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(HAROLD) DAVID KAHN

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A topological semigroup is a Hausdorff space S together with a continuous associative multiplication $m: S \times S \to S$. The lifting of the group structure of a topological group to its simply connected covering space is a technique used in the theory of Lie groups. In this paper we investigate the lifting of the multiplication of a topological semigroup S to its simply connected covering space $(\bar{S}, \varphi)$. A general theory is developed and applications to examples are discussed.

1. Covering spaces. Let $\bar{S}$ and S be locally connected topological spaces and $\varphi: \bar{S} \to S$ a continuous map. If C is a subset of S, then C is *evenly covered* if $\varphi \mid \bar{C}: \bar{C} \to C$ is a homeomorphism for each component $\bar{C}$ of $\varphi^{-1}(C)$. If each point in S has an evenly covered open neighborhood, then φ is called a *covering map*. If φ is a covering map and $\bar{S}$ is connected, then $(\bar{S}, \varphi)$ is called a *covering space* of S. A covering space is called *trivial* if the covering map is a homeomorphism, and if S admits only trivial covering spaces, then S is called *simply connected*. If $(\bar{S}_1, \varphi_1)$ and $(\bar{S}_2, \varphi_2)$ are simply connected covering spaces of S and $\psi: \bar{S}_1 \to \bar{S}_2$ is a homeomorphism such that $\varphi_2 \circ \psi = \varphi_1$, then ψ is called a *covering space isomorphism*. An *automorphism* of $(\bar{S}, \varphi)$ is an isomorphism of $(\bar{S}, \varphi)$ with itself.

LEMMA 1. *Let $(\bar{S}, \varphi)$ be a covering space of S and T a connected space. If $\alpha, \beta: T \to \bar{S}$ are continuous maps with $\varphi \circ \alpha = \varphi \circ \beta$, then α and β agree everywhere or nowhere.*

LEMMA 2. *Let P be a topological space. Then P is simply connected if and only if* (a) *P is connected and locally connected and* (b) *if $\varphi: \bar{S} \to S$ is a covering map, $\psi: P \to S$ is continuous, p is in P, s is in $\bar{S}$ with $\psi(p) = \varphi(s)$, then there exists unique continuous $\bar{\psi}: P \to \bar{S}$ such that $\psi = \varphi \circ \bar{\psi}$ and $\bar{\psi}(p) = s$.*

LEMMA 3. *Let (P, ψ) and $(\bar{S}, \varphi)$ be covering spaces of S with p in P and s in $\bar{S}$ with $\psi(p) = \varphi(s)$. If P is simply connected and $\bar{\psi}: P \to \bar{S}$ is the unique lifting of ψ with $\bar{\psi}(p) = s$, then $\bar{\psi}$ is a covering map.*

LEMMA 4. *If $(\bar{S}_1, \varphi_1)$ and $(\bar{S}_2, \varphi_2)$ are simply connected covering spaces of S and s_i is in $\bar{S}_i$, $i = 1, 2$ with $\varphi_1(s_1) = \varphi_2(s_2)$, then there exists a unique covering space isomorphism $\psi: \bar{S}_1 \to \bar{S}_2$ such that $\psi(s_1) = s_2$.*

LEMMA 5. *Let* $(\bar{S}, \varphi)$ *be a simply connected covering space of* S. *We define the set of all automorphisms of* $(\bar{S}, \varphi)$ *to be the Poincare group or fundamental group of* S *and denote it by* $P(S)$. *The orbits of* $P(S)$ *are the discrete subspaces* $\varphi^{-1}(x)$, x *in* S, *and* $P(S)$ *is simply transitive on these orbits, i.e., a given point can be mapped into a given point in the same orbit by precisely one automorphism in* $P(S)$.

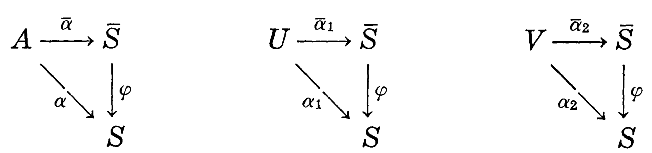
LEMMA 6. $(\bar{S}, \varphi)$ *be a covering space of* S. *If* A *is a connected, locally connected subspace of* S *and* $\bar{A}$ *is a component of* $\varphi^{-1}(A)$, *then* $(\bar{A}, \varphi \mid \bar{A})$ *is a covering space of* A.

LEMMA 7. *If* S *and* T *are topological spaces admitting simply connected covering spaces* $(\bar{S}, \varphi_1)$ *and* $(\bar{T}, \varphi_2)$, *then* $S \times T$ *admits the simply connected covering space* $(\bar{S} \times \bar{T}, \varphi_1 \times \varphi_2)$ *and* $P(S \times T) \cong P(S) \times P(T)$. *It follows that the product of two topological spaces is simply connected if and only if both are.*

The proofs of the above lemmas can be found in either Chevalley [2], Hochschild [4], Hofmann [5], or Pontrjagin [10]. Theorem 8 seems to be of a van Kampen type.

THEOREM 8. *Let* U, V *be simply connected subsets of a space* A. *If* $U \backslash V$ *and* $V \backslash U$ *are separated and if* $U \cap V$ *is nonvoid and connected, then* $U \cup V$ *is simply connected.*

Proof. We may assume $A = U \cup V$. Then A is trivially connected and is locally connected by a proof identical to the first paragraph of Lemma 1.3 on page 45 of Hochschild [4]. Now let $\varphi: \bar{S} \to S$ be a covering map, α a continuous map of A into S, a_0 a point of A, s_0 a point of $\bar{S}$ with $\alpha(a_0) = \varphi(s_0)$. We may assume a_0 is in U. Define $\alpha_1 = \alpha \mid U: U \to S$. Since U is simply connected and

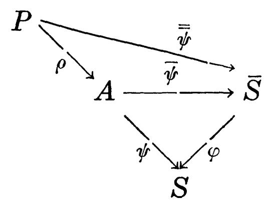


$\alpha_1(a_0) = \alpha(a_0) = \varphi(s_0)$, there is continuous $\bar{\alpha}_1: U \to \bar{S}$ with $\varphi \circ \bar{\alpha}_1 = \alpha_1$ and $\bar{\alpha}_1(a_0) = s_0$. Fix b_0 in $U \cap V$ and define $y_0 = \bar{\alpha}_1(b_0)$ in $\bar{S}$. Then $\varphi(y_0) = \varphi \circ \bar{\alpha}_1(b_0) = \alpha_1(b_0) = \alpha_2(b_0)$, where $\alpha_2 = \alpha \mid V: V \to S$. Since V is simply connected, there is continuous $\bar{\alpha}_2: V \to \bar{S}$ with $\varphi \circ \bar{\alpha}_2 = \alpha_2$ and $\bar{\alpha}_2(b_0) = y_0$. We now define the maps $\beta_i = \bar{\alpha}_i \mid U \cap V: U \cap V \to \bar{S}$, $i = 1, 2$. We note that $\varphi \circ \beta_1 = \varphi \circ (\bar{\alpha}_1 \mid U \cap V) = (\varphi \circ \bar{\alpha}_1) \mid U \cap V =$

$\alpha_1 \mid U \cap V = \alpha_2 \mid U \cap V = (\varphi \circ \bar{\alpha}_2) \mid U \cap V = \varphi \circ (\bar{\alpha}_2 \mid U \cap V) = \varphi \circ \beta_2$ and that $\beta_1(b_0) = \bar{\alpha}_1(b_0) = y_0 = \bar{\alpha}_2(b_0) = \beta_2(b_0)$. Since $U \cap V$ is connected, we have $\bar{\alpha}_1 \mid U \cap V = \beta_1 = \beta_2 = \bar{\alpha}_2 \mid U \cap V$. We can now define $\bar{\alpha}: A \to \bar{S}$ with $\bar{\alpha}(a) = \bar{\alpha}_1(a)$, when a is in U, and $= \bar{\alpha}_2(a)$, when a is in V. The continuity of $\bar{\alpha}$ follows by Exercise 3B of Kelley [7], and it is clear that $\varphi \circ \bar{\alpha} = \alpha$ and that $\bar{\alpha}(a_0) = s_0$. Finally, the uniqueness of $\bar{\alpha}$ follows again by the connectedness of $U \cap V$.

LEMMA 9. *If P is a simply connected topological space and A is a retract of P, then A is simply connected.*

Proof. It is clear that A is connected and locally connected. Let $\varphi: \bar{S} \to S$ be a covering map, $\psi: A \to S$ be continuous, a in A and s in $\bar{S}$ with $\psi(a) = \varphi(s)$. Moreover, let $\rho: P \to A$ be the retraction map. Then $\psi \circ \rho: P \to S$ is continuous and $\psi \circ \rho(a) = \psi(a) = \varphi(s)$.



Since P is simply connected, there is continuous $\bar{\bar{\psi}}: P \to \bar{S}$ with $\psi \circ \rho = \varphi \circ \bar{\bar{\psi}}$ and $\bar{\bar{\psi}}(a) = s$. It is now straightforward to show that if $\bar{\psi} = \bar{\bar{\psi}} \mid A$, then $\varphi \circ \bar{\psi} = \psi$ and $\bar{\psi}(a) = s$. Uniqueness of $\bar{\psi}$ follows from the connectedness of A.

LEMMA 10. *Let $(\bar{S}, \varphi)$ be a simply connected covering space of S and A a retract of S. If $\bar{A}$ is a component of $\varphi^{-1}(A)$, then $\bar{A}$ is a retract of $\bar{S}$ and $(\bar{A}, \varphi \mid \bar{A})$ is a simply connected covering space of A.*

Proof. Let $\rho: S \to S$ be the retract and $\bar{a}$ be in $\bar{A}$. Since $\varphi(\bar{a})$ is in A, we have $\rho \circ \varphi(\bar{a}) = \varphi(\bar{a})$ and ρ lifts to continuous $\bar{\rho}: \bar{S} \to \bar{S}$ with $\bar{\rho}(\bar{a}) = \bar{a}$ and $\varphi \circ \bar{\rho} = \rho \circ \varphi$. Now let $j: \bar{A} \subseteq \bar{S}$ and $\bar{\rho} \mid \bar{A}: \bar{A} \to \bar{S}$. Then it is straightforward to show that $\varphi \circ (\bar{\rho} \mid \bar{A}) = \varphi \circ j$ and that $(\bar{\rho} \mid \bar{A})(\bar{a}) = j(\bar{a})$, which implies that $\bar{\rho} \mid \bar{A} = j$. Since $\varphi(\bar{\rho}(\bar{S})) = \rho(\varphi(\bar{S})) = \rho(S) = A$, we have $\rho(\bar{S})$ a connected subset of $\varphi^{-1}(A)$. Observing that $\bar{a}$ is in $\bar{A} \cap \bar{\rho}(\bar{S})$, we have $\bar{\rho}(\bar{S}) \subseteq \bar{A}$. Therefore, $\bar{\rho}$ is a retraction of $\bar{S}$ onto $\bar{A}$. Moreover, $(\bar{A}, \varphi \mid \bar{A})$ is a simply connected covering space of A by Lemmas 6 and 9 of this section.

LEMMA 11. *If the topological product of two spaces admits a simply connected covering space, then so do both of them.*

Proof. Let (P, φ) be a simply connected covering space of $S \times T$. If t is in T and $\bar{S}$ is a component of $\varphi^{-1}(S \times t)$, then $(\bar{S}, \theta \circ (\varphi \mid \bar{S}))$ is a simply connected covering space of S, where $\theta: S \times t \to S$ is the natural homeomorphism. Indeed, $S \times t$ is obviously a retract of $S \times T$, and we apply Lemma 10.

LEMMA 12. *Let $(\bar{S}, \varphi)$ be a simply connected covering space of S, A a connected, locally connected subset of S, and $\bar{A}$ a component of $\varphi^{-1}(A)$. If $\bar{A}$ is simply connected, and we let $P(S)$ and $P(A)$ be the automorphism groups of $(\bar{S}, \varphi)$ and $(\bar{A}, \varphi \mid \bar{A})$, respectively, then there exists a monomorphism $\theta: P(A) \to P(S)$ such that if ψ is in $P(A)$, then $\theta(\psi) = \bar{\psi}$ is the unique extension of ψ to $\bar{\psi}$ in $P(S)$. Moreover, θ is an isomorphism if and only if $\varphi^{-1}(A)$ is connected, i.e., if and only if $\bar{A} = \varphi^{-1}(A)$.*

Proof. Suppose ψ is in $P(A)$. Fix a_1 in $\bar{A}$. Let $\psi(a_1) = a_2$ in $\bar{A}$. Now, $\varphi(a_1) = (\varphi \mid \bar{A})(a_1) = (\varphi \mid \bar{A}) \circ \psi(a_1) = (\varphi \mid \bar{A})(a_2) = \varphi(a_2)$. Thus, there exists unique $\bar{\psi}$ in $P(S)$ such that $\bar{\psi}(a_1) = a_2$.

We show that $\bar{\psi}$ is an extension of ψ. We first show that $\bar{\psi}(\bar{A}) = \bar{A}$. Clearly, $\bar{\psi}(\varphi^{-1}(A)) = \varphi^{-1}(A)$. We see that $\bar{\psi}(\bar{A})$ is a connected subset of $\varphi^{-1}(A)$ with a_2 in $\bar{A} \cap \bar{\psi}(\bar{A})$. Therefore, $\bar{\psi}(\bar{A}) \subseteqq \bar{A}$. Let η be the inverse of ψ in $P(A)$. As before, we find $\bar{\eta}$ in $P(S)$ such that $\bar{\eta}(a_2) = a_1$ and $\bar{\eta}(\bar{A}) \subseteqq \bar{A}$. Now, $\bar{\psi} \circ \bar{\eta}$ is in $P(S)$ and fixes a_2. Thus, $\bar{\psi} \circ \bar{\eta}$ is the identity of $P(S)$, and $\bar{A} = \bar{\psi} \circ \bar{\eta}(\bar{A}) \subseteqq \bar{\psi}(\bar{A}) \subseteqq \bar{A}$. Therefore, $\bar{\psi}(\bar{A}) = \bar{A}$. Since $\bar{\psi}: \bar{S} \to \bar{S}$ is a homeomorphism, so is $\bar{\psi} \mid \bar{A}: \bar{A} \to \bar{A}$. Moreover, $(\varphi \mid \bar{A}) \circ (\bar{\psi} \mid \bar{A})(a) = \varphi \circ \bar{\psi}(a) = \varphi(a) = (\varphi \mid \bar{A})(a)$, for all a in $\bar{A}$. So, $\bar{\psi} \mid \bar{A}$ is in $P(A)$. But ψ is in $P(A)$, and $\psi(a_1) = a_2 = (\bar{\psi} \mid \bar{A})(a_1)$. Thus we have $\psi = \bar{\psi} \mid \bar{A}$, as described.

Now that we have θ a well-defined function, we observe that it is trivially injective. A simple computational argument shows that θ is a homomorphism.

We next show that $\bar{A} = \varphi^{-1}(A)$ if and only if θ is surjective. Suppose $\bar{A} = \varphi^{-1}(A)$. Let ψ be in $P(S)$. Then $\psi(\bar{A}) = \psi(\varphi^{-1}(A)) = \varphi^{-1}(A) = \bar{A}$. As above, we see that $\psi \mid \bar{A}$ is in $P(A)$. Moreover, $\theta(\psi \mid \bar{A}) = \psi$. Therefore, θ is surjective. Conversely, suppose θ is surjective. Let $\bar{a}_1$ be in $\varphi^{-1}(A)$. Let $\varphi(\bar{a}_1) = a$ in A. There exists $\bar{a}_2$ in $\bar{A}$ such that $\varphi(\bar{a}_2) = a = \varphi(\bar{a}_1)$. Thus, there is $\bar{\psi}$ in $P(S)$ with $\bar{\psi}(\bar{a}_2) = \bar{a}_1$. Since θ is onto, there is ψ in $P(A)$ with $\theta(\psi) = \bar{\psi}$, i.e., $\psi = \bar{\psi} \mid \bar{A}$. Then $\bar{a}_1 = \bar{\psi}(\bar{a}_2) = \psi(\bar{a}_2)$ in $\bar{A}$. Since $\bar{a}_1$ was arbitrary in $\varphi^{-1}(A)$, we have $\varphi^{-1}(A) \subseteqq \bar{A}$, and they are equal.

2. General theory of covering semigroups. Let $\bar{S}$ and S be topological semigroups and $\varphi: \bar{S} \to S$ a homomorphism. If, moreover, $(\bar{S}, \varphi)$ is a covering space of S, then we say that $(\bar{S}, \varphi)$ is a *covering* *semigroup* of S. The proofs of the first two of the following theorems are omitted, as they are similar to the development of covering groups. See [2], [4], [5].

THEOREM 1. *Let S be a topological semigroup with topological space structure admitting a simply connected covering space $(\bar{S}, \varphi)$. Let e be an idempotent in S and fix some point $\bar{e}$ in $\bar{S}$ such that $\varphi(\bar{e}) = e$. There exists a unique topological semigroup multiplication on $\bar{S}$ such that $\bar{e}$ is an idempotent and φ is a homomorphism. If e is an identity for S, then $\bar{e}$ is an identity for $\bar{S}$. If S is a topological group, then so is $\bar{S}$.*

THEOREM 2. *Let $(\bar{S}_1, \varphi_1)$ and $(\bar{S}_2, \varphi_2)$ be covering semigroups of S with idempotents $\bar{e}_1$ in $\bar{S}_1$ and $\bar{e}_2$ in $\bar{S}_2$ such that $\varphi_1(\bar{e}_1) = \varphi_2(\bar{e}_2)$. If $\bar{S}_1$ is simply connected, then there exists a unique homomorphism and covering map $\psi: \bar{S}_1 \to \bar{S}_2$ with $\varphi_2 \circ \psi = \varphi_1$ and $\psi(\bar{e}_1) = \bar{e}_2$. Moreover, if $\bar{S}_2$ is also simply connected, then ψ is a covering space and semigroup isomorphism.*

THEOREM 3. *Let $[X, G, Y]_\sigma$ be a topological paragroup (Hofmann and Mostert [6]) where $X(Y)$ is a left (right) zero semigroup and G is a group. If X, G, and Y admit simply connected covering spaces $(\bar{X}, \varphi_1)$, $(\bar{G}, \varphi_2)$ and $(\bar{Y}, \varphi_3)$, then the left (right) zero multiplication of $X(Y)$ lifts to a left (right) zero multiplication on $\bar{X}(\bar{Y})$ and the group multiplication of G lifts to a group multiplication on $\bar{G}$. Moreover, the sandwich function $\sigma: Y \times X \to G$ lifts to a sandwich function $\bar{\sigma}: \bar{Y} \times \bar{X} \to \bar{G}$ such that $([\bar{X}, \bar{G}, \bar{Y}]_{\bar{\sigma}}, \varphi_1 \times \varphi_2 \times \varphi_3)$ is a simply connected covering paragroup of $[X, G, Y]_\sigma$.*

Proof. Note that $\varphi_1(\varphi_3)$ is automatically a homomorphism if we give $\bar{X}(\bar{Y})$ the left (right) zero multiplication. Any lifting of σ to $\bar{\sigma}$ allows us to form the paragraph $[\bar{X}, \bar{G}, \bar{Y}]_{\bar{\sigma}}$. A straightforward computation, making use of the equation $\sigma \circ (\varphi_3 \times \varphi_1) = \varphi_2 \circ \bar{\sigma}$, shows that $\varphi_1 \times \varphi_2 \times \varphi_3: [\bar{X}, \bar{G}, \bar{Y}]_{\bar{\sigma}} \to [X, G, Y]_\sigma$ is a homomorphism. We omit further details.

THEOREM 4. *If $(\bar{S}, \varphi)$ is a covering semigroup of S, then φ^{-1} (center S) = center $\bar{S}$.*

Proof. Clearly, center $\bar{S} \subseteqq \varphi^{-1}$ (center S). Let s be any element of φ^{-1} (center S). Define $\alpha, \beta: \bar{S} \to \bar{S}$ with $\alpha(x) = sx$ and $\beta(x) = xs$. Straightforward computations show that $\varphi \circ \alpha = \varphi \circ \beta$ and that $\alpha(s) = \beta(s)$. Thus, $\alpha = \beta$, i.e., s is in center $\bar{S}$.

For the rest of this section we assume that $(\bar{S}, \varphi)$ is a simply connected covering semigroup of S. Moreover, $\bar{S}$ and S have identities $\bar{1}$ and 1, respectively. We define Ker φ to be $\varphi^{-1}(1)$. Although this is not standard semigroup terminology, we feel that Theorem 6 of this section is ample motivation.

COROLLARY 5. *Ker φ is central.*

Proof. Note that 1 is central.

THEOREM 6. *If s is in* Ker φ *and we define $\psi: \bar{S} \to \bar{S}$ by $\psi(x) = sx$, then ψ is in $P(S)$. This defines an isomorphism between* Ker φ *and $P(S)$. Therefore, $P(S)$ is commutative.*

Proof. Let s be in Ker φ and define ψ as above. There exists η in $P(S)$ with $\eta(\bar{1}) = s$. Straightforward computation shows that $\varphi \circ \psi = \varphi \circ \eta$ and $\psi(\bar{1}) = \eta(\bar{1})$. So, $\psi = \eta$, and ψ is in $P(S)$. Since $\bar{S}$ has an identity, we conclude that mapping s into ψ gives a monomorphism of Ker φ into $P(S)$. We show that the mapping is onto. Let ψ be in $P(S)$. Define $s = \psi(\bar{1})$. Then s is in Ker φ, and we define $\eta = \theta(s)$ in $P(S)$. But then ψ and η agree at $\bar{1}$ and, therefore, are equal.

COROLLARY 7. *If a and b are in $\bar{S}$ with $\varphi(a) = \varphi(b)$, then there exists unique s in* Ker φ *with $sa = b$.*

Material from here through Corollary 18 is independent and completely algebraic in nature, providing we define $(\bar{S}, \varphi)$ to be an *algebraic covering of S with group $P(S)$* if:

(a) $\bar{S}$ and S are purely algebraic semigroups with identities $\bar{1}$ and 1, respectively.

(b) The map $\varphi: \bar{S} \to S$ is a surmorphism with Ker $\varphi = \varphi^{-1}(1)$ being a central subgroup of $\bar{S}$.

(c) Ker φ acts on $\bar{S}$ with orbits $\varphi^{-1}(x)$, x in S, and is simply transitive on these orbits.

(d) $P(S)$ is a faithful functional representation of Ker φ on $\bar{S}$.

LEMMA 8. *If x is in S, $\bar{x}$ is in $\varphi^{-1}(x)$, and A, B are subsets of S, then $\varphi^{-1}(AxB) = \varphi^{-1}(A)\bar{x}\varphi^{-1}(B)$. Also $\varphi^{-1}(Ax) = \varphi^{-1}(A)\bar{x}$, $\varphi^{-1}(xB) = \bar{x}\varphi^{-1}(B)$, and $\varphi^{-1}(AB) = \varphi^{-1}(A)\varphi^{-1}(B)$.*

Proof. It is trivial that $\varphi^{-1}(A)\bar{x}\varphi^{-1}(B) \subseteqq \varphi^{-1}(AxB)$. Conversely, let y be in $\varphi^{-1}(AxB)$. There exists a in A, b in B with $\varphi(y) = axb$. If we pick $\bar{a}$, $\bar{b}$, in $\bar{S}$ with $\varphi(\bar{a}) = a$ and $\varphi(\bar{b}) = b$, then $\varphi(\bar{a}\bar{x}\bar{b}) = axb = \varphi(y)$. Thus, there exists s in Ker φ with $s(\bar{a}\bar{x}\bar{b}) = y$. Observing that $s\bar{a}$ is in $\varphi^{-1}(A)$, we have $y = (s\bar{a})\bar{x}\bar{b}$ in $\varphi^{-1}(A)\bar{x}\varphi^{-1}(B)$, as desired.

The remaining equations follow easily from the equation $\varphi^{-1}(AxB) = \varphi^{-1}(A)\bar{x}\varphi^{-1}(B)$. Indeed, if $\bar{x} = \bar{1}$ and $x = 1$, we have $\varphi^{-1}(AB) = \varphi^{-1}(A)\varphi^{-1}(B)$, and if B or A is $\{1\}$, then the remaining equations result.

THEOREM 9. *If H is a subgroup of S, then $\varphi^{-1}(H)$ is a subgroup of $\bar{S}$. In particular, if e is an idempotent in S, then $\varphi^{-1}(e)$ is subgroup of $\bar{S}$. Moreover, if θ: $\operatorname{Ker}\varphi \to \varphi^{-1}(e)$ by $\theta(s) = s\bar{e}$, where $\bar{e}$ is the identity of $\varphi^{-1}(e)$, then θ is an isomorphism. Thus, $\varphi^{-1}(e) \cong P(S)$. Note that it follows that $\varphi^{-1}(H)$ is an extension of $P(S)$ by H, in the sense of Kurosh* [8], p. 76.

Proof. Let $\bar{x}$ be in $\varphi^{-1}(H)$, $\varphi(\bar{x}) = x$ in H. Then $\bar{x}\varphi^{-1}(H) = \varphi^{-1}(xH) = \varphi^{-1}(H)$ and $\varphi^{-1}(H)\bar{x} = \varphi^{-1}(Hx) = \varphi^{-1}(H)$. Therefore, $\varphi^{-1}(H)$ is a group.

We show θ is an isomorphism. Since $\bar{e}$ is idempotent and $\operatorname{Ker}\varphi$ is central, $\theta(st) = (st)\bar{e} = (s\bar{e})(t\bar{e}) = \theta(s)\theta(t)$, for all s, t in $\operatorname{Ker}\varphi$. Moreover, if x is in $\varphi^{-1}(e)$ then there exists unique s in $\operatorname{Ker}\varphi$ with $s\bar{e} = x$, i.e., $\theta(s) = x$. Therefore, θ is an isomorphism.

THEOREM 10. *If $\bar{E}$ and E are the sets of idempotents of $\bar{S}$ and S, respectively, then $\varphi \mid \bar{E}$: $\bar{E} \to E$ is bijective. In particular, if S has no idempotents other than 1, then $\bar{S}$ has no idempotents other than $\bar{1}$.*

Proof. If e is in E, then $\varphi^{-1}(e)$ is a group and thus contains exactly one idempotent.

In the next few pages we deal with $\mathscr{L}$-, $\mathscr{R}$-, $\mathscr{H}$-, $\mathscr{D}$-, and $\mathscr{J}$-classes of a semigroup. Notation and terminology are as in Clifford and Preston [3].

LEMMA 11. *Let a, b be in S and $\bar{a}$, $\bar{b}$ in $\varphi^{-1}(a)$, $\varphi^{-1}(b)$, respectively. Then $a\mathscr{L}b$ if and only if $\bar{a}\mathscr{L}\bar{b}$, and similarly for $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$.*

Proof. The fact that $\bar{a}\mathscr{L}\bar{b}$ implies $a\mathscr{L}b$ is automatic algebraically, and likewise for $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$. All that is needed is that $\bar{S}$ and S be algebraic semigroups and that φ be an epimorphism. Conversely, let $a\mathscr{L}b$. Then $\bar{S}\bar{a} = \varphi^{-1}(S)\bar{a} = \varphi^{-1}(Sa) = \varphi^{-1}(Sb) = \varphi^{-1}(S)\bar{b} = \bar{S}\bar{b}$ gives $\bar{a}\mathscr{L}\bar{b}$. Symmetrically, $a\mathscr{R}b$ implies $\bar{a}\mathscr{R}\bar{b}$. As for $\mathscr{H}$-classes, we have $a\mathscr{H}b$ if and only if $a\mathscr{L}b$ and $a\mathscr{R}b$ if and only if $\bar{a}\mathscr{L}\bar{b}$ and $\bar{a}\mathscr{R}\bar{b}$ if and only if $\bar{a}\mathscr{H}\bar{b}$. As for $\mathscr{D}$-classes, we use the fact that for any semigroup S, $\mathscr{D} = \mathscr{L}\circ\mathscr{R}$, [3], page 47.

Thus, suppose $a\mathscr{D}b$. Then there is c in S with $a\mathscr{L}c$ and $c\mathscr{R}b$. If $\bar{c}$ is in $\varphi^{-1}(c)$, then $\bar{a}\mathscr{L}\bar{c}$ and $\bar{c}\mathscr{R}\bar{b}$, i.e., $\bar{a}\mathscr{D}\bar{b}$. Finally, for $\mathscr{J}$-classes we have $a\mathscr{J}b$ implies $\bar{S}\bar{a}\bar{S} = \varphi^{-1}(SaS) = \varphi^{-1}(SbS) = \bar{S}\bar{b}\bar{S}$, i.e., $\bar{a}\mathscr{J}\bar{b}$.

THEOREM 12. *φ induces a bijective correspondence between the $\mathscr{L}$ classes of $\bar{S}$ and the $\mathscr{L}$-classes of S. More precisely, if $\bar{a}$ is in $\bar{S}$ and $a = \varphi(\bar{a})$, then $\varphi^{-1}(L_a) = L_{\bar{a}}$. This holds similarly for R_a, H_a, D_a, and J_a.*

Proof. x is in $\varphi^{-1}(L_a)$ if and only if $\varphi(x)$ is in L_a if and only if $\varphi(x)\mathscr{L}a$ if and only if $x\mathscr{L}\bar{a}$ if and only if x is in $L_{\bar{a}}$. Similar proofs hold for R_a, H_a, D_a, and J_a.

COROLLARY 13. *φ induces a bijective correspondence between the maximal subgroups of $\bar{S}$ and the maximal subgroups of S. More precisely, if $\bar{H}$ is a maximal subgroup of $\bar{S}$, then $\varphi(\bar{H})$ is a maximal subgroup of S; if H is a maximal subgroup of S, then $\varphi^{-1}(H)$ is a maximal subgroup of $\bar{S}$.*

Proof. This is immediate if we observe that the maximal subgroups of a semigroup are precisely the $\mathscr{H}$-classes containing idempotents [3], p. 61.

Let S be a semigroup, H an $\mathscr{H}$-class of S, and s an element of S such that $sH \subseteqq H$. Then we denote by γ_s the element of $\Gamma(H)$, the left Schützenberger group [3] of H, such that $\gamma_s(x) = sx$, for all x in H. The following theorem generalizes Theorem 9.

THEOREM 14. *If H is an $\mathscr{H}$-class in S and $\bar{H} = \varphi^{-1}(H)$ is the corresponding $\mathscr{H}$-class in $\bar{S}$, then the left Schützenberger group $\Gamma(\bar{H})$ is an extension of $P(S)$ by the left Schützenberger group $\Gamma(H)$.*

Proof. Let $T(\bar{H})$ be the subsemigroup of $\bar{S}$ of all s in $\bar{S}$ with $s\bar{H} \subseteqq \bar{H}$, and let $T(H)$ be similar in S. Let $\bar{\nu}\colon T(\bar{H}) \to \Gamma(\bar{H})$ and $\nu\colon T(H) \to \Gamma(H)$ be the natural homomorphisms. It is straightforward to show that $\varphi^{-1}(T(H)) = T(\bar{H})$ and that φ induces epimorphisms $\varphi_H\colon T(\bar{H}) \to T(H)$ and $\varphi^H\colon \Gamma(\bar{H}) \to \Gamma(H)$ with $\varphi^H \circ \bar{\nu} = \nu \circ \varphi_H$. Moreover, $\operatorname{Ker}\varphi$ is contained in $T(\bar{H})$, and $\bar{\nu}(\operatorname{Ker}\varphi)$ is contained in $\operatorname{Ker}\varphi^H$. Thus $\bar{\nu}$ induces a homomorphism $\bar{\nu}_0\colon \operatorname{Ker}\varphi \to \operatorname{Ker}\varphi^H$. Since the image of $\bar{\nu}_0$ is the restriction of all the functions in $P(S)$ to $\bar{H}$, it follows that $\bar{\nu}_0$ is injective. We next show that $\bar{\nu}_0$ is surjective. Let ψ be in $\operatorname{Ker}\varphi^H$. There is s in $T(\bar{H})$ with $\psi = \bar{\nu}(s)$. Let $\bar{x}$ be in $\bar{H}$. If $\varphi(\bar{x}) = x$ in H, then $\varphi(s\bar{x}) = \varphi(s)x = \gamma_{\varphi(s)}(x) = [\nu \circ \varphi_H(s)](x) = [\varphi^H \circ \bar{\nu}(s)](x) = [\varphi^H(\psi)](x) = \gamma_1(x) = x = \varphi(\bar{x})$. Thus, there is t in $\operatorname{Ker}\varphi$

with $t\bar{x} = s\bar{x}$, and we have γ_t and γ_s in $\Gamma(\bar{H})$ agreeing at $\bar{x}$. But $\Gamma(\bar{H})$ is simply transitive on $\bar{H}$, and thus $\bar{\nu}_0(t) = \gamma_t = \gamma_s = \psi$, as desired.

We recall that an element a of a semigroup S is called *regular* if $axa = a$ for some x in S, and S is called *regular* if every element of S is regular. Moreover, a and b are *inverses* of each other if $aba = a$ and $bab = b$, and S is an *inverse semigroup* if every element of S has a unique inverse. The following are equivalent for an element a of a semigroup S: (1) the element a is regular, (2) the element a has an inverse b, (3) the principal left ideal generated by a has an idempotent generator, and (4) the principal right ideal generated by a has an idempotent generator [3], p. 27.

THEOREM 15. *If a is a regular element of S and $\bar{a}$ is in $\varphi^{-1}(a)$, then $\bar{a}$ is regular. Therefore, if S is regular then so is $\bar{S}$.*

Proof. Since a is regular, there is an idempotent e in S with $Se = Sa$. Let $\bar{e}$ be the idempotent in $\varphi^{-1}(e)$. Then $\bar{S}\bar{e} = \varphi^{-1}(Se) = \varphi^{-1}(Sa) = \bar{S}\bar{a}$, and thus $\bar{a}$ is regular.

THEOREM 16. *If S is an inverse semigroup, then so is $\bar{S}$.*

Proof. We recall that a semigroup is inverse if and only if every principal right ideal and every principal left ideal has a unique idempotent generator. Let S be an inverse semigroup. By the above theorem, every principal right ideal and every principal left ideal has at least one idempotent generator. Suppose $\bar{e}$ and $\bar{f}$ are idempotents in $\bar{S}$ with $\bar{S}\bar{e} = \bar{S}\bar{f}$. Then $\varphi(\bar{e})$ and $\varphi(\bar{f})$ are idempotents generating the same principal left ideal in S. Since S is an inverse semigroup, we have $\varphi(\bar{e}) = \varphi(\bar{f})$, which implies $\bar{e} = \bar{f}$, by Theorem 10. Principal right ideals are treated symmetrically.

THEOREM 17. *If I is a left ideal (right ideal) (ideal) in S, then $\varphi^{-1}(I)$ is a left ideal (right ideal) (ideal) in $\bar{S}$. If $\bar{I}$ is a left ideal (right ideal) (ideal) in $\bar{S}$, then $\varphi^{-1}\varphi(\bar{I}) = \bar{I}$. Therefore, φ induces a bijective, inclusion preserving correspondence between the left ideals (right ideals) (ideals) of $\bar{S}$ and those of S.*

Proof. Let I be a left ideal in S. Then $\bar{S}\varphi^{-1}(I) = \bar{\varphi}^{-1}(SI) \subseteq \varphi^{-1}(I)$, i.e., $\varphi^{-1}(I)$ is a left ideal in $\bar{S}$. Now, let x be in $\varphi^{-1}\varphi(\bar{I})$ where $\bar{I}$ is a left ideal in $\bar{S}$. There is y in $\bar{I}$ with $\varphi(x) = \varphi(y)$. So, there is s in $\operatorname{Ker}\varphi$ with $x = sy$ in $\bar{I}$. The proof for right ideals or ideals is similar.

COROLLARY 18. *If I is a minimal left ideal (right ideal) (ideal) in S, then $\varphi^{-1}(I)$ is a minimal left ideal (right ideal) (ideal) in $\bar{S}$.*

THEOREM 19. *If S has a minimal ideal K then $P(S) \cong P(K)$.*

Proof. By Proposition 1.9 of [1] we have that K is a retract of S, and thus K is connected and locally connected. Let $\bar{K} = \varphi^{-1}(K)$. By Corollary 18, $\bar{K}$ is the minimal ideal of $\bar{S}$ and, hence, is connected. By Lemma 10 of the previous section, $\bar{K}$ is simply connected. Then by Lemma 12 of that section $P(K) \cong P(S)$.

THEOREM 20. *Let S have a minimal ideal K. Moreover, let e be a primitive idempotent in K. Let $X = E(Se)$, $Y = E(eS)$ be the sets of idempotents in Se and eS, respectively, and let $G = eSe$, a maximal subgroup of K. Let $\sigma: Y \times X \rightarrow G$ such that $\sigma(y, x) = yx$. Let $\theta: [X, G, Y]_\sigma \rightarrow K$ be the canonical map, i.e., $\theta(x, g, y) = xgy$. Now, θ is an algebraic isomorphism and continuous [6]. If θ is also a homeomorphism, then X and Y are simply connected and thus $P(K) \cong P(G)$.*

Proof. From Proposition 1.9 of [1], p. 47, we have that K is a retract of S. Let $\bar{K} = \varphi^{-1}(K)$. By Lemma 10 of the previous section, $(\bar{K}, \varphi \mid \bar{K})$ is a simply connected covering space of K. The topological space structure of $[X, G, Y]_\sigma$ is $X \times G \times Y$ with the product topology. By Lemma 11 of the previous section and the fact that θ is a homeomorphism, X, G, and Y have simply connected covering spaces $(\bar{X}, \varphi_1)$, $(\bar{G}, \varphi_2)$, and $(\bar{Y}, \varphi_3)$. By Theorem 3, $([\bar{X}, \bar{G}, \bar{Y}]_{\bar{\sigma}}, \varphi')$ is a simply connected covering paragroup of $[X, G, Y]_\sigma$, where $\varphi' = \varphi_1 \times \varphi_2 \times \varphi_3$. In lifting σ to $\bar{\sigma}$ we

$$\begin{array}{ccc} \bar{Y} \times \bar{X} & \xrightarrow{\bar{\sigma}} & \bar{G} \\ {\scriptstyle \varphi_3 \times \varphi_1}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_2} \\ Y \times X & \xrightarrow{\sigma} & G \end{array}$$

can choose $\bar{\sigma}$ such that $\bar{\sigma}(\bar{e}_3, \bar{e}_1) = \bar{e}_2$, where $\bar{e}_2$ is the identity of $\bar{G}$ and $\bar{e}_3$ and $\bar{e}_1$ are fixed in $\bar{Y}$ and $\bar{X}$, respectively, such that $\varphi_3(\bar{e}_3) = e$ and $\varphi_1(\bar{e}_1) = e$.

Now $\theta \circ \varphi'(\bar{e}_1, \bar{e}_2, \bar{e}_3) = \theta(e, e, e) = e^3 = e = (\varphi \mid K)(\bar{e})$, where $\bar{e}$ is the idempotent of $\bar{K}$ such that $\varphi(\bar{e}) = e$. By Theorem 2, we can lift θ to a semigroup and covering space isomorphism $\bar{\theta}$ so that $\bar{\theta}(\bar{e}_1, \bar{e}_2, \bar{e}_3) = \bar{e}$ and $(\varphi \mid \bar{K}) \circ \bar{\theta} = \theta \circ \varphi'$.

$$\begin{array}{ccc} [\bar{X}, \bar{G}, \bar{Y}]_{\bar{\sigma}} & \xrightarrow{\bar{\theta}} & \bar{K} \\ \varphi' \downarrow & & \downarrow \varphi \mid \bar{K} \\ [X, G, Y]_{\sigma} & \xrightarrow{\theta} & K \end{array}$$

We now show that all the elements of $\bar{X} \times \bar{e}_2 \times \bar{e}_3$ are idempotent. Now, $\varphi_2(\bar{\sigma}(\bar{e}_3 \times \bar{X})) = \sigma((\varphi_3 \times \varphi_1)(\bar{e} \times \bar{X})) = \sigma(e \times X) = eX = e$, since X is a left zero semigroup. This means that $\bar{\sigma}(\bar{e}_3 \times \bar{X})$ is a connected subset of the discrete set Ker φ_2. Moreover, $\bar{e}_2 = \bar{\sigma}(\bar{e}_3, \bar{e}_1)$ is in $\bar{\sigma}(\bar{e}_3 \times \bar{X})$. Therefore, $\bar{\sigma}(\bar{e}_3 \times \bar{X}) = \{\bar{e}_2\}$. Thus, if x is in $\bar{X}$, then $(x, \bar{e}_2, \bar{e}_3)^2 = (x, \bar{e}_2\bar{\sigma}(\bar{e}_3, x)\bar{e}_2, \bar{e}_3) = (x, \bar{e}_2^3, \bar{e}_3) = (x, \bar{e}_2, \bar{e}_3)$, as desired.

We show that $\varphi_1: \bar{X} \to X$ is one-to-one. Let x_1, x_2 be in $\bar{X}$ with $\varphi_1(x_1) = \varphi_1(x_2)$. Then $\varphi(\bar{\theta}(x_i, \bar{e}_2, \bar{e}_3)) = (\varphi \mid \bar{K}) \circ \bar{\theta}(x_i, \bar{e}_2, \bar{e}_3) = \theta \circ \varphi'(x_i, \bar{e}_2, \bar{e}_3) = \theta(\varphi_1(x_i), e, e) = \varphi_1(x_i)ee = \varphi_1(x_i)$, $i = 1, 2$, since $\varphi_1(x_i)$ and e are in X, a left zero semigroup. Hence, $\varphi(\bar{\theta}(x_1, \bar{e}_2, \bar{e}_3)) = \varphi_1(x_1) = \varphi_1(x_2) = \varphi(\bar{\theta}(x_2, \bar{e}_2, \bar{e}_3))$. Since $(x_1, \bar{e}_2, \bar{e}_3)$ and $(x_2, \bar{e}_2, \bar{e}_3)$ are idempotents, so are $\bar{\theta}(x_1, \bar{e}_2, \bar{e}_3)$ and $\bar{\theta}(x_2, \bar{e}_2, \bar{e}_3)$. By Theorem 10, $\bar{\theta}(x_1, \bar{e}_2, \bar{e}_3) = \bar{\theta}(x_2, \bar{e}_2, \bar{e}_3)$. Hence, $(x_1, \bar{e}_2, \bar{e}_3) = (x_2, \bar{e}_2, \bar{e}_3)$ and $x_1 = x_2$.

Therefore, X is simply connected, and symmetrically, Y is simply connected. Moreover, $P(K) \cong P(X \times G \times Y) \cong P(X) \times P(G) \times P(Y) \cong P(G)$.

Let $(\bar{G}, \beta)$ be a simply connected covering group of a compact Lie group G. It is known [4] that the following are equivalent: (a) G is semisimple, (b) $P(G)$ is finite, (c) $\bar{G}$ is compact. The following corollary follows easily.

COROLLARY 21. *Using the hypotheses and notation of Theorem* 20 *and assuming that S is compact and that G is a Lie group, we have that the following are equivalent*: (a) *G is semisimple*, (b) *$P(S)$ is finite*, (c) *$\bar{S}$ is compact.*

3. Applications and examples.

(A) *Semigroups on the cylinder.* Mostert and Shields [9] proved that a topological semigroup on the plane with an identity and no other idempotents must be a group. The cylinder can be handled as follows.

THEOREM. *Let S be a topological semigroup with identity* 1 *and with the cylinder $S^1 \times R$ as topological space structure. Here R is the line and $S^1 = \{(x, y): (x, y)$ in R^2 and $x^2 + y^2 = 1\}$. If S has no idempotents other than* 1, *then S is a group.*

Proof. S has a simply connected covering semigroup $(\bar{S}, \varphi)$ with identity $\bar{1}$ and space the plane. Moreover, $\bar{S}$ has no other idempotents.

By Mostert and Shields, $\bar{S}$ is a group. Being the homomorphic image of a group, S is a group.

(B) *A non-locally connected example.* In this section we discuss one type of cylindrical semigroup [6], p. 67. Following [6], we define $H = [0, \infty)$ and $H^* = [0, \infty]$, both under addition.

THEOREM 1. *Let $(\bar{A}, \varphi)$ be a covering group of the group A, and let $f: H \rightarrow A$ be a continuous homomorphism. Define $f^+: H \rightarrow H^* \times A$ by $f^+(p) = (p, f(p))$. Since H is simply connected, there exists a unique homomorphism $\bar{f}: H \rightarrow \bar{A}$ such that $\varphi \circ \bar{f} = f$. Now define $\bar{f}^+: H \rightarrow H^* \times \bar{A}$ by $\bar{f}^+(p) = (p, \bar{f}(p))$. Let $S = f^+(H) \cup \infty \times A$ and $\bar{S} = \bar{f}^+(H) \cup \infty \times \bar{A}$.*

Then S and $\bar{S}$ are closed subsemigroups of $H^* \times A$ and $H^* \times \bar{A}$, respectively, and $\bar{f}^+(H)$ is the component of $(1 \times \varphi)^{-1}(f^+(H))$ that contains $(0, \bar{1})$, where $1 \times \varphi: H^* \times \bar{A} \rightarrow H^* \times A$. Moreover, $(\bar{S}, (1 \times \varphi) \mid \bar{S})$ is a sort of "not necessarily connected (at most two components) covering semigroup" of S in the sense that $(\bar{f}^+(H), (1 \times \varphi) \mid \bar{f}^+(H))$ is a trivial covering semigroup of $f^+(H)$ and $(\infty \times \bar{A}, (1 \times \varphi) \mid \infty \times \bar{A})$ is a covering semigroup of $\infty \times A$.

Proof. The fact that S and $\bar{S}$ are closed subsemigroups of $H^* \times A$ and $H^* \times \bar{A}$ follows as in [6], as does the fact that $f^+(H)$ and $\bar{f}^+(H)$ are copies of H as subsemigroups of S and $\bar{S}$. Observing that $(1 \times \varphi) \circ \bar{f}^+ = f^+$, we have that $\bar{f}^+(H)$ is a connected subsemigroup of $(1 \times \varphi)^{-1}(f^+(H))$. Let C be the component of $(1 \times \varphi)^{-1}(f^+(H))$ containing $\bar{f}^+(H)$. Then $(C, (1 \times \varphi) \mid C)$ is a covering semigroup of the simply connected $f^+(H)$. Thus C is a copy of H, and we must have $\bar{f}^+(H) = C$. The rest of the theorem is now obvious.

THEOREM 2. *Let A be a connected topological group and $f: H \rightarrow A$ a continuous homomorphism. Define $f^+: H \rightarrow H^* \times A$ and S as in Theorem 1. Then S is not connected if and only if f is an imbedding onto a closed subset of A.*

Proof. S is not connected if and only if $f^+(H)$ is closed in S and, therefore, if and only if $f^+(H)$ is closed in $H^* \times A$. This means that for each point a in A, there is a p_a in H and a neighborhood N_a of a such that $(p_a, \infty] \times N_a$ is disjoint from $f^+(H)$, i.e., $(p, f(p))$ is not in $(p_a, \infty] \times N_a$ for all p in H. Thus, S is not connected is equivalent to the existence of a neighborhood N_a of each point a of A such that $f(p)$ is not in N_a for sufficiently large p. This last is equivalent to $f(H)$ being closed in A and the local finiteness of the collection of all sets of the form $f([k, k + 1])$, k a non-negative integer. The remainder of the proof is straightforward.

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