ABSOLUTE SUMMABILITY BY RIESZ MEANS

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In this paper we ensure the absolute Riesz summability of Lebesgue-Fourier series under more liberal conditions imposed upon the generating function of Lebesgue-Fourier series and by taking more general type of Riesz means than whatever the present author has previously taken in proving the corresponding result. Also we give a refinement over the criterion previously proved by author himself.

1. Definitions and notations. Let \( \sum_{n=0}^{\infty} a_n \) be a given infinite series with the sequence of partial sums \( \{s_n\} \). Throughout the paper we suppose that
\[
\lambda_n = \mu_0 + \mu_1 + \mu_2 + \cdots + \mu_n \to \infty, \text{ as } n \to \infty .
\]
The sequence-to-sequence transformation
\[
t_n = \frac{1}{\lambda_n} \sum_{\nu=0}^{n} \mu_\nu s_\nu ,
\]
defines the Riesz means of sequence \( \{s_n\} \) (or the series \( \sum_{n=0}^{\infty} a_n \)) of the type \( \{\lambda_{n-1}\} \) and order unity.\(^1\) If \( t_n \to s \), as \( n \to \infty \), the sequence \( \{s_n\} \) is said to be summable \( (R, \lambda_{n-1}, 1) \) to the sum \( s \) and if, in addition, \( \{t_n\} \in BV \),\(^2\) then it is said to be absolutely summable \( (R, \lambda_{n-1}, 1) \), or summable \( |R, \lambda_{n-1}, 1| \) and symbolically we write \( \sum_{n=0}^{\infty} a_n \in |R, \lambda_{n-1}, 1| \).

The series \( \sum_{n=1}^{\infty} a_n \in |R, \lambda_{n-1}, 1| \), if
\[
\sum_{m=0}^{\infty} \left| \frac{\Delta \lambda_m}{\lambda_n \lambda_{n+1}} \sum_{\nu=0}^{n} \lambda_\nu a_{\nu+1} \right| < \infty .
\]

Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable in the sense of Lebesgue over \( (-\pi, \pi) \). Without any loss of generality the constant term of the Lebesgue-Fourier series of \( f(t) \) can be taken to be zero, so that
\[
\int_{-\pi}^{\pi} f(t)dt = 0 ,
\]
and
\[
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) .
\]

\(^1\) It is some-times called \((N, \mu_n)\) mean, or \((R, \mu_n)\) mean, or Riesz's discrete mean of 'type' \( \lambda_{n-1} \) and 'order' unity and is, in fact, equivalent to the usually known \((R, \lambda_{n-1}, 1)\) mean. An explicit proof of it is contained in Iyer [6]. Also see Dikshit [3].

\(^2\) \{'t_n\} \in BV\' means \( \sum_{n} |\Delta t_n| < \infty \), when \( \Delta t_n = t_n - t_{n+1} \).
We use the following notations:

\[(1.3) \quad \phi(t) = \frac{1}{2} \{f(x + t) + f(x - t)\} .\]

\[(1.4) \quad \Lambda(t) = \frac{1}{t} \int_0^t u d\phi(u) .\]

\[(1.5) \quad K(n, t) = \sum_{\nu=0}^n \frac{\lambda_\nu}{(\nu + 1)} \sin (\nu + 1)t .\]

2. Introduction. Recently the present author [2] has established the following theorem concerning the absolute Riesz summability of Lebesgue-Fourier series of the type \(\exp (n^\alpha)(0 < \alpha < 1)\) and order unity.

**Theorem A.** If (i) \(\phi(t) \in BV(0, \pi)\) and (ii) \(\Lambda(t)(\log k/t)^{1+\varepsilon} \in BV(0, \pi)\), where \(\varepsilon > 0\) and \(k \geq \pi e^2\), then \(\sum_{n=1}^\infty A_n(x) \in |R, \exp (n^\alpha), 1|(0 < \alpha < 1)\).

By using the technique, which Mohanty [7] used in establishing the criterion for the absolute convergence of a Lebesgue-Fourier series at a point, which is the analogue for absolute convergence of the classical Hardy-Littlewood convergence criterion [4, 5], we have recently established the following:

**Theorem B.** If (i) \(\phi(t) \in BV(0, \pi)\), (ii) \(\Lambda(t)(\log k/t)^{1+\varepsilon} \in BV(0, \pi)\), where \(\varepsilon > 0\), \(k \geq \pi e^2\) and (iii) \(\{n^\alpha A_n(x)\} \in BV\), for \(0 < \alpha < 1\), then \(\sum_{n=1}^\infty |A_n(x)| < \infty\).

The purpose of this paper is to ensure the absolute Riesz summability of Lebesgue-Fourier series under more liberal condition imposed upon the generating function of Lebesgue-Fourier series and taking more general type of Riesz means.

We first prove the following general theorem.

**Theorem 1.** Let, for \(0 < \alpha < 1\), the strictly increasing sequences \(\{\lambda_n\}\) and \(\{g(n)\}\), of nonnegative terms, tending to infinity with \(n\), satisfy the following conditions:

\[(2.1) \quad \log \frac{\pi}{t} = O[g(k/t)]; \text{ as } t \to 0 ,\]

\[(2.2) \quad \{\lambda_n/(n + 1)\}/ with n \geq n_0 ,\]

\[(2.3) \quad n^{1-\alpha} A_{\lambda_n} = O[\lambda_{n+1}]; \text{ as } n \to \infty ,\]
(2.4) \[
\begin{align*}
(\text{i}) & \quad \{\frac{x}{g(x)}\} \to \text{with } x, \\
(\text{ii}) & \quad x \frac{d}{dx} \left( \frac{1}{g(k/x)} \right) \to \text{with } x, \\
(\text{iii}) & \quad \frac{d}{dx} \left( \frac{1}{g(k/x)} \right) \to \text{with } x.
\end{align*}
\]

(2.5) \[
\begin{align*}
(\text{i}) & \quad \left[ \frac{d}{dt} \left( \frac{1}{g(k/t)} \right) \right]_{t=1/n} = O\{n/g(n)\}, \\
(\text{ii}) & \quad \sum_{n=1}^{\infty} (ng(n))^{-1} < \infty.
\end{align*}
\]

Then, if \( \phi(t) \in BV(0, \pi) \) and \( \Lambda(t)g(k/t) \in BV(0, \pi) \), the series
\[
\sum_{n=1}^{\infty} A_n(x) \in |R, \lambda_{n-1}, 1|,
\]
where \( k \) is a suitable positive constant such that \( g(k/t) > 0 \) for \( t > 0 \).

3. We shall use the following order-estimates, uniformly in \( 0 < t \leq \pi \).

(3.1) \[
K(n, t) = O\{t^{-1}\lambda_n/(n + 1)\}.
\]

(3.2) \[
\int_0^t \frac{\sin (n + 1)u}{ug(k/u)} du = O\{1/g(n + 1)\}.
\]

(3.3) \[
\int_0^t \sin (n + 1)u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du = O\{1/g(n + 1)\}.
\]

**Proof of 3.1.** By using Abel’s Lemma and (2.2), the proof follows.

**Proof of 3.2.** Case (I). When \((n + 1)^{-1} \leq t\), we have
\[
\int_0^t \frac{\sin (n + 1)u}{ug(k/u)} du = \left( \int_0^{(n+1)^{-1}} + \int_{(n+1)^{-1}}^t \right) \frac{\sin (n + 1)u}{ug(k/u)} du
\]
\[
= I_1 + I_2, \text{ say}.
\]

Now, since \(|\sin (n + 1)u| \leq (n + 1)u\), we have
\[
I_1 = O\{(n + 1) \int_0^{(n+1)^{-1}} \frac{1}{g(k/u)} du\} = O\{1/g(n + 1)\}.
\]

And, by the second mean value theorem and (2.4)(i) we have
\[
I_2 = O\{1/g(n + 1)\}.
\]

Case (II). When \((n + 1)^{-1} > t\), we have
\[ \int_0^t \sin(n + 1)u \, du = \left( (n + 1)^{-1} - \int_0^{(n + 1)^{-1}} \right) \sin(n + 1)u \, du = I_1 - I_2', \text{ say.} \]

Proceeding as in \( I_i \), for \( I_i' \), we obtain

\[ I_i' = O\{1/g(n + 1)\}. \]

This completes the proof.

**Proof of (3.3).** In view of (2.4)(ii), (2.4)(iii) and (2.5)(i), the proof runs parallel to that of (3.2).

4. We require the following lemmas, for the proof of the theorems.

**Lemma 1.** If \( F(x) \in BV(a, b) \), then it can be expressed as \((F_1(x) - F_2(x))\) where \( F_1(x) \) and \( F_2(x) \) are positive, bounded and monotonic increasing functions in \((a, b)\) (see Carslaw [1], p. 83).

**Lemma 2 (Pati [8]).** If (i) \( \sum_{n=1}^{\infty} a_n \in |R, \lambda_n, k| (k > 0) \), (ii) \( \{\lambda_n/\lambda_{n+1}\} \in BV \) and (iii) \( \{a_n/\lambda_n - \lambda_{n-1}\} \in BV \), then \( \sum_{n=1}^{\infty} |a_n| < \infty \).

5. **Proof of Theorem 1.** We have

\[ A_n(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \, dt \]
\[ = \frac{2}{\pi} \left[ \sin nt \phi(t) \right]_0^\pi - \frac{2}{\pi} \int_0^\pi \sin nt \, d\phi(t) \]
\[ = -\frac{2}{\pi} \int_0^\pi \frac{\sin nt}{n} d\phi(t) \]
\[ = -\frac{2}{\pi} \int_0^\pi \sin nt \, A(t) \, dt + \frac{2}{\pi} \int_0^\pi A(t) t \frac{\sigma}{\partial t} \left( \frac{\sin nt}{nt} \right) \, dt \]
\[ = \frac{2}{\pi} \int_0^\pi A(t) g(k/t) \frac{t}{g(k/t)} \frac{\sigma}{\partial t} \left( \frac{\sin nt}{nt} \right) \, dt, \]

integrating by parts.

In view of Lemma 1 and second mean value theorem, the series \( \sum_{n=1}^{\infty} A_n(x) \in |R, \lambda_{n-1}, 1| \), if

\[ \sum = \sum_{n=0}^{\infty} \left| \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} \sum_{v=0}^{n} \frac{\lambda_v}{(\nu + 1)} \int_{0}^{t} u \frac{\partial}{\partial u} \left( \frac{\sin (\nu + 1)u}{u} \right) du \right| = O(1), \]

uniformly in \( 0 < t \leq \pi \). And, now
\[
\int_0^t \frac{u}{g(k/u)} \frac{\partial}{\partial u} \left( \frac{\sin (\nu + 1)u}{u} \right) \, du = \frac{\sin (\nu + 1)t}{g(k/t)} - \int_0^t \frac{\sin (\nu + 1)u}{ug(k/u)} \, du.
\]

Therefore
\[
\sum \leq \frac{1}{g(k/t)} \sum_{\nu=0}^{\infty} \left\{ \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} k(n, t) \right\} + \sum_{\nu=0}^{\infty} \sum_{\nu=0}^{n} \frac{\lambda_n}{(\nu + 1)} \int_0^t \frac{\sin (\nu + 1)u}{ug(k/u)} \, du \leq \sum_1 + \sum_2 + \sum_3, \text{ say.}
\]

Now, we write, for \( T = [t^{-1/(1-\alpha)}] \)
\[
\sum_1 = \sum_{n=0}^{T-1} + \sum_{n=T}^{\infty} = \sum_{1,1} + \sum_{1,2}, \text{ say.}
\]

Since \( \sin (\nu + 1) = O(1) \), we have
\[
\sum_{1,1} = O\left\{ \frac{1}{g(k/t)} \sum_{n=0}^{T-1} \left\{ \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} \sum_{\nu=0}^{n} \frac{\lambda_n}{(\nu + 1)} \right\} \right\}
\]
\[
= O\left\{ \frac{1}{g(k/t)} \sum_{n=0}^{T-1} \left\{ \frac{\lambda_n}{(\nu + 1)} \sum_{n=0}^{T-1} \left\{ \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right\} \right\} \right\}
\]
\[
= O\left\{ \frac{1}{g(k/t)} \sum_{n=0}^{T-1} \left\{ \frac{1}{\nu + 1} \right\} \right\}
= O(1),
\]
by (2.1), uniformly in \( 0 < t \leq \pi \). And, by (3.1),
\[
\sum_{1,2} = O\left\{ \frac{t^{-1}}{g(k/t)} \sum_{n=T}^{\infty} \left\{ \frac{\Delta \lambda_n}{(n + 1)\lambda_{n+1}} \right\} \right\}
\]
\[
= O\left\{ \frac{t^{-1}}{g(k/t)} \sum_{n=T}^{\infty} \left\{ (n + 1)\lambda_{n+1} \right\} \right\}
\]
\[
= O\left\{ \frac{t^{-1}}{g(k/t)} T^{\alpha - 1} \right\}
\]
\[
= O(1),
\]
uniformly in \( 0 < t \leq \pi \). And, by (3.2), we have
\[
\sum_2 = O\left\{ \sum_{n=0}^{\infty} \left\{ \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} \sum_{\nu=0}^{n} \frac{\lambda_n}{(\nu + 1)g(\nu + 1)} \right\} \right\}
\]
= O\left\{ \sum_{\nu=0}^{\infty} \frac{\lambda_{\nu}}{(\nu + 1)g(\nu + 1)} \sum_{n=\nu}^{\infty} \left( \frac{1}{\lambda_{n}} - \frac{1}{\lambda_{n+1}} \right) \right\} \\
= O\left\{ \sum_{\nu=0}^{\infty} \frac{1}{(\nu + 1)g(\nu + 1)} \right\} \\
= O(1) ,

by (2.5)(ii), uniformly in 0 < t \leq \pi. Also, by using (3.3), we get \\
\sum_{g} = O(1) ,

uniformly in 0 < t \leq \pi.

This terminates the proof of Theorem 1.

6. In this section we give a criterion for the absolute convergence of Lebesgue-Fourier series at a point. First we consider the following corollary of Theorem 1.

**COROLLARY.** If (i) \( \phi(t) \in BV(0, \pi) \) and (ii) \( \Lambda(t)g(k/t) \in BV(0, \pi) \), then \\
\( \sum_{n=1}^{\infty} A_n(x) \in \mathbb{R}, \exp(n^\alpha) \cdot 1(0 < \alpha < 1) \), whenever \( g(k/t) \) stands for any one of the following functions:

\[
\left( \log \frac{k}{t} \right)^{1+\epsilon}, \log \frac{k}{t} \left( \log_{2} \frac{k}{t} \right)^{1+\epsilon}, \ldots, \log \frac{k}{t} \log \frac{k}{t} \ldots \left( \log_{p} \frac{k}{t} \right)^{1+\epsilon}
\]

where \( \log_{r} k/t = \log \log_{r-1} k/t, \log_{1} k/t = \log k/t, c > 0 \), and \( k \) is any suitable positive constant such that \( g(k/\pi) > 0 \).

**THEOREM 2.** If (i) \( \phi(t) \in BV(0, \pi) \), (ii) \( \Lambda(t)g(k/t) \in BV(0, \pi) \) and (iii) \( \{n^{1-\alpha}A_n(x)\} \in BV \) for \( 0 < \alpha < 1 \), then \\
\( \sum_{n=1}^{\infty} |A_n(x)| < \infty \), where \( g(k/t) \) is as defined as in the above corollary.

**Proof of Theorem 2.** Mohanty (7) observed that for \( \lambda_n = e^{n^\alpha} \) sequences (i) \( \{\lambda_n/\lambda_{n+1}\} \in BV \) and (ii) \( \{n^{\alpha-1}\lambda_n\} \in BV \) and hence the conditions (ii) and (iii) of Lemma 2 are satisfied. Thus, in view of the above corollary, the proof follows by Lemma 2.

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