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A NOTE ON THE CHARACTERIZATION OF CONDITIONAL EXPECTATION OPERATORS

DANIEL ELIOT WULBERT

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Let (X, Σ, μ) be an arbitrary measure space. A complete characterization is presented for the norm one positive projections P of $L^1(X, \Sigma, \mu)$ into itself such that $\|f\|_\infty \geq \|Pf\|_\infty$ for each essentially bounded, summable function f .

If (X, Σ, μ) is a probability measure space it is known [1], [2], and [4] that such operators coincide precisely with the conditional expectation operators defined on L^1 (see definitions below). In this note we show that this characterization extends to an arbitrary measure space. The proof presented here is a direct, constructive proof requiring only basic measure theory. Although this extension is not unexpected, it does not seem to be a consequence of the methods used in the case for the finite measure spaces. An independent proof for this result, using the ergodic theory of Markov processes, was found by S. Foguel. Also extensions to arbitrary measure spaces of related theorems in [1], have recently been done by L. Tzafriri [5].

DEFINITION. Let (X, Σ, μ) be a measure space. Let Σ_0 be a σ -subring of Σ . We call a projection P on $L^1(X, \Sigma, \mu)$ a *conditional expectation operator* with respect to Σ_0 if Pf is Σ_0 measurable for all f , and if for each U in Σ_0 we have

$$\int_U Pf d\mu = \int_U f d\mu.$$

Clearly a conditional expectation operator is positive projection of norm one.

Notation. Let P be a norm one positive projection from $L^1(X, \Sigma, \mu)$ onto E . Let

$$\Sigma_0 = \{K \subseteq X : K = \text{supp } f, f \geq 0, f \text{ in } E\}.$$

We use $\text{supp } f$ to denote the support of a function f . The characteristic function of a set A is written 1_A .

LEMMA 1. *E is a lattice.*

Proof. Let f and g be in E . Since $f \vee g$ dominates f and g , $P(f \vee g)$ dominates f , g and hence $f \vee g$. Since $\|f \vee g\| \geq \|P(f \vee g)\|$,

we have that $f \vee g = P(f \vee g)$.

LEMMA 2. Σ_0 is a σ -ring.

Proof. We need to show that countable unions of members of Σ_0 are in Σ_0 , and differences of members in Σ_0 are in Σ_0 . Suppose U_i is in Σ_0 for $i = 1, 2, \dots$. Let f_i be norm one positive functions in E whose support in U_i . Then $f = \sum_{i=1}^{\infty} (1/2)^i f_i$ is a positive function in E whose support is $\bigcup_{i=1}^{\infty} U_i$.

Suppose U and V are in Σ_0 , and are the supports of positive function f and g in E .

Let

$$f_n = (f - ng) \vee 0.$$

Let

$$f'(x) = \begin{cases} f(x) & \text{for } x \text{ in } U - V \\ 0 & \text{otherwise.} \end{cases}$$

From the dominated convergence theorem f_n converges to f' . Thus $U - V$ is in Σ_0 .

LEMMA 3. Suppose f vanishes off some member of Σ_0 . The following is true.

- (i) $P(|f|) = 0$ implies $f = 0$.
- (ii) if K is in Σ_0 then $P(1_K f) = 1_K P(f)$.
- (iii) $\int_K P f d\mu = \int_K f d\mu$ for all K in Σ_0 .

Proof. We will assume that f is nonnegative. Suppose g is a nonnegative number of E such that f vanishes off $\text{supp}(g)$. Since $f \wedge ng$ increases monotonically to f , it suffices to assume that f is bounded by a member of E . We will assume therefore that $0 \leq f \leq g$.

(i) $P(g - f) = g$ so $\|g\| \leq \|g - f\|$, but $0 \leq g - f \leq g$. Therefore $f = 0$.

(ii) Suppose that h is a nonnegative member of E such that $\text{supp } h = K$. Since $g \wedge nh$ converges monotonically to $1_K g$, it follows that $1_K g$ is in K .

Now $0 \leq P(1_K f) \leq P(1_K g) = 1_K g$. Hence $P(1_K f)$ vanishes off K , and $P(1_K f) \leq 1_K P(f)$. We also have $1_K g - P(1_K f) = P(1_K(g - f)) \leq 1_K P(g - f) = 1_K g - 1_K P(f)$. Thus $1_K P f = P(1_K f)$.

(iii) $\int_K f d\mu = \|1_K f\| \geq \|P(1_K f)\| = \int_K P f d\mu$. Similarly

$$\begin{aligned} \int g d\mu - \int_K f d\mu &= \int_K (g - f) d\mu = \|1_K(g - f)\| \\ &\geq \|P(1_K(g - f))\| = \int_K (Pg - Pf) d\mu. \end{aligned}$$

Hence $\int_K Pf d\mu = \int_K f d\mu$.

For the remainder of the paper we will also assume that for each essentially bounded f in L^1 , $\|f\|_\infty \geq \|Pf\|_\infty$.

LEMMA 4. *Each member of E is Σ_0 -measurable.*

Proof. We first show that $g \wedge c$ is in E for each constant function c and each g in E . It suffices to prove this assertion for positive functions g and for $c > 0$. However this is almost obvious for since P is positive $g \geq P(g \wedge c) \geq 0$, and from the hypothesis $c \geq P(g \wedge c)$. Thus $g \wedge c \geq P(g \wedge c)$. Now with $K = \text{supp } g$ Lemma 3 (iii) implies that $g \wedge c = P(g \wedge c)$.

It follows that for any c , $g - g \wedge c$ is in E . Hence if g is a positive function in E , the set $\{x \text{ in } X : g(x) > c\}$ is also the support of $g - g \wedge c$, and thus is in Σ_0 . Hence Pf is Σ_0 measurable for each f in L^1 .

PROPOSITION. *P is the conditional expectation operator with respect to Σ_0 .*

Proof. Let f be in L^1 . Let $r = \sup\left\{\int_K |f| d\mu : K \text{ in } \Sigma_0\right\}$. Let K be a member of Σ_0 such that $\int_K |f| d\mu = r$. Writing $f = 1_K f + (f - 1_K f)$ we see that f is the sum of a function which vanishes off a member of Σ_0 and a function which vanishes on each member of Σ_0 . Thus in view of all the previous lemmas it remains only to show that $Pf = 0$ if f vanishes on each member of Σ_0 . We may assume that f is bounded and nonnegative. Since the support of nPf is in Σ_0 , and since f vanishes on all members of Σ_0 , we have

$$\|f + nPf\|_\infty = \max(\|f\|_\infty, n\|Pf\|_\infty),$$

but

$$\|f + nPf\|_\infty \geq \|P(f + nPf)\|_\infty = (n + 1)\|Pf\|_\infty.$$

This implies that $\|Pf\|_\infty = 0$.

REMARKS. The hypothesis that $\|f\|_\infty \geq \|Pf\|_\infty$ is equivalent to the assumption that $P(1_A) \leq 1$ for all sets A of finite measure.

The referee has pointed out that the main result in this note is valid for norm one projections defined on L_p spaces. Besides the proofs presented here, one would also use the fact that there do not exist two distinct norm one projections of a smooth space onto a subspace. (For L_p spaces this result is in [1]. For smooth spaces a proof is in [6, Lemma 1]). The organization of this note was also suggested by the referee, and adapts to L_p operators more readily than the original.

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