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A DENSITY WHICH COUNTS MULTIPLICITY

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P. Erdős, using analytic theorems, has proven the following results: Let $f(x)$ be the number of integers m such that $\phi(m) \leq x$, where ϕ is the Euler function, and let $g(x)$ be the number of integers n such that $\sigma(n) \leq x$, where σ is the usual sum of divisors function. Then there are positive (but undetermined) constants c_1 and c_2 such that $f(x) = c_1x + o(x)$ and $g(x) = c_2x + o(x)$. The constants c_1 and c_2 can be calculated using complex analysis including the Wiener-Ikehara Theorem. A major purpose of this paper is to give an elementary proof that $\lim_{x \rightarrow \infty} f(x)/x$ exists and, in the process, calculate the value of the limit. These considerations of multiplicity motivate a generalization of natural density which counts multiplicity. This paper contains an investigation of this generalization.

Let $A = \{a_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers ≥ 1 . For a positive integer j , define $\#(A, j)$ to be the number of integers i such that $a_i \leq j$ (that is, the number of elements of A counting multiplicity which are $\leq j$). If $\liminf_{j \rightarrow \infty} \#(A, j)/j = \alpha$ (we allow $\alpha = \infty$) we say A has Δ -asymptotic density α and we define $\underline{\Delta}(A) = \alpha$. We also define $\bar{\Delta}(A) = \limsup_{j \rightarrow \infty} \#(A, j)/j$. If $\underline{\Delta}(A) = \bar{\Delta}(A)$ we say A has Δ -natural density α and we define $\Delta(A) = \alpha$. It is clear that a reordering of A does not affect $\underline{\Delta}(A)$ or $\bar{\Delta}(A)$. It is also clear that $\underline{\Delta}(A) = \underline{\Delta}(\{[a_i]\}_{i=1}^{\infty})$ and $\bar{\Delta}(A) = \bar{\Delta}(\{[a_i]\}_{i=1}^{\infty})$ where $[a_i]$ is the greatest integer which does not exceed a_i . Unless otherwise specified all sequences in this paper will be of positive real numbers.

Throughout this paper d will denote natural density, i.e., the classical analog of Δ where multiplicity is not counted; Z^+ will denote the set of positive integers; Q^+ will denote the positive rational numbers; R^+ will denote the set of positive real numbers; p will always be a prime; and $P = \{p_i\}_{i=1}^{\infty}$ will be the sequence, in the natural order, of primes.

If $\gamma: Z^+ \rightarrow R^+$ then to γ there corresponds the unique sequence $\gamma(1), \gamma(2), \dots$. We will write γ in place of this sequence. Thus, for example, in the notation of this paper $\Delta(\phi)$ and $\Delta(\sigma)$ exist and are positive [5]. If for instance $\gamma = \tau$, where $\tau(n) =$ the number of positive integer divisors of the positive integer n , then it is clear that $\Delta(\tau) = \infty$.

If $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_j\}_{j=1}^{\infty}$ are sequences then define $A + B$ to be the sequence, in the natural order, of positive real numbers x such that there exist i and $j \in Z^+$ with $a_i + b_j = x$, and x appears in this

sequence the precise number of distinct ways we can write $x = a_{i_1} + b_{j_1}$. Note that it is possible to have $x = a_{i_1} + b_{j_1}$ and yet for x not to be a member of $A + B$. This happens precisely when some positive number $y < x$ is representable infinitely often in the form $y = a_i + b_j$. Finally if A and B are sets of positive reals then define $A \setminus B$ to be the complement of B in A .

1. Number theoretic functions. In this section we investigate the densities of certain sequences related to the ϕ function and other functions.

We first prove some lemmas which we will use to calculate $\Delta(\phi)$.

DEFINITION 1.1. For each $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$ define

$$\phi_k(n) = n \prod_{\substack{p|n \\ p \leq p_k}} \frac{p-1}{p} ;$$

cf. [8, p. 56].

LEMMA 1.1.1. $\Delta(\phi_k) = \prod_{p \leq p_k} (1 + (1/p(p-1)))$ for each $k \in \mathbb{Z}^+$.

Proof. Pick $k \in \mathbb{Z}^+$ and define $P^k = \{p_1, p_2, \dots, p_k\}$. To each subset P_j^k ($j = 1, 2, \dots, 2^k$) of P^k there corresponds the sequence of positive integers which are divisible by each member of P_j^k and by no member of $P^k \setminus P_j^k$. These sequences are pairwise disjoint and their union is \mathbb{Z}^+ .

For a subset P_j^k of P^k say $\{n_{j,i}\}_{i=1}^\infty$ is the corresponding sequence. It is clear that

$$(*) \quad \#(\phi_k, n) = \sum_{j=1}^{2^k} \#(\{n_{j,i}\}_{i=1}^\infty, n) \text{ for each } n \in \mathbb{Z}^+.$$

Now for a fixed P_j^k the density of $\{n_{j,i}\}_{i=1}^\infty$ is clearly

$$\prod_{p \in P_j^k} \frac{1}{p} \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p} .$$

Also for each integer m in this sequence we have

$$\phi_k(m) = m \prod_{p \in P_j^k} \frac{p-1}{p} .$$

Therefore

$$\Delta(\{\phi_k(m)\}_m \text{ in the sequence defined by } P_j^k) = \left(\prod_{p \in P_j^k} \frac{p}{p-1} \right) \left(\prod_{p \in P_j^k} \frac{1}{p} \right) \left(\prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p} \right) = \prod_{p \in P_j^k} \frac{1}{p-1} \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p} .$$

So by (*) we have

$$\begin{aligned} \Delta(\phi_k) &= \sum_{j=1}^{2^k} \left(\prod_{p \in P_j^k} \frac{1}{p-1} \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p} \right) \\ &= \sum_{j=1}^{2^k} \frac{\prod_{p \in P^k \setminus P_j^k} \frac{(p-1)^2}{p}}{\prod_{p \in P^k} (p-1)} = \frac{\prod_{p \in P^k} \left(1 + \frac{(p-1)^2}{p} \right)}{\prod_{p \in P^k} (p-1)} = \prod_{p \in P^k} \left(1 + \frac{1}{p(p-1)} \right) \end{aligned}$$

and the lemma is proved.

Note. $\lim_{k \rightarrow \infty} \Delta(\phi_k) = \prod_{p \in P} \left(1 + \frac{1}{p(p-1)} \right) < \infty .$

LEMMA 1.1.2. Choose $n \in \mathbb{Z}^+$, $n > 1$, and say $r \in \mathbb{Z}^+$ satisfies $p_1 p_2 \cdots p_r \leq n$. Then $\#(\phi_r, n) \leq n(\Delta(\phi_r) + 1)$. In fact if

$$n = t p_1 p_2 \cdots p_r, \quad t \geq 1, \quad t \in \mathbb{Q}^+,$$

then $\#(\phi_r, n) \leq n(\Delta(\phi_r) + 1/t)$.

Proof. Say $n = t p_1 \cdots p_r$ ($t \geq 1$). Then if

$$P_j^r = \{q_1, \dots, q_s\} \subset \{p_1, \dots, p_r\}$$

we have $R_{j,r} \stackrel{\text{def}}{=} \text{the number of integers } m \text{ such that } \phi_r(m) \leq n \text{ and } q_1 \cdots q_s \mid m \text{ and none of the members of } P^r \setminus P_j^r \text{ divide } m = \text{the number of integers } m \leq n(q_1/q_1 - 1) \cdots (q_s/q_s - 1) \text{ which are divisible by } q_1 \cdots q_s \text{ and divisible by no member of } P^r \setminus P_j^r. \text{ Say } T_{j,r} \text{ is the smallest integer } \geq t(q_1/q_1 - 1) \cdots (q_s/q_s - 1). \text{ Then clearly } R_{j,r} \leq \text{the number of integers } m \text{ which do not exceed } p_1 \cdots p_r T_{j,r} \text{ and which are divisible by } q_1 \cdots q_s \text{ and divisible by no member of } P^r \setminus P_j^r. \text{ But since } T_{j,r} \text{ is an integer we have}$

$$\begin{aligned} R_{j,r} &\leq (p_1 \cdots p_r T_{j,r}) \frac{1}{q_1 \cdots q_s} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \\ &\leq p_1 \cdots p_r \left(t \frac{q_1}{q_1 - 1} \cdots \frac{q_s}{q_s - 1} + 1 \right) \frac{1}{q_1 \cdots q_s} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} . \end{aligned}$$

Now $\#(\phi_r, n) = \sum_{j=1}^{2^r} R_{j,r}$. So

$$\begin{aligned} \#(\phi_r, n) &\leq \sum_{j=1}^{2^r} \left(p_1 \cdots p_r \left(t \prod_{p \in P_j^r} \frac{p}{p-1} + 1 \right) \prod_{p \in P_j^r} \frac{1}{p} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \right) \\ &= t p_1 \cdots p_r \sum_{j=1}^{2^r} \left(\prod_{p \in P_j^r} \frac{1}{p-1} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \right) \\ &\quad + p_1 \cdots p_r \sum_{j=1}^{2^r} \left(\prod_{p \in P_j^r} \frac{1}{p} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \right) = n \left(\Delta(\phi_r) + \frac{1}{t} \right) \end{aligned}$$

and the lemma is proved.

LEMMA 1.1.3. Choose $n \in \mathbb{Z}^+$, $n > 1$, and say $r \in \mathbb{Z}^+$ is defined by $p_1 \cdots p_r \leq n < p_1 \cdots p_{r+1}$. Then we have

$$\phi(m) \leq n \Rightarrow \phi_r(m) \leq \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1} n.$$

Thus

$$\#(\phi, n) \leq \# \left(\phi_r, \left[\frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1} n \right] \right).$$

Proof. Suppose m has more than $r + 1$ distinct prime divisors. Then $\phi(m) \geq (p_{r+2} - 1)(p_{r+1} - 1) \cdots (p_1 - 1) \geq p_1 \cdots p_{r+1} > n$, a contradiction. So m has at most $r + 1$ distinct prime divisors.

Now

$$\phi_r(m) = \phi(m) \prod_{\substack{p|m \\ p > p_r}} \frac{p}{p-1} \leq n \prod_{\substack{p|m \\ p > p_r}} \frac{p}{p-1} \leq n \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1}$$

since m has at most $r + 1$ distinct prime divisors and the lemma is proved.

THEOREM 1.1.

$$\Delta(\phi) = \prod_{p \in P} \left(1 + \frac{1}{p(p-1)} \right) = \frac{\zeta(2) \cdot \zeta(3)}{\zeta(6)},$$

where ζ denotes the Riemann Zeta function.

Proof. It is well known [7, p. 246] that $\zeta(s) = \prod_{p \in P} (1/1 - p^{-s})$ for $s > 1$. Thus it follows that $\prod_{p \in P} (1 + (1/p(p-1))) = (\zeta(2) \cdot \zeta(3)/\zeta(6))$. So it only remains to show that $\Delta(\phi) = \prod_{p \in P} (1 + (1/p(p-1)))$.

For $r \in \mathbb{Z}^+$ let $g_r = (p_{r+1}/p_{r+1} - 1) \cdots (p_{2r+1}/p_{2r+1} - 1)$. It follows from Mertens' Theorem and Tchebychef's Theorem [7, pp. 351 and 9] that $\lim_{r \rightarrow \infty} g_r = 1$. Choose $n \in \mathbb{Z}^+$, $n > 1$, and say $r \in \mathbb{Z}^+$ is defined by $p_1 \cdots p_r \leq n = tp_1 \cdots p_r < p_1 p_2 \cdots p_{r+1}$, where $t \geq 1$.

Now, $\#(\phi_r, n) = \#(\phi_{r-1}, n) + (\#(\phi_r, n) - \#(\phi_{r-1}, n))$. But

$$\#(\phi_r, n) - \#(\phi_{r-1}, n)$$

is the number of integers m such that $p_r | m$ and

$$n < \phi_{r-1}(m) \leq \frac{p_r}{p_{r-1}} n.$$

This number is the sum (over $j = 1, 2, \dots, 2^{r-1}$) of the number of integers less than or equal to

$$\left(\prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) \frac{np_r}{p_r-1}$$

and greater than

$$\left(\prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) n$$

which are divisible by p_r and each $p \in P_j^{r-1}$ and not divisible by any $p \in P^{r-1} \setminus P_j^{r-1}$. It then follows that

$$\begin{aligned} & \#(\phi_r, n) - \#(\phi_{r-1}, n) \\ & \leq \sum_{j=1}^{2^{r-1}} \left\{ \left(\frac{2np_r}{p_r(p_r-1)} \left(\prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) - \frac{n}{2p_r} \left(\prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) \right) \right. \\ & \quad \left. \times \prod_{p \in P_j^{r-1}} \frac{1}{p} \prod_{p \in P^{r-1} \setminus P_j^{r-1}} \frac{p-1}{p} \right\} = n\Delta(\phi_{r-1}) \left(\frac{2}{p_r-1} - \frac{1}{2p_r} \right) = o(n). \end{aligned}$$

So $\#(\phi_r, n) = \#(\phi_{r-1}, n) + o(n)$.

By Lemma 1.1.2 we have

$$\#(\phi_{r-1}, n) \leq n \left(\Delta(\phi_{r-1}) + \frac{1}{p_r} \right) = n\Delta(\phi_r) + o(n).$$

So $\#(\phi_r, n) \leq n\Delta(\phi_r) + o(n)$. By Lemma 1.1.3 we have $\#(\phi, n) \leq \#(\phi_r[g_r n])$.

So $\#(\phi, n) \leq [g_r n]\Delta(\phi_r) + o([g_r n]) = n\Delta(\phi_r) + o(n)$. Divide by n and let $n \rightarrow \infty$ to get $\lim_{n \rightarrow \infty} \#(\phi, n)/n \leq \lim_{r \rightarrow \infty} \Delta(\phi_r)$.

Finally $\Delta(\phi) \geq \lim_{k \rightarrow \infty} \Delta(\phi_k)$ because if we choose $k \in \mathbb{Z}^+$ then for n large we have $\#(\phi, n) \geq \#(\phi_k, n) \geq n(\Delta(\phi_k) - 1/k)$ and so

$$\liminf_{n \rightarrow \infty} \#(\phi, n)/n \geq \Delta(\phi_k) - 1/k$$

for each $k \in \mathbb{Z}^+$. Thus $\Delta(\phi) = \lim_{r \rightarrow \infty} \Delta(\phi_r) = \prod_{p \in P} (1 + (1/p(p-1)))$ and the theorem is proved.

A related result due to P. Erdős may be found in [4, pp. 211-213].

DEFINITION 1.2. For $t \geq 1$, t a real number, a positive integer n is said to be t -abundant if $\sigma(n) \geq tn$.

H. Davenport [3] has shown that for t as above, the sequence of t -abundant positive integers has a natural density.

THEOREM 1.2. For each $k \in \mathbb{Z}^+$ let d_k = the natural density of the k -abundant integers. Then $\sum_{k=1}^{\infty} d_k \leq \Delta(\phi) = (\zeta(2) \cdot \zeta(3) / \zeta(6))$.

Proof. It is known that $\phi(n)\sigma(n)/n^2 < 1$ for each integer $n > 1$

[7, p. 267]. So if $n \in [(k - 1)N, kN]$ and $\sigma(n) \geq kn$ then $\phi(n) \leq N$. Thus for $k \in \mathbb{Z}^+$ and for N large, depending on k , we have

$$\begin{aligned} \#(\phi, N) &\geq N + d_2(2N - N) + d_3(3N - 2N) + \dots \\ &\quad + d_k(kN - (k - 1)N) - \frac{N}{k} \\ &= N(1 + d_2 + d_3 + \dots + d_k - 1/k) \\ &= N(d_1 + d_2 + \dots + d_k - 1/k). \end{aligned}$$

Now divide by N and let $N \rightarrow \infty$. We then have

$$\Delta(\phi) \geq \lim_{k \rightarrow \infty} (d_1 + d_2 + \dots + d_k - 1/k) = \sum_{k=1}^{\infty} d_k$$

and the theorem is proved.

2. **General theorems.** We begin this section by stating some results whose proofs are not difficult.

1. If $A = \{a_i\}_{i=1}^{\infty}$ is a sequence such that $\Delta(A) = \infty$ then there exists a sequence $\{i_j\}_{j=1}^{\infty}$ of positive integers with $\sum_{j=1}^{\infty} a_{i_j}/i_j < \infty$.

2. If $A = \{a_i\}_{i=1}^{\infty}$ is a sequence such that $\Delta(A) = 0$ then $\sum_{a_i \leq r} a_i = o(r^2)$ and $\sum_{a_i \leq r} 1/a_i = o(\log r)$.

3. If $A = \{a_i\}_{i=1}^{\infty}$ is a sequence such that $\infty > \Delta(A) > 0$ then $\sum_{a_i \leq r} a_i \sim \Delta(A)r^2/2$ and $\sum_{a_i \leq r} 1/a_i \sim \Delta(A) \log r$.

THEOREM 2.1. *Let $A = \{a_i\}_{i=1}^{\infty}$ be a sequence such that $\Delta(A) = \infty$. Then there exists a strictly increasing sequence $\{i_j\}_{j=1}^{\infty}$ of positive integers with $d(\{i_j\}_{j=1}^{\infty}) = 0$ and $\Delta(\{a_{i_j}\}_{j=1}^{\infty}) = \infty$.*

Proof. It suffices to assume $\lim_{i \rightarrow \infty} a_i = \infty$ because otherwise the proof is immediate.

Case I. $a_1 \leq a_2 \leq a_3 \leq \dots$

First, there is no loss of generality in supposing $a_1 < a_2 < a_3 < \dots$ because if $a_i = a_{i+1} = \dots = a_{i+r-1} < a_{i+r}$ for some i then define

$$\varepsilon = \min \left(a_{i+r} - a_i, \begin{array}{l} \text{the distance from } a_i \text{ to the} \\ \text{smallest integer greater than } a_i \end{array} \right)$$

and replace a_{i+t} by $a_i + t\varepsilon/r$ for $t = 0, 1, \dots, r - 1$.

We now define a subsequence B of A by induction. Let $a_1 \in B$. If each of a_1, a_2, \dots, a_{k-1} has already been either included in B or excluded from B , place a_k in B if

$$\frac{\#(B, a_{k-1}) + 1}{a_k} \leq \sqrt{\frac{\#(A, a_k)}{a_k}}$$

and exclude a_k from B if the inequality fails. It then follows that $\#(B, a_k)/a_k \sim \sqrt{\#(A, a_k)/a_k}$ and so $\Delta(B) = \infty$. Also if we write $B = \{a_{i_j}\}_{j=1}^\infty$ then we have $d(\{i_j\}_{j=1}^\infty) = 0$ because

$$\begin{aligned} \frac{n}{i_n} &= \frac{\#\{i_j\}_{j=1}^\infty, i_n}{i_n} = \frac{\#(B, a_{i_n})}{\#(A, a_{i_n})} = \frac{a_{i_n}}{\#(A, a_{i_n})} \frac{\#(B, a_{i_n})}{a_{i_n}} \\ &\sim \sqrt{\frac{a_{i_n}}{\#(A, a_{i_n})}} \left(\sqrt{\frac{a_{i_n}}{\#(A, a_{i_n})}} \frac{\#(B, a_{i_n})}{a_{i_n}} \right) \end{aligned}$$

which tends to $0.1 = 0$ as $n \rightarrow \infty$.

Case II. We make no assumptions about the monotonicity of A . However, without loss of generality, we may still assume $a_i = a_j \Rightarrow i = j$, for we can always order A by size, deal with A as in Case I, and then apply the inverse of the permutation used to order A to the new sequence which is derived from A by use of the ϵ 's.

Now order A by size and call this sequence $A^* = \{a_i^*\}_{i=1}^\infty$. We have $a_i^* < a_{i+1}^*$ for all $i \in \mathbb{Z}^+$. It follows immediately that if any $n - 1$ elements are deleted from A the minimum of the remaining elements is $\leq a_n^*$. It is also clear that if $A_1^* = \{a_{2i-1}\}_{i=1}^\infty$ then $\Delta(A_1^*) = \infty$.

Apply Case I to A^* to get a subsequence $B^* = \{a_{i_j}^*\}_{j=1}^\infty$ of A^* such that $\Delta(B^*) = \infty$ and $d(\{i_j\}_{j=1}^\infty) = 0$. Now define t_1 by $a_{t_1} = \min(\{a_{i_1}, a_{i_1+1}, a_{i_1+2}, \dots\})$. It follows that $t_1 \geq i_1$ and $a_{t_1} \leq a_{i_1}^*$. Define t_2 by $a_{t_2} = \min(\{a_{i_2}, a_{i_2+1}, a_{i_2+2}, \dots\} \setminus \{a_{t_1}\})$. It follows that $t_2 \geq i_2$ and $a_{t_2} \leq a_{i_2+1}^*$. In general define t_j by

$$a_{t_j} = \min(\{a_{i_j}, a_{i_j+1}, a_{i_j+2}, \dots\} \setminus \{a_{t_1}, a_{t_2}, \dots, a_{t_{j-1}}\}) .$$

It follows that $t_j \geq i_j$ and $a_{t_j} \leq a_{i_j+j-1}^*$.

Since $t_j \geq i_j$ for all $j \in \mathbb{Z}^+$, it follows that $d(\{t_j\}_{j=1}^\infty) = 0$. Also $\Delta(\{a_{i_j}^*\}_{j=1}^\infty) = \infty$ so $\Delta(\{a_{i_{2j-1}}^*\}_{j=1}^\infty) = \infty$ so $\Delta(\{a_{i_j+j-1}^*\}_{j=1}^\infty) = \infty$. It then follows that $\Delta(\{a_{t_j}\}_{j=1}^\infty) = \infty$ and the theorem is proved.

To emphasize that care must be taken in the choice of $\{i_j\}_{j=1}^\infty$ in the above theorem we note the following result.

THEOREM 2.2. *Suppose $\{i_j\}_{j=1}^\infty$ is a sequence of positive integers such that $d(\{i_j\}_{j=1}^\infty) = 0$. Then there exists a strictly increasing sequence $A = \{a_i\}_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} a_i = \infty$, $\Delta(A) = \infty$, and $\Delta(\{a_{i_j}\}_{j=1}^\infty) = 0$.*

THEOREM 2.3. *For each number α such that $0 \leq \alpha \leq \infty$ there exist two sequences A and B such that $\Delta(A) = \Delta(B) = 0$ and $\Delta(A + B) = \alpha$.*

Proof. If $\alpha = 0$ choose $A = B$ to be the sequence of factorials.

If $\alpha = \infty$ choose $A = B = P$. Then by the Prime Number Theorem $\Delta(A + B) = \infty$.

Suppose $0 < \alpha < \infty$. Choose β and $\gamma \in R^+$ so that $(1/4)\pi\beta\gamma = \alpha$. Let $A = \{n^2/\beta^2\}_{n=1}^{\infty}$ and $B = \{n^2/\gamma^2\}_{n=1}^{\infty}$. Clearly $\Delta(A) = 0 = \Delta(B)$. Also, the number of elements in $A + B$ which are $\leq n$ is the number of lattice points (k, m) in the positive quadrant of the ellipse

$$k^2/\beta^2 + m^2/\gamma^2 \leq n.$$

This number is $(1/4)\pi\beta\gamma n + O(\sqrt{n})$. Thus $\Delta(A + B) = (1/4)\pi\beta\gamma = \alpha$ and the theorem is proved.

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Vol. 34, No. 2

June, 1970

Shair Ahmad, <i>On the oscillation of solutions of a class of linear fourth order differential equations</i>	289
Leonard Asimow and Alan John Ellis, <i>Facial decomposition of linearly compact simplexes and separation of functions on cones</i>	301
Kirby Alan Baker and Albert Robert Stralka, <i>Compact, distributive lattices of finite breadth</i>	311
James W. Cannon, <i>Sets which can be missed by side approximations to spheres</i>	321
Prem Chandra, <i>Absolute summability by Riesz means</i>	335
Francis T. Christoph, <i>Free topological semigroups and embedding topological semigroups in topological groups</i>	343
Henry Bruce Cohen and Francis E. Sullivan, <i>Projecting onto cycles in smooth, reflexive Banach spaces</i>	355
John Dauns, <i>Power series semigroup rings</i>	365
Robert E. Dressler, <i>A density which counts multiplicity</i>	371
Kent Ralph Fuller, <i>Primary rings and double centralizers</i>	379
Gary Allen Gislason, <i>On the existence question for a family of products</i>	385
Alan Stuart Gleit, <i>On the structure topology of simplex spaces</i>	389
William R. Gordon and Marvin David Marcus, <i>An analysis of equality in certain matrix inequalities. I</i>	407
Gerald William Johnson and David Lee Skoug, <i>Operator-valued Feynman integrals of finite-dimensional functionals</i>	415
(Harold) David Kahn, <i>Covering semigroups</i>	427
Keith Milo Kendig, <i>Fibrations of analytic varieties</i>	441
Norman Yeomans Luther, <i>Weak denseness of nonatomic measures on perfect, locally compact spaces</i>	453
Guillermo Owen, <i>The four-person constant-sum games; Discriminatory solutions on the main diagonal</i>	461
Stephen Parrott, <i>Unitary dilations for commuting contractions</i>	481
Roy Martin Rakestraw, <i>Extremal elements of the convex cone A_n of functions</i>	491
Peter Lewis Renz, <i>Intersection representations of graphs by arcs</i>	501
William Henry Ruckle, <i>Representation and series summability of complete biorthogonal sequences</i>	511
F. Dennis Sentilles, <i>The strict topology on bounded sets</i>	529
Saharon Shelah, <i>A note on Hanf numbers</i>	541
Harold Simmons, <i>The solution of a decision problem for several classes of rings</i>	547
Kenneth S. Williams, <i>Finite transformation formulae involving the Legendre symbol</i>	559