

# Pacific Journal of Mathematics

## **ON THE EXISTENCE QUESTION FOR A FAMILY OF PRODUCTS**

GARY ALLEN GISLASON

## ON THE EXISTENCE QUESTION FOR A FAMILY OF PRODUCTS

GARY A. GISLASON

Let  $X$  be a topological space and let  $P$  and  $Q$  be finite dimensional linear subspaces of  $C(X)$ . Since the set  $PQ = \{pq: p \in P, q \in Q\}$  is a subset of a finite dimensional linear subspace of  $C(X)$ , existence of best approximations from  $PQ$  is assured if and only if  $PQ$  is closed. If  $p \in P, q \in Q$ , and  $pq=0$  imply that  $p=0$  or  $q=0$ , then  $PQ$  is shown to be closed. An example shows that  $PQ$  is not closed in general.

In an important paper by B. Boehm, [1], the problem of existence of best approximations from certain nonlinear families is discussed. One of these families is formed by taking products of elements of two finite dimensional linear subspaces of  $C(X)$ , where  $X$  is a topological space. In a seminar Professor H. Loeb pointed out an error in Boehm's existence proof for this family. Professor Loeb then posed the question whether the result was correct and if it was not to develop reasonable hypotheses to insure existence.

In this paper we consider these questions and show that Boehm's hypotheses have to be modified.

Throughout this paper  $X$  will denote a topological space and  $C(X)$  the vector space of all bounded, continuous, real-valued functions defined on  $X$ . The symbol  $\|\dots\|$  will represent the uniform norm defined on  $C(X)$  by  $\|f\| = \sup \{|f(x)|: x \in X\}$ .

One question under consideration is that of existence of best approximations from the set

$$PQ = \{pq: p \in P \text{ and } q \in Q\}$$

where  $P$  and  $Q$  are finite dimensional linear subspaces of  $C(X)$ . (Here  $(pq)(x) = p(x) \cdot q(x)$ .) First of all it is easy to see that  $PQ$  is a subset of a finite dimensional linear subspace of  $C(X)$ . In fact, if  $\{g_1, \dots, g_m\}$  is a basis of  $P$  and  $\{h_1, \dots, h_n\}$  is a basis of  $Q$  then it is clear that  $PQ$  is contained in the linear space generated by the set of products of functions  $\{g_i h_j: i = 1, \dots, m \text{ and } j = 1, \dots, n\}$ . The dimension of this linear space is at most equal to  $mn$ . Now, it is known that best approximations exist from a subset of a finite dimensional linear subspace if and only if the subset is closed. But, as the following example shows, the set  $PQ$  is not closed in general.

EXAMPLE 1. Let  $P = \{ag_1 + bg_2: a, b \text{ real}\}$  and  $Q = \{ch_1 + dh_2: c, d \text{ real}\}$  be finite dimensional linear subspaces of  $C[0, 5]$  with basis functions described by the graphs in Figure 1.

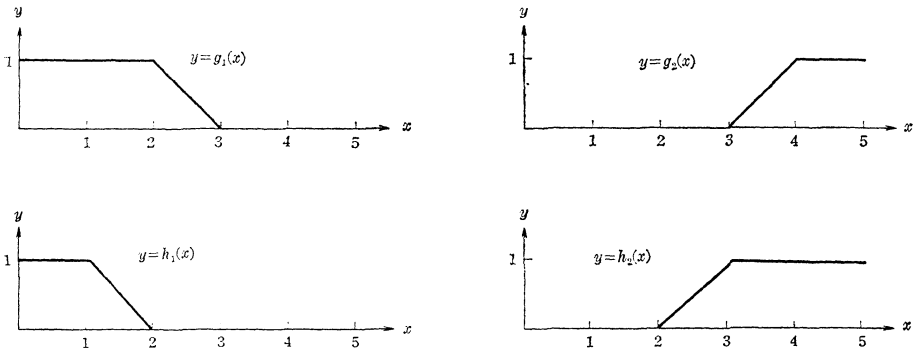


FIGURE 1.

Observe that

(1)  $g_1 h_1 = h_1$ ,  $g_2 h_1 = 0$ , and  $g_2 h_2 = g_2$  and that

(2)  $h_1$ ,  $g_1 h_2$  and  $g_2$  are linearly independent.

Now, consider the sequence  $\{p_n q_n\}_{n=1}^{\infty}$  in  $PQ$  in which

$$p_n = g_1 + n g_2$$

and

$$q_n = h_1 + \frac{1}{n} h_2$$

for each  $n$ . By (1) it follows that

$$\begin{aligned} p_n q_n &= (g_1 + n g_2) \left( h_1 + \frac{1}{n} h_2 \right) \\ (*) \quad &= g_1 h_1 + \frac{1}{n} g_1 h_2 + n g_2 h_1 + g_2 h_2 \\ &= h_1 + \frac{1}{n} g_1 h_2 + g_2 \end{aligned}$$

for each  $n$ . Thus

$$p_n q_n \longrightarrow h_1 + g_2$$

as  $n \rightarrow \infty$  so  $h_1 + g_2$  is in the closure of  $PQ$ . If  $h_1 + g_2$  is contained in  $PQ$  then there exist real coefficients  $a$ ,  $b$ ,  $c$  and  $d$  such that

$$(a g_1 + b g_2)(c h_1 + d h_2) = h_1 + g_2.$$

By (1) the above equation reduces to

$$a c h_1 + a d g_1 h_2 + b d g_2 = h_1 + g_2$$

and by (2) to the system

$$ac = 1$$

$$ad = 0$$

$$bd = 1$$

which has no solution. Thus  $h_1 + g_2$  is not in  $PQ$  and  $PQ$  is not closed.

The above result shows that Boehm's Theorem 4, [1], is incorrect. The fact that  $g_2 h_1 = 0$  even though  $g_2 \neq 0$  and  $h_1 \neq 0$  plays an important role in Example 1, as is apparent from (\*). As it turns out, one modification to Boehm's hypotheses which implies that  $PQ$  is closed is to rule out the possibility of a product in  $PQ$  being zero when neither of the factors is zero.

**THEOREM 1.** *Let  $P$  and  $Q$  be finite dimensional linear subspaces of  $C(X)$ . If  $p \in P$ ,  $q \in Q$ , and  $pq = 0$  imply that  $p = 0$  or  $q = 0$  then  $PQ$  is closed.*

*Proof.* Let  $g$  be a function in the closure of  $PQ$ , and let  $\{p_n q_n\}_{n=1}^{\infty}$  be a sequence in  $PQ$  such that

$$(3) \quad \|g - p_n q_n\| \longrightarrow 0$$

as  $n \rightarrow \infty$ . If  $g = 0$  then certainly  $g \in PQ$  since  $0 \in P$  and  $0 \in Q$ . So, assume that  $g \neq 0$ . Then with no loss of generality it may be assumed that  $p_n q_n \neq 0$  and that  $\|p_n\| = 1$  for each  $n$ . Since closed and bounded subsets of finite dimensional linear spaces are compact, it suffices to show that a subsequence of the sequence  $\{q_n\}_{n=1}^{\infty}$  is bounded. Assume the contrary. Then by going to subsequences, if necessary, and letting  $n \rightarrow \infty$  it follows that

$$(4) \quad \frac{p_n q_n}{\|q_n\|} \longrightarrow 0.$$

By the preceding remark on compactness there exist nonzero functions  $p \in P$  and  $q \in Q$  such that

$$(5) \quad \frac{p_n q_n}{\|q_n\|} \longrightarrow pq$$

by going to further subsequences, if necessary, and letting  $n \rightarrow \infty$ . But (4) implies that  $pq = 0$  which is a contradiction.

Clearly, in Theorem 1,  $C(X)$  can be replaced by any real or complex normed algebra.

A counter example to Boehm's Theorem 5, [1], exists using ordinary polynomials and corresponding rational functions.

In a forthcoming paper we will discuss further the existence problem and also the characterization question for this setting.

#### BIBLIOGRAPHY

1. B. W. Boehm, *Existence of best rational Tchebycheff approximations*, Pacific J. Math. **15** (1965), 19-27.

Received October 30, 1969. Supported in part by NSF Grant 8686.

UNIVERSITY OF OREGON

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON  
Stanford University  
Stanford, California 94305

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

RICHARD PIERCE  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

|  |     |
|--|-----|
| Shair Ahmad, <i>On the oscillation of solutions of a class of linear fourth order differential equations</i> .....                       | 289 |
| Leonard Asimow and Alan John Ellis, <i>Facial decomposition of linearly compact simplexes and separation of functions on cones</i> ..... | 301 |
| Kirby Alan Baker and Albert Robert Stralka, <i>Compact, distributive lattices of finite breadth</i> .....                                | 311 |
| James W. Cannon, <i>Sets which can be missed by side approximations to spheres</i> .....   | 321 |
| Prem Chandra, <i>Absolute summability by Riesz means</i> .....   | 335 |
| Francis T. Christoph, <i>Free topological semigroups and embedding topological semigroups in topological groups</i> .....                | 343 |
| Henry Bruce Cohen and Francis E. Sullivan, <i>Projecting onto cycles in smooth, reflexive Banach spaces</i> .....                        | 355 |
| John Dauns, <i>Power series semigroup rings</i> .....  | 365 |
| Robert E. Dressler, <i>A density which counts multiplicity</i> .....   | 371 |
| Kent Ralph Fuller, <i>Primary rings and double centralizers</i> .....  | 379 |
| Gary Allen Gislason, <i>On the existence question for a family of products</i> .....   | 385 |
| Alan Stuart Gleit, <i>On the structure topology of simplex spaces</i> .....  | 389 |
| William R. Gordon and Marvin David Marcus, <i>An analysis of equality in certain matrix inequalities. I</i> .....                        | 407 |
| Gerald William Johnson and David Lee Skoug, <i>Operator-valued Feynman integrals of finite-dimensional functionals</i> .....             | 415 |
| (Harold) David Kahn, <i>Covering semigroups</i> .....  | 427 |
| Keith Milo Kendig, <i>Fibrations of analytic varieties</i> .....   | 441 |
| Norman Yeomans Luther, <i>Weak denseness of nonatomic measures on perfect, locally compact spaces</i> .....                              | 453 |
| Guillermo Owen, <i>The four-person constant-sum games; Discriminatory solutions on the main diagonal</i> .....                           | 461 |
| Stephen Parrott, <i>Unitary dilations for commuting contractions</i> .....   | 481 |
| Roy Martin Rakestraw, <i>Extremal elements of the convex cone <math>A_n</math> of functions</i> .....                                    | 491 |
| Peter Lewis Renz, <i>Intersection representations of graphs by arcs</i> .....  | 501 |
| William Henry Ruckle, <i>Representation and series summability of complete biorthogonal sequences</i> .....                              | 511 |
| F. Dennis Sentilles, <i>The strict topology on bounded sets</i> .....  | 529 |
| Saharon Shelah, <i>A note on Hanf numbers</i> .....  | 541 |
| Harold Simmons, <i>The solution of a decision problem for several classes of rings</i> .....   | 547 |
| Kenneth S. Williams, <i>Finite transformation formulae involving the Legendre symbol</i> .....   | 559 |