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**AN ANALYSIS OF EQUALITY IN CERTAIN MATRIX
INEQUALITIES. I**

WILLIAM R. GORDON AND MARVIN DAVID MARCUS

AN ANALYSIS OF EQUALITY IN CERTAIN MATRIX INEQUALITIES, I

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In this paper we are concerned with analyzing the cases of equality in certain inequalities that relate the eigenvalues and main diagonal elements of hermitian matrices.

Let E_r denote the r^{th} elementary symmetric function of k variables ($E_0 = 1$). If $H = (h_{ij})$ is an n -square positive semi-definite hermitian matrix with eigenvalues $\gamma_1 \leq \dots \leq \gamma_n$ and if $1 \leq r \leq k \leq n$, then it is known that

$$(1.1) \quad E_r(h_{11}, \dots, h_{kk}) \geq E_r(\gamma_1, \dots, \gamma_k).$$

If $r > 1$ and at least r of h_{11}, \dots, h_{kk} are positive then (1.1) can be equality if and only if there exists a permutation $\varphi \in S_k$ such that

$$(1.2) \quad H = \text{diag}(\gamma_{\varphi(1)}, \dots, \gamma_{\varphi(k)}) \dot{+} H_{n-k}$$

where H_{n-k} is $(n-k)$ -square and $\dot{+}$ denotes direct sum. Of course, if $r = k = n$ then (1.1) is the Hadamard determinant theorem:

$$(1.3) \quad \prod_{i=1}^n h_{ii} \geq \det(H).$$

If some $h_{ii} = 0$, then H is singular and (1.3) is equality. If $h_{ii} > 0, i=1, \dots, n$, then the condition (1.2) yields the well-known criterion for equality in (1.3), namely $H = \text{diag}(h_{11}, \dots, h_{nn})$.

2. Results. Let $f(x) = f(x_1, \dots, x_k)$ be a function defined for all nonnegative vectors $x \geq 0$ (i.e., $x_i \geq 0, i = 1, \dots, k$). We shall assume that f is symmetric: $f(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = f(x)$ for all $\sigma \in S_k$, the symmetric group of degree k . Let C_r denote the cone consisting of all $x \geq 0$ with at least r positive components. The function f is said to be strictly C_r -concave if f is concave for $x \in C_r$ and if for x and y in C_r and $0 < \theta < 1$ the equality

$$(2.1) \quad f(\theta x + (1 - \theta)y) = \theta f(x) + (1 - \theta)f(y)$$

holds then it follows that $x \sim y$, i.e., x is a positive multiple of y . The usual definition of strict concavity requires that f be concave and that (2.1) holds if and only if $x = y$. We say that f is C_r -positive if: $f(x) > 0$ if and only if $x \in C_r$. Also, f is strictly C_r -monotone if $f(x + u) > f(x), x \in C_r, u \geq 0, u \neq 0$.

THEOREM 1. Let $H = (h_{ij})$ be an n -square positive semi-definite

hermitian matrix with eigenvalues $0 \leq \gamma_1 \leq \dots \leq \gamma_n$. Let $1 \leq r \leq k \leq n$. Assume that f is symmetric, concave and nondecreasing in each variable. Let $h_{\omega_t \omega_t}$, $t = 1, \dots, k$, be k main diagonal entries of H . Then

$$(2.2) \quad f(h_{\omega_1 \omega_1}, \dots, h_{\omega_k \omega_k}) \geq f(\gamma_1, \dots, \gamma_k) .$$

Assume in addition that f is strictly C_r -monotone, strictly C_r -concave and C_r -positive. If at least r of the $h_{\omega_t \omega_t}$, $t = 1, \dots, k$, are positive then equality holds in (2.2) if and only if for some $\varphi \in S_k$

$$(2.3) \quad h_{\omega_t \omega_t} = \gamma_{\varphi(t)} , \quad t = 1, \dots, k ,$$

and, in fact, in row and column ω_t , H is 0 off the main diagonal, $t = 1, \dots, k$.

The inequality (2.2) is found in [3].

Proof. To begin with we can assume that $\omega_t = t$, $t = 1, \dots, k$, and $h_{11} \leq \dots \leq h_{kk}$. For, we can rearrange the main diagonal entries with a permutation similarity without affecting the eigenvalues. A trivial induction shows that for f strictly C_r -concave, $a^t \in C_r$, and $\theta_t > 0$, $t = 1, \dots, m$, $\sum_{t=1}^m \theta_t = 1$, then

$$(2.4) \quad f\left(\sum_{t=1}^m \theta_t a^t\right) \geq \sum_{t=1}^m \theta_t f(a^t)$$

and equality implies that $a^s \sim a^t$, $s, t = 1, \dots, m$. Now there exists a unitary U such that $U^* \text{diag}(\gamma_1, \dots, \gamma_n)U = H$ and hence

$$(2.5) \quad h_{ii} = \sum_{j=1}^n |u_{ji}|^2 \gamma_j , \quad i = 1, \dots, n .$$

Since the matrix U is unitary we know that the matrix S whose (i, j) entry is $|u_{ji}|^2$, is doubly stochastic (d.s.). Thus (2.5) becomes

$$(2.6) \quad (h_{11}, \dots, h_{nn}) = S(\gamma_1, \dots, \gamma_n) .$$

Let $d = (h_{11}, \dots, h_{nn})$, $\gamma = (\gamma_1, \dots, \gamma_n)$, and for any n -tuple x let $x[k]$ denote the truncated vector (x_1, \dots, x_k) . If $\sigma \in S_n$ then $x^\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. By Birkhoff's theorem [1] let

$$(2.7) \quad S = \sum_{\sigma \in G} c_\sigma P_\sigma$$

where G is a subset of S_n , $c_\sigma > 0$, $\sigma \in G$, P_σ is an n -square permutation matrix corresponding to σ and $\sum_{\sigma \in G} c_\sigma = 1$. From (2.6), (2.7) and (2.4) we have

$$(2.8) \quad \begin{aligned} f(d[k]) &= f\left(\sum_{\sigma \in G} c_\sigma \gamma^\sigma[k]\right) \\ &\geq \sum_{\sigma \in G} c_\sigma f(\gamma^\sigma[k]) . \end{aligned}$$

Consider a summand in (2.8) and choose $\mu_\sigma \in S_k$ so that

$$\sigma(\mu_\sigma(1)) < \dots < \sigma(\mu_\sigma(k))$$

and hence

$$(2.9) \quad \gamma_{\sigma(\mu_\sigma(1))} \leq \dots \leq \gamma_{\sigma(\mu_\sigma(k))} .$$

The symmetry of f implies that

$$f(\gamma^\sigma[k]) = f(\gamma^{\sigma\mu\sigma}[k]) .$$

Now since $\sigma\mu_\sigma(t) \geq t, t = 1, \dots, k$, we know that

$$(2.10) \quad \gamma_{\sigma\mu\sigma}(t) \geq \gamma_t ,$$

$t = 1, \dots, k$. Then since f is nondecreasing in each variable we have

$$(2.11) \quad f(\gamma^\sigma[k]) \geq f(\gamma_1, \dots, \gamma_k)$$

and hence (2.8) becomes

$$(2.12) \quad f(d[k]) \geq f(\gamma_1, \dots, \gamma_k) ,$$

the required inequality (2.2).

Suppose equality holds in (2.12). Since $d[k] \in C_r$ we know that $f(d[k]) > 0$ and hence $f(\gamma[k]) > 0$. Thus $\gamma[k] \in C_r$. We also know that $f(\gamma^{\sigma\mu\sigma}[k]) = f(\gamma[k])$ and in view of (2.10) it follows that

$$(2.13) \quad \gamma^{\sigma\mu\sigma}[k] = \gamma[k] .$$

Setting $\mu_\sigma^{-1} = \nu_\sigma \in S_k$ in (2.13) we have

$$(2.14) \quad \gamma^\sigma[k] = (\gamma[k])^{\nu_\sigma} .$$

We must also have equality in (2.8) which because of the strict C_r -concavity implies that $\gamma^\sigma[k] \sim \gamma^\theta[k], \sigma, \theta$ in G . In other words,

$$\gamma^\sigma[k] = a_\sigma \gamma^\tau[k]$$

for some fixed $\tau \in G, a_\sigma > 0$ all $\sigma \in G$. In view of (2.14)

$$\gamma^\sigma[k] = a_\sigma (\gamma[k])^{\nu_\sigma}$$

so that

$$\begin{aligned} d[k] &= \sum_{\sigma \in G} c_\sigma \gamma^\sigma[k] \\ &= \sum_{\sigma \in G} c_\sigma a_\sigma (\gamma[k])^{\nu_\sigma} \\ &= c (\gamma[k])^{\nu_\tau}, c > 0 . \end{aligned}$$

The equality in (2.12) implies that

$$\begin{aligned} f(d[k]) &= f(\gamma[k]) \\ &= f(\gamma[k])^{\nu_\tau} \end{aligned}$$

and thus

$$f(c(\gamma[k])^{\nu_\tau}) = f((\gamma[k])^{\nu_\tau})$$

or

$$f(c\gamma[k]) = f(\gamma[k]) .$$

Now $\gamma[k] \in C_r$ and hence by (2.1) $c = 1$. Thus

$$(2.15) \quad d[k] = (\gamma[k])^{\nu_\tau} .$$

Since $h_{11} \leq \dots \leq h_{kk}$, (2.15) implies that

$$\gamma_{\nu_\tau(1)} \leq \dots \leq \gamma_{\nu_\tau(k)} .$$

But $\gamma_1 \leq \dots \leq \gamma_k$ and $\nu_\tau \in S_k$ and hence $\gamma_{\nu_\tau(t)} = \gamma_t$, $t = 1, \dots, k$. In other words,

$$(2.16) \quad h_{ii} = \gamma_i , \quad i = 1, \dots, k .$$

Now we assert that (2.16) implies that the first k rows and columns of H are 0 off the main diagonal. To see this we observe that if $e_1 = (\delta_{11}, \dots, \delta_{n1})$ and u_1, \dots, u_n are orthonormal eigenvectors of H corresponding to $\gamma_1, \dots, \gamma_n$ respectively, then using the standard inner product in the vector space of complex n -tuples,

$$(2.17) \quad \begin{aligned} h_{11} &= (He_1, e_1) \\ &= \sum_{j=1}^n |(e_1, u_j)|^2 \gamma_j . \end{aligned}$$

Since $\gamma_1 = h_{11}$ we conclude from (2.17) that $(e_1, u_j) = 0$, if $\gamma_j > \gamma_1$. Suppose $\gamma_1 = \dots = \gamma_r < \gamma_{r+1} \leq \dots \leq \gamma_n$. Then $(e_1, u_j) = 0$, $j = r+1, \dots, n$, and hence $e_1 \in \langle u_1, \dots, u_r \rangle$, the space spanned by u_1, \dots, u_r . But then $He_1 = \gamma_1 e_1$ and we conclude that the first column (and row) of H is 0 off the main diagonal. Since $\gamma_2, \dots, \gamma_n$ are the eigenvalues of the submatrix obtained from H by deleting row and column 1, an obvious induction completes the proof.

Make the following choice for f :

$$(2.18) \quad f(x_1, \dots, x_k) = E_r^{1/r}(x_1^q, \dots, x_k^q)$$

where $0 < q \leq 1$. We assert that for $r > 1$ or $r = 1, q < 1$, f is strictly C_r -concave. For $0 < \theta < 1$ consider

$$(2.19) \quad \begin{aligned} f(\theta x + (1 - \theta)y) &= E_r^{1/r}((\theta x_1 + (1 - \theta)y_1)^q, \dots, (\theta x_k + (1 - \theta)y_k)^q) \\ &\geq E_r^{1/r}(\theta x_1^q + (1 - \theta)y_1^q, \dots, \theta x_k^q + (1 - \theta)y_k^q) \\ &\geq \theta E_r^{1/r}(x_1^q, \dots, x_k^q) + (1 - \theta)E_r^{1/r}(y_1^q, \dots, y_k^q) \\ &= \theta f(x) + (1 - \theta)f(y) . \end{aligned}$$

In (2.19) we have used the monotonicity and C_r -concavity of $E_r^{1/r}$ [4], $r > 1$, and the strict concavity of $t^q, t \geq 0$, for $r = 1$. When $q < 1$ the first inequality in (2.19) is strict unless $x = y$. If $q = 1, r > 1$, then the second inequality is strict unless $x \sim y$. In either event if (2.19) is equality then $x \sim y$ so that f is indeed strictly C_r -concave. Also, f is obviously strictly C_r -monotone and C_r -positive. We have

COROLLARY 1. *Let H satisfy the hypotheses of Theorem 1 and let $0 < q \leq 1$. Then*

$$(2.20) \quad E_r(h_{\omega_1 \omega_1}^q, \dots, h_{\omega_k \omega_k}^q) \geq E_r(\gamma_1^q, \dots, \gamma_k^q).$$

If at least r of the $h_{\omega_t \omega_t}$ are positive, $t = 1, \dots, k$, then equality holds in (2.20) if and only if for some $\varphi \in S_k$,

$$h_{\omega_t \omega_t} = \gamma_{\varphi(t)}, \quad t = 1, \dots, k,$$

and H is 0 off the main diagonal in row and column $\omega_t, t = 1, \dots, k$.

We remark that if fewer than r of the $h_{\omega_t \omega_t}$ are positive then the left side of (2.20) is 0 and hence fewer than r of $\gamma_1, \dots, \gamma_k$ are positive. If $r = k = n$ then (2.20) becomes

$$(2.21) \quad \prod_{j=1}^n h_{jj} \geq \det H,$$

the Hadamard determinant theorem. If H is nonsingular and equality holds in (2.21) then Corollary 1 implies (since $h_{jj} > 0, j = 1, \dots, n$) that $H = \text{diag}(h_{11}, \dots, h_{nn})$. If H is singular and equality holds in (2.21) then some $h_{jj} = 0$ and H has a zero row and column.

As another example consider the function

$$f(x) = E_r(x_1, \dots, x_k) / E_{r-1}(x_1, \dots, x_k)$$

for $x \in C_r$. We assert that f is strictly C_r -monotone, C -positive, and strictly C_r -concave. The C_r -positivity is obvious and the strict C_r -concavity is a result in [4]. To verify the strict C_r -monotonicity we show that for $x \in C_r$

$$(2.22) \quad \frac{\partial f}{\partial x_j} > 0, \quad j = 1, \dots, k.$$

This will suffice since we are only interested in showing that $f(x + u) > f(x), x \in C_r, u \geq 0, u \neq 0$.

First observe that

$$(2.23) \quad E_r(x) = x_j E_{r-1}(\hat{x}_j) + E_r(\hat{x}_j)$$

where $E_r(\hat{x}_j)$ indicates the r^{th} elementary symmetric function of

$x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k$. Thus the sign of $\partial f/\partial x_j$ is the same as the sign of

$$(2.24) \quad E_{r-1}(x)E_{r-1}(\hat{x}_j) - E_r(x)E_{r-2}(\hat{x}_j) .$$

From (2.23) we see that (2.24) is equal to

$$\begin{aligned} & (x_j E_{r-2}(\hat{x}_j) + E_{r-1}(\hat{x}_j))E_{r-1}(\hat{x}_j) - (x_j E_{r-1}(\hat{x}_j) + E_r(\hat{x}_j))E_{r-2}(\hat{x}_j) \\ & = E_{r-1}^2(\hat{x}_j) - E_r(\hat{x}_j)E_{r-2}(\hat{x}_j) . \end{aligned}$$

Now it is known [2] that

$$E_{r-1}^2(\hat{x}_j) > E_r(\hat{x}_j)E_{r-2}(\hat{x}_j)$$

since at least $r - 1$ of the components of \hat{x}_j are positive. We can now state

COROLLARY 2. *Let H satisfy the hypotheses of Theorem 1 and assume that at least $r - 1$ of $\gamma_1, \dots, \gamma_k$ are positive. Then*

$$(2.25) \quad \frac{E_r(h_{\omega_1\omega_1}, \dots, h_{\omega_k\omega_k})}{E_{r-1}(h_{\omega_1\omega_1}, \dots, h_{\omega_k\omega_k})} \geq \frac{E_r(\gamma_1, \dots, \gamma_k)}{E_{r-1}(\gamma_1, \dots, \gamma_k)} .$$

If at least r of $\gamma_1, \dots, \gamma_k$ are positive then the inequality (2.25) is equality if and only if for some $\varphi \in S_k$

$$h_{\omega_t\omega_t} = \gamma_{\varphi(t)} , \quad t = 1, \dots, k$$

and H is 0 off the main diagonal in row and column $\omega_t, t = 1, \dots, k$.

Proof. First observe that if p of $\gamma_1, \dots, \gamma_k$ are positive then H has at least $n - k + p$ positive eigenvalues. Hence since H is positive semi-definite we know that at most $n - (n - k + p) = k - p$ of the main diagonal elements can be 0. We conclude that any set of k main diagonal elements must contain at least p positive elements. It follows that both sides of (2.25) are defined. Also, if $p = r$ we obtain the stated conditions for equality by applying Theorem 1.

We can derive an immediate consequence of Theorem 1 by replacing the matrix H by X^*HX where X is any n -square unitary matrix. The main diagonal entries of X^*HX are $(Hx_j, x_j), j = 1, \dots, n$ where x_j is the j th column of X .

COROLLARY 3. *Let H and f be as in Theorem 1. Then for any set of k orthonormal vectors $x_1, \dots, x_k,$*

$$(2.26) \quad f((Hx_1, x_1), \dots, (Hx_k, x_k)) \geq f(\gamma_1, \dots, \gamma_k) .$$

If at least r of the inner products $(Hx_j, x_j), j = 1, \dots, k,$ are positive

then (2.26) is equality if and only if

$$(2.27) \quad Hx_j = \gamma_{\varphi(j)}x_j, \quad j = 1, \dots, k,$$

for some $\varphi \in S_k$, i.e., x_1, \dots, x_k are an orthonormal set of eigenvectors corresponding to $\gamma_1, \dots, \gamma_k$ in some order.

Proof. Let X be a unitary matrix whose first k columns are x_1, \dots, x_k . The result (2.26) follows from Theorem 1 applied to X^*HX . If equality holds and if r of the inner products $(Hx_1, x_1), \dots, (Hx_k, x_k)$ are positive then X^*HX is 0 off the main diagonal in row and column j , $j = 1, \dots, k$, and $(X^*HX)_{jj} = \gamma_{\varphi(j)}$, $j = 1, \dots, k$, for an appropriate $\varphi \in S_k$. This completes the proof.

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