PROJECTING ONTO CYCLES IN SMOOTH, REFLEXIVE BANACH SPACES

Henry Bruce Cohen and Francis E. Sullivan
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H. B. COHEN AND F. E. SULLIVAN

This paper deals with operator algebras generated by certain classes of norm 1 projections on smooth, reflexive Banach spaces. For a strictly increasing continuous function \( J \) on the nonnegative reals, the set of "\( J \)-projections" gives rise to operator algebras equal to their second commutants. The principal result is that the closed subspace generated by the set of elements \( Ex \), where \( x \) is fixed and \( E \) runs through a Boolean algebra of \( J \)-projections, is the range of a norm 1 projection that commutes with each projection in the Boolean algebra. Sufficient conditions using Clarkson type norm inequalities are given for the commutativity of the set of all \( J \)-projections. Examples in Orlicz spaces are given.

1. Projections in smooth spaces. A normer of a nonzero element \( x \) in a Banach space \( X \) is a functional \( x^* \) in the dual \( X^* \) such that \( \|x^*\| = 1 \) and \( \|x\| = x^*(x) \). A normer for \( x \) always exists; we say that \( X \) is smooth if every nonzero \( x \) has but one normer, denoted \( N(x) \). We make the definition \( N(0) = 0 \).

Proof of the following three lemmas is left to the reader; see, for instance, [5; p. 447].

**Lemma 1.** In a smooth space \( X \), the norming map \( N: X \to S^* \cup \{0\} \) has the following properties, where \( S^* \) is the unit sphere of \( X^* \).

1. \( N(x) \) is the only element of \( S^* \) such that \( N(x)(x) = \|x\| \) if \( x \neq 0 \).
2. \( N(\lambda x) = (|\lambda|/\lambda)N(x) \) for all scalars \( \lambda \neq 0 \); in particular, \( N(\lambda x) = N(x) \) for \( \lambda > 0 \).
3. In the real case, \( N(x)(y) = \lim (\lambda \to 0)(\|x + \lambda y\| - \|x\|)/\lambda \) for \( x, y \in X \) and \( x \neq 0 \).

**Lemma 2.** If \( X \) is a smooth complex Banach space, \( \text{Re} \ X \) is also smooth; indeed, for each \( x \neq 0 \), \( \text{Re} \ N(x) \) is the normer of \( x \) in \( (\text{Re} \ X)^* \).

A vector \( x \) is said to be James-orthogonal to \( y \) if \( \|x + \lambda y\| \geq \|x\| \) for all real numbers \( \lambda \).

**Lemma 3.** If \( X \) is a smooth space, then \( N(x)(y) = 0 \) if and only if \( x \) is James-orthogonal to \( y \) in the real case and James-orthogonal to both \( y \) and \( iy \) in the complex case. If \( Y \) is a subspace, then \( N(x)(y) = 0(y \in Y) \) if and only if \( \|x + y\| \geq \|x\| \) \((y \in Y)\).
**Lemma 4.** If $E$ is a norm one projection in a normed linear space $X$, then $\|a + b\| \geq \|a\|$ for every $a \in EX$ and $b \in (I - E)X$.

*Proof.* $\|a\| = \|E(a + b)\| \leq \|a + b\|$. 

**Lemma 5.** If $E$ is a norm one projection on a smooth space $X$, $N(Ex)(Ey) = N(Ex)(y)(x, y \in X)$.

*Proof.* This is an immediate consequence of Lemmas 3 and 4.

**Theorem 6.** A subspace of a smooth space $X$ can be the range of at most one norm 1 projection.

*Proof.* Suppose $E$ and $F$ are norm 1 projections on $X$ with $EX = FX$. Then $EF = F$ and $FE = E$ so that $E - F = E(I - F') = F(E - I)$. If $E \neq F$, there is an $x$ such that

$$0 \neq \|Ex - Fx\| = N(Ex - Fx)(Ex - Fx)$$

$$= N(E(I - F)x)(Ex) - N(F(E - I)x)(Fx)$$

$$= N(E(I - F)x)(x) - N(F(E - I)x)(x) = 0 ,$$

a contradiction. We wish to thank the referee for sharpening the following two lemmas into their present form and for suggesting lines of proof.

**Theorem 7.** A subspace of a rotund space can be the null manifold of at most one norm 1 projection.

*Proof.* Suppose $E$ and $F$ are distinct norm 1 projections on a rotund space $X$, with the same null manifold $N$. Then there is an element $x$ in the range of $E$ that is not in the range of $F$. Then $x = y + w$ where $y$ is the range of $F$, $w$ is in $N$, and $x$ and $y$ are not linearly dependent.

$$\|x\| = \|E(x - 1/2w)\| \leq \|x - 1/2w\| = \|1/2(x + y)\|$$

$$\|y\| = \|F(y + 1/2w)\| \leq \|y + 1/2w\| = \|1/2(x + y)\|$$

so that $1/2(\|x\| + \|y\|) \leq 1/2(x + y) \| \leq 1/2(\|x\| + \|y\|)$, $\|x + y\| = \|x\| + \|y\|$, and $X$ is not rotund.

**Theorem 8.** For any norm 1 projection $E$ on a smooth space $X$, $N(EX \cap S) \subseteq E^*X^* \cap N(S)$, with equality if $X$ is smooth and rotund. If $X$ is reflexive, then $N(S) = S^*$, but in any case $N(S)$ is dense in $S^*$.
Proof. If \( x^* \in N(EX \cap S) \), then there is a norm 1 vector \( x \) such that \( x^* = N(x) \) and \( Ex = x \). Then \( E^*N(x)(y) = N(Ex)(Ey) = N(Ex)(y) = x^*(y) \) by Lemma 5 for all \( y \) in \( X \); hence, \( x^* \in E^*X^* \cap N(S) \).

If \( X \) is rotund and \( x^* \in E^*X^* \cap N(S) \), then \( x^* = N(x) \) where \( ||x|| = 1 \) and \( E^*(N(x)) = N(x) \). Then

\[
||x + Ex|| \leq ||x|| + ||Ex|| \leq ||x|| + ||x||
\]

\[
= N(x)(x) + N(x)(x) = N(x)(x) + (E^*N(x))(x) = N(x)(x + Ex) \leq ||x + Ex||.
\]

Then \( ||x|| + ||Ex|| = ||x + Ex|| = x \) and \( x = Ex \) by rotundity and the fact that \( E \) is a projection.

The last statement follows from results of James [7] and Bishop-Phelps [2].

2. \( \mathcal{F} \)-projections. Throughout this section, \( \mathcal{F} \) denotes a fixed, but arbitrary, strictly increasing continuous function from the set of nonnegative real numbers into itself.

Definition. An \( \mathcal{F} \)-projection on a Banach space \( X \) is a projection \( E \) on \( X \) for which \( \mathcal{F}(||x||) = \mathcal{F}(||Ex||) + \mathcal{F}(||(I - E)x||) \) for all \( x \) in \( X \).

Lemma 9. (1) An \( \mathcal{F} \)-projection has norm 1 or 0; (2) If \( E \) is an \( \mathcal{F} \)-projection, \( \mathcal{F}(||a + b||) = \mathcal{F}(||a||) + \mathcal{F}(||b||) \) and \( ||a + b|| = ||a - b|| \) for all \( a \) in \( E[X] \), \( b \) in \( (I - E)[X] \); (3) the product of two commuting \( \mathcal{F} \)-projections is an \( \mathcal{F} \)-projection.

Proof. (1) If \( E \) is an \( \mathcal{F} \)-projection,

\[
\mathcal{F}(||EX||) \leq \mathcal{F}(||Ex||) + \mathcal{F}(||(I - E)x||) = \mathcal{F}(||x||).
\]

Since \( \mathcal{F} \) is strictly increasing, \( ||Ex|| \leq ||x||. \)

\[
\mathcal{F}(||a + b||) = \mathcal{F}(||Ea + (I - E)b||)
\]

(2) \( = \mathcal{F}(||E(a + (I - E)b)||) + \mathcal{F}(||(I - E)(Ea + (I - E)b)||)
\]

\( = \mathcal{F}(||Ea||) + \mathcal{F}(||(I - E)b||) \),

and

\[
||a + b|| = \mathcal{F}^{-1}(\mathcal{F}(||a + b||)) = \mathcal{F}^{-1}(\mathcal{F}(||a||) + \mathcal{F}(||b||))
\]

\( = \mathcal{F}^{-1}(\mathcal{F}(||a||) + \mathcal{F}(||-b||)) = \mathcal{F}^{-1}(\mathcal{F}(||a - b||)) = ||a - b||.
\]

(3) If \( E \) and \( F \) are commuting \( \mathcal{F} \)-projections,

\[
\mathcal{F}(||x||) = \mathcal{F}(||Fx||) + \mathcal{F}(||(I - F)x||)
\]

\( = \mathcal{F}(||EFx||) + \mathcal{F}(||(I - E)Fx||) + \mathcal{F}(||(I - F)x||) \).
\[ \mathcal{F} (\| \mathcal{E} F x \|) + \mathcal{F} (\| F(I - E)x + (I - F)x \|) = \mathcal{F} (\| \mathcal{E} F x \|) + \mathcal{F} (\| (I - EF)x \|) \]

for all \( x \) in \( X \).

**Remark.** If \( E \) is an \( \mathcal{F} \)-projection, then \( \| a + b \| \), where \( a \) is any norm 1 vector in \( E X \) and \( b \) is any norm 1 vector in \( (I - E)X \), is constant at \( \mathcal{F}^{-1}(2\mathcal{F}(1)) \). For

\[ \| a + b \| = \mathcal{F}^{-1}(\| a + b \|) = \mathcal{F}^{-1}(\mathcal{F}(\| a \|) + \mathcal{F}(\| b \|)) . \]

**Theorem 10.** A maximal family \( \mathcal{P} \) of commuting \( \mathcal{F} \)-projections is a complete-Boolean algebra of norm 1 projections.

**Proof.** Clearly 0 and \( I \) are in \( \mathcal{P} \) and if \( E \) is in \( \mathcal{P} \), so is \( I - E \) by the symmetry of the definition of an \( \mathcal{F} \)-projection. If \( E \) and \( F \) are in \( \mathcal{P} \), \( EF \) is an \( \mathcal{F} \)-projection by Lemma 9, and it commutes with \( \mathcal{P} \). Therefore, \( EF \) is in \( \mathcal{P} \). Thus \( \mathcal{P} \) is a Boolean algebra of projections on \( X \) as defined by Bade [1]. Now suppose \( E_\alpha \) is an increasing net of projections in \( \mathcal{P} \). For each \( x \) in \( X \) and for \( \alpha \leq \beta \), \( E_\alpha x = E_\alpha E_\beta x \). So \( \| E_\alpha x \| \leq \| x \| \); thus, \( \mathcal{F}(\| E_\alpha x \|) \) is an increasing net of real numbers bounded above by \( \mathcal{F}(\| x \|) \); hence, convergent. This implies \( E_\alpha x \) is Cauchy, as follows. Given \( \varepsilon \geq 0 \), choose \( \theta \) such that

\[ \mathcal{F}(\| E_\alpha x \|) \geq \lim_\alpha \mathcal{F}(\| E_\alpha x \|) - \mathcal{F}(\varepsilon/2) \]

for all \( \alpha \geq \theta \). If \( \beta \geq \theta \),

\[ \mathcal{F}(\| E_\beta x - E_\theta x \|) + \mathcal{F}(\| E_\theta x \|) = \mathcal{F}(\| E_\beta x - E_\beta E_\theta x \|) + \mathcal{F}(\| E_\theta E_\beta x \|) = \mathcal{F}(\| (I - E_\theta)E_\beta x \|) + \mathcal{F}(\| E_\beta E_\theta x \|) = \mathcal{F}(\| E_\beta x \|) . \]

Thus,

\[ \mathcal{F}(\| E_\beta x - E_\theta x \|) = \mathcal{F}(\| E_\beta x \|) - \mathcal{F}(\| E_\theta x \|) . \]

And from this

\[ \mathcal{F}(\varepsilon/2) \geq \lim_\alpha \mathcal{F}(\| E_\alpha x \|) - \mathcal{F}(\| E_\theta x \|) \]

\[ \geq \mathcal{F}(\| E_\beta x \|) - \mathcal{F}(\| E_\theta x \|) = \mathcal{F}(\| E_\beta x - E_\theta x \|); \]

hence, \( \varepsilon/2 \geq \| E_\beta x - E_\theta x \| \) because \( \mathcal{F} \) is increasing. If \( \alpha, \beta \geq \theta \),

\[ \| E_\alpha x - E_\beta x \| \leq \| E_\alpha x - E_\theta x \| + \| E_\beta x - E_\theta x \| \leq \varepsilon . \]

Define \( E x = \lim_\alpha E_\alpha x \) for every \( x \) in \( X \). Then \( E \) is surely a projection and, since \( \mathcal{F} \) is continuous, \( E \) is an \( \mathcal{F} \)-projection; since \( E \)
commutes with \( P \), it is in \( P \). This completes the argument.

By Zorn's lemma, complete Boolean algebras of \( P \)-projections always exist, although they may be trivial. Nontrivial examples are given later.

**Theorem 11.** Suppose that all vectors \( v \) and \( w \) in \( X \) satisfy the (Clarkson) inequality

\[
\frac{1}{2}F(||v + w||) + \frac{1}{2}F(||v - w||) \leq F(||v||) + F(||w||)
\]

and suppose \( F(2) \neq 4, F(1) = 1 \). Then any two \( F \)-projections commute (and so the set of all \( F \)-projections form a complete Boolean algebra of projections). The same result holds for the reverse inequality.

**Proof.** Let \( E \) and \( F \) be two \( F \)-projections and \( x \in X \). Then decomposing \( Ex \) into \( F \) and then \( E \) components, applying Clarkson's inequality, and simplifying (using Lemma 9) we obtain

\[
F(||Ex||) = F(||EFEx||) + F(||E(I - F)Ex||)
+ F(||(I - E)FEx||) + F(||(I - E)(I - F)Ex||)
\geq 1/2F(||EFEx + E(I - F)Ex||) + 1/2F(||EFEx - E(I - F)Ex||)
+ 1/2F(||(I - E)FEx + (I - E)(I - F)Ex||)
+ 1/2F(||(I - E)FEx - (I - E)(I - F)Ex||)
= 1/2F(||Ex||) + 1/2F(||EFEx - E(I - F)Ex||)
+ (I - E)FEx - (I - E)(I - F)Ex||)
+ 1/2F(||EFEx - (I - E)(I - F)Ex||)
+ 1/2F(||(I - E)FEx + (I - F)Ex||)
= F(||Ex||).
\]

This implies equality in Clarkson's inequality for the vectors \((I - E)FEx\) and \((I - E)(I - F)Ex\):

\[
F(||(I - E)FEx||) + F(||(I - E)(I - F)Ex||)
= 1/2F(||(I - E)FEx + (I - E)(I - F)Ex||)
+ 1/2F(||(I - E)FEx - (I - E)(I - F)Ex||).
\]

Since the first term on the right is zero, we can define \( Z = Z(x) \equiv (I - E)FEx = -(I - E)(I - F)Ex \) and obtain \( 4F(||z||) = F(2||z||) \).

What if \( Z(x) \neq 0 \)? Then \( ||Z(x)|| = 1 \), and we have

\[
4 = 4F(||Z(x)||) = F(2||Z(x)||) = F(2)
\]

which contradicts the hypothesis. Thus \( Z = 0 \) and so \( FEx = EFEx \).
for any $x$ and any two $\mathcal{F}$-projections $E$ and $F$. Replacing $E$ and $F$ by $(I - E)$ and $F$ yields $F(I - E)x = (I - E)F(I - E)x$; whence $EFx = EFEx$. Therefore $FEx = EFx$ and so $E$ and $F$ commute.

**Remark.** Consider $\mathcal{F}(\lambda) = \lambda^p$ for a fixed $p, 1 \leq p < \infty$. An $\mathcal{F}$-projection for such an $\mathcal{F}$ is called an $L^p$-projection. Cunningham [4] showed that the $L^1$ projections always commute in any Banach space. The above theorem shows that for $p \neq 2$, the $L^p$ projections in an $L^p$ space commute.

**Definition.** A net $T_\alpha$ of projections on a Banach space $X$ is said to be increasing if $\alpha < \beta$ implies $T_\alpha T_\beta = T_\alpha = T_\beta T_\alpha$.

**Theorem 12.** If $T_\alpha$ is an increasing net of norm 1 projections on a reflexive Banach space $X$, then $T_\alpha$ converges in the strong operator topology of $X$ to a norm 1 projection $T$ that commutes with each $T_\alpha$ and whose range is the norm closure of $\bigcup \alpha T_\alpha[X]$.

**Proof.** The essentials of a proof can be found in [8; p. 223].

3. Projecting onto cycle subspaces.

**Definition.** If $\mathcal{P}$ is a Boolean algebra of projections on $X$ and $x$ is in $X$, let $S(x; \mathcal{P})$ denote the cycle generated by $x$ and $\mathcal{P}$; that is, the closed subspace of $X$ generated by $\{Ex: E \in \mathcal{P}\}$.

**Theorem 13.** Let $\mathcal{P}$ be a Boolean algebra of $\mathcal{F}$-projections on a Banach space $X$ that is smooth and reflexive, and let $x \in X$. Then $S(x; \mathcal{P})$ is the range of a (unique) norm 1 projection that commutes with $\mathcal{P}$.

**Proof.** Let $\pi$ denote the set of all partitions of $x$ by $\mathcal{P}$; that is, finite subsets $\{E_1, \ldots, E_n\}$ of $\mathcal{P}$ such that $E_iE_j = 0$ if $i \neq j$ and $(V_iE_i)(x) = \sum_i E_ix = x$. The set $\{I\}$ is such a partition. Order $\pi$ by setting $\mathcal{F} r \mathcal{A}$ if, given $A$ in $\mathcal{A}$ there is an $E$ in $\mathcal{E}$ such that $AE = A$. This “is refined by” relation $r$ is reflexive, anti-symmetric, transitive, and it directs the set $\pi$. Indeed, if $\{E_1, \ldots, E_n\}$ and $\{A_1, \ldots, A_m\}$ are partitions of $x$, then one common refinement is the set of $E_iA_j$ such that $E_iA_jx \neq 0$.

For each partition $\mathcal{E}$ of $x$, define $T(\mathcal{E})(y) = \sum (E \in \mathcal{E})(N(Ex)(y)/\|Ex\||)Ex$ for all $y$ in $X$. The transformation $T(\mathcal{E})$ is obviously linear; that it is a projection on $X$ is an immediate consequence of the fact that for $E$ and $F$ in $\mathcal{P}$ with $EF = 0$, $N(Ex)(Fy) = N(Ex)(EFy) = 0$. We now show that the norm of $T(\mathcal{E})$ is 1. It is not 0, first of all,
because the projection leaves $x$ fixed. Proceeding, let $y \in X$.

$$||N(Ex)(y)/||Ex||Ex|| = |N(Ex)(y)| = |N(Ex)(Ey)| \leq ||Ey||.$$ 

From this,

$$\mathcal{F} (||y||) \geq \mathcal{F} (||V(E \in \mathcal{C})Ey||) = \mathcal{F} (||\sum (E \in \mathcal{C})Ey||)$$

$$= \sum (E \in \mathcal{C}) \mathcal{F} (||Ey||) \geq \sum (E \in \mathcal{C}) \mathcal{F} (||N(Ex)(y)/||Ex||Ex||)$$

$$= \mathcal{F} (||\sum (E \in \mathcal{C})(N(Ex)(y)/||Ex||Ex||)Ex||) = \mathcal{F} (||T(\mathcal{C})y||).$$

Consequently $||T(\mathcal{C})y|| \leq ||y||$.

In order to apply Theorem 12, we must show that $T(\mathcal{A})T(\mathcal{C}) = T(\mathcal{C})T(\mathcal{A})$ under the assumption that $\mathcal{C} r \mathcal{A}$. It is a routine matter to use Lemma 5 to check that $T(\mathcal{A})(Ax) = Ax$ for any $A$ in $\mathcal{A}$, that $T(\mathcal{A})(Ex) = Ex$ for any $E$ in $\mathcal{C}$, and that, therefore, $T(\mathcal{C}) = T(\mathcal{A})T(\mathcal{C})$. Let $z$ be a given element of the null manifold of $T(\mathcal{A})$. Then for each $A$ in $\mathcal{A}$, $(N(Ax)(x)/||Ax||)Ax = AT(\mathcal{A})z = 0$ so that $N(Ax)(Ax) = N(Ax)(z) = 0$. Then $Ax$ is James orthogonal to $Ax$:

$$||Ax + Az|| \geq ||Ax||.$$ 

Then

$$\mathcal{F} (||Ex + Ez||) = \mathcal{F} (||\sum (AE = A)A(x + z)||)$$

$$= \sum (AE = A) \mathcal{F} (||Ax + Az||) \geq \sum (AE = A) \mathcal{F} (||Ax||)$$

$$= \mathcal{F} (||\sum (AE = A)Ax||) = \mathcal{F} (||Ex||),$$

for every $E$ in $\mathcal{C}$. Therefore, $||Ex + Ez|| \geq ||Ex||$ and, similarly, $||Ex + iEz|| \geq ||Ex||$ if $X$ is complex. In any case, $N(Ex)(z) = N(Ex)(Ez) = 0$ for all $E$ in $\mathcal{C}$ and, therefore, $z$ is in the null manifold of $T(\mathcal{C})$. Since the null manifold of $T(\mathcal{C})$ contains that of $T(\mathcal{A})$, we have $T(\mathcal{C})T(\mathcal{A}) = T(\mathcal{C})$.

By Theorem 12, there is a norm $1$ projection $T$ commuting with every $T(\mathcal{C})$ that is the limit in the strong operator topology of the net $T(\mathcal{C})$ and whose range is the subspace cl $\cup (E \in \pi)T(\mathcal{C})[X]$. Let us show that $T$ commutes with the projections in $\mathcal{P}$. Let $E \in \mathcal{P}$. If $Ex \neq 0$, let $\mathcal{C}$ denote the set $\{E\}$ or $\{E, I - E\}$ that is a partition of $x$. Given $\mathcal{A} \in \pi$ such that $\mathcal{C} r \mathcal{A}$,

$$T(\mathcal{A})Ey = \sum (A \in \mathcal{A})(N Ax)(Ey)/||Ax||Ax$$

$$= \sum (AE = A)(N Ax)(Ey)/||Ax||Ax$$

$$= \sum (AE = A)(N Ax)(y)/||Ax||EAx$$

$$= E(\sum (AE = A)(N Ax)(y)/||Ax||Ax)Ax$$

$$= E(\sum (A \in \mathcal{A})(N Ax)(y)/||Ax||Ax)Ax$$

$$= ET(\mathcal{A})y.$$
for all $y$ in $X$. Consequently, for each $y$ in $X$,
\[
TEy = \lim (E rA) T(A) Ey = \lim (E rA) ET(A)y = E \lim (E rA) T(A)y = ETy.
\]
Therefore, $TE = ET$ provided $Ex \neq 0$. If $Ex = 0$, then $(I - E)x \neq 0$ and $T(I - E) = (I - E)T$ by the same argument. From this, $TE = ET$ when $Ex = 0$.

For all $A$ in $\pi$, $T(A)[X] \subseteq S(x; P)$; hence, $T[X] \subseteq S(x; P)$. And given $E \in P$, if $Ex \neq 0$, then, letting $C$ be the above partition of $x$, $S(x; C) \subseteq T[X]$. This completes the proof of Theorem 13.

**THEOREM 14.** Let $P$ be a complete Boolean algebra of $F$-projections on a Banach space that is reflexive and smooth. Then the weakly closed algebra $W(P)$ of operators on $X$ generated by $P$ is equal to its second commutant.

**Proof.** Bade [1] shows that if $P$ is complete, then $W(P)$ is the uniformly closed algebra of operators generated by $P$ and it consists, furthermore, of exactly those (bounded linear) operators of $X$ which leave invariant every closed linear manifold invariant under $P$.

Suppose $A$ is in the second commutant of $W(P)$. For each $x$ in $X$, let $T^x$ denote the norm one projection whose range is $S_x = S(x; P)$. Then $T^x$ commutes with $W(P)$ so that $AT^x = T^xA$ for all $x$ in $X$. From this, we have that $A$ leaves each $S_x$ invariant: $AS_x = AT^x X = T^x AX \subseteq T^x X = S_x$. If $M$ is a closed subspace left invariant under $P$, then $S_m \subseteq M$ for all $m$ in $M$; whence, $A(m) \in AS_m \subseteq S_m \subseteq M$ for each $m$ in $M$. Therefore, $A$ leaves $M$ invariant. Therefore, $A \in W(P)$.

4. A class of examples. Let $(S, \Sigma, \mu)$ be a measure space with the property FSP (a measurable set of infinite measure contains a measurable subset of finite positive measure). This condition is discussed in [9]. We consider an Orlicz space $L_M$ over $(S, \Sigma, \mu)$ where the complimentary Young's functions $M$ and $N$ are normalized ($M(1) + N(1) = 1$), satisfy $\Delta_2$ conditions, and have continuous, strictly increasing derivatives denoted $m$ and $n$, respectively. Then $L_M$ is reflexive and [9; Corollary 2.1] the Luxemburg norms in both $L_M$ and $L_N$ are strongly differentiable. Furthermore, the weak derivative of a norm 1 function $f_0$ in $L_M$ is given by $f \rightarrow \int f m(f_0) d\mu$.

**LEMMA 15.** If $0 \leq f \in L_M$, then $m\left(\frac{f(x)}{\|f\|}\right) = \frac{m(f(x))}{\|mf\|}$ for almost
all \( x \in S \).

**Proof.** If \( h = \alpha g \) for \( \alpha \geq 0 \) and if \( h, g \geq 0 \) a.e., we have equality for \( h \) and \( m(g) \) in Holder's inequality: \( \|h\| \|mg\| = \int hm(g)d\mu. \) Then
\[
\int \frac{fm(f)}{\|f\|}d\mu = ||f|| = \left[ \int \frac{m(f)}{||m(f)||} \right]d\mu \text{ so } m(f/\|f\|) \text{ and } m(f)/\|mf\| \\
\text{are normers for } f. \text{ Since } L_M \text{ is smooth, normers are unique.}
\]

**Lemma 16.** Assume the existence of sets of arbitrarily small positive measure. If \( f, g \in L_M \) with \( 0 < \|f\| < \|g\|, \) then \( 0 < \|mf\| < \|mg\| \).

**Proof.** Set \( K = \|g\|/\|f\| > 1 \). Choose \( x \in S \) such that \( 0 < m(g(x))/\|m(g)\| = m(g(x))/\|g\| \). Set \( a = |g(x)|/K > 0 \). For any measurable set \( E \), let \( f_E \) be the function constant on \( E \) at the value \( a \), and agreeing with \( |f| \) outside of \( E \). By diminishing the measure of \( E \), the function \( f_E \) may be brought in the norm of \( L_M \) as close to \( |f| \) as desired. Furthermore, \( \|m(Kf_E)\| - \|mf\| \) approaches \( \|m(Kf)| - \|mf\| \) as \( E \) decreases. It is therefore, possible to choose a set \( E \) of positive measure so small that
\[
m(g(x)/\|g\|)(\|f\|/\|f_E\|)\|m(Kf_E)\| > m(g(x)/\|g\|)\|mf\| .
\]
Select \( y \in E \) such that \( m(Kf_E(y)) = m(Kf_E(y)/\|Kf_E\|)\|m(Kf_E)\|. \) Computing, we have
\[
m(g(x)/\|g\|)\|mg\| = m(g(x)) = m(Ka) = m(Kf_E(y))
\]
\[
= m(f_E(y)/\|f_E\|)\|m(Kf_E)\| = m(a/\|f_E\|)\|m(Kf_E)\|
\]
\[
= m((g(x)/\|g\|)(\|f\|/\|f_E\|))\|m(Kf_E)\| > m(g(x)/\|g\|)\|mf\| .
\]
Cancelling \( m(g(x)/\|g\|) \) finishes the argument.

Perhaps Lemma 16 is true without restrictions on the measure space. We have not settled this.

Define \( \mathcal{F}(\lambda) = \|f\| \|mf\| = \int |f| m(f)d\mu \) where \( f \) is any function in \( L_M \) of norm \( \lambda \). From Lemma 16, it is clear that \( \mathcal{F} \) is well defined and strictly increasing. To show continuity, let \( E \) be any set of finite positive measure and a \( (\lambda) = \lambda/\|\chi_E\| \). Then \( a(\lambda) \) is continuous and
\[
\mathcal{F}(\lambda) = \int a(\lambda)\chi_E m(a(\lambda)\chi_E)d\mu = \int a(\lambda)m(a(\lambda))\chi_E d\mu = a(\lambda)m(a(\lambda))\mu E ,
\]
a continuous function.

Each measurable set \( E \) gives rise to the characteristic projection \( f \rightarrow \chi_E f \).
LEMMA 17. Every characteristic projection is an $F$-projection.

Proof.

$$F(||f||) = \int f \, m(f) \, d\mu = \int_E f \, m(f) \, d\mu + \int_{S \setminus E} f \, m(f) \, d\mu$$

$$= \int (\chi_E f) m(\chi_E f) \, d\mu + \int (\chi_{S \setminus E} f) m(\chi_{S \setminus E} f) \, d\mu$$

$$= F(||\chi_E f||) + F(||\chi_{S \setminus E} f||).$$

REFERENCES


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