A DENSITY WHICH COUNTS MULTIPLICITY

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P. Erdős, using analytic theorems, has proven the following results: Let \( f(x) \) be the number of integers \( m \) such that \( \phi(m) \leq x \), where \( \phi \) is the Euler function, and let \( g(x) \) be the number of integers \( n \) such that \( \sigma(n) \leq x \), where \( \sigma \) is the usual sum of divisors function. Then there are positive (but undetermined) constants \( c_1 \) and \( c_2 \) such that \( f(x) = c_1 x + o(x) \) and \( g(x) = c_2 x + o(x) \). The constants \( c_1 \) and \( c_2 \) can be calculated using complex analysis including the Wiener-Ikehara Theorem. A major purpose of this paper is to give an elementary proof that \( \lim_{x \to \infty} f(x)/x \) exists and, in the process, calculate the value of the limit. These considerations of multiplicity motivate a generalization of natural density which counts multiplicity. This paper contains an investigation of this generalization.

Let \( A = \{a_i\}_{i=1}^{\infty} \) be a sequence of positive real numbers \( \geq 1 \). For a positive integer \( j \), define \( \#(A, j) \) to be the number of elements of \( A \) counting multiplicity which are \( \leq j \). If \( \liminf_{j \to \infty} \#(A, j)/j = \alpha \) (we allow \( \alpha = \infty \)) we say \( A \) has \( \mathcal{A} \)-asymptotic density \( \alpha \) and we define \( \mathcal{A}(A) = \alpha \). We also define \( \overline{\mathcal{A}}(A) = \limsup_{j \to \infty} \#(A, j)/j \). If \( \mathcal{A}(A) = \overline{\mathcal{A}}(A) \) we say \( A \) has \( \mathcal{A} \)-natural density \( \alpha \) and we define \( \mathcal{A}(A) = \alpha \). It is clear that a reordering of \( A \) does not affect \( \mathcal{A}(A) \) or \( \overline{\mathcal{A}}(A) \). It is also clear that \( \mathcal{A}(A) = \mathcal{A}([a_i])_{i=1}^{\infty} \) and \( \overline{\mathcal{A}}(A) = \overline{\mathcal{A}}([a_i])_{i=1}^{\infty} \) where \( [a_i] \) is the greatest integer which does not exceed \( a_i \). Unless otherwise specified all sequences in this paper will be of positive real numbers.

Throughout this paper \( d \) will denote natural density, i.e., the classical analog of \( \mathcal{A} \) where multiplicity is not counted; \( Z^+ \) will denote the set of positive integers; \( Q^+ \) will denote the positive rational numbers; \( R^+ \) will denote the set of positive real numbers; \( p \) will always be a prime; and \( P = \{p_i\}_{i=1}^{\infty} \) will be the sequence, in the natural order, of primes.

If \( \gamma : Z^+ \to R^+ \) then to \( \gamma \) there corresponds the unique sequence \( \gamma(1), \gamma(2), \ldots \). We will write \( \gamma \) in place of this sequence. Thus, for example, in the notation of this paper \( \mathcal{A}(\phi) \) and \( \mathcal{A}(\sigma) \) exist and are positive [5]. If for instance \( \gamma = \tau \), where \( \tau(n) \) = the number of positive integer divisors of the positive integer \( n \), then it is clear that \( \mathcal{A}(\tau) = \infty \).

If \( A = \{a_i\}_{i=1}^{\infty} \) and \( B = \{b_j\}_{j=1}^{\infty} \) are sequences then define \( A + B \) to be the sequence, in the natural order, of positive real numbers \( x \) such that there exist \( i \) and \( j \in Z^+ \) with \( a_i + b_j = x \), and \( x \) appears in this
sequence the precise number of distinct ways we can write $x = a_i + b_j$. Note that it is possible to have $x = a_i + b_j$ and yet for $x$ not to be a member of $A + B$. This happens precisely when some positive number $y < x$ is representable infinitely often in the form $y = a_i + b_j$. Finally if $A$ and $B$ are sets of positive reals then define $A \setminus B$ to be the complement of $B$ in $A$.

1. Number theoretic functions. In this section we investigate the densities of certain sequences related to the $\phi$ function and other functions.

We first prove some lemmas which we will use to calculate $\Delta(\phi)$.

**Definition 1.1.** For each $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$ define

$$\phi_k(n) = n \prod_{p \leq p_k} \frac{p - 1}{p};$$

cf. [8, p. 56].

**Lemma 1.1.1.** $\Delta(\phi_k) = \prod_{p \leq p_k} (1 + \frac{1}{(p(p - 1))})$ for each $k \in \mathbb{Z}^+$.

**Proof.** Pick $k \in \mathbb{Z}^+$ and define $P^k = \{p_1, p_2, \ldots, p_k\}$. To each subset $P_j^k$ ($j = 1, 2, \ldots, 2^k$) of $P^k$ there corresponds the sequence of positive integers which are divisible by each member of $P_j^k$ and by no member of $P^k \setminus P_j^k$. These sequences are pairwise disjoint and their union is $\mathbb{Z}^+$. For a subset $P_j^k$ of $P^k$ say $\{n_{j,i}\}_{i=1}^{2^k}$ is the corresponding sequence. It is clear that

$$(\ast) \quad \#(\phi_k, n) = \sum_{j=1}^{2^k} \#(\{\phi_k(n_{j,i})\}_{i=1}^{2^k}, n) \quad \text{for each } n \in \mathbb{Z}^+.$$

Now for a fixed $P_j^k$ the density of $\{n_{j,i}\}_{i=1}^{2^k}$ is clearly

$$\prod_{p \in P_j^k} \frac{1}{p} \prod_{p \in P^k \setminus P_j^k} \frac{p - 1}{p}.$$

Also for each integer $m$ in this sequence we have

$$\phi_k(m) = m \prod_{p \in P_j^k} \frac{p - 1}{p}.$$

Therefore

$$\Delta(\{\phi_k(m)\} \text{ in the sequence defined by } P_j^k)$$

$$= \left( \prod_{p \in P_j^k} \frac{p}{p - 1} \right) \left( \prod_{p \in P_j^k} \frac{1}{p} \right) \left( \prod_{p \in P^k \setminus P_j^k} \frac{p - 1}{p} \right) = \prod_{p \in P_j^k} \frac{1}{p - 1} \prod_{p \in P^k \setminus P_j^k} \frac{p - 1}{p}.$$
So by (*) we have
\[
\Delta(\phi_r) = \sum_{j=1}^{z_k} \left( \prod_{p \in P_j^k} \frac{1}{p - 1} \cdot \prod_{p \in P_j^k \setminus P_j^*} \frac{p - 1}{p} \right)
\]
\[
= \sum_{j=1}^{z_k} \prod_{p \in P_j^k} \frac{(p - 1)^2}{p} \prod_{p \in P_j^k \setminus P_j^*} \left( 1 + \frac{(p - 1)^2}{p} \right)
= \prod_{p \in P_j^k} \left( 1 + \frac{1}{p(p - 1)} \right)
\]
and the lemma is proved.

**Note.** \( \lim_{k \to \infty} \Delta(\phi_r) = \prod_{p \in P} \left( 1 + \frac{1}{p(p - 1)} \right) < \infty \).

**LEMMA 1.1.2.** Choose \( n \in \mathbb{Z}^+ \), \( n > 1 \), and say \( r \in \mathbb{Z}^+ \) satisfies \( p_1, p_2, \ldots, p_r \leq n \). Then \( \#(\phi_r, n) \leq n(\Delta(\phi_r) + 1) \). In fact if
\[
n = t p_1, p_2, \ldots, p_r, \ t \geq 1, \ t \in \mathbb{Q}^+,
\]
then \( \#(\phi_r, n) \leq n(\Delta(\phi_r) + 1/t) \).

**Proof.** Say \( n = t p_1 \cdots p_r \) (\( t \geq 1 \)). Then if
\[
P_j^r = \{q_1, \cdots, q_s\} \subset \{p_1, \cdots, p_r\}
\]
we have \( R_{j,r} \) \( \overset{\text{def}}{=} \) the number of integers \( m \) such that \( \phi_r(m) \leq n \) and \( q_1 \cdots q_s \mid m \) and none of the members of \( P_j \setminus P_j^r \) divide \( m = \) the number of integers \( m \leq n(q_1/q_1 - 1) \cdots (q_s/q_s - 1) \) which are divisible by \( q_1 \cdots q_s \) and divisible by no member of \( P_j^r \). Say \( T_{j,r} \) is the smallest integer \( \geq t(q_1/q_1 - 1) \cdots (q_s/q_s - 1) \). Then clearly \( R_{j,r} \leq \) the number of integers \( m \) which do not exceed \( p_1 \cdots p_r T_{j,r} \) and which are divisible by \( q_1 \cdots q_s \) and divisible by no member of \( P_j \setminus P_j^r \). But since \( T_{j,r} \) is an integer we have
\[
R_{j,r} \leq \frac{1}{q_1 \cdots q_s} \prod_{p \in P_j \setminus P_j^r} \frac{p - 1}{p}
\]
\[
\leq p_1 \cdots p_r \left( t \frac{q_1}{q_1 - 1} \cdots \frac{q_s}{q_s - 1} + 1 \right) \frac{1}{q_1 \cdots q_s} \prod_{p \in P \setminus P_j^r} \frac{p - 1}{p}.
\]
Now \( \#(\phi_r, n) = \sum_{j=1}^{z_r} R_{j,r} \). So
\[
\#(\phi_r, n) \leq \sum_{j=1}^{z_r} \left( p_1 \cdots p_r \left( t \prod_{p \in P_j} \frac{p}{p - 1} + 1 \right) \prod_{p \in P_j \setminus P_j^r} \frac{p - 1}{p} \right)
\]
\[
= t p_1 \cdots p_r \sum_{j=1}^{z_r} \left( \prod_{p \in P_j} \frac{1}{p - 1} \prod_{p \in P \setminus P_j^r} \frac{p - 1}{p} \right)
\]
\[
+ p_1 \cdots p_r \sum_{j=1}^{z_r} \left( \prod_{p \in P_j \setminus P_j^r} \frac{1}{p} \prod_{p \in P \setminus P_j^r} \frac{p - 1}{p} \right) = n(\Delta(\phi_r) + \frac{1}{t})
\]
LEMMA 1.1.3. Choose \( n \in \mathbb{Z}^+ \), \( n > 1 \), and say \( r \in \mathbb{Z}^+ \) is defined by \( p_1 \cdots p_r \leq n < p_1 \cdots p_{r+1} \). Then we have

\[
\phi(m) \leq n \Rightarrow \phi_r(m) \leq \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1} n.
\]

Thus

\[
\#(\phi, n) \leq \#(\phi_r, \left[ \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1} n \right]).
\]

Proof. Suppose \( m \) has more than \( r + 1 \) distinct prime divisors. Then \( \phi(m) \geq (p_{r+2} - 1)(p_{r+1} - 1) \cdots (p_1 - 1) \geq p_1 \cdots p_{r+1} > n \), a contradiction. So \( m \) has at most \( r + 1 \) distinct prime divisors.

Now

\[
\phi_r(m) = \phi(m) \prod_{p \leq p_r} \frac{p}{p - 1} \leq n \prod_{p > p_r} \frac{p}{p - 1} \leq n \frac{p_{r+1}}{p_{r+1} - 1} \cdots \frac{p_{2r+1}}{p_{2r+1} - 1}
\]
since \( m \) has at most \( r + 1 \) distinct prime divisors and the lemma is proved.

THEOREM 1.1.

\[
A(\phi) = \prod_{p \in \mathbb{P}} \left( 1 + \frac{1}{p(p - 1)} \right) = \frac{\zeta(2) \cdot \zeta(3)}{\zeta(6)},
\]

where \( \zeta \) denotes the Riemann Zeta function.

Proof. It is well known [7, p. 246] that \( \zeta(s) = \prod_{p \in \mathbb{P}} (1/1 - p^{-s}) \) for \( s > 1 \). Thus it follows that \( \prod_{p \in \mathbb{P}} (1 + (1/p(p - 1))) = (\zeta(2) \cdot \zeta(3))/\zeta(6)) \).

So it only remains to show that \( A(\phi) = \prod_{p \in \mathbb{P}} (1 + (1/p(p - 1))) \).

For \( r \in \mathbb{Z}^+ \) let \( g_r = (p_{r+1}/p_{r+1} - 1) \cdots (p_{2r+1}/p_{2r+1} - 1) \). It follows from Mertens’ Theorem and Tchebychev’s Theorem [7, pp. 351 and 9] that \( \lim_{r \to \infty} g_r = 1 \). Choose \( n \in \mathbb{Z}^+ \), \( n > 1 \), and say \( r \in \mathbb{Z}^+ \) is defined by \( p_1 \cdots p_r \leq n = tp_1 \cdots p_r < p_1 p_2 \cdots p_{r+1} \), where \( t \geq 1 \).

Now, \( \#(\phi_r, n) = \#(\phi_{r-1}, n) + (\#(\phi_r, n) - \#(\phi_{r-1}, n)) \). But

\[
\#(\phi_r, n) - \#(\phi_{r-1}, n)
\]
is the number of integers \( m \) such that \( p_r \mid m \) and

\[
n < \phi_{r-1}(m) \leq \frac{p_r}{p_{r-1}} n.
\]

This number is the sum (over \( j = 1, 2, \cdots, 2^{r-1} \)) of the number of integers less than or equal to
and greater than
\[
\left( \prod_{p \in P_j r^{-1}} \frac{p}{p - 1} \right)^n \frac{np_r}{p_r - 1}
\]
which are divisible by \( p_r \) and each \( p \in P_j r^{-1} \) and not divisible by any \( p \in P_j P_{j-1} \). It then follows that
\[
\#(\phi_r, n) - \#(\phi_{r-1}, n) \leq \sum_{j=1}^{z_r} \left( \frac{2np_r}{p_r(p_r - 1)} \left( \prod_{p \in P_j r^{-1}} \frac{p}{p - 1} \right) - \frac{n}{2p_r} \left( \prod_{p \in P_j r^{-1}} \frac{p}{p - 1} \right) \right)
\]
\[
\times \left( \prod_{p \in P_j r^{-1}} \frac{1}{p} \prod_{p \in P_{j-1} P_{j-2} r^{-1}} \frac{p - 1}{p} \right) = n \Delta(\phi_{r-1}) \left( \frac{2}{p_r - 1} - \frac{1}{2p_r} \right) = o(n).
\]
So \( \#(\phi_r, n) = \#(\phi_{r-1}, n) + o(n) \).

By Lemma 1.1.2 we have
\[
\#(\phi_{r-1}, n) \leq n \left( \Delta(\phi_{r-1}) + \frac{1}{p_r} \right) = n \Delta(\phi_r) + o(n).
\]
So \( \#(\phi_r, n) \leq n \Delta(\phi_r) + o(n) \). By Lemma 1.1.3 we have \( \#(\phi, n) \leq \#(\phi, [g, n]) \).

So \( \#(\phi, n) \leq [g, n] \Delta(\phi_r) + o([g, n]) = n \Delta(\phi_r) + o(n) \). Divide by \( n \) and let \( n \to \infty \) to get \( \lim_{n \to \infty} \#(\phi, n)/n \leq \lim_{r \to \infty} \Delta(\phi_r) \).

Finally \( \Delta(\phi) = \lim_{r \to \infty} \Delta(\phi_r) = \prod_{p \in P} (1 + (1/p(p - 1)) \)
and the theorem is proved.

A related result due to P. Erdös may be found in [4, pp. 211–213].

**Definition 1.2.** For \( t \geq 1 \), \( t \) a real number, a positive integer \( n \) is said to be \( t \)-abundant if \( \sigma(n) \geq tn \).

H. Davenport [3] has shown that for \( t \) as above, the sequence of \( t \)-abundant positive integers has a natural density.

**Theorem 1.2.** For each \( k \in \mathbb{Z}^+ \) let \( d_k = \) the natural density of the \( k \)-abundant integers. Then \( \sum_{k=1}^{\infty} d_k \leq \Delta(\phi) = (\zeta(2) \cdot \zeta(3) / \zeta(6)) \).

**Proof.** It is known that \( \phi(n)\sigma(n)/n^2 < 1 \) for each integer \( n > 1 \)
[7, p. 267]. So if \( n \in [(k-1)N, kN] \) and \( \sigma(n) \geq kn \) then \( \phi(n) \leq N \). Thus for \( k \in \mathbb{Z}^+ \) and for \( N \) large, depending on \( k \), we have

\[
\#(\phi, N) \geq N + d_2(2N - N) + d_3(3N - 2N) + \ldots \\
+ d_k(kN - (k - 1)N) - \frac{N}{k} \\
= N(1 + d_2 + d_3 + \ldots + d_k - 1/k) \\
= N(d_1 + d_2 + \ldots + d_k - 1/k) .
\]

Now divide by \( N \) and let \( N \to \infty \). We then have

\[
\Delta(\phi) \geq \lim_{k \to \infty} (d_1 + d_2 + \ldots + d_k - 1/k) = \sum_{k=1}^{\infty} d_k
\]

and the theorem is proved.

2. General theorems. We begin this section by stating some results whose proofs are not difficult.

1. If \( A = \{a_i\}_{i=1}^{\infty} \) is a sequence such that \( \Delta(A) = \infty \) then there exists a sequence \( \{i_j\}_{j=1}^{\infty} \) of positive integers with \( \sum_{j=1}^{\infty} a_{i_j}/i_j < \infty \).

2. If \( A = \{a_i\}_{i=1}^{\infty} \) is a sequence such that \( \Delta(A) = 0 \) then \( \sum_{i=1}^{\infty} a_i = o(r^2) \) and \( \sum_{i=1}^{\infty} 1/a_i = o(\log r) \).

3. If \( A = \{a_i\}_{i=1}^{\infty} \) is a sequence such that \( \infty > \Delta(A) > 0 \) then \( \sum_{i=1}^{\infty} a_i \sim \Delta(A)r^2/2 \) and \( \sum_{i=1}^{\infty} 1/a_i \sim \Delta(A) \log r \).

**Theorem 2.1.** Let \( A = \{a_i\}_{i=1}^{\infty} \) be a sequence such that \( \Delta(A) = \infty \). Then there exists a strictly increasing sequence \( \{i_j\}_{j=1}^{\infty} \) of positive integers with \( d(\{i_j\}_{j=1}^{\infty}) = 0 \) and \( \Delta(\{a_{i_j}\}_{j=1}^{\infty}) = \infty \).

**Proof.** It suffices to assume \( \lim_{i \to \infty} a_i = \infty \) because otherwise the proof is immediate.

**Case I.** \( a_1 \leq a_2 \leq a_3 \leq \cdots \).

First, there is no loss of generality in supposing \( a_1 < a_2 < a_3 < \cdots \) because if \( a_i = a_{i+1} = \cdots = a_{i+r-1} < a_{i+r} \) for some \( i \) then define

\[
\varepsilon = \min\left( a_{i+r} - a_i, \text{the distance from } a_i \text{ to the smallest integer greater than } a_i \right)
\]

and replace \( a_{i+t} \) by \( a_i + t\varepsilon/r \) for \( t = 0, 1, \ldots, r-1 \).

We now define a subsequence \( B \) of \( A \) by induction. Let \( a_i \in B \). If each of \( a_1, a_2, \ldots, a_{k-1} \) has already been either included in \( B \) or excluded from \( B \), place \( a_k \) in \( B \) if

\[
\frac{\#(B, a_{k-1}) + 1}{a_k} \leq \sqrt{\frac{\#(A, a_k)}{a_k}}
\]
and exclude \( a_k \) from \( B \) if the inequality fails. It then follows that 
\[
\frac{\#(B, a_k)}{a_k} \sim \sqrt{\frac{\#(A, a_k)}{a_k}}
\]
and so \( \Delta(B) = \infty \). Also if we write \( B = \{ a_{ij} \}_{j=1}^{\infty} \) then we have 
\[
d(\{i_j\}_{j=1}^{\infty}) = 0
\]
because 
\[
\frac{n}{i_n} = \frac{\#(i_{j, n=1}^{\infty}, i_n)}{i_n} = \frac{\#(B, a_{i_n})}{\#(A, a_{i_n})} = \frac{a_{i_n}}{\#(B, a_{i_n})}
\]
which tends to 0.1 = 0 as \( n \to \infty \).

**Case II.** We make no assumptions about the monotonicity of \( A \). However, without loss of generality, we may still assume \( a_i = a_j \Rightarrow i = j \), for we can always order \( A \) by size, deal with \( A \) as in Case I, and then apply the inverse of the permutation used to order \( A \) to the new sequence which is derived from \( A \) by use of the \( \varepsilon \)'s.

Now order \( A \) by size and call this sequence \( A^* = \{a_{i^*}\}_{i=1}^{\infty} \). We have 
\[
a_i^* < a_{i+1}^*
\]
for all \( i \in \mathbb{Z}^+ \). It follows immediately that if any \( n - 1 \) elements are deleted from \( A \) the minimum of the remaining elements is \( \leq a_x^* \). It is also clear that if \( A^* = \{a_{2i-1}\}_{i=1}^{\infty} \) then \( \Delta(A^*) = \infty \).

Apply Case I to \( A^* \) to get a subsequence \( B^* = \{a_{i^*_j}\}_{j=1}^{\infty} \) of \( A^* \) such that 
\[
\Delta(B^*) = \infty \quad \text{and} \quad d(\{i_j\}_{j=1}^{\infty}) = 0.
\]
Now define \( t_1 \) by 
\[
a_{t_1} = \min (\{a_{i_1}, a_{i_1+1}, a_{i_1+2}, \ldots\})
\]
It follows that 
\[
t_1 \geq i_1 \quad \text{and} \quad a_{t_1} \leq a_{i_1}^*.
\]
Define \( t_2 \) by 
\[
a_{t_2} = \min (\{a_{i_2}, a_{i_2+1}, a_{i_2+2}, \ldots\}) \setminus \{a_{t_1}\}
\]
It follows that 
\[
t_2 \geq i_2 \quad \text{and} \quad a_{t_2} \leq a_{i_2+1}^*.
\]
In general define \( t_j \) by 
\[
a_{t_j} = \min (\{a_{i_j}, a_{i_{j+1}}, a_{i_{j+2}}, \ldots\}) \setminus \{a_{t_1}, a_{t_2}, \ldots, a_{t_{j-1}}\}
\]
It follows that 
\[
t_j \geq i_j \quad \text{and} \quad a_{t_j} \leq a_{i_j+j-1}^*.
\]
Since 
\[
t_j \geq i_j \quad \text{for all} \quad j \in \mathbb{Z}^+,
\]
it follows that 
\[
d(\{t_j\}_{j=1}^{\infty}) = 0.
\]
Also 
\[
\Delta(\{a_{i_j}^{(2)}\}_{j=1}^{\infty}) = \infty \quad \text{so} \quad \Delta(\{a_{i_{j+1}}^{(2)}\}_{j=1}^{\infty}) = \infty \quad \text{so} \quad \Delta(\{a_{i_{j+1}+j-1}^{(2)}\}_{j=1}^{\infty}) = \infty.
\]
It then follows that 
\[
\Delta(\{a_{i_j}^{(2)}\}_{j=1}^{\infty}) = \infty \quad \text{and} \quad \text{the theorem is proved.}
\]

To emphasize that care must be taken in the choice of \( \{i_j\}_{j=1}^{\infty} \) in the above theorem we note the following result.

**Theorem 2.2.** Suppose \( \{i_j\}_{j=1}^{\infty} \) is a sequence of positive integers such that 
\[
d(\{i_j\}_{j=1}^{\infty}) = 0.
\]
Then there exists a strictly increasing sequence \( A = \{a_i\}_{i=1}^{\infty} \) such that \( \lim_{i \to \infty} a_i = \infty \), \( \Delta(A) = \infty \), and \( \Delta(\{a_i\}_{i=1}^{\infty}) = 0 \).

**Theorem 2.3.** For each number \( \alpha \) such that \( 0 \leq \alpha \leq \infty \) there exist two sequences \( A \) and \( B \) such that \( \Delta(A) = \Delta(B) = 0 \) and 
\[
\Delta(A + B) = \alpha.
\]

**Proof.** If \( \alpha = 0 \) choose \( A = B \) to be the sequence of factorials.
If $\alpha = \infty$ choose $A = B = P$. Then by the Prime Number Theorem $\Delta(A + B) = \infty$.

Suppose $0 < \alpha < \infty$. Choose $\beta$ and $\gamma \in \mathbb{R}^+$ so that $(1/4)\pi \beta \gamma = \alpha$. Let $A = \{n^2/\beta^2\}_{n=1}^{\infty}$ and $B = \{n^2/\gamma^2\}_{n=1}^{\infty}$. Clearly $\Delta(A) = 0 = \Delta(B)$. Also, the number of elements in $A + B$ which are $\leq n$ is the number of lattice points $(k, m)$ in the positive quadrant of the ellipse

$$k^2/\beta^2 + m^2/\gamma^2 \leq n.$$ 

This number is $(1/4)\pi \beta \gamma n + O(\sqrt{n})$. Thus $\Delta(A + B) = (1/4)\pi \beta \gamma = \alpha$ and the theorem is proved.

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