PROJECTING ONTO CYCLES IN SMOOTH, REFLEXIVE BANACH SPACES

HENRY BRUCE COHEN AND FRANCIS E. SULLIVAN
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This paper deals with operator algebras generated by certain classes of norm 1 projections on smooth, reflexive Banach spaces. For a strictly increasing continuous function $J: \mathbb{R}^+ \to \mathbb{R}^+$ on the nonnegative reals, the set of \"$J$-projections\" gives rise to operator algebras equal to their second commutants. The principal result is that the closed subspace generated by the set of elements $Ex$, where $x$ is fixed and $E$ runs through a Boolean algebra of $J$-projections, is the range of a norm 1 projection that commutes with each projection in the Boolean algebra. Sufficient conditions using Clarkson type norm inequalities are given for the commutativity of the set of all $J$-projections. Examples in Orlicz spaces are given.

1. Projections in smooth spaces. A \textit{normer} of a nonzero element $x$ in a Banach space $X$ is a functional $x^*$ in the dual $X^*$ such that $\|x^*\| = 1$ and $\|x\| = x^*(x)$. A normer for $x$ always exists; we say that $X$ is \textit{smooth} if every nonzero $x$ has but one normer, denoted $N(x)$. We make the definition $N(0) = 0$.

Proof of the following three lemmas is left to the reader; see, for instance, [5; p. 447].

LEMMA 1. In a smooth space $X$, the norming map $N: X \to S^* \cup \{0\}$ has the following properties, where $S^*$ is the unit sphere of $X^*$.

(1) $N(x)$ is the only element of $S^*$ such that $N(x)(x) = \|x\|$ if $x \neq 0$.

(2) $N(\lambda x) = (|\lambda|/\lambda)N(x)$ for all scalars $\lambda \neq 0$; in particular, $N(\lambda x) = N(x)$ for $\lambda > 0$.

(3) In the real case, $N(x)(y) = \lim (\lambda \to 0)((\|x + \lambda y\| - \|x\|)/\lambda$ for $x, y \in X$ and $x \neq 0$.

LEMMA 2. If $X$ is a smooth complex Banach space, $\text{Re } X$ is also smooth; indeed, for each $x \neq 0$, $\text{Re } N(x)$ is the normer of $x$ in $(\text{Re } X)^*$.

A vector $x$ is said to be \textit{James-orthogonal} to $y$ if $\|x + \lambda y\| \geq \|x\|$ for all real numbers $\lambda$.

LEMMA 3. If $X$ is a smooth space, then $N(x)(y) = 0$ if and only if $x$ is James-orthogonal to $y$ in the real case and James-orthogonal to both $y$ and $iy$ in the complex case. If $Y$ is a subspace, then $N(x)(y) = 0(y \in Y)$ if and only if $\|x + y\| \geq \|x\|(y \in Y)$.
**Lemma 4.** If $E$ is a norm one projection in a normed linear space $X$, then $\|a + b\| \geq \|a\|$ for every $a \in EX$ and $b \in (I - E)X$.

**Proof.** $\|a\| = \|E(a + b)\| \leq \|a + b\|$.

**Lemma 5.** If $E$ is a norm one projection on a smooth space $X$, $N(Ex)(Ey) = N(Ex)(y)(x, y \in X)$.

**Proof.** This is an immediate consequence of Lemmas 3 and 4.

**Theorem 6.** A subspace of a smooth space $X$ can be the range of at most one norm 1 projection.

**Proof.** Suppose $E$ and $F$ are norm 1 projections on $X$ with $EX = FX$. Then $EF = F$ and $FE = E$ so that $E - F = E(I - F') = F(E - I)$. If $E \neq F$, there is an $x$ such that

$$0 \neq \|Ex - Fx\| = N(Ex - Fx)(Ex - Fx)$$

$$= N(E(I - F)x)(Ex) - N(F(E - I)x)(Fx)$$

$$= N(E(I - F)x)(x) - N(F(E - I)x)(x) = 0,$$

a contradiction.

We wish to thank the referee for sharpening the following two lemmas into their present form and for suggesting lines of proof.

**Theorem 7.** A subspace of a rotund space can be the null manifold of at most one norm 1 projection.

**Proof.** Suppose $E$ and $F$ are distinct norm 1 projections on a rotund space $X$, with the same null manifold $N$. Then there is an element $x$ in the range of $E$ that is not in the range of $F$. Then $x = y + w$ where $y$ is the range of $F$, $w$ is in $N$, and $x$ and $y$ are not linearly dependent.

$$\|x\| = \|Ex - 1/2w\| \leq \|x - 1/2w\| = \|1/2(x + y)\|$$

$$\|y\| = \|F(y + 1/2w)\| \leq \|y + 1/2w\| = \|1/2(x + y)\|$$

so that $1/2(\|x\| + \|y\|) \leq 1/2(\|x + y\|) \leq 1/2(\|x\| + \|y\|), \|x + y\| = \|x\| + \|y\|$, and $X$ is not rotund.

**Theorem 8.** For any norm 1 projection $E$ on a smooth space $X$, $N(EX \cap S) \subseteq E^*X^* \cap N(S)$, with equality if $X$ is smooth and rotund. If $X$ is reflexive, then $N(S) = S^*$, but in any case $N(S)$ is dense in $S^*$. 


Proof. If \( x^* \in N(EX \cap S) \), then there is a norm 1 vector \( x \) such that \( x^* = N(x) \) and \( Ex = x \). Then \( E^*N(x)(y) = N(Ex)(Ey) = N(Ex)(y) = x^*(y) \) by Lemma 5 for all \( y \) in \( X \); hence, \( x^* \in E^*X^* \cap N(S) \).

If \( X \) is rotund and \( x^* \in E^*X^* \cap N(S) \), then \( x^* = N(x) \) where \( \|x\| = 1 \) and \( E^*(N(x)) = N(x) \). Then

\[
\|x + Ex\| \leq \|x\| + \|Ex\| \leq \|x\| + \|x\| = N(x)(x) + N(x)(x) = N(x)(x) + (E^*N(x))(x) = N(x)(x + Ex) \leq \|x + Ex\|.
\]

Then \( \|x\| + \|Ex\| = \|x + Ex\| \) and \( x = Ex \) by rotundity and the fact that \( E \) is a projection.

The last statement follows from results of James [7] and Bishop-Phelps [2].

2. \( \mathcal{F} \)-projections. Throughout this section, \( \mathcal{F} \) denotes a fixed, but arbitrary, strictly increasing continuous function from the set of nonnegative real numbers into itself.

**Definition.** An \( \mathcal{F} \)-projection on a Banach space \( X \) is a projection \( E \) on \( X \) for which \( \mathcal{F}(\|x\|) = \mathcal{F}(\|Ex\|) + \mathcal{F}(\|(I - E)x\|) \) for all \( x \) in \( X \).

**Lemma 9.** (1) An \( \mathcal{F} \)-projection has norm 1 or 0; (2) If \( E \) is an \( \mathcal{F} \)-projection, \( \mathcal{F}(\|a + b\|) = \mathcal{F}(\|a\|) + \mathcal{F}(\|b\|) \) and \( \|a + b\| = \|a - b\| \) for all \( a \) in \( E[X] \), \( b \) in \( (I - E)[X] \); (3) the product of two commuting \( \mathcal{F} \)-projections is an \( \mathcal{F} \)-projection.

**Proof.** (1) If \( E \) is an \( \mathcal{F} \)-projection,

\[
\mathcal{F}(\|Ex\|) \leq \mathcal{F}(\|Ex\|) + \mathcal{F}(\|(I - E)x\|) = \mathcal{F}(\|x\|).
\]

Since \( \mathcal{F} \) is strictly increasing, \( \|Ex\| \leq \|x\| \).

\[
\mathcal{F}(\|a + b\|) = \mathcal{F}(\|Ea + (I - E)b\|)
\]

(2) \( = \mathcal{F}(\|E(Ea + (I - E)b)\|) + \mathcal{F}(\|(I - E)(Ea + (I - E)b)\|) = \mathcal{F}(\|Ea\|) + \mathcal{F}(\|(I - E)b\|) \),

and

\[
\|a + b\| = \mathcal{F}^{-1}(\mathcal{F}(\|a + b\|) = \mathcal{F}^{-1}(\mathcal{F}(\|a\|) + \mathcal{F}(\|b\|)) = \mathcal{F}^{-1}(\mathcal{F}(\|a - b\|)) = \|a - b\|.
\]

(3) If \( E \) and \( F \) are commuting \( \mathcal{F} \)-projections,

\[
\mathcal{F}(\|x\|) = \mathcal{F}(\|Fx\|) + \mathcal{F}(\|(I - F)x\|) = \mathcal{F}(\|EFx\|) + \mathcal{F}(\|(I - E)Fx\|) + \mathcal{F}(\|(I - F)x\|)
\]
\[= \mathcal{F}(\|EFx\|) + \mathcal{F}(\|I-E\|x + (I-F)x\|)\]
\[= \mathcal{F}(\|EFx\|) + \mathcal{F}(\|(I-E)Fx\|)\]
for all \(x\) in \(X\).

**Remark.** If \(E\) is an \(\mathcal{F}\)-projection, then \(\|a + b\|\), where \(a\) is any norm 1 vector in \(EX\) and \(b\) is any norm 1 vector in \((I-E)X\), is constant at \(\mathcal{F}^{-1}(2\mathcal{F}(1))\). For
\[\|a + b\| = \mathcal{F}^{-1}\mathcal{F}(\|a + b\|) = \mathcal{F}^{-1}(\mathcal{F}(\|a\|) + \mathcal{F}(\|b\|))\]

**Theorem 10.** A maximal family \(\mathcal{P}\) of commuting \(\mathcal{F}\)-projections is a complete-Boolean algebra of norm 1 projections.

**Proof.** Clearly 0 and \(I\) are in \(\mathcal{P}\) and if \(E\) is in \(\mathcal{P}\), so is \((I-E)\) by the symmetry of the definition of an \(\mathcal{F}\)-projection. If \(E\) and \(F\) are in \(\mathcal{P}\), \(EF\) is an \(\mathcal{F}\)-projection by Lemma 9, and it commutes with \(\mathcal{P}\). Therefore, \(EF\) is in \(\mathcal{P}\). Thus \(\mathcal{P}\) is a Boolean algebra of projections on \(X\) as defined by Bade [1]. Now suppose \(E_a\) is an increasing net of projections in \(\mathcal{P}\). For each \(x\) in \(X\) and for \(\alpha \leq \beta\), \(E_{a\alpha} = E_{a\beta}\). So \(\|E_{a\alpha}x\| \leq \|x\|\); thus, \(\mathcal{F}(\|E_{a\alpha}x\|)\) is an increasing net of real numbers bounded above by \(\mathcal{F}(\|x\|)\); hence, convergent. This implies \(E_{a\alpha}x\) is Cauchy, as follows. Given \(\varepsilon \geq 0\), choose \(\theta\) such that
\[\mathcal{F}(\|E_{a\alpha}x\|) \geq \lim_\alpha \mathcal{F}(\|E_{a\alpha}x\|) - \mathcal{F}(\varepsilon/2)\]
for all \(\alpha \geq \theta\). If \(\beta \geq \theta\),
\[\mathcal{F}(\|E_{a\beta}x - E_{a\alpha}x\|) + \mathcal{F}(\|E_{a\alpha}x\|)\]
\[= \mathcal{F}(\|E_{a\beta}x - E_{a\beta}E_{a\alpha}x\|) + \mathcal{F}(\|E_{a\beta}E_{a\alpha}x\|)\]
\[= \mathcal{F}(\|(I - E_{a\beta})E_{a\alpha}x\|) + \mathcal{F}(\|E_{a\beta}E_{a\alpha}x\|) = \mathcal{F}(\|E_{a\beta}x\|)\]
Thus,
\[\mathcal{F}(\|E_{a\beta}x - E_{a\alpha}x\|) = \mathcal{F}(\|E_{a\beta}x\|) - \mathcal{F}(\|E_{a\alpha}x\|)\]
And from this
\[\mathcal{F}(\varepsilon/2) \geq \lim_\alpha \mathcal{F}(\|E_{a\alpha}x\|) - \mathcal{F}(\|E_{a\beta}x\|)\]
\[\geq \mathcal{F}(\|E_{a\alpha}x\|) - \mathcal{F}(\|E_{a\alpha}x\|) = \mathcal{F}(\|E_{a\alpha}x - E_{a\beta}x\|)\]
hence, \(\varepsilon/2 \geq \|E_{a\alpha}x - E_{a\beta}x\|\) because \(\mathcal{F}\) is increasing. If \(\alpha, \beta \geq \theta\),
\[\|E_{a\alpha}x - E_{a\beta}x\| \leq \|E_{a\alpha}x - E_{a\alpha}x\| + \|E_{a\alpha}x - E_{a\beta}x\| \leq \varepsilon\]

Define \(Ex = \lim_\alpha E_{a\alpha}x\) for every \(x\) in \(X\). Then \(E\) is surely a projection and, since \(\mathcal{F}\) is continuous, \(E\) is an \(\mathcal{F}\)-projection; since \(E\)
commutes with $\mathcal{P}$, it is in $\mathcal{P}$. This completes the argument.

By Zorn’s lemma, complete Boolean algebras of $\mathcal{F}$-projections always exist, although they may be trivial. Nontrivial examples are given later.

**Theorem 11.** Suppose that all vectors $v$ and $w$ in $X$ satisfy the (Clarkson) inequality

$$1/2 \mathcal{F}(||v + w||) + 1/2 \mathcal{F}(||v - w||) \leq \mathcal{F}(||v||) + \mathcal{F}(||w||)$$

and suppose $\mathcal{F}(2) \neq 4$, $\mathcal{F}(1) = 1$. Then any two $\mathcal{F}$-projections commute (and so the set of all $\mathcal{F}$-projections form a complete Boolean algebra of projections). The same result holds for the reverse inequality.

**Proof.** Let $E$ and $F$ be two $\mathcal{F}$-projections and $x \in X$. Then decomposing $Ex$ into $F$ and then $E$ components, applying Clarkson’s inequality, and simplifying (using Lemma 9) we obtain

$$\mathcal{F}(||Ex||) = \mathcal{F}(||EFEx||) + \mathcal{F}(||E(I-F)Ex||)$$

$$+ \mathcal{F}(||I-E)FEx||) + \mathcal{F}(||I-E)(I-F)Ex||$$

$$\geq 1/2 \mathcal{F}(||EFEx + E(I-F)Ex||) + 1/2 \mathcal{F}(||EFEx - E(I-F)Ex||)$$

$$+ 1/2 \mathcal{F}(||I-E)FEx + (I-E)(I-F)Ex||)$$

$$+ 1/2 \mathcal{F}(||I-E)FEx - (I-E)(I-F)Ex||)$$

$$= 1/2 \mathcal{F}(||Ex||) + 1/2 \mathcal{F}(||EFEx - E(I-F)Ex$$

$$+ (I-E)FEx - (I-E)(I-F)Ex||)$$

$$= 1/2 \mathcal{F}(||Ex||) + 1/2 \mathcal{F}(||FEx - (I-F)Ex||)$$

$$= 1/2 \mathcal{F}(||Ex||) + 1/2 \mathcal{F}(||FEx + (I-F)Ex||)$$

$$= \mathcal{F}(||Ex||).$$

This implies equality in Clarkson’s inequality for the vectors $(I-E)FEx$ and $(I-E)(I-F)Ex$:

$$\mathcal{F}(||I-E)FEx||) + \mathcal{F}(||I-E)(I-F)Ex||)$$

$$= 1/2 \mathcal{F}(||I-E)FEx + (I-E)(I-F)Ex||)$$

$$+ 1/2 \mathcal{F}(||I-E)FEx - (I-E)(I-F)Ex||) .$$

Since the first term on the right is zero, we can define $Z \equiv Z(x) \equiv (I-E)FEx \equiv -(I-E)(I-F)Ex$ and obtain $4 \mathcal{F}(||z||) = \mathcal{F}(2||z||)$.

What if $Z(x) \neq 0$? Then $||Z(x)|| = 1$, and we have

$$4 = 4 \mathcal{F}(||Z(x)|| ||Z(x)||) = \mathcal{F}(2||Z(x)|| ||Z(x)||) = \mathcal{F}(2)$$

which contradicts the hypothesis. Thus $Z = 0$ and so $FEx = EFEx$
for any $x$ and any two $\mathcal{F}$-projections $E$ and $F$. Replacing $E$ and $F$ by $(I - E)$ and $F$ yields $F(I - E)x = (I - E)F(I - E)x$; whence $EFx = EFEx$. Therefore $FEx = EFx$ and so $E$ and $F$ commute.

**Remark.** Consider $\mathcal{F}(\lambda) = \lambda^p$ for a fixed $p, 1 \leq p < \infty$. An $\mathcal{F}$-projection for such an $\mathcal{F}$ is called an $L^p$-projection. Cunningham [4] showed that the $L^1$ projections always commute in any Banach space. The above theorem shows that for $p \neq 2$, the $L^p$ projections in an $L^p$ space commute.

**Definition.** A net $T_\alpha$ of projections on a Banach space $X$ is said to be increasing if $\alpha < \beta$ implies $T_\alpha T_\beta = T_\alpha$. 

**Theorem 12.** If $T_\alpha$ is an increasing net of norm 1 projections on a reflexive Banach space $X$, then $T_\alpha$ converges in the strong operator topology of $X$ to a norm 1 projection $T$ that commutes with each $T_\alpha$ and whose range is the norm closure of $\bigcup_\alpha T_\alpha[X]$.

**Proof.** The essentials of a proof can be found in [8; p. 223].

3. Projecting onto cycle subspaces.

**Definition.** If $\mathcal{P}$ is a Boolean algebra of projections on $X$ and $x$ is in $X$, let $S(x; \mathcal{P})$ denote the cycle generated by $x$ and $\mathcal{P}$; that is, the closed subspace of $X$ generated by $x$.

**Theorem 13.** Let $\mathcal{P}$ be a Boolean algebra of $\mathcal{F}$-projections on a Banach space $X$ that is smooth and reflexive, and let $x \in X$. Then $S(x; \mathcal{P})$ is the range of a (unique) norm 1 projection that commutes with $\mathcal{P}$.

**Proof.** Let $\pi$ denote the set of all partitions of $x$ by $\mathcal{P}$; that is, finite subsets $\{E_1, \ldots, E_n\}$ of $\mathcal{P}$ such that $E_i E_j = 0$ if $i \neq j$ and $(V_i E_i)(x) = \sum E_i x = x$. The set $\{I\}$ is such a partition. Order $\pi$ by setting $\mathcal{E} r \mathcal{M}$ if, given $A$ in $\mathcal{M}$ there is an $E$ in $\mathcal{E}$ such that $AE = A$. This “is refined by” relation $r$ is reflexive, anti-symmetric, transitive, and it directs the set $\pi$. Indeed, if $\{E_1, \ldots, E_n\}$ and $\{A_1, \ldots, A_m\}$ are partitions of $x$, then one common refinement is the set of $E_i A_j$ such that $E_i A_j x \neq 0$.

For each partition $\mathcal{E}$ of $x$, define $T(\mathcal{E})(y) = \sum (E \in \mathcal{E})(N(E x)(y)/\|Ex\|)Ex$ for all $y$ in $X$. The transformation $T(\mathcal{E})$ is obviously linear; that it is a projection on $X$ is an immediate consequence of the fact that for $E$ and $F$ in $\mathcal{P}$ with $EF = 0$, $N(Ez)(Fy) = N(Ez)(EFy) = 0$. We now show that the norm of $T(\mathcal{E})$ is 1. It is not 0, first of all,
because the projection leaves \( x \) fixed. Proceeding, let \( y \in X \).

\[
\| N(Ex)(y) \|/\| Ex \| \| Ex \| = |N(Ex)(y)| = |N(Ex)(Ey)| \leq \| Ey \| .
\]

From this,

\[
\mathcal{T}(\| y \|) \geq \mathcal{T}(\| V(E \in \mathcal{E})Ey \|) = \mathcal{T}(\| \sum (E \in \mathcal{E})Ey \|)
\]

\[
= \sum (E \in \mathcal{E}) \mathcal{T}(\| Ey \|) \geq \sum (E \in \mathcal{E}) \mathcal{T}(\| N(Ex)(y)/\| Ex \| ) Ex \|
\]

\[
= \mathcal{T}(\| \sum (E \in \mathcal{E})(N(Ex)(y)/\| Ex \| ) Ex \|) = \mathcal{T}(\| T(\mathcal{E})y \|).
\]

Consequently \( \| T(\mathcal{E})y \| \leq \| y \| .\)

In order to apply Theorem 12, we must show that \( T(\mathcal{A})T(\mathcal{E}) = T(\mathcal{E}) = T(\mathcal{E})T(\mathcal{A}) \) under the assumption that \( \mathcal{E} \cap \mathcal{A} \). It is a routine matter to use Lemma 5 to check that \( T(\mathcal{A})(Ax) = Ax \) for any \( A \) in \( \mathcal{A} \), that \( T(\mathcal{E})(Ex) = Ex \) for any \( E \) in \( \mathcal{E} \), and that, therefore, \( T(\mathcal{E}) = T(\mathcal{A})T(\mathcal{E}) \). Let \( z \) be a given element of the null manifold of \( T(\mathcal{A}) \). Then for each \( A \) in \( \mathcal{A} \), \( (N(Ax)(z)/\| Ax \| )Ax = AT(\mathcal{A})z = 0 \) so that \( N(Ax)(z) = N(Ax)(z) = 0 \). Then \( Ax \) is James orthogonal to \( Ax \):

\[
\| Ax + Az \| \geq \| Ax \| .
\]

Then

\[
\mathcal{T}(\| Ex + Ez \|) = \mathcal{T}(\| \sum (AE = A)A(x + z) \|)
\]

\[
= \sum (AE = A) \mathcal{T}(\| Ax + Az \|) \geq \sum (AE = A) \mathcal{T}(\| Ax \|)
\]

\[
= \mathcal{T}(\| \sum (AE = A)Ax \|) = \mathcal{T}(\| Ex \|),
\]

for every \( E \) in \( \mathcal{E} \). Therefore, \( \| Ex + Ez \| \geq \| Ex \| \) and, similarly, \( \| Ex + iEz \| \geq \| Ex \| \) if \( X \) is complex. In any case, \( N(Ex)(z) = N(Ex)(Ex) = 0 \) for all \( E \) in \( \mathcal{E} \) and, therefore, \( z \) is in the null manifold of \( T(\mathcal{E}) \). Since the null manifold of \( T(\mathcal{E}) \) contains that of \( T(\mathcal{A}) \), we have \( T(\mathcal{E})T(\mathcal{A}) = T(\mathcal{E}) \).

By Theorem 12, there is a norm 1 projection \( T \) commuting with every \( T(\mathcal{E}) \) that is the limit in the strong operator topology of the net \( T(\mathcal{E}) \) and whose range is the subspace cl \( \cup (E \in \pi)T(\mathcal{E})[X] \). Let us show that \( T \) commutes with the projections in \( \mathcal{P} \). Let \( E \in \mathcal{P} \). If \( Ex \neq 0 \), let \( \mathcal{E} \) denote the set \( \{ E \} \) or \( \{ E, I - E \} \) that is a partition of \( x \). Given \( \mathcal{A} \in \pi \) such that \( \mathcal{E} \cap \mathcal{A} \),

\[
T(\mathcal{A})Ey = \sum (AE = A)(N(Ax)(Ey)/\| Ax \| )Ax
\]

\[
= \sum (AE = A)(N(Ax)(Ey)/\| Ax \| )Ax
\]

\[
= \sum (AE = A)(N(Ax)(y)/\| Ax \| )EAx
\]

\[
= E(\sum (AE = A)(N(Ax)(y)/\| Ax \| )Ax)
\]

\[
= E(\sum (A \in \mathcal{A})(N(Ax)(y)/\| Ax \| )Ax)
\]

\[
= ET(\mathcal{A})y
\]
for all \( y \) in \( X \). Consequently, for each \( y \) in \( X \),

\[
TEy = \lim (E r, \mathcal{A}) T(\mathcal{A}) Ey = \lim (E r, \mathcal{A}) ET(\mathcal{A}) y
= E \lim (E r, \mathcal{A}) T(\mathcal{A}) y = ETy.
\]

Therefore, \( TE = ET \) provided \( Ex \neq 0 \). If \( Ex = 0 \), then \( (I - E)x \neq 0 \) and \( T(I - E) = (I - E) T \) by the same argument. From this, \( TE = ET \) when \( Ex = 0 \).

For all \( \mathcal{A} \) in \( \pi \), \( T(\mathcal{A})[X] \subseteq S(x; \mathcal{P}) \); hence, \( T[X] \subseteq S(x; \mathcal{P}) \). And given \( E \in \mathcal{P} \), if \( Ex \neq 0 \), then, letting \( \mathcal{E} \) be the above partition of \( x, S(x; \mathcal{E}) \subseteq T[X] \). This completes the proof of Theorem 13.

**Theorem 14.** Let \( \mathcal{P} \) be a complete Boolean algebra of \( \mathcal{F} \)-projections on a Banach space that is reflexive and smooth. Then the weakly closed algebra \( \mathcal{W}(\mathcal{P}) \) of operators on \( X \) generated by \( \mathcal{P} \) is equal to its second commutant.

**Proof.** Bade [1] shows that if \( \mathcal{P} \) is complete, then \( \mathcal{W}(\mathcal{P}) \) is the uniformly closed algebra of operators generated by \( \mathcal{P} \) and it consists, furthermore, of exactly those (bounded linear) operators of \( X \) which leave invariant every closed linear manifold invariant under \( \mathcal{P} \).

Suppose \( A \) is in the second commutant of \( \mathcal{W}(\mathcal{P}) \). For each \( x \) in \( X \), let \( T^* \) denote the norm one projection whose range is \( S_x = S(x; \mathcal{P}) \). Then \( T^* \) commutes with \( \mathcal{W}(\mathcal{P}) \) so that \( AT^* = T^* A \) for all \( x \) in \( X \). From this, we have that \( A \) leaves each \( S_x \) invariant; \( AS_x = AT^* X = T^* AX \subseteq T^* X = S_x \). If \( M \) is a closed subspace left invariant under \( \mathcal{P} \), then \( S_m \subseteq M \) for all \( m \) in \( M \); whence, \( A(m) \in AS_m \subseteq S_m \subseteq M \) for each \( m \) in \( M \). Therefore, \( A \) leaves \( M \) invariant. Therefore, \( A \in \mathcal{W}(\mathcal{P}) \).

4. A class of examples. Let \( (S, \Sigma, \mu) \) be a measure space with the property FSP (a measurable set of infinite measure contains a measurable subset of finite positive measure). This condition is discussed in [9]. We consider an Orlicz space \( L_M \) over \( (S, \Sigma, \mu) \) where the complimentary Young's functions \( M \) and \( N \) are normalized \((M(1) + N(1) = 1)\), satisfy \( \Delta_2 \) conditions, and have continuous, strictly increasing derivatives denoted \( m \) and \( n \), respectively. Then \( L_M \) is reflexive and [9; Corollary 2.1] the Luxemburg norms in both \( L_M \) and \( L_N \) are strongly differentiable. Furthermore, the weak derivative of a norm 1 function \( f_0 \) in \( L_M \) is given by \( f \rightarrow \int f m(f_0) d\mu \).

**Lemma 15.** If \( 0 \leq f \in L_M \), then \( \frac{m(f(x))}{\|f\|} = \frac{m(f_0(x))}{\|m,f\|} \) for almost
all $x \in S$.

Proof. If $h = \alpha g$ for $\alpha \geq 0$ and if $h, g \geq 0$ a.e., we have equality for $h$ and $m(g)$ in Holder's inequality: $\|h\| \|mg\| = \int hm(g)d\mu$. Then

$$\left\{ \int m \left( \frac{f}{\|f\|} \right) d\mu = \|f\| = \int f \left( \frac{m(f)}{\|m(f)\|} \right) d\mu \right\}$$

so $m(\|f\|/\|f\|)$ and $m(\|f\|/\|mf\|)$ are normers for $f$. Since $L_\mu$ is smooth, normers are unique.

**Lemma 16.** Assume the existence of sets of arbitrarily small positive measure. If $f, g \in L_\mu$ with $0 < \|f\| < \|g\|$, then $0 < \|mf\| < \|mg\|$.

Proof. Set $K = \|g\|/\|f\| > 1$. Choose $x \in S$ such that $0 < m(g(x))/m(g) = m(g(x))/\|g\|$. Set $a = |g(x)|/K > 0$. For any measurable set $E$, let $f_E$ be the function constant on $E$ at the value $a$, and agreeing with $|f|$ outside of $E$. By diminishing the measure of $E$, the function $f_E$ may be brought in the norm of $L_\mu$ as close to $|f|$ as desired. Furthermore, $\|m(Kf_E)\| - \|mf\|$ approaches $\|m(Kf)\| - \|mf\| > 0$ as $E$ decreases. It is therefore, possible to choose a set $E$ of positive measure so small that

$$m(g(x)/\|g\|)(\|f\|/\|f_E\|)\|m(Kf_E)\| > m(g(x)/\|g\|)\|mf\|.$$

Select $y \in E$ such that $m(Kf_E(y)) = m(Kf_E(y)/\|Kf_E\|)\|m(Kf_E)\|$. Computing, we have

$$m(g(x)/\|g\|)\|mg\| = m(g(x)) = m(Ka) = m(Kf_E(y)) = m(f_E(y)/\|f_E\|)\|m(Kf_E)\| = m(a/\|f_E\|)\|m(Kf_E)\| = m(g(x)/\|g\|)(\|f\|/\|f_E\|)\|m(Kf_E)\| > m(g(x)/\|g\|)\|mf\|.$$

Cancelling $m(g(x)/\|g\|)$ finishes the argument.

Perhaps Lemma 16 is true without restrictions on the measure space. We have not settled this.

Define $\mathcal{F}(\lambda) = \|f\|\|mf\| = \left\{ \int f \left( \frac{m(f)}{\|m(f)\|} \right) d\mu \right\}$ where $f$ is any function in $L_\mu$ of norm $\lambda$. From Lemma 16, it is clear that $\mathcal{F}$ is well defined and strictly increasing. To show continuity, let $E$ be any set of finite positive measure and $a(\lambda) = \lambda/\|\chi_E\|$. Then $a(\lambda)$ is continuous and

$$\mathcal{F}(\lambda) = \int a(\lambda)\chi_E m(a(\lambda)\chi_E) d\mu = \int a(\lambda)m(a(\lambda))\chi_E d\mu = a(\lambda)m(a(\lambda))\mu E,$$

a continuous function.

Each measurable set $E$ gives rise to the characteristic projection $f \rightarrow \chi_E f$. 

...
LEMMA 17. Every characteristic projection is an $\mathcal{F}$-projection.

Proof.

$$\mathcal{F} (\| f \|) = \int \chi_E f \, d\mu = \int_E \chi_E f \, d\mu + \int_{S \setminus E} \chi_E f \, d\mu$$

$$= \int (\chi_E f) m(\chi_E f) \, d\mu + \int (\chi_{S \setminus E} f) m(\chi_{S \setminus E} f) \, d\mu$$

$$= \mathcal{F} (\| \chi_E f \|) + \mathcal{F} (\| \chi_{S \setminus E} f \|).$$

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