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AN ANALYSIS OF EQUALITY IN CERTAIN MATRIX INEQUALITIES, I

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In this paper we are concerned with analyzing the cases of equality in certain inequalities that relate the eigenvalues and main diagonal elements of hermitian matrices.

Let E_r denote the r^{th} elementary symmetric function of k variables ($E_0 = 1$). If $H=(h_{ij})$ is an n-square positive semi-definite hermitian matrix with eigenvalues $\gamma_1 \leqq \cdots \leqq \gamma_n$ and if $1 \leqq r \leqq k \leqq n$, then it is known that

$$E_r(h_{11}, \cdots, h_{kk}) \geqq E_r(\gamma_1, \cdots, \gamma_k) . \tag{1.1}$$

If $r > 1$ and at least r of $h_{11}, \cdots, h_{kk}$ are positive then (1.1) can be equality if and only if there exists a permutation $\varphi \in S_k$ such that

$$H = \operatorname{diag}(\gamma_{\varphi(1)}, \cdots, \gamma_{\varphi(k)}) \dotplus H_{n-k} \tag{1.2}$$

where H_{n-k} is $(n-k)$-square and $\dotplus$ denotes direct sum. Of course, if $r = k = n$ then (1.1) is the Hadamard determinant theorem:

$$\prod_{i=1}^{n} h_{ii} \geqq \det(H) . \tag{1.3}$$

If some $h_{ii} = 0$, then H is singular and (1.3) is equality. If $h_{ii} > 0, i=1, \cdots, n$, then the condition (1.2) yields the well-known criterion for equality in (1.3), namely $H = \operatorname{diag}(h_{11}, \cdots, h_{nn})$.

2. Results. Let $f(x) = f(x_1, \cdots, x_k)$ be a function defined for all nonnegative vectors $x \geqq 0$ (i.e., $x_i \geqq 0$, $i = 1, \cdots, k$). We shall assume that f is symmetric: $f(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) = f(x)$ for all $\sigma \in S_k$, the symmetric group of degree k. Let C_r denote the cone consisting of all $x \geqq 0$ with at least r positive components. The function f is said to be *strictly C_r-concave* if f is concave for $x \in C_r$ and if for x and y in C_r and $0 < \theta < 1$ the equality

$$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y) \tag{2.1}$$

holds then it follows that $x \sim y$, i.e., x is a positive multiple of y. The usual definition of *strict concavity* requires that f be concave and that (2.1) holds if and only if $x = y$. We say that f is *C_r-positive* if: $f(x) > 0$ if and only if $x \in C_r$. Also, f is strictly *C_r-monotone* if $f(x+u) > f(x)$, $x \in C_r$, $u \geqq 0$, $u \neq 0$.

THEOREM 1. *Let $H = (h_{ij})$ be an n-square positive semi-definite* *hermitian matrix with eigenvalues* $0 \leqq \gamma_1 \leqq \cdots \leqq \gamma_n$. *Let* $1 \leqq r \leqq k \leqq n$. *Assume that* f *is symmetric, concave and nondecreasing in each variable. Let* $h_{\omega_t\omega_t}$, $t = 1, \cdots, k$, *be* k *main diagonal entries of* H. *Then*

$$f(h_{\omega_1\omega_1}, \cdots, h_{\omega_k\omega_k}) \geqq f(\gamma_1, \cdots, \gamma_k) . \tag{2.2}$$

Assume in addition that f *is strictly* C_r*-monotone, strictly* C_r*-concave and* C_r*-positive. If at least* r *of the* $h_{\omega_t\omega_t}$, $t = 1, \cdots, k$, *are positive then equality holds in* (2.2) *if and only if for some* $\varphi \in S_k$

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \qquad t = 1, \cdots, k , \tag{2.3}$$

and, in fact, in row and column ω_t, H *is* 0 *off the main diagonal,* $t = 1, \cdots, k$.

The inequality (2.2) is found in [3].

Proof. To begin with we can assume that $\omega_t = t$, $t = 1, \cdots, k$, and $h_{11} \leqq \cdots \leqq h_{kk}$. For, we can rearrange the main diagonal entries with a permutation similarity without affecting the eigenvalues. A trivial induction shows that for f strictly C_r-concave, $a^t \in C_r$, and $\theta_t > 0$, $t = 1, \cdots, m$, $\sum_{t=1}^m \theta_t = 1$, then

$$f\left(\sum_{t=1}^m \theta_t a^t\right) \geqq \sum_{t=1}^m \theta_t f(a^t) \tag{2.4}$$

and equality implies that $a^s \sim a^t$, $s, t = 1, \cdots, m$. Now there exists a unitary U such that $U^* \operatorname{diag}(\gamma_1, \cdots, \gamma_n)U = H$ and hence

$$h_{ii} = \sum_{j=1}^n |u_{ji}|^2 \gamma_j , \qquad i = 1, \cdots, n . \tag{2.5}$$

Since the matrix U is unitary we know that the matrix S whose (i, j) entry is $|u_{ji}|^2$, is doubly stochastic (d.s.). Thus (2.5) becomes

$$(h_{11}, \cdots, h_{nn}) = S(\gamma_1, \cdots, \gamma_n) . \tag{2.6}$$

Let $d = (h_{11}, \cdots, h_{nn})$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, and for any n-tuple x let $x[k]$ denote the truncated vector $(x_1, \cdots, x_k)$. If $\sigma \in S_n$ then $x^\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. By Birkhoff's theorem [1] let

$$S = \sum_{\sigma \in G} c_\sigma P_\sigma \tag{2.7}$$

where G is a subset of S_n, $c_\sigma > 0$, $\sigma \in G$, P_σ is an n-square permutation matrix corresponding to σ and $\sum_{\sigma \in G} c_\sigma = 1$. From (2.6), (2.7) and (2.4) we have

$$\begin{aligned} f(d[k]) &= f\left(\sum_{\sigma \in G} c_\sigma \gamma^\sigma[k]\right) \\ &\geqq \sum_{\sigma \in G} c_\sigma f(\gamma^\sigma[k]) . \end{aligned} \tag{2.8}$$

Consider a summand in (2.8) and choose $\mu_\sigma \in S_k$ so that

$$\sigma(\mu_\sigma(1)) < \cdots < \sigma(\mu_\sigma(k))$$

and hence

$$\gamma_{\sigma(\mu_\sigma(1))} \leqq \cdots \leqq \gamma_{\sigma(\mu_\sigma(k))} \,. \tag{2.9}$$

The symmetry of f implies that

$$f(\gamma^\sigma[k]) = f(\gamma^{\sigma\mu_\sigma}[k]) \,.$$

Now since $\sigma\mu_\sigma(t) \geqq t$, $t = 1, \cdots, k$, we know that

$$\gamma_{\sigma\mu_\sigma}(t) \geqq \gamma_t \,, \tag{2.10}$$

$t = 1, \cdots, k$. Then since f is nondecreasing in each variable we have

$$f(\gamma^\sigma[k]) \geqq f(\gamma_1, \cdots, \gamma_k) \tag{2.11}$$

and hence (2.8) becomes

$$f(d[k] \geqq f(\gamma_1, \cdots, \gamma_k) \,, \tag{2.12}$$

the required inequality (2.2).

Suppose equality holds in (2.12). Since $d[k] \in C_r$ we know that $f(d[k]) > 0$ and hence $f(\gamma[k]) > 0$. Thus $\gamma[k] \in C_r$. We also know that $f(\gamma^{\sigma\mu_\sigma}[k]) = f(\gamma[k])$ and in view of (2.10) it follows that

$$\gamma^{\sigma\mu_\sigma}[k] = \gamma[k] \,. \tag{2.13}$$

Setting $\mu_\sigma^{-1} = \nu_\sigma \in S_k$ in (2.13) we have

$$\gamma^\sigma[k] = (\gamma[k])^{\nu_\sigma} \,. \tag{2.14}$$

We must also have equality in (2.8) which because of the strict C_r-concavity implies that $\gamma^\sigma[k] \sim \gamma^\theta[k]$, σ, θ in G. In other words,

$$\gamma^\sigma[k] = a_\sigma \gamma^\tau[k]$$

for some fixed $\tau \in G$, $a_\sigma > 0$ all $\sigma \in G$. In view of (2.14)

$$\gamma^\sigma[k] = a_\sigma(\gamma[k])^{\nu_\tau}$$

so that

$$\begin{aligned} d[k] &= \sum_{\sigma \in G} c_\sigma \gamma^\sigma[k] \\ &= \sum_{\sigma \in G} c_\sigma a_\sigma (\gamma[k])^{\nu_\tau} \\ &= c(\gamma[k])^{\nu_\tau}, \; c > 0 \,. \end{aligned}$$

The equality in (2.12) implies that

$$\begin{aligned} f(d[k]) &= f(\gamma[k]) \\ &= f(\gamma[k])^{\nu_\tau}) \end{aligned}$$

and thus

$$f(c(\gamma[k])^{\nu_\tau}) = f((\gamma[k])^{\nu_\tau})$$

or

$$f(c\gamma[k]) = f(\gamma[k]) .$$

Now $\gamma[k] \in C_r$ and hence by (2.1) $c = 1$. Thus

$$d[k] = (\gamma[k])^{\nu_\tau} . \tag{2.15}$$

Since $h_{11} \leqq \cdots \leqq h_{kk}$, (2.15) implies that

$$\gamma_{\nu_\tau(1)} \leqq \cdots \gamma_{\nu_\tau(k)} .$$

But $\gamma_1 \leqq \cdots \leqq \gamma_k$ and $\nu_\tau \in S_k$ and hence $\gamma_{\nu_\tau(t)} = \gamma_t$, $t = 1, \cdots, k$. In other words,

$$h_{ii} = \gamma_i , \qquad i = 1, \cdots, k . \tag{2.16}$$

Now we assert that (2.16) implies that the first k rows and columns of H are 0 off the main diagonal. To see this we observe that if $e_1 = (\delta_{11}, \cdots, \delta_{n1})$ and $u_1, \cdots, u_n$ are orthonormal eigenvectors of H corresponding to $\gamma_1, \cdots, \gamma_n$ respectively, then using the standard inner product in the vector space of complex n-tuples,

$$\begin{aligned} h_{11} &= (He_1, e_1) \\ &= \sum_{j=1}^{n} |(e_1, u_j)|^2 \gamma_j . \end{aligned} \tag{2.17}$$

Since $\gamma_1 = h_{11}$ we conclude from (2.17) that $(e_1, u_j) = 0$, if $\gamma_j > \gamma_1$. Suppose $\gamma_1 = \cdots = \gamma_r < \gamma_{r+1} \leqq \cdots \leqq \gamma_n$. Then $(e_1, u_j) = 0$, $j = r+1, \cdots, n$, and hence $e_1 \in \langle u_1, \cdots, u_r \rangle$, the space spanned by $u_1, \cdots, u_r$. But then $He_1 = \gamma_1 e_1$ and we conclude that the first column (and row) of H is 0 off the main diagonal. Since $\gamma_2, \cdots, \gamma_n$ are the eigenvalues of the submatrix obtained from H by deleting row and column 1, an obvious induction completes the proof.

Make the following choice for f:

$$f(x_1, \cdots, x_k) = E_r^{1/r}(x_1^q, \cdots, x_k^q) \tag{2.18}$$

where $0 < q \leqq 1$. We assert that for $r > 1$ or $r = 1$, $q < 1$, f is strictly C_r-concave. For $0 < \theta < 1$ consider

$$\begin{aligned} f(\theta x + (1-\theta)y) &= E_r^{1/r}((\theta x_1 + (1-\theta)y_1)^q, \cdots, (\theta x_k + (1-\theta)y_k)^q) \\ &\geqq E_r^{1/r}(\theta x_1^q + (1-\theta)y_1^q, \cdots, \theta x_k^q + (1-\theta)y_k^q) \\ &\geqq \theta E_r^{1/r}(x_1^q, \cdots, x_k^q) + (1-\theta)E_r^{1/r}(y_1^q, \cdots, y_k^q) \\ &= \theta f(x) + (1-\theta)f(y) . \end{aligned} \tag{2.19}$$

In (2.19) we have used the monotonicity and C_r-concavity of $E_r^{1/r}$ [4], $r > 1$, and the strict concavity of t^q, $t \geqq 0$, for $r = 1$. When $q < 1$ the first inequality in (2.19) is strict unless $x = y$. If $q = 1$, $r > 1$, then the second inequality is strict unless $x \sim y$. In either event if (2.19) is equality then $x \sim y$ so that f is indeed strictly C_r-concave. Also, f is obviously strictly C_r-monotone and C_r-positive. We have

COROLLARY 1. *Let H satisfy the hypotheses of Theorem 1 and let $0 < q \leqq 1$. Then*

$$E_r(h^q_{\omega_1\omega_1}, \cdots, h^q_{\omega_k\omega_k}) \geqq E_r(\gamma_1^q, \cdots, \gamma_k^q) . \tag{2.20}$$

If at least r of the $h_{\omega_t\omega_t}$ are positive, $t = 1, \cdots, k$, then equality holds in (2.20) if and only if for some $\varphi \in S_k$,

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \qquad t = 1, \cdots, k ,$$

and H is 0 off the main diagonal in row and column ω_t, $t = 1, \cdots, k$.

We remark that if fewer than r of the $h_{\omega_t\omega_t}$ are positive then the left side of (2.20) is 0 and hence fewer than r of $\gamma_1, \cdots, \gamma_k$ are positive. If $r = k = n$ then (2.20) becomes

$$\prod_{j=1}^{n} h_{jj} \geqq \det H , \tag{2.21}$$

the Hadamard determinant theorem. If H is nonsingular and equality holds in (2.21) then Corollary 1 implies (since $h_{jj} > 0$, $j = 1, \cdots, n$) that $H = \operatorname{diag}(h_{11}, \cdots, h_{nn})$. If H is singular and equality holds in (2.21) then some $h_{jj} = 0$ and H has a zero row and column.

As another example consider the function

$$f(x) = E_r(x_1, \cdots, x_k)/E_{r-1}(x_1, \cdots, x_k)$$

for $x \in C_r$. We assert that f is strictly C_r-monotone, C-positive, and strictly C_r-concave. The C_r-positivity is obvious and the strict C_r-concavity is a result in [4]. To verify the strict C_r-monotonicity we show that for $x \in C_r$

$$\frac{\partial f}{\partial x_j} > 0 , \qquad j = 1, \cdots, k . \tag{2.22}$$

This will suffice since we are only interested in showing that $f(x + u) > f(x)$, $x \in C_r$, $u \geqq 0$, $u \neq 0$.

First observe that

$$E_r(x) = x_j E_{r-1}(\hat{x}_j) + E_r(\hat{x}_j) \tag{2.23}$$

where $E_r(\hat{x}_j)$ indicates the r^{th} elementary symmetric function of

$x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k$. Thus the sign of $\partial f/\partial x_j$ is the same as the sign of

$$E_{r-1}(x)E_{r-1}(\hat{x}_j) - E_r(x)E_{r-2}(\hat{x}_j) . \tag{2.24}$$

From (2.23) we see that (2.24) is equal to

$$\begin{aligned}(x_j E_{r-2}(\hat{x}_j) + E_{r-1}(\hat{x}_j))E_{r-1}(\hat{x}_j) - (x_j E_{r-1}(\hat{x}_j) + E_r(\hat{x}_j))E_{r-2}(\hat{x}_j) \\ = E^2_{r-1}(\hat{x}_j) - E_r(\hat{x}_j)E_{r-2}(\hat{x}_j) .\end{aligned}$$

Now it is known [2] that

$$E^2_{r-1}(\hat{x}_j) > E_r(\hat{x}_j)E_{r-2}(\hat{x}_j)$$

since at least $r-1$ of the components of $\hat{x}_j$ are positive. We can now state

COROLLARY 2. *Let H satisfy the hypotheses of Theorem* 1 *and assume that at least $r-1$ of $\gamma_1, \cdots, \gamma_k$ are positive. Then*

$$\frac{E_r(h_{\omega_1\omega_1}, \cdots, h_{\omega_k\omega_k})}{E_{r-1}(h_{\omega_1\omega_1}, \cdots, h_{\omega_k\omega_k})} \geqq \frac{E_r(\gamma_1, \cdots, \gamma_k)}{E_{r-1}(\gamma_1, \cdots, \gamma_k)} . \tag{2.25}$$

If at least r of $\gamma_1, \cdots, \gamma_k$ are positive then the inequality (2.25) *is equality if and only if for some $\varphi \in S_k$*

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \qquad t = 1, \cdots, k$$

and H is 0 *off the main diagonal in row and column ω_t, $t = 1, \cdots, k$.*

Proof. First observe that if p of $\gamma_1, \cdots, \gamma_k$ are positive then H has at least $n - k + p$ positive eigenvalues. Hence since H is positive semi-definite we know that at most $n - (n - k + p) = k - p$ of the main diagonal elements can be 0. We conclude that any set of k main diagonal elements must contain at least p positive elements. It follows that both sides of (2.25) are defined. Also, if $p = r$ we obtain the stated conditions for equality by applying Theorem 1.

We can derive an immediate consequence of Theorem 1 by replacing the matrix H by X^*HX where X is any n-square unitary matrix. The main diagonal entries of X^*HX are (Hx_j, x_j), $j = 1, \cdots, n$ where x_j is the jth column of X.

COROLLARY 3. *Let H and f be as in Theorem* 1. *Then for any set of k orthonormal vectors $x_1, \cdots, x_k$,*

$$f((Hx_1, x_1), \cdots, (Hx_k, x_k)) \geqq f(\gamma_1, \cdots, \gamma_k) . \tag{2.26}$$

If at least r of the inner products (Hx_j, x_j), $j = 1, \cdots, k$, are positive *then* (2.26) *is equality if and only if*

$$Hx_j = \gamma_{\varphi(j)} x_j \,, \qquad j = 1, \cdots, k \,, \tag{2.27}$$

for some $\varphi \in S_k$, i.e., $x_1, \cdots, x_k$ *are an orthonormal set of eigenvectors corresponding to* $\gamma_1, \cdots, \gamma_k$ *in some order.*

Proof. Let X be a unitary matrix whose first k columns are $x_1, \cdots, x_k$. The result (2.26) follows from Theorem 1 applied to X^*HX. If equality holds and if r of the inner products $(Hx_1, x_1), \cdots, (Hx_k, x_k)$ are positive then X^*HX is 0 off the main diagonal in row and column j, $j = 1, \cdots, k$, and $(X^*HX)_{jj} = \gamma_{\varphi(j)}$, $j = 1, \cdots, k$, for an appropriate $\varphi \in S_k$. This completes the proof.

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