FACIAL DECOMPOSITION OF LINEARLY COMPACT SIMPLEXES AND SEPARATION OF FUNCTIONS ON CONES

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Necessary and sufficient conditions for a linearly compact simplex $K$ to be uniquely decomposable at a face are given. If $P$ is a cone having the Riesz decomposition property and if $-f, g$ are subadditive homogeneous functions on $P$ with $f \geq g$ then it is shown that there is an additive homogeneous function $h$ on $P$ with $f \geq h \geq g$. If $P$ is a lattice cone for the dual space of an ordered Banach space $X$ and if $-f, g$ are also $w^*$-continuous then, under certain conditions, it is possible to choose $h \in X$; a consequence of this result is Andô's theorem, that an ordered Banach space has the Riesz decomposition property if its dual space is a lattice. A nonmeasure theoretic proof of Edwards' separation theorem for compact simplexes is also deduced from these results.

Let $K$ be a linearly compact simplex in a real vector space $E$. Without loss of generality we will assume that $K$ is contained in a hyperplane $e^{-1}(1)$ and that $E = \text{lin } K$, where $\text{lin } K$ denotes the linear hull of $K$. Then it is well known that $E$ is a vector lattice relative to the cone with base $K$, and that $\text{co } (K \cup -K)$ is the closed unit ball for a norm making $E$ a pre-AL-space. In fact if $K$ is compact for a locally convex Hausdorff topology on $E$ then $E$ is the Banach dual space of $A(K)$, the space of all affine continuous functions on $K$ with supremum norm (cf. [5]). (We refer to [10] as a general reference for the lattice theory and terminology that is used.)

The set $K$ is said to be decomposable at a face $F$ if there exists a complementary face $F'$ of $K$ such that $F \cap F' = \emptyset$ while $\text{co } (F \cup F') = K$. If a complementary face to $F$ exists then it is evident that it is uniquely determined; moreover, in this case, Alfsen [1] has shown that the decomposition is unique in the sense that each $k \in K$ has a unique decomposition $k = \lambda x + (1 - \lambda) y$ with $x \in F$, $y \in F'$ and $0 \leq \lambda \leq 1$. Alfsen has also given a necessary and sufficient condition for $K$ to be decomposable at $F$; we give here other necessary and sufficient conditions which are perhaps more closely tied to the order and norm structure of $E$.

**Theorem 1.** Let $K$ be a linearly compact simplex and $F, F'$ disjoint faces of $K$. Then $F$ and $F'$ are complementary faces for a
(necessarily unique) decomposition of \( K \) if and only if \( E \) is the order-direct sum of \( \text{lin} \ F \) and \( \text{lin} \ F' \). Consequently, if \( E \) is complete in its norm then \( K \) is uniquely decomposable at \( F \) if and only if \( F \) is norm-closed.

**Proof.** Since \( F \) is a face of \( K \) it is easily verified that \( \text{lin} \ F \) is a lattice ideal in \( E \) and that \((\text{lin} \ F) \cap K = F\). If \( E \) is the order-direct sum of \( \text{lin} \ F \) and \( \text{lin} \ F' \) then each \( x \in K \) has a unique decomposition \( x = y + z \) with \( y, z \geq 0, \ y \in \text{lin} \ F, \ z \in \text{lin} \ F' \); hence \( K = \text{co}(F \cup F') \), and the decomposition is unique.

Suppose conversely that \( K = \text{co}(F \cup F') \). Then, since \( E = \text{lin} \ F + \text{lin} \ F' \), it will follow that \( E \) is the order-direct sum of \( \text{lin} \ F \) and \( \text{lin} \ F' \) if we prove that \( \text{lin} \ F' = (\text{lin} \ F)^\perp = \{y \in E : |x| \land |y| = 0, \ \forall x \in \text{lin} \ F\} \) (cf. [10, p. 38]). Since \( (\text{lin} \ F)^\perp \) is a lattice ideal the set \( G = K \cap (\text{lin} \ F)^\perp \) is a face of \( K \) disjoint from \( F \), and hence \( G \subseteq F' \). However if \( x \in \text{lin} \ F' \cap G \) then there exists a \( y \in \text{lin} \ F \) such that \( x \land y = z \neq 0 \); but, since \( F \) and \( F' \) are faces of \( K \) and \( x = z + (x - z), \ y = z + (y - z) \), this implies that \( z/\|z\| \in F' \cap F'' \) which is impossible. Therefore we have \( (\text{lin} \ F)^\perp = \text{lin} \ G = \text{lin} \ F' \).

If \( E \) is complete in its norm then it is an \( AL \)-space. If \( F \) is norm-closed then the continuity of the lattice operations in \( E \) shows that \( \text{lin} \ F \) is also norm-closed, and hence is a band. Therefore, by a theorem of Riesz (cf. [10, p. 39]), \( \text{lin} \ F \) has an order-direct complement in \( E \), and so \( K \) is uniquely decomposable at \( F \). If, conversely, \( K \) is uniquely decomposable at \( F \) then there exists a natural affine function \( f \) on \( K \) such that \( F = f^{-1}(0), \ F' = f^{-1}(1) \). The function \( f \) has an obvious extension to a continuous linear functional \( g \) on \( E \) and, since \( F = K \cap g^{-1}(0) \), it follows that \( F \) is norm-closed.

If \( K \) is a compact simplex then \( E \) is certainly a Banach space, and so the following result is immediate.

**Corollary.** If \( K \) is a compact simplex and \( F \) a face of \( K \), then \( K \) is uniquely decomposable at \( F \) if and only if \( F \) is norm-closed.

The corollary generalizes Alfsen’s result that a compact simplex is uniquely decomposable at each closed (i.e., compact) face. When \( K \) is an arbitrary compact convex set Alfsen and Andersen [2] characterize the decomposable faces of \( K \). However it is not true that every linearly compact simplex is decomposable at every norm-closed face, as the following example shows.

**Example.** Let \( K \) denote the continuous nonnegative functions \( f \) on
Then $K$ is a base for the lattice cone in $C[0, 1]$, and hence is a linearly compact simplex, and it is clear that $F$ is a face of $K$. The norm in $C[0, 1]$ associated with $K$ is the $L_1[0, 1]$-norm, and hence $F$ is norm-closed. Suppose that there exists a face $F'$ complementary to $F$ in $K$. Then, since $f(1/2) = 0$ for all $f \in F'$, there exists a $u \in F'$ such that $u(1/2) > 0$. However it is easy to decompose $u$ nontrivially $u = \lambda g + (1-\lambda)h$ with $g \in F'$, $h \in F''$ and $0 < \lambda < 1$. Since $F''$ is a face of $K$ it follows that $g \in F \cap F''$, which is a contradiction. Therefore $K$ is not decomposable at the norm-closed face $F$.

It has been shown by Asimow [4] that the state space of a function algebra is decomposable at every extreme point, and so such a property does not characterize simplexes among compact convex sets; this property does however characterize simplexes among finite-dimensional compact convex sets as the following slightly more general result shows.

**Proposition.** If $K$ is a compact convex set which is decomposable at each extreme point $x$, and such that each complementary face $\{x\}'$ is closed, then $K$ is a finite-dimensional simplex.

**Proof.** If the set $K_*$ of extreme points of $K$ is infinite then there exists an accumulation point $u \in K$. For each $x \in K_*$ the set $K_*$ consists of $x$ together with the extreme points of the closed set $\{x\}'$. Consequently $u \in \{x\}'$ for all $x \in K_*$. Therefore the intersection of the faces $\{x\}'$ forms a closed face $F$ of $K$ which is not empty, since $u \in F$. However if $y$ is an extreme point of $F$ then $y \in K_*$, and also $y \in \{y\}'$ which is impossible. Hence $K_*$ is finite.

If $K_*$ has $m$ points and $K$ has dimension $n$ then, for each $x \in K_*$, it is clear that $\{x\}'$ has $m-1$ extreme points and has dimension $n-1$, and $\{x\}'$ has a similar decomposition property to $K$. Reducing in this way we see eventually that $m = n + 1$, that is $K$ is an $n$-dimensional simplex.

If $K$ is a compact simplex then the above result shows that not all faces $\{x\}'$ can be closed. For example, for the simplex $\{x \in l^1; x \geq 0, \|x\| \leq 1\}$ all but one of the faces $\{x\}'$ are closed, while for the simplex of probability measures on $[0, 1]$ none of the faces $\{x\}'$ are closed.

2. We prove an analogue for linearly compact simplexes of Edwards’ separation theorem [6], which characterizes compact simplexes; this is a corollary of the following result.
THEOREM 2. Let $P$ be a cone possessing the Riesz decomposition property, and let $-f$, $g$ be subadditive homogeneous functionals on $P$ with $f \geq g$. Then there exists an additive homogeneous functional $h$ on $P$ such that $f \geq h \geq g$.

Proof. If we define $h$ on $P$ by

$$h(x) = \inf \left\{ \sum_{i=1}^{n} f(x_i) : x = \sum_{i=1}^{n} x_i, x_i \in P \right\}$$

then it is clear that $f \geq h \geq g$, and hence $h$ is finite-valued. Moreover, $h$ is positive-homogeneous and subadditive. If $x = y + z$ with $x, y, z \in P$, and if $\varepsilon > 0$ choose $x_i \in P$ such that $x = \sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} f(x_i) \leq h(x) + \varepsilon$. Then there exist $a_{i_1} \in P$ such that $\sum_{i=1}^{n} a_{i_1} = y$, $a_{i_1} + a_{i_2} = x_i$ for $i = 1, 2, \ldots, n$. We have

$$h(x) \geq \sum_{i=1}^{n} f(x_i) - \varepsilon \geq \sum_{i=1}^{n} f(a_{i_1}) + \sum_{i=1}^{n} f(a_{i_2}) - \varepsilon \geq h(y) + h(z) - \varepsilon,$$

so that $h$ is additive and homogeneous.

In the corollary below $K$ will denote a linearly compact subset of $E$, again contained in a hyperplane $e^{-i}(1)$ and such that $\text{lin} K = E$. By $A^b(K)$ we will denote the Banach space of all bounded real-valued affine functions on $K$ with the supremum norm. If $\text{co}(K \cup -K)$ is linearly bounded then its Minkowski functional is a norm in $E$ and $A^b(K)$ is simply the Banach dual space of $E$ for this norm. In the particular case when $K$ is compact for some locally convex Hausdorff topology on $E$, $A^b(K)$ is the second dual space of $A(K)$.

COROLLARY. The following statements are equivalent.

(i) $K$ is a linearly compact simplex.

(ii) $\text{co}(K \cup -K)$ is linearly compact and, if $-f$, $g$ are bounded convex functions on $K$ with $f \geq g$, there exists an $h \in A^b(K)$ such that $f \geq h \geq g$.

Proof. (i) $\rightarrow$ (ii). That $\text{co}(K \cup -K)$ is linearly compact was proved in [5, Th. 2]. If $P$ is the cone generated by $K$ as a base then $P$ is a lattice-cone. If $f$ and $g$ are extended homogeneously to the rest of $P$ then the existence of the required $h \in A^b(K)$ follows from the theorem.

(ii) $\rightarrow$ (i). If $u_i, u_2, v_1, v_2 \in A^b(K)$ and $u_i, u_2 \leq v_1, v_2$ then, putting $g(x) = \max \{ u_i(x), u_2(x) \}$, $f(x) = \min \{ v_1(x), v_2(x) \}$ for all $x \in K$, we obtain a function $h \in A^b(K)$ such that $u_i, u_2 \leq h \leq v_1, v_2$. The $w^*$-compactness of order intervals in $A^b(K)$ now shows that $A^b(K)$ is a vector lattice, in fact an $AM$-space. Therefore $E$ is an $AL$-space and, in
particular, each \( \eta \in E \) has a unique decomposition \( \eta = \eta_1 - \eta_2 \) with \( \eta_1 \geq 0 \) and \( ||\eta|| = ||\eta_1|| + ||\eta_2|| \), namely for \( \eta_1 = \eta^+, \eta_2 = \eta^- \). Since, by hypothesis, \( \text{co} (K \cup -K) \) is the closed unit ball of \( E \) it follows that \( E \) is a sublattice of \( E \). Therefore \( K \) is a linearly compact simplex.

It is perhaps surprising that the linear compactness condition on \( \text{co} (K \cup -K) \) cannot be dropped, as the following two simple examples show.

Examples. (i) Let \( E \) be the linear subspace of \( l_1 \), spanned by those elements with only finitely many nonzero coordinates, together with the two elements \( \{2^{-n}\}^\infty_{n=1}, \{(-3)^{-n}\} \), and let \( K = \{x \in E : x \geq 0, \|x\| \leq 1\} \). If \( S = \{x \in E : \|x\| \leq 1\} \) then it is obvious that for each \( \varepsilon > 0 \) we have \( \text{co} (K \cup -K) \subseteq S \subseteq (1 + \varepsilon) \text{co} (K \cup -K) \). If \( x = \{(-3)^{-n}\} \) then \( x^+ \in E \), so that \( 2x \in S \) but \( 2x \not\in \text{co} (K \cup -K) \). Therefore \( \text{co} (K \cup -K) \) is not linearly closed; in other terminology \( E \) has a \( (1 + \varepsilon) \)-generating cone for all \( \varepsilon > 0 \) but not a 1-generating cone. However, a straightforward verification shows that \( E \) has the Riesz decomposition property and hence, by Theorem 2, \( K \) has the separation property stated in part (ii) of the Corollary. However \( K \) is not a simplex.

(ii) Let \( K \) denote the polynomials \( p \) nonnegative on \([0, 1]\) and such that \( \int_0^1 p(x) \, dx = 1 \). It is clear that \( \text{co} (K \cup -K) \) is not linearly compact because the polynomials do not constitute a sublattice of \( L_1[0, 1] \). It is true, but less obvious, that \( \text{lin} K \) has the Riesz decomposition property (cf. [7]). We are grateful to Professor W. A. J. Luxemburg for bringing this fact and reference to our notice.

By an ordered Banach space we shall mean a partially ordered Banach space which has a closed, normal, generating cone. If \( X \) is an ordered Banach space then so is \( X^* \) (cf. [8]). The following lemma now follows from a result of Kadison ([9, Lemma 4.3]).

**Lemma 1.** Let \( X \) be an ordered Banach space and let
\[
K = \{f \in X^*: f \geq 0, \|f\| \leq 1\} ,
\]
equipped with the \( w^* \)-topology. Then \( X \) is order and topologically isomorphic to
\[
\text{A}_0(K) = \{f \in A(K): f(0) = 0\} .
\]

**Lemma 2.** Let \( C \) be a cone in a vector space \( V \), let \( p \) be a function homogeneous on \( C \) and let \( f \) be a function affine on \( V \) such that \( f(x) \leq p(x) \) for all \( x \in C \). Then the linear function \( g = f - f(0) \) satisfies \( g(x) \leq p(x) \) for all \( x \in C \).
Proof. It is simple to check that \( g \) is linear on \( V \). Suppose that there is a point \( x \in C \) such that \( g(x) > p(x) \). Then if \( \varepsilon = -f(0) \) and \( \delta = f(x) - p(x) \) we have \( \varepsilon \leq 0 \) and \( g(x) - p(x) = \delta + \varepsilon > 0 \). Hence there exists an \( r \geq 1 \) such that \( r(\delta + \varepsilon) > \varepsilon \), and we have

\[
f(x) = r^{-1} f(rx) + (1 - r^{-1}) f(0) .
\]

Therefore

\[
f(rx) - p(rx) = r(f(x) - p(x)) + (r - 1) \varepsilon = r(\delta + \varepsilon) - \varepsilon > 0 ,
\]

which gives a contradiction.

The following theorem is the main result of this section and is a topological version of Theorem 2.

**Theorem 3.** Let \( X \) be an ordered Banach space such that the dual cone \( P^* \) is a lattice cone in \( X^* \), and let \(-f, g\) be \( w^*\)-continuous subadditive homogeneous functionals on \( P^* \) with \( f \geq g \). If either (i) \( f = u_1 \wedge u_2, g = v_1 \vee v_2 \), where \( u_1, u_2, v_1, v_2 \in X \), or (ii) the dual cone in \( X^{**} \) possesses an interior point, then there exists an \( h \in X \) such that \( f \geq h \geq g \).

**Proof.** If \( K = \{ x \in P^* : \| x \| \leq 1 \} \) then Lemma 1 shows that we can assume that \( X = A_0(K) \), and it is sufficient to find an \( h \in X \) such that \( f(x) \geq h(x) \geq g(x) \) for all \( x \in K \).

Let \( G \) denote the \( w^*\)-closed convex hull of the graph of \( f \) in \( K \times R \) and define \( \hat{f}(x) = \sup \{ u(x) : u \in A(K), u \leq f \} \) for all \( x \in K \). A straightforward calculation shows that \( \hat{f}(x) \leq \inf \{ r : (x, r) \in G \} \) for each \( x \in K \). If \( \mu < \inf \{ r : (x, r) \in G \} \) then by separating \( (x, \mu) \) from \( G \) we obtain a \( v \in A(K) \) such that \( v \leq f \) while \( v(x) > \mu \); therefore \( \hat{f}(x) = \inf \{ r : (x, r) \in G \} \). Given \( \varepsilon > 0 \), for each \( x \in K \) let \( N_x \) be a \( w^*\)-compact convex neighbourhood of \( x \) such that \( |f(x) - f(y)| < \varepsilon \) for each \( y \in N_x \), and let \( K \subseteq \bigcup_{i=1} \{ N_{x_i} \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon] \} \),

and so we can write \( x = \sum_{i=1}^n \lambda_i y_i, \hat{f}(x) = \sum_{i=1}^n \lambda_i r_i \) with \( y_i \in N_{x_i} \) and \( r_i \in [f(x_i) - \varepsilon, f(x_i) + \varepsilon] \) for each \( i \). If we now define for each \( x \in P^* \)

\[
\tilde{f}(x) = \inf \left\{ \sum_{i=1}^n f(x_i) : x_i \in P^*, \sum_{i=1}^n x_i = x \right\}
\]

then, for each \( x \in K \),

\[
\tilde{f}(x) \leq \sum_{i=1}^n \lambda_i \hat{f}(y_i) \leq \sum_{i=1}^n \lambda_i f(x_i) + \varepsilon \leq \sum_{i=1}^n \lambda_i r_i + 2\varepsilon = \hat{f}(x) + 2\varepsilon .
\]
Therefore $\tilde{f}(x) \leq \bar{f}(x)$ for each $x \in K$. If $\alpha > 0$ and if $\hat{f}_a(x) = \sup \{u(x) : u \in A(\alpha K), u \leq f\}$ then the argument shows that $\tilde{f}(x) \leq \hat{f}_a(x)$ for all $x \in \alpha K$; in particular $\hat{f}_a(0) = 0 = \bar{f}(0)$.

If condition (i) holds then we have

$$\bar{f}(x) = \inf \{u_i(x_i) + u_a(x_a) : x = x_i + x_a, x_i \in P^*\}.$$ 

Since $P^*$ is a normal cone we can choose $\alpha > 0$ such that $\|x_i\| + \|x_a\| \leq \alpha \|x\|$ whenever $x = x_i + x_a$ with $x_i \in P^*$.

If condition (ii) holds and if $\zeta$ is an interior point of the dual cone in $X^{**}$ then the order interval $[-\zeta, \zeta]$ is the unit ball for an equivalent norm in $X^{**}$, and hence $X^*$ has an equivalent norm which is additive on $P^*$. Therefore there exists an $\alpha > 0$ such that $\sum_{i=1}^n \|x_i\| \leq \alpha \|\sum_{i=1}^n x_i\|$ whenever $x_i \in P^*$.

Now let $x = \sum_{i=1}^n x_i = 0$, where $x_i \in P^*$, and with $n = 2$ if (i) holds. If $\lambda = \sum_{i=1}^n \|x_i\|$, and if $y_i = 0$ when $x_i = 0$, $y_i = \lambda x_i / \|x_i\|$ when $x_i \neq 0$, then $y_i \in \alpha K$ for each $i$. Since $\hat{f}_a$ is convex on $\alpha K$ we have

$$\hat{f}_a(x) = \hat{f}_a\left(\sum_{i=1}^n \|x_i\| \lambda \frac{y_i}{\lambda} \right) \leq \sum_{i=1}^n \frac{\|x_i\|}{\lambda} f_a(y_i) \leq \sum_{i=1}^n \frac{\|x_i\|}{\lambda} f(y_i)$$

$$= \sum_{i=1}^n f(x_i).$$

In case (ii) this inequality gives $\hat{f}_a(x) \leq \tilde{f}(x)$ for each $x \in K$, while in case (i) we have $\hat{f}_a(x) \leq u_i(x_i) + u_a(x_a)$ which again gives $\hat{f}_a(x) \leq \bar{f}(x)$; in either case therefore we have proved that $\tilde{f}(x) = \hat{f}_a(x)$ for each $x \in K$. If we define $\|f\| = \sup \{|f(x)| : x \in K\}$, and $\|\tilde{f}\|$ similarly, then we have $|f(x)| \leq \sum_{i=1}^n \|f\| \|x_i\| \leq \alpha \|f\| \|x\|$ for each $x \in K$, so that $\|\tilde{f}\| \leq \alpha \|f\|$.

By Theorem 2 $\tilde{f}$ is additive on $P^*$ and the above argument shows that $\tilde{f}$ is w*-l.s.c. on $\beta K$ for each $\beta > 0$. The set $\{x \in P^* : \tilde{f}(x) \leq r\}$ is convex and its intersection with each multiple of the unit ball of $X^*$ is w*-closed; hence $\tilde{f}$ is w*-l.s.c. on $P^*$. If we write $\bar{g}$ for $-(-\bar{g})$ then $\bar{g}$ is w*-u.s.c. on $P^*$ and is additive, homogeneous and satisfies $g \leq \bar{g} \leq \tilde{f}$. If $\epsilon > 0$ and $r > \alpha$ then, by separating the sets $\{(x, t) \in P^* \times R : t > \tilde{f}(x)\}$ and $\{(y, s - \epsilon/r) \in K \times R : s \leq \bar{g}(y)\}$ and applying Lemma 2, we obtain a $\omega_0 \in X$ such that $\omega_0 \leq \tilde{f}$ and $\omega_0(x) > g(x) - \epsilon/r$ for all $x \in K$. Hence if $z_i = (g - \omega_0) \vee 0$, $z_i$ is homogeneous, subadditive and w*-continuous on $P^*$ with $\|z_i\| < \epsilon/r$. The above argument shows that $\tilde{\omega}$ is w*-u.s.c. on $P^*$ and that $\|\tilde{\omega}\| \leq \alpha \|z_i\| < \epsilon$. Since the set $K \times \{\epsilon/r\}$ is disjoint from the w*-closed cone $\{(x, t) : x \in P^*, t \leq \tilde{\omega}(x)\}$ the separation theorem gives a $p_i \in X$ such that $p_i \geq \tilde{\omega}_i \geq g - \omega_i$, 0 and $\|p_i\| \leq \epsilon$.

Using the procedure of the preceding paragraph choose $f_i, g_i \in X$.
such that \( f_i \leq f \), \( g_i \geq 0 \), \( g \leq f_i + g_i \) and \( \| g_i \| < 1/2 \), in particular we have \( f \wedge (f_i + g_i) \geq g \vee f_i \). By induction there exist sequences \( \{f_n\} \) and \( \{g_n\} \) in \( X \) such that (a) \( g_n \geq 0 \), \( \| g_n \| < 2^{-n} \), (b) \( g \vee f_n \leq f_{n+1} + g_{n+1} \), (c) \( f_{n+1} \leq f \wedge (f_n + g_n) \). Properties (b) and (c) give \( -g_{n+1} \leq f_{n+1} \leq f_n + g_n \) so that \( \| f_{n+1} - f_n \| < 2^{-n} \). Therefore \( \{f_n\} \) converges to \( h \in X \) such that \( h \leq f \) by (c), and \( h \geq g \) by (b).

**Corollary 1 (Andô [3]).** If \( X \) is an ordered Banach space such that \( X^* \) is a lattice for the dual ordering then \( X \) has the Riesz decomposition property.

**Corollary 2 (Edwards [6]).** If \( K \) is a compact simplex and if \( -f, g \) are u. s. c. convex functions on \( K \) with \( f \geq g \) then there exists an \( h \in A(K) \) such that \( f \geq h \geq g \).

**Proof.** By truncating if necessary we may assume that \( f \) and \( g \) are bounded, say \( |f(x)|, \ |g(x)| \leq \lambda \) for all \( x \in K \). First suppose that the strict inequality \( f > g \) holds; then the set \( G = \{(x,t) : \lambda \leq t \leq g(x)\} \) is compact in \( K \times R \) and is a subset of the convex set \( H = \{(y,s) : s < f(y)\} \) which is relatively open in \( K \times R \). Therefore, taking the convex hull of a finite covering of \( G \) by compact convex neighbourhoods in \( H \), we see that \( H \) contains the closed convex hull of \( G \). Hence for each \( x \in K \) there is an \( f_x \in A(K) \) and a neighbourhood \( U_x \) of \( x \) such that \( g < f_x \) while \( f_x(y) < f(y) \) for all \( y \in U_x \). If \( K \leq \bigcup_{i=1}^{u} U_{x_i} \) and if \( f' = f_{x_1} \wedge \cdots \wedge f_{x_n} \) then \( f' \) is continuous and concave on \( K \) with \( g < f' < f \). Similarly we can construct a continuous convex function \( g' \) on \( K \) such that \( g < g' < f' < f \). The functions \( -f', g' \) have natural extensions to \( w^* \)-continuous subadditive homogeneous functions on the positive cone \( P^* \) of \( A(K)^* \) such that \( g' \leq f' \), and so Theorem 3 gives an \( h' \in A(K) \) such that \( g < g' \leq h' \leq f' \leq f \).

In the general case \( f \geq g \) there exists an \( h_1 \in A(K) \) such that \( f + 1 > h_1 > g - 1 \). By considering the functions \( (f \wedge h_1) + 1/2 \) and \( (g \vee h_1) - 1/2 \) we similarly obtain an \( h_2 \in A(K) \) such that \( f + 1/2 > h_2 > g - 1/2 \) while \( \| h_2 - h_1 \| < 1/2 \). Proceeding in this way we obtain a sequence \( \{h_n\} \) which converges in \( A(K) \) to \( h \) such that \( g \leq h \leq f \).

**References**


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