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**A DENSITY WHICH COUNTS MULTIPLICITY**

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## A DENSITY WHICH COUNTS MULTIPLICITY

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**P. Erdős, using analytic theorems, has proven the following results: Let  $f(x)$  be the number of integers  $m$  such that  $\phi(m) \leq x$ , where  $\phi$  is the Euler function, and let  $g(x)$  be the number of integers  $n$  such that  $\sigma(n) \leq x$ , where  $\sigma$  is the usual sum of divisors function. Then there are positive (but undetermined) constants  $c_1$  and  $c_2$  such that  $f(x) = c_1x + o(x)$  and  $g(x) = c_2x + o(x)$ . The constants  $c_1$  and  $c_2$  can be calculated using complex analysis including the Wiener-Ikehara Theorem. A major purpose of this paper is to give an elementary proof that  $\lim_{x \rightarrow \infty} f(x)/x$  exists and, in the process, calculate the value of the limit. These considerations of multiplicity motivate a generalization of natural density which counts multiplicity. This paper contains an investigation of this generalization.**

Let  $A = \{a_i\}_{i=1}^{\infty}$  be a sequence of positive real numbers  $\geq 1$ . For a positive integer  $j$ , define  $\#(A, j)$  to be the number of integers  $i$  such that  $a_i \leq j$  (that is, the number of elements of  $A$  counting multiplicity which are  $\leq j$ ). If  $\liminf_{j \rightarrow \infty} \#(A, j)/j = \alpha$  (we allow  $\alpha = \infty$ ) we say  $A$  has  $\Delta$ -asymptotic density  $\alpha$  and we define  $\underline{\Delta}(A) = \alpha$ . We also define  $\bar{\Delta}(A) = \limsup_{j \rightarrow \infty} \#(A, j)/j$ . If  $\underline{\Delta}(A) = \bar{\Delta}(A)$  we say  $A$  has  $\Delta$ -natural density  $\alpha$  and we define  $\Delta(A) = \alpha$ . It is clear that a reordering of  $A$  does not affect  $\underline{\Delta}(A)$  or  $\bar{\Delta}(A)$ . It is also clear that  $\underline{\Delta}(A) = \underline{\Delta}(\{[a_i]\}_{i=1}^{\infty})$  and  $\bar{\Delta}(A) = \bar{\Delta}(\{[a_i]\}_{i=1}^{\infty})$  where  $[a_i]$  is the greatest integer which does not exceed  $a_i$ . Unless otherwise specified all sequences in this paper will be of positive real numbers.

Throughout this paper  $d$  will denote natural density, i.e., the classical analog of  $\Delta$  where multiplicity is not counted;  $Z^+$  will denote the set of positive integers;  $Q^+$  will denote the positive rational numbers;  $R^+$  will denote the set of positive real numbers;  $p$  will always be a prime; and  $P = \{p_i\}_{i=1}^{\infty}$  will be the sequence, in the natural order, of primes.

If  $\gamma: Z^+ \rightarrow R^+$  then to  $\gamma$  there corresponds the unique sequence  $\gamma(1), \gamma(2), \dots$ . We will write  $\gamma$  in place of this sequence. Thus, for example, in the notation of this paper  $\Delta(\phi)$  and  $\Delta(\sigma)$  exist and are positive [5]. If for instance  $\gamma = \tau$ , where  $\tau(n) =$  the number of positive integer divisors of the positive integer  $n$ , then it is clear that  $\Delta(\tau) = \infty$ .

If  $A = \{a_i\}_{i=1}^{\infty}$  and  $B = \{b_j\}_{j=1}^{\infty}$  are sequences then define  $A + B$  to be the sequence, in the natural order, of positive real numbers  $x$  such that there exist  $i$  and  $j \in Z^+$  with  $a_i + b_j = x$ , and  $x$  appears in this

sequence the precise number of distinct ways we can write  $x = a_{i_1} + b_{j_1}$ . Note that it is possible to have  $x = a_{i_1} + b_{j_1}$  and yet for  $x$  not to be a member of  $A + B$ . This happens precisely when some positive number  $y < x$  is representable infinitely often in the form  $y = a_i + b_j$ . Finally if  $A$  and  $B$  are sets of positive reals then define  $A \setminus B$  to be the complement of  $B$  in  $A$ .

**1. Number theoretic functions.** In this section we investigate the densities of certain sequences related to the  $\phi$  function and other functions.

We first prove some lemmas which we will use to calculate  $\Delta(\phi)$ .

**DEFINITION 1.1.** For each  $n \in \mathbb{Z}^+$  and  $k \in \mathbb{Z}^+$  define

$$\phi_k(n) = n \prod_{\substack{p|n \\ p \leq p_k}} \frac{p-1}{p};$$

cf. [8, p. 56].

**LEMMA 1.1.1.**  $\Delta(\phi_k) = \prod_{p \leq p_k} (1 + (1/p(p-1)))$  for each  $k \in \mathbb{Z}^+$ .

*Proof.* Pick  $k \in \mathbb{Z}^+$  and define  $P^k = \{p_1, p_2, \dots, p_k\}$ . To each subset  $P_j^k$  ( $j = 1, 2, \dots, 2^k$ ) of  $P^k$  there corresponds the sequence of positive integers which are divisible by each member of  $P_j^k$  and by no member of  $P^k \setminus P_j^k$ . These sequences are pairwise disjoint and their union is  $\mathbb{Z}^+$ .

For a subset  $P_j^k$  of  $P^k$  say  $\{n_{j,i}\}_{i=1}^\infty$  is the corresponding sequence. It is clear that

$$(*) \quad \#(\phi_k, n) = \sum_{j=1}^{2^k} \#(\{n_{j,i}\}_{i=1}^\infty, n) \quad \text{for each } n \in \mathbb{Z}^+.$$

Now for a fixed  $P_j^k$  the density of  $\{n_{j,i}\}_{i=1}^\infty$  is clearly

$$\prod_{p \in P_j^k} \frac{1}{p} \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p}.$$

Also for each integer  $m$  in this sequence we have

$$\phi_k(m) = m \prod_{p \in P_j^k} \frac{p-1}{p}.$$

Therefore

$$\Delta(\{\phi_k(m)\}_m \text{ in the sequence defined by } P_j^k) = \left( \prod_{p \in P_j^k} \frac{p}{p-1} \right) \left( \prod_{p \in P_j^k} \frac{1}{p} \right) \left( \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p} \right) = \prod_{p \in P_j^k} \frac{1}{p-1} \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p}.$$

So by (\*) we have

$$\begin{aligned} \Delta(\phi_k) &= \sum_{j=1}^{2^k} \left( \prod_{p \in P_j^k} \frac{1}{p-1} \prod_{p \in P^k \setminus P_j^k} \frac{p-1}{p} \right) \\ &= \sum_{j=1}^{2^k} \frac{\prod_{p \in P^k \setminus P_j^k} \frac{(p-1)^2}{p}}{\prod_{p \in P^k} (p-1)} = \frac{\prod_{p \in P^k} \left( 1 + \frac{(p-1)^2}{p} \right)}{\prod_{p \in P^k} (p-1)} = \prod_{p \in P^k} \left( 1 + \frac{1}{p(p-1)} \right) \end{aligned}$$

and the lemma is proved.

*Note.*  $\lim_{k \rightarrow \infty} \Delta(\phi_k) = \prod_{p \in P} \left( 1 + \frac{1}{p(p-1)} \right) < \infty$ .

LEMMA 1.1.2. Choose  $n \in \mathbb{Z}^+$ ,  $n > 1$ , and say  $r \in \mathbb{Z}^+$  satisfies  $p_1 p_2 \cdots p_r \leq n$ . Then  $\#(\phi_r, n) \leq n(\Delta(\phi_r) + 1)$ . In fact if

$$n = t p_1 p_2 \cdots p_r, \quad t \geq 1, \quad t \in \mathbb{Q}^+,$$

then  $\#(\phi_r, n) \leq n(\Delta(\phi_r) + 1/t)$ .

*Proof.* Say  $n = t p_1 \cdots p_r$  ( $t \geq 1$ ). Then if

$$P_j^r = \{q_1, \dots, q_s\} \subset \{p_1, \dots, p_r\}$$

we have  $R_{j,r} \stackrel{\text{def}}{=} \text{the number of integers } m \text{ such that } \phi_r(m) \leq n \text{ and } q_1 \cdots q_s \mid m \text{ and none of the members of } P^r \setminus P_j^r \text{ divide } m = \text{the number of integers } m \leq n(q_1/q_1 - 1) \cdots (q_s/q_s - 1) \text{ which are divisible by } q_1 \cdots q_s \text{ and divisible by no member of } P^r \setminus P_j^r$ . Say  $T_{j,r}$  is the smallest integer  $\geq t(q_1/q_1 - 1) \cdots (q_s/q_s - 1)$ . Then clearly  $R_{j,r} \leq \text{the number of integers } m \text{ which do not exceed } p_1 \cdots p_r T_{j,r} \text{ and which are divisible by } q_1 \cdots q_s \text{ and divisible by no member of } P^r \setminus P_j^r$ . But since  $T_{j,r}$  is an integer we have

$$\begin{aligned} R_{j,r} &\leq (p_1 \cdots p_r T_{j,r}) \frac{1}{q_1 \cdots q_s} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \\ &\leq p_1 \cdots p_r \left( t \frac{q_1}{q_1 - 1} \cdots \frac{q_s}{q_s - 1} + 1 \right) \frac{1}{q_1 \cdots q_s} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p}. \end{aligned}$$

Now  $\#(\phi_r, n) = \sum_{j=1}^{2^r} R_{j,r}$ . So

$$\begin{aligned} \#(\phi_r, n) &\leq \sum_{j=1}^{2^r} \left( p_1 \cdots p_r \left( t \prod_{p \in P_j^r} \frac{p}{p-1} + 1 \right) \prod_{p \in P_j^r} \frac{1}{p} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \right) \\ &= t p_1 \cdots p_r \sum_{j=1}^{2^r} \left( \prod_{p \in P_j^r} \frac{1}{p-1} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \right) \\ &\quad + p_1 \cdots p_r \sum_{j=1}^{2^r} \left( \prod_{p \in P_j^r} \frac{1}{p} \prod_{p \in P^r \setminus P_j^r} \frac{p-1}{p} \right) = n \left( \Delta(\phi_r) + \frac{1}{t} \right) \end{aligned}$$

and the lemma is proved.

LEMMA 1.1.3. Choose  $n \in \mathbb{Z}^+$ ,  $n > 1$ , and say  $r \in \mathbb{Z}^+$  is defined by  $p_1 \cdots p_r \leq n < p_1 \cdots p_{r+1}$ . Then we have

$$\phi(m) \leq n \Rightarrow \phi_r(m) \leq \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1} n.$$

Thus

$$\#(\phi, n) \leq \# \left( \phi_r, \left[ \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1} n \right] \right).$$

*Proof.* Suppose  $m$  has more than  $r + 1$  distinct prime divisors. Then  $\phi(m) \geq (p_{r+2} - 1)(p_{r+1} - 1) \cdots (p_1 - 1) \geq p_1 \cdots p_{r+1} > n$ , a contradiction. So  $m$  has at most  $r + 1$  distinct prime divisors.

Now

$$\phi_r(m) = \phi(m) \prod_{\substack{p|m \\ p > p_r}} \frac{p}{p-1} \leq n \prod_{\substack{p|m \\ p > p_r}} \frac{p}{p-1} \leq n \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1}$$

since  $m$  has at most  $r + 1$  distinct prime divisors and the lemma is proved.

THEOREM 1.1.

$$\Delta(\phi) = \prod_{p \in P} \left( 1 + \frac{1}{p(p-1)} \right) = \frac{\zeta(2) \cdot \zeta(3)}{\zeta(6)},$$

where  $\zeta$  denotes the Riemann Zeta function.

*Proof.* It is well known [7, p. 246] that  $\zeta(s) = \prod_{p \in P} (1/1 - p^{-s})$  for  $s > 1$ . Thus it follows that  $\prod_{p \in P} (1 + (1/p(p-1))) = (\zeta(2) \cdot \zeta(3) / \zeta(6))$ . So it only remains to show that  $\Delta(\phi) = \prod_{p \in P} (1 + (1/p(p-1)))$ .

For  $r \in \mathbb{Z}^+$  let  $g_r = (p_{r+1}/p_{r+1} - 1) \cdots (p_{2r+1}/p_{2r+1} - 1)$ . It follows from Mertens' Theorem and Tchebychef's Theorem [7, pp. 351 and 9] that  $\lim_{r \rightarrow \infty} g_r = 1$ . Choose  $n \in \mathbb{Z}^+$ ,  $n > 1$ , and say  $r \in \mathbb{Z}^+$  is defined by  $p_1 \cdots p_r \leq n = tp_1 \cdots p_r < p_1 p_2 \cdots p_{r+1}$ , where  $t \geq 1$ .

Now,  $\#(\phi_r, n) = \#(\phi_{r-1}, n) + (\#(\phi_r, n) - \#(\phi_{r-1}, n))$ . But

$$\#(\phi_r, n) - \#(\phi_{r-1}, n)$$

is the number of integers  $m$  such that  $p_r | m$  and

$$n < \phi_{r-1}(m) \leq \frac{p_r}{p_{r-1}} n.$$

This number is the sum (over  $j = 1, 2, \dots, 2^{r-1}$ ) of the number of integers less than or equal to

$$\left( \prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) \frac{np_r}{p_r-1}$$

and greater than

$$\left( \prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) n$$

which are divisible by  $p_r$  and each  $p \in P_j^{r-1}$  and not divisible by any  $p \in P^{r-1} \setminus P_j^{r-1}$ . It then follows that

$$\begin{aligned} & \#(\phi_r, n) - \#(\phi_{r-1}, n) \\ & \leq \sum_{j=1}^{2^{r-1}} \left\{ \left( \frac{2np_r}{p_r(p_r-1)} \left( \prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) - \frac{n}{2p_r} \left( \prod_{p \in P_j^{r-1}} \frac{p}{p-1} \right) \right) \right. \\ & \quad \left. \times \prod_{p \in P_j^{r-1}} \frac{1}{p} \prod_{p \in P^{r-1} \setminus P_j^{r-1}} \frac{p-1}{p} \right\} = n\Delta(\phi_{r-1}) \left( \frac{2}{p_r-1} - \frac{1}{2p_r} \right) = o(n). \end{aligned}$$

So  $\#(\phi_r, n) = \#(\phi_{r-1}, n) + o(n)$ .

By Lemma 1.1.2 we have

$$\#(\phi_{r-1}, n) \leq n \left( \Delta(\phi_{r-1}) + \frac{1}{p_r} \right) = n\Delta(\phi_r) + o(n).$$

So  $\#(\phi_r, n) \leq n\Delta(\phi_r) + o(n)$ . By Lemma 1.1.3 we have  $\#(\phi, n) \leq \#(\phi_r[g_r n])$ .

So  $\#(\phi, n) \leq [g_r n] \Delta(\phi_r) + o([g_r n]) = n\Delta(\phi_r) + o(n)$ . Divide by  $n$  and let  $n \rightarrow \infty$  to get  $\overline{\lim}_{n \rightarrow \infty} \#(\phi, n)/n \leq \lim_{r \rightarrow \infty} \Delta(\phi_r)$ .

Finally  $\underline{\Delta}(\phi) \geq \lim_{k \rightarrow \infty} \Delta(\phi_k)$  because if we choose  $k \in Z^+$  then for  $n$  large we have  $\#(\phi, n) \geq \#(\phi_k, n) \geq n(\Delta(\phi_k) - 1/k)$  and so

$$\liminf_{n \rightarrow \infty} \#(\phi, n)/n \geq \Delta(\phi_k) - 1/k$$

for each  $k \in Z^+$ . Thus  $\Delta(\phi) = \lim_{r \rightarrow \infty} \Delta(\phi_r) = \prod_{p \in P} (1 + (1/p(p-1)))$  and the theorem is proved.

A related result due to P. Erdős may be found in [4, pp. 211-213].

**DEFINITION 1.2.** For  $t \geq 1$ ,  $t$  a real number, a positive integer  $n$  is said to be  $t$ -abundant if  $\sigma(n) \geq tn$ .

H. Davenport [3] has shown that for  $t$  as above, the sequence of  $t$ -abundant positive integers has a natural density.

**THEOREM 1.2.** For each  $k \in Z^+$  let  $d_k =$  the natural density of the  $k$ -abundant integers. Then  $\sum_{k=1}^{\infty} d_k \leq \Delta(\phi) = (\zeta(2) \cdot \zeta(3) / \zeta(6))$ .

*Proof.* It is known that  $\phi(n)\sigma(n)/n^2 < 1$  for each integer  $n > 1$

[7, p. 267]. So if  $n \in [(k-1)N, kN]$  and  $\sigma(n) \geq kn$  then  $\phi(n) \leq N$ . Thus for  $k \in \mathbb{Z}^+$  and for  $N$  large, depending on  $k$ , we have

$$\begin{aligned} \#(\phi, N) &\geq N + d_2(2N - N) + d_3(3N - 2N) + \cdots \\ &\quad + d_k(kN - (k-1)N) - \frac{N}{k} \\ &= N(1 + d_2 + d_3 + \cdots + d_k - 1/k) \\ &= N(d_1 + d_2 + \cdots + d_k - 1/k). \end{aligned}$$

Now divide by  $N$  and let  $N \rightarrow \infty$ . We then have

$$\Delta(\phi) \geq \lim_{k \rightarrow \infty} (d_1 + d_2 + \cdots + d_k - 1/k) = \sum_{k=1}^{\infty} d_k$$

and the theorem is proved.

**2. General theorems.** We begin this section by stating some results whose proofs are not difficult.

1. If  $A = \{a_i\}_{i=1}^{\infty}$  is a sequence such that  $\Delta(A) = \infty$  then there exists a sequence  $\{i_j\}_{j=1}^{\infty}$  of positive integers with  $\sum_{j=1}^{\infty} a_{i_j}/i_j < \infty$ .

2. If  $A = \{a_i\}_{i=1}^{\infty}$  is a sequence such that  $\Delta(A) = 0$  then  $\sum_{a_i \leq r} a_i = o(r^2)$  and  $\sum_{a_i \leq r} 1/a_i = o(\log r)$ .

3. If  $A = \{a_i\}_{i=1}^{\infty}$  is a sequence such that  $\infty > \Delta(A) > 0$  then  $\sum_{a_i \leq r} a_i \sim \Delta(A)r^2/2$  and  $\sum_{a_i \leq r} 1/a_i \sim \Delta(A) \log r$ .

**THEOREM 2.1.** *Let  $A = \{a_i\}_{i=1}^{\infty}$  be a sequence such that  $\Delta(A) = \infty$ . Then there exists a strictly increasing sequence  $\{i_j\}_{j=1}^{\infty}$  of positive integers with  $d(\{i_j\}_{j=1}^{\infty}) = 0$  and  $\Delta(\{a_{i_j}\}_{j=1}^{\infty}) = \infty$ .*

*Proof.* It suffices to assume  $\lim_{i \rightarrow \infty} a_i = \infty$  because otherwise the proof is immediate.

*Case I.*  $a_1 \leq a_2 \leq a_3 \leq \cdots$ .

First, there is no loss of generality in supposing  $a_1 < a_2 < a_3 < \cdots$  because if  $a_i = a_{i+1} = \cdots = a_{i+r-1} < a_{i+r}$  for some  $i$  then define

$$\varepsilon = \min \left( a_{i+r} - a_i, \begin{array}{l} \text{the distance from } a_i \text{ to the} \\ \text{smallest integer greater than } a_i \end{array} \right)$$

and replace  $a_{i+t}$  by  $a_i + t\varepsilon/r$  for  $t = 0, 1, \dots, r-1$ .

We now define a subsequence  $B$  of  $A$  by induction. Let  $a_1 \in B$ . If each of  $a_1, a_2, \dots, a_{k-1}$  has already been either included in  $B$  or excluded from  $B$ , place  $a_k$  in  $B$  if

$$\frac{\#(B, a_{k-1}) + 1}{a_k} \leq \sqrt{\frac{\#(A, a_k)}{a_k}}$$

and exclude  $a_k$  from  $B$  if the inequality fails. It then follows that  $\#(B, a_k)/a_k \sim \sqrt{\#(A, a_k)/a_k}$  and so  $\Delta(B) = \infty$ . Also if we write  $B = \{a_{i_j}\}_{j=1}^\infty$  then we have  $d(\{i_j\}_{j=1}^\infty) = 0$  because

$$\begin{aligned} \frac{n}{i_n} &= \frac{\#(\{i_j\}_{j=1}^\infty, i_n)}{i_n} = \frac{\#(B, a_{i_n})}{\#(A, a_{i_n})} = \frac{a_{i_n}}{\#(A, a_{i_n})} \frac{\#(B, a_{i_n})}{a_{i_n}} \\ &\sim \sqrt{\frac{a_{i_n}}{\#(A, a_{i_n})}} \left( \sqrt{\frac{a_{i_n}}{\#(A, a_{i_n})}} \frac{\#(B, a_{i_n})}{a_{i_n}} \right) \end{aligned}$$

which tends to  $0.1 = 0$  as  $n \rightarrow \infty$ .

*Case II.* We make no assumptions about the monotonicity of  $A$ . However, without loss of generality, we may still assume  $a_i = a_j \Rightarrow i = j$ , for we can always order  $A$  by size, deal with  $A$  as in Case I, and then apply the inverse of the permutation used to order  $A$  to the new sequence which is derived from  $A$  by use of the  $\epsilon$ 's.

Now order  $A$  by size and call this sequence  $A^* = \{a_i^*\}_{i=1}^\infty$ . We have  $a_i^* < a_{i+1}^*$  for all  $i \in \mathbb{Z}^+$ . It follows immediately that if any  $n - 1$  elements are deleted from  $A$  the minimum of the remaining elements is  $\leq a_n^*$ . It is also clear that if  $A_1^* = \{a_{2i-1}\}_{i=1}^\infty$  then  $\Delta(A_1^*) = \infty$ .

Apply Case I to  $A^*$  to get a subsequence  $B^* = \{a_{i_j}^*\}_{j=1}^\infty$  of  $A^*$  such that  $\Delta(B^*) = \infty$  and  $d(\{i_j\}_{j=1}^\infty) = 0$ . Now define  $t_1$  by  $a_{t_1} = \min(\{a_{i_1}, a_{i_1+1}, a_{i_1+2}, \dots\})$ . It follows that  $t_1 \geq i_1$  and  $a_{t_1} \leq a_{i_1}^*$ . Define  $t_2$  by  $a_{t_2} = \min(\{a_{i_2}, a_{i_2+1}, a_{i_2+2}, \dots\} \setminus \{a_{i_1}\})$ . It follows that  $t_2 \geq i_2$  and  $a_{t_2} \leq a_{i_2+1}^*$ . In general define  $t_j$  by

$$a_{t_j} = \min(\{a_{i_j}, a_{i_j+1}, a_{i_j+2}, \dots\} \setminus \{a_{i_1}, a_{i_2}, \dots, a_{i_{j-1}}\}).$$

It follows that  $t_j \geq i_j$  and  $a_{t_j} \leq a_{i_j+1}^*$ .

Since  $t_j \geq i_j$  for all  $j \in \mathbb{Z}^+$ , it follows that  $d(\{t_j\}_{j=1}^\infty) = 0$ . Also  $\Delta(\{a_{i_j}^*\}_{j=1}^\infty) = \infty$  so  $\Delta(\{a_{t_j}^*\}_{j=1}^\infty) = \infty$  so  $\Delta(\{a_{t_j+j-1}^*\}_{j=1}^\infty) = \infty$ . It then follows that  $\Delta(\{a_{i_j}\}_{j=1}^\infty) = \infty$  and the theorem is proved.

To emphasize that care must be taken in the choice of  $\{i_j\}_{j=1}^\infty$  in the above theorem we note the following result.

**THEOREM 2.2.** *Suppose  $\{i_j\}_{j=1}^\infty$  is a sequence of positive integers such that  $d(\{i_j\}_{j=1}^\infty) = 0$ . Then there exists a strictly increasing sequence  $A = \{a_i\}_{i=1}^\infty$  such that  $\lim_{i \rightarrow \infty} a_i = \infty$ ,  $\Delta(A) = \infty$ , and  $\Delta(\{a_{i_j}\}_{j=1}^\infty) = 0$ .*

**THEOREM 2.3.** *For each number  $\alpha$  such that  $0 \leq \alpha \leq \infty$  there exist two sequences  $A$  and  $B$  such that  $\Delta(A) = \Delta(B) = 0$  and  $\Delta(A + B) = \alpha$ .*

*Proof.* If  $\alpha = 0$  choose  $A = B$  to be the sequence of factorials.



If  $\alpha = \infty$  choose  $A = B = P$ . Then by the Prime Number Theorem  $\Delta(A + B) = \infty$ .

Suppose  $0 < \alpha < \infty$ . Choose  $\beta$  and  $\gamma \in R^+$  so that  $(1/4)\pi\beta\gamma = \alpha$ . Let  $A = \{n^2/\beta^2\}_{n=1}^{\infty}$  and  $B = \{n^2/\gamma^2\}_{n=1}^{\infty}$ . Clearly  $\Delta(A) = 0 = \Delta(B)$ . Also, the number of elements in  $A + B$  which are  $\leq n$  is the number of lattice points  $(k, m)$  in the positive quadrant of the ellipse

$$k^2/\beta^2 + m^2/\gamma^2 \leq n.$$

This number is  $(1/4)\pi\beta\gamma n + O(\sqrt{n})$ . Thus  $\Delta(A + B) = (1/4)\pi\beta\gamma = \alpha$  and the theorem is proved.

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