true for s, i.e., if s maps S into $\mathfrak{R}$, the space of open sets of R.

The topology on $R \times S$, induced by the adjoining system $\mathscr{E}(S)$, can now be described as follows: a subset $[\phi]$ of $R \times S$ is open in this topology if and only if ϕ and its reciprocal ψ are topologically reciprocal.

The filters $\mathfrak{F}(\{s\})(s \in S)$ constitute an open subbasis for the topology given by $\mathscr{E}$ on $\mathfrak{S}$. Because $[\phi]$ is open if and only if $\phi: R \rightarrow \mathfrak{S}$ is continuous, $[\phi]$ will be open if and only if $\phi^{-1}\mathfrak{F}(\{s\}) = \{r \in R: s \in \phi r\} = \psi s$ is open in R. This is exactly what we have claimed.

Another description of the topology on $R \times S$ has been given by R. Brown (see [2]), who also briefly comments on its connection with the point-open topology on Con (S, T). ([2], Remark 1.15).

To obtain maximal adjoining systems, we first show how to construct new systems out of a family of given ones. Let $\mathscr{F}_i (i \in I)$ be a family of adjoining systems on a space $\mathfrak{S}$. Define $\cup^* \mathscr{F}_i (i \in I)$ as the set of all filters which can be written as finite intersections of filters belonging to $\cup \mathscr{F}_i$. By a straightforward application of the definitions involved one proves easily: $\cup^* \mathscr{F}_i (i \in I)$ is an adjoining system; if all filters belonging to $\cup \mathscr{F}_i$ are compactly generated, the same holds for the filters of $\cup^* \mathscr{F}_i$.

This construction allows to define two distinguished adjoining systems on any space $\mathfrak{S}$. The first one is the system $\mathscr{M}$, obtained by applying our construction to the family of all adjoining systems on $\mathfrak{S}$. $\mathscr{M}$ is of course the maximal adjoining system on $\mathfrak{S}$; furthermore, $\mathscr{M}(S)$ is functorial.

The second distinguished system will be denoted by $\mathscr{K}$; it is constructed from all adjoining systems consisting of compactly generated filters only. Since $\mathscr{E}$ is such a system, $\mathscr{K}$ is not empty. $\mathscr{K}(S)$ is functorial: for any continuous $q: S \to S'$ and any compactly generated filter $\mathfrak{F}(K)$ we have $q'\mathfrak{F}(K) = \mathfrak{F}(qK)$; qK is compact.

Note that other adjoining systems satisfying the functorial requirement can be obtained by cardinality arguments. Consider for instance, on a space $\mathfrak{S}$, the family of all adjoining systems of compactly generated filters, where for each such filter a generator may be found with cardinality $\leqq m$, m a fixed infinite cardinal. Application of our construction to this family yields an adjoining and functorial system $\mathscr{K}_m(S)$ for each space S.

5. The compact-open topology.

The compact-open topology on a space $\mathrm{Con}(S, T)$ is defined by the system $\mathscr{K}_0$ of all compactly generated filters on $\mathfrak{S}$. $\mathscr{K}_0$ need not be adjoining, as will be shown presently; hence the compact-open topology does not always provide a solution to our problem. However, due to the importance of this topology for the theory of function spaces an investigation of spaces S for which the compact-open topology on $\mathrm{Con}(S, T)$ is induced, for any T, by some adjoining system is indicated.

Let $\mathscr{F}$ be such a system. According to Theorem 4, we have $\overline{\mathscr{F}} = \overline{\mathscr{K}_0}$; conversely, a system $\mathscr{F}$ satisfying this equality defines, for any T, the compact-open topology on $\mathrm{Con}(S, T)$.

THEOREM 7. *If $\overline{\mathscr{F}} = \overline{\mathscr{K}_0}$ for some adjoining system $\mathscr{F}$ on $\mathfrak{S}$, the space S satisfies*:

(D) *If K is a compact subset, and if A_1, A_2 are open subsets of S such that $K \subset A_1 \cup A_2$, there are compact sets K_1, K_2 in S with $K_1 \subset A_1$, $K_2 \subset A_2$, $K \subset K_1 \cup K_2$.*

Conversely, if (D) *holds, the adjoining system $\mathscr{K}$ defined in* § 4 *equals $\mathscr{K}_0$.*

Proof. Suppose $\overline{\mathscr{F}} = \overline{\mathscr{K}_0}$ and let K, A_1, A_2 satisfy the hypothesis of condition (D). Then $A_1 \cup A_2 \in \mathfrak{F}(K)$ and $\mathfrak{F}(K) \in \overline{\mathscr{F}}$. Since $\overline{\mathscr{F}}$ is adjoining, there exist filters $\mathfrak{F}_1, \mathfrak{F}_2 \in \overline{\mathscr{F}}$ with $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$, $\mathfrak{F}_1 \cap \mathfrak{F}_2 \subset$

$\mathfrak{F}(K)$. $\mathfrak{F}_1$, $\mathfrak{F}_2$ being members of $\overline{\mathscr{K}_0}$, there are compact sets L_{1i}, L_{2j} $(i \in I, j \in J)$ such that $\mathfrak{F}_1 = \cup \mathfrak{F}(L_{1i})$, $\mathfrak{F}_2 = \cup \mathfrak{F}(L_{2j})(i \in I, j \in J)$. $A_1 \in \mathfrak{F}_1$ implies $L_1 \subset A_1$, where L_1 is one of the sets L_{1i}; similarly, $L_2 \subset A_2$. Furthermore, $\mathfrak{F}_1 \cap \mathfrak{F}_2 = \cup (\mathfrak{F}(L_{1i}) \cap \mathfrak{F}(L_{2j})) = \cup \mathfrak{F}(L_{1i} \cup L_{2j})$, thus $\mathfrak{F}(L_1 \cup L_2) \subset \mathfrak{F}(K)$. As was shown in §2, the latter relation is equivalent to $K \subset K^* \subset L_1^* \cup L_2^*$. By letting $K_1 = L_1^*$, $K_2 = L_2^*$ we have established the first part of the theorem.

Suppose (D) holds in a space S; then it also holds if "compact" is replaced by "fully compact". Theorem 3 now shows $\mathscr{K}_0$ to be adjoining. Obviously, $\overline{\mathscr{K}_0} = \overline{\mathscr{K}}$.

Examples of spaces satisfying (D) are easily provided: discrete, indiscrete and pseudo-finite spaces (all compact sets finite; see [4] or [5]) are instances in case. We give a sufficient condition for (D) to hold, which covers some important classes of spaces.

We shall use the following notation: $T_i (i = 1, 2, 3, 4, 5)$ denotes the usual separation axioms, where for $i \geqq 2$ we do not assume T_1. A space is called a KC space (according to [5]), if every compact set is closed; it will be called a K^*C space if every fully compact set is closed. This is equivalent to: if K is compact, A open, and if $K \subset A$, then $\bar{K} \subset A$ (where $\bar{K}$ is the closure of K). Finally, a space will be said to be KT_4, if every compact subspace is a T_4 space.

THEOREM 8. *Any KT_4 space satisfies condition* (D). *If the space is a K^*C space, KT_4 is equivalent to* (D).

Proof. Let K be a compact subset, A_1, A_2 open subsets of a KT_4 space S, such that $K \subset A_1 \cup A_2$. The sets $K - A_1$, $K - A_2$ are closed subsets of the subspace K and $(K - A_1) \cap (K - A_2) = \varnothing$. By assumption, there are open sets B_1, B_2 in S such that $K - A_1 \subset B_1$, $K - A_2 \subset B_2$, $K \cap B_1 \cap B_2 = \varnothing$. Writing $K_1 = K - B_1$, $K_2 = K - B_2$, we have $K_1 \subset A_1$, $K_2 \subset A_2$ and $K_1 \cup K_2 = K$. The sets K_1, K_2 being closed in the compact subspace K, they are compact in S.

Let S be a K^*C space satisfying (D), and let K be a compact subspace of S. If C_1, C_2 are closed disjoint subsets of K, write D_1, D_2 for their complements in S. Because $K \subset D_1 \cup D_2$ and D_1, D_2 are open, there are compact sets L_1, L_2 in S with $L_1 \subset D_1$, $L_2 \subset D_2$, $K \subset L_1 \cup L_2$. By assumption, $\bar{L}_1 \subset D_1$, $\bar{L}_2 \subset D_2$. Writing M_1, M_2 for the complements of $\bar{L}_1, \bar{L}_2$ in S, we have $C_1 \subset M_1$, $C_2 \subset M_2$, $K \cap M_1 \cap M_2 = \varnothing$. Hence K is a T_4 space.

It is easy to see that T_2, T_3 and T_5 spaces are KT_4 spaces and thus satisfy (D). This is trivial for T_5. Any subspace of a $T_2(T_3)$ space is itself $T_2(T_3)$; as is well known, a compact $T_2(T_3)$ space is also a T_4 space.

We shall see in the next section that (D) holds also in locally compact spaces.

On the other hand, neither T_1 (or the stronger KC) nor compactness imply (D). Consider the space Q^+, the one-point-compactification of the space Q of rationals. Q^+ is a compact KC space (see e.g., [5]), hence also T_1 and K^*C. By Theorem 8, Q^+ will satisfy (D) only if it is a T_4 space. Because T_1 and T_4 imply T_2, Q^+ would have to be a Hausdorff space, which it is not.

6. **The product topology.** The topology induced on the space $R \times S$ by an adjoining system on $\mathfrak{S}$ need not be the product topology. In this section we give necessary and sufficient condition in order that a given adjoining system induces the product topology on $R \times S$, for any R.

Let $\mathfrak{F}$ be a filter on $\mathfrak{S}$ and let $F_i (i \in I)$ be the family of its members. We write $\mathfrak{F}^0$ for the set $(\cap F_i)^0 (i \in I)$. (X^0 denotes the interior of X in the space S). If $\mathfrak{F} = \mathfrak{F}(K)$ is compactly generated, then $\mathfrak{F}^0 = K^{*0}$.

Let R, S be two topological spaces, A an open subset of R, B an open subset of S. We define a mapping $\psi(A, B)\colon R \rightarrow \mathfrak{S}$ by $\psi(A, B)r = B$ $(r \in A)$, $\psi(A, B)r = \emptyset (r \in R - A)$. Obviously, with the notation of §2, $[\psi(A, B)] = A \times B$. Let $\mathfrak{F}$ be any filter on $\mathfrak{S}$. It is easy to see that $\psi(A, B)^{-1}\mathfrak{F}$ is equal either to A or to $\emptyset$ or to R. Hence $\psi(A, B)$ is continuous for all A, B and $A \times B$ is open no matter what adjoining system is chosen on $\mathfrak{S}$ to determine a topology on $R \times S$. This shows that any such topology is finer than the product topology.

Since the family of sets $A \times B$ is an open basis for the product topology on $R \times S$, an adjoining system on $\mathfrak{S}$ determines this topology if and only if to any continuous map $\phi\colon R \rightarrow \mathfrak{S}$ there correspond open sets $A_i \subset R$, $B_i \subset S (i \in I)$ such that $[\phi] = \cup [\psi(A_i, B_i)] (i \in I)$.

THEOREM 9. *An adjoining system $\mathscr{F}$ on $\mathfrak{S}$ induces the product topology on $R \times S$, for any R, if and only if any open set $A \subset S$ satisfies $A = \cup \mathfrak{F}^0$, where $\mathfrak{F}$ runs through all filters of $\mathscr{F}$ containing A as an element.*

Proof. The condition is necessary. Let R be the space $\mathfrak{S}$ with the topology given by $\mathscr{F}$. We claim: the family of sets $[\psi(\mathfrak{F}, B)]$ $(\mathfrak{F} \in \mathscr{F}, B$ open in $S)$ is an open basis for the product topology on $\mathfrak{S} \times S$. Indeed, an open set in $\mathfrak{S}$ is given by $\mathfrak{A} = \cup \mathfrak{F}_j (\mathfrak{F}_j \in \mathscr{F}, j \in J)$ and $[\psi(\mathfrak{A}, B)] = \cup [\psi(\mathfrak{F}_j, B)] (j \in J)$. Consider the identical mapping $\varepsilon\colon \mathfrak{S} \rightarrow \mathfrak{S}$. Since ε is continuous, $[\varepsilon]$ is open in $\mathfrak{S} \times S$ and there exist filters $\mathfrak{F}_i \in \mathscr{F}$ and open sets $B_i \subset S (i \in I)$ such that $[\varepsilon] = \cup [\psi(\mathfrak{F}_i, B_i)]$ $(i \in I)$, or equivalently, $\varepsilon = \cup \psi(\mathfrak{F}_i, B_i)$ (according to the notation introduced in §2). For any open $A \subset S$ we have $\varepsilon A = A = \cup \psi(\mathfrak{F}_i, B_i)A =$ $\cup B_j (i \in I, j \in J(A))$, where the subset $J(A) \subset I$ is determined by $i \in J(A)$ if and only if $A \in \mathfrak{F}_i$. Hence for any $i \in I$ and any $A \in \mathfrak{F}_i$ we have

$B_i \subset A$ and consequently $B_i \subset \mathfrak{F}_i^0$. Fix an open $A \subset S$ and consider the index set $J(A)$. Then $\mathfrak{F}_j^0 \subset A$ for all $j \in J(A)$, hence $\bigcup \mathfrak{F}_j^0 \subset A$, but also $A = \bigcup B_j \subset \bigcup \mathfrak{F}_j^0$. This proves $A = \bigcup \mathfrak{F}_j^0$ and also $A = \bigcup \mathfrak{F}^0$, as stated in the theorem.

The condition is sufficient. Let R be any topological space, $\phi: R \to \mathfrak{S}$ a continuous map. For any $\mathfrak{F} \in \mathscr{F}$, define $\phi(\mathfrak{F}) = \psi(\phi^{-1}\mathfrak{F}, \mathfrak{F}^0)$. By definition, for $r \in R$, $\phi(\mathfrak{F})r = \mathfrak{F}^0$ if $\phi r \in \mathfrak{F}$, otherwise $\phi(\mathfrak{F})r = \varnothing$. Taking the union of all the sets $\phi(\mathfrak{F})r$ for all $\mathfrak{F} \in \mathscr{F}$, r fixed, one obtains $\bigcup \phi(\mathfrak{F})r = \bigcup \mathfrak{G}^0$, where $\mathfrak{G}$ runs through all filters of $\mathscr{F}$ containing ϕr. By assumption, this is equal to ϕr and we have $\phi = \bigcup \phi(\mathfrak{F}) = \bigcup \psi(\phi^{-1}\mathfrak{F}, \mathfrak{F}^0)(\mathfrak{F} \in \mathscr{F})$. Hence $[\phi]$ is open in the product topology on $R \times S$.

We shall call a space S satisfying the condition of Theorem 9, *locally* $\mathscr{F}$. The reason for this terminology is the following: let A be open in S, p a point of A. The condition is satisfied if and only if there is a filter $\mathfrak{F} \in \mathscr{F}$ containing A and such that $p \in \mathfrak{F}^0 \subset A$.

If $\mathscr{F}$ consists of compactly generated filters and if $\mathfrak{K}(\mathscr{F})$ is the set of its fully compact generators, S is locally $\mathscr{F}$ if and only if for any open $A \subset S$ and any point $p \in A$ there is $K \in \mathfrak{K}(\mathscr{F})$ such that $p \in K^0 \subset K \subset A$, i.e., if and only if $\mathfrak{K}(\mathscr{F})$ is a local neighborhood base for each point of S. If $\mathfrak{K}(\mathscr{F})$ is the family of all fully compact sets of S, this is just (a version of) local compactness, as is easily seen.

As was shown above, the product topology on $R \times S$ is always coarser than any topology induced by an adjoining system. This remark leads to.

THEOREM 10. *Let $\mathscr{F}$ be a family of compact filters on $\mathfrak{S}$ and let the space S be locally $\mathscr{F}$. Then $\overline{\mathscr{F}}$ consists of all compact filters on $\mathfrak{S}$ and is adjoining; in fact, $\overline{\mathscr{F}}$ equals the maximal adjoining system $\mathscr{M}$ on $\mathfrak{S}$.*

Proof. Let $\mathfrak{G}$ be any compact filter on $\mathfrak{S}$ and let $A \in \mathfrak{G}$. By assumption, $A = \bigcup \mathfrak{F}^0$, where $\mathfrak{F}$ runs through all filters of $\mathscr{F}$ containing A. Since $\mathfrak{G}$ is compact, there is a finite number of such filters, say $\mathfrak{F}_1, \cdots, \mathfrak{F}_n$, such that $\mathfrak{F}_1^0 \cup \cdots \cup \mathfrak{F}_n^0 \in \mathfrak{G}$. Consider $\mathfrak{H} = \mathfrak{F}_1 \cap \cdots \cap \mathfrak{F}_n$; it is easy to check that $\mathfrak{H}^0 \supset \mathfrak{F}_1^0 \cup \cdots \cup \mathfrak{F}_n^0$. Consequently, $\mathfrak{H}^0 \in \mathfrak{G}$ and $\mathfrak{H} \subset \mathfrak{G}$. For any $A \in \mathfrak{G}$ we have found a filter $\mathfrak{H}(A) \in \mathscr{F}$ with $A \in \mathfrak{H}(A) \subset \mathfrak{G}$; hence $\mathfrak{G} = \bigcup \mathfrak{H}(A)(A \in \mathfrak{G})$ and $\mathfrak{G} \in \overline{\mathscr{F}}$.

To prove $\overline{\mathscr{F}} = \mathscr{M}$, we show that $\mathscr{F}$ is adjoining, i.e., satisfies condition (A) of §2. Let $\mathfrak{F} \in \mathscr{F}$, A, B open in S, $A \cup B \in \mathfrak{F}$. Writing, as above, $A = \bigcup \mathfrak{G}^0$, $B = \bigcup \mathfrak{H}^0(A \in \mathfrak{G} \in \mathscr{F}, B \in \mathfrak{H} \in \mathscr{F})$ we can find filters $\mathfrak{G}(A)$, $\mathfrak{H}(B) \in \mathscr{F}$ with $A \in \mathfrak{G}(A)$, $B \in \mathfrak{H}(B)$, $\mathfrak{G}(A)^0 \cup \mathfrak{H}(B)^0 \in \mathfrak{F}$. The latter relation implies $\mathfrak{G}(A) \cap \mathfrak{H}(B) \subset \mathfrak{F}$, as is easily seen by using the formula $\mathfrak{G}^0 \cup \mathfrak{H}^0 \subset (\mathfrak{G} \cap \mathfrak{H})^0$.

COROLLARY. *If S is locally compact (in the sense introduced above, i.e., locally $\mathscr{K}_0$), then $\overline{\mathscr{K}_0} = \mathscr{M}$ and the compact-open topology on* Con (S, T) *corresponds to the product topology on $R \times S$.*

This corollary together with Theorem 7 implies that a locally compact space satisfies condition (D) of Theorem 7. This fact can of course be shown directly.

We shall now give an example of a space S which is not locally $\mathscr{F}$ for any system of compact filters. By our results the product topology on $R \times S$ will not, in general, be induced by any adjoining system on $\mathfrak{S}$.

Let S be an uncountable set with the countable topology (a set is closed if and only if it is at most countable or the whole space). As is well known, the resulting space is pseudo-finite (compact = finite). We want to show that every compact filter $\mathfrak{F}$ on $\mathfrak{S}$ is compactly generated.

Let K be the intersection of all members of $\mathfrak{F}$ and suppose $K \subset A$, where A is open in S. We shall prove $A \in \mathfrak{F}$. A is either cofinite or cocountable. In the first case, let $A = S - \{v_1, \cdots, v_n\}$. For each i, $1 \leqq i \leqq n$, there is $F_i \in \mathfrak{F}$ with $v_i \notin F_i$. Then $F_1 \cap \cdots \cap F_n \subset S - \{v_1, \cdots, v_n\} = A$ and $A \in \mathfrak{F}$. In the second case, let $A = S - V$, where $V = \{v_1, v_2, \cdots\}$. For each natural i, there is again $F_i \in \mathfrak{F}$ with $v_i \notin F_i$. We use, for any i, the notation $V_i = \{v_1, \cdots, v_i\}$, $A_i = A \cup V_i$. Obviously, $F_1 \cap F_2 \cap \cdots \cap F_i \subset S - V_i$, which implies $S - V_i \in \mathfrak{F}$. The union of all A_i is equal to S and therefore belongs to the filter $\mathfrak{F}$. Since $\mathfrak{F}$ is compact, there is a finite number of sets A_i whose union is a member of $\mathfrak{F}$. As is easily seen, this union is equal to some A_j. Then $A_j \cap (S - V_j) \in \mathfrak{F}$; but $A_j \cap (S - V_j) = A$.

Our result implies $\mathfrak{F} = \mathfrak{F}(K)$, thus $\mathfrak{F}$ is compactly generated. It is now evident that S cannot be locally $\mathscr{F}$ for any system of compact filters $\mathscr{F}$, for this would imply the existence of finite open sets on S.

The author wishes to dedicate this paper to Professor Hugo Hadwiger of the University of Bern on the occasion of his sixtieth birthday.

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