ON THE EXISTENCE QUESTION FOR A FAMILY OF PRODUCTS

GARY ALLEN GISLASON
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Let $X$ be a topological space and let $P$ and $Q$ be finite dimensional linear subspaces of $C(X)$. Since the set $PQ = \{pq: p \in P, q \in Q\}$ is a subset of a finite dimensional linear subspace of $C(X)$, existence of best approximations from $PQ$ is assured if and only if $PQ$ is closed. If $p \in P$, $q \in Q$, and $pq = 0$ imply that $p = 0$ or $q = 0$, then $PQ$ is shown to be closed. An example shows that $PQ$ is not closed in general.

In an important paper by B. Boehm, [1], the problem of existence of best approximations from certain nonlinear families is discussed. One of these families is formed by taking products of elements of two finite dimensional linear subspaces of $C(X)$, where $X$ is a topological space. In a seminar Professor H. Loeb pointed out an error in Boehm's existence proof for this family. Professor Loeb then posed the question whether the result was correct and if it was not to develop reasonable hypotheses to insure existence.

In this paper we consider these questions and show that Boehm's hypotheses have to be modified.

Throughout this paper $X$ will denote a topological space and $C(X)$ the vector space of all bounded, continuous, real-valued functions defined on $X$. The symbol $\| \cdots \|$ will represent the uniform norm defined on $C(X)$ by $\|f\| = \sup \{|f(x)|: x \in X\}$.

One question under consideration is that of existence of best approximations from the set

$$PQ = \{pq: p \in P \text{ and } q \in Q\}$$

where $P$ and $Q$ are finite dimensional linear subspaces of $C(X)$. (Here $(pq)(x) = p(x) \cdot q(x)$.) First of all it is easy to see that $PQ$ is a subset of a finite dimensional linear subspace of $C(X)$. In fact, if $\{g_1, \cdots, g_m\}$ is a basis of $P$ and $\{h_1, \cdots, h_n\}$ is a basis of $Q$ then it is clear that $PQ$ is contained in the linear space generated by the set of products of functions $\{g_i h_j: i = 1, \cdots, m \text{ and } j = 1, \cdots, n\}$. The dimension of this linear space is at most equal to $mn$. Now, it is known that best approximations exist from a subset of a finite dimensional linear subspace if and only if the subset is closed. But, as the following example shows, the set $PQ$ is not closed in general.

**Example 1.** Let $P = \{ag_1 + bg_2: a, b \text{ real}\}$ and $Q = \{ch_1 + dh_2: c, d \text{ real}\}$ be finite dimensional linear subspaces of $C[0, 5]$ with basis functions described by the graphs in Figure 1.
Observe that

(1) \( g_1 h_1 = h_1, g_2 h_1 = 0 \), and \( g_2 h_2 = g_2 \) and that

(2) \( h_1, g_2 h_2 \text{ and } g_2 \) are linearly independent.

Now, consider the sequence \( \{p_n q_n\}_{n=1}^{\infty} \) in \( PQ \) in which

\[ p_n = g_1 + n g_2 \]

and

\[ q_n = h_1 + \frac{1}{n} h_2 \]

for each \( n \). By (1) it follows that

\[
\begin{align*}
p_n q_n &= (g_1 + n g_2) \left( h_1 + \frac{1}{n} h_2 \right) \\
&= g_1 h_1 + \frac{1}{n} g_1 h_2 + n g_2 h_1 + g_2 h_2 \\
&= h_1 + \frac{1}{n} g_1 h_2 + g_2
\end{align*}
\]

for each \( n \). Thus

\[ p_n q_n \longrightarrow h_1 + g_2 \]

as \( n \to \infty \) so \( h_1 + g_2 \) is in the closure of \( PQ \). If \( h_1 + g_2 \) is contained in \( PQ \) then there exist real coefficients \( a, b, c \) and \( d \) such that

\[
(a g_1 + b g_2)(c h_1 + d h_2) = h_1 + g_2 .
\]

By (1) the above equation reduces to

\[ ach_1 + ad g_1 h_2 + bd g_2 = h_1 + g_2 \]

and by (2) to the system
which has no solution. Thus $h_1 + g_2$ is not in $PQ$ and $PQ$ is not closed.

The above result shown that Boehm’s Theorem 4, [1], is incorrect. The fact that $g_2h_1 = 0$ even though $g_2 \neq 0$ and $h_1 \neq 0$ plays an important role in Example 1, as is apparent from (*). As it turns out, one modification to Boehm’s hypotheses which implies that $PQ$ is closed is to rule out the possibility of a product in $PQ$ being zero when neither of the factors is zero.

**Theorem 1.** Let $P$ and $Q$ be finite dimensional linear subspaces of $C(X)$. If $p \in P$, $q \in Q$, and $pq = 0$ imply that $p = 0$ or $q = 0$ then $PQ$ is closed.

**Proof.** Let $g$ be a function in the closure of $PQ$, and let \( \{p_n q_n\}_{n=1}^{\infty} \) be a sequence in $PQ$ such that

\[
\|g - p_n q_n\| \to 0
\]

as $n \to \infty$. If $g = 0$ then certainly $g \in PQ$ since $0 \in P$ and $0 \in Q$. So, assume that $g \neq 0$. Then with no loss of generality it may be assumed that $p_n q_n \neq 0$ and that $\|p_n\| = 1$ for each $n$. Since closed and bounded subsets of finite dimensional linear spaces are compact, it suffices to show that a subsequence of the sequence $\{q_n\}_{n=1}^{\infty}$ is bounded. Assume the contrary. Then by going to subsequences, if necessary, and letting $n \to \infty$ it follows that

\[
\frac{p_n q_n}{\|q_n\|} \to 0.
\]

By the preceding remark on compactness there exist nonzero functions $p \in P$ and $q \in Q$ such that

\[
\frac{p_n q_n}{\|q_n\|} \to pq
\]

by going to further subsequences, if necessary, and letting $n \to \infty$. But (4) implies that $pq = 0$ which is a contradiction.

Clearly, in Theorem 1, $C(X)$ can be replaced by any real or complex normed algebra.

A counter example to Boehm’s Theorem 5, [1], exists using ordinary polynomials and corresponding rational functions.
In a forthcoming paper we will discuss further the existence problem and also the characterization question for this setting.

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